CHAPTER IV

THE APPLICATION OF HYPERCOMPLEX NUMBER SYSTEMS

Hypercomplex numbers may be used to prove the following identities for real numbers.

$$(4.1)\quad (a_0^2 + a_1^2)(b_0^2 + b_1^2) = (a_0b_0 - a_1b_1)^2 + (a_1b_0 + a_0b_1)^2$$

(10, p. 224)

$$(4.2)\quad (a_0^2 + a_1^2 + a_2^2 + a_3^2)(b_0^2 + b_1^2 + b_2^2 + b_3^2)$$
$$\begin{aligned} = {} & (a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3)^2 \\ & +(a_1b_0 + a_0b_1 - a_3b_2 + a_2b_3)^2 \\ & +(a_2b_0 + a_3b_1 + a_0b_2 - a_1b_3)^2 \\ & +(a_3b_0 - a_2b_1 + a_1b_2 + a_0b_3)^2 \end{aligned}$$

(10, p.277)

$$(4.3)\quad (a_0^2 + a_1^2 + \ldots\ldots a_7^2)(b_0^2 + b_1^2 + \ldots\ldots b_7^2)$$
$$\begin{aligned} = {} & (a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3 - a_4b_4 - a_5b_5 - a_6b_6 - a_7b_7)^2 \\ & + (a_1b_0 + a_0b_1 - a_3b_2 + a_2b_3 - a_5b_4 + a_4b_5 - a_7b_6 + a_6b_7)^2 \\ & + (a_2b_0 + a_3b_1 + a_0b_2 - a_1b_3 + a_6b_4 - a_7b_5 - a_4b_6 + a_5b_7)^2 \\ & + (a_3b_0 - a_2b_1 + a_1b_2 + a_0b_3 - a_7b_4 - a_6b_5 + a_5b_6 + a_4b_7)^2 \\ & + (a_4b_0 + a_5b_1 - a_6b_2 + a_7b_3 + a_0b_4 - a_1b_5 + a_2b_6 - a_3b_7)^2 \\ & + (a_5b_0 - a_4b_1 + a_7b_2 + a_6b_3 + a_1b_4 + a_0b_5 - a_3b_6 - a_2b_7)^2 \\ & + (a_6b_0 + a_7b_1 + a_4b_2 - a_5b_3 - a_2b_4 + a_3b_5 + a_0b_6 - a_1b_7)^2 \\ & + (a_7b_0 - a_6b_1 - a_5b_2 - a_4b_3 + a_3b_4 + a_2b_5 + a_1b_6 + a_0b_7)^2 \end{aligned}$$

1. Proof of 4.1 with the help of complex numbers. Let

Let $x = a_0 + ia_1$ and $y = b_0 + ib_1$ be two complex numbers, then their complex conjugates are $\bar{x} = a_0 - ia_1$ and $\bar{y} = b_0 - ib_1$.

$$\text{Since}\quad (a_0^2 + a_1^2)(b_0^2 + b_1^2) = |X|^2\,|Y|^2$$
$$= (X\bar{X})(Y\bar{Y}) \quad \ldots\ldots\ldots (\text{ch.II})$$
$$= X(\bar{X}Y)\bar{Y}$$
$$= X(Y\bar{X})\bar{Y}$$
$$= (XY)\overline{(XY)}$$
$$= |XY|^2$$
$$\text{Therefore}\quad (a_0^2 + a_1^2)(b_0^2 + b_1^2) = (a_0b_0 - a_1b_1)^2 + (a_1b_0 + a_0b_1)^2$$

Other equivalent formulae may also be obtained by changing the sign of one or more of the numbers, a_0, a_1, b_0, b_1.

For example, if b_1 is replaced by $(-b_1)$ we otain

$$(a_0^2 + a_1^2)(b_0^2 + b_1^2) = (a_0b_0 + a_1b_1)^2 + (a_1b_0 - a_0b_1)^2.$$

2. Proof of (4.2) with the help of quaternions.

Let $x = a_0 + ia_1 + ja_2 + ka_3$ and $y = b_0 + ib_1 + jb_2 + kb_3$ be two quaternions, then their quaternion conjugates are

$$\bar{x} = a_0 - ia_1 - ja_2 - ka_3 \quad\text{and}\quad \bar{y} = b_0 - ib_1 - jb_2 - kb_3\,.$$

$$\text{Since}\ (a_0^2 + a_1^2 + a_2^2 + a_3^2)(b_0^2 + b_1^2 + b_2^2 + b_3^2) = |X|^2\,|Y|^2$$
$$= (X\bar{X})(Y\bar{Y}) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\text{ch. II})$$
$$= \{Y(X\bar{X})\}\bar{Y} \quad \ldots\ldots\ldots\ldots (\text{commute with real numbers})$$
$$= \{(YX)(\bar{X}\bar{Y})\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (\text{associative})$$
$$= (YX)\overline{(YX)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (\text{ch. II})$$
$$= |YX|^2$$

$$\text{Therefore}\quad (a_0^2 + \ldots\ldots\ldots a_3^2)(b_0^2 + \ldots\ldots\ldots\ldots b_3^2)$$
$$= (a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3)^2$$
$$+ (a_1b_0 + a_0b_1 - a_3b_2 + a_2b_3)^2$$
$$+ (a_2b_0 + a_3b_1 + a_0b_2 - a_1b_3)^2$$
$$+ (a_3b_0 - a_2b_1 + a_1b_2 + a_0b_3)^2$$

Other equivalent formulae may also be obtained by changing the sign of one or more of the numbers.

For example, if b_1, b_2, b_3 are replaced by $-b_1$, $-b_2$, $-b_3$, we get

$$\begin{aligned} |X|^2 |Y|^2 &= (a_0b_0 + a_1b_1 + a_2b_2 + a_3b_3)^2 \\ &+ (a_1b_0 - a_0b_1 + a_3b_2 - a_2b_3)^2 \\ &+ (a_2b_0 - a_3b_1 - a_0b_2 + a_1b_3)^2 \\ &+ (a_3b_0 + a_2b_1 - a_1b_2 - a_0b_3)^2 \end{aligned}$$

3. Proof of (4.3) with the help of Cayley numbers.

Let $x = \sum_0^7 e_i a_i$ and $y = \sum_0^7 e_i b_i$ be two Cayley numbers, then their conjugates are $\bar{x} = e_0 a_0 - \sum_1^7 e_i a_i$ and $\bar{y} = e_0 b_0 - \sum_1^7 e_i b_i$.

First we have, $\sum_0^7 a_1^2 \sum_0^7 b_2^2 = |X|^2 |Y|^2$

$$= (X\bar{X})(Y\bar{Y}) = \left\{(X\bar{X})\, Y\right\} \bar{Y}$$

$$= \left\{Y\,(X\bar{X})\right\} \bar{Y} ;$$

Note:

The last two steps depend on the fact that $X\bar{X}$ is real and that the associative and commutative laws hold for the multiplication of $X\bar{X}$ with Cayley numbers.

$$\begin{aligned} Y\,(X\bar{X})^{*} &= \sum_0^7 e_i b_i \left[\sum_0^7 e_i a_i \left(e_0 a_0 - \sum_1^7 e_i a_i\right)\right] \\ &= \sum_0^7 e_i b_i \left[\sum_0^7 e_i a_i \, e_0 a_0 - \sum_0^7 e_i a_i \sum_1^7 e_i a_i\right] \\ &= \left[\sum_0^7 e_i b_i \left(\sum_0^7 e_i a_i \, e_0 a_0\right)\right] - \left[\sum_0^7 e_i b_i \left(\sum_0^7 e_i a_i \sum_1^7 e_i a_i\right)\right] \\ &= \left[\left(\sum_0^7 e_i b_i \sum_0^7 e_i a_i\right) e_0 a_0\right] - \left[\left(\sum_0^7 e_i b_i \sum_0^7 e_i a_i\right) \sum_1^7 e_i a_i\right]^{*} \\ &= \left(\sum_0^7 e_i b_i \sum_0^7 e_i a_i\right) \left(e_0 a_0 - \sum_1^7 e_i a_i\right) \\ &= (Y X)\, \bar{X} \end{aligned}$$

$$\left\{(Y\,X)\,X\right\}\bar{Y} = \left\{\left(\sum_0^7 e_ib_i \sum_0^7 e_ia_i\right)\left(e_oa_o - \sum_1^7 e_ia_i\right)\right\}\left(e_ob_o - \sum_1^7 e_ib_i\right)$$

$$= \left\{\left(\sum_0^7 e_ib_i \sum_0^7 e_ia_i\right) e_oa_o - \left(\sum_0^7 e_ib_i \sum_0^7 e_ia_i\right) \sum_1^7 e_ia_i\right\}$$

$$\left(e_ob_o - \sum_1^7 e_ib_i\right)$$

$$= \left\{\left(\sum_0^7 e_ib_i \sum_0^7 e_ia_i\right)e_oa_o\right\} e_ob_o - \left\{\left(\sum_0^7 e_ib_i \sum_0^7 e_ia_i\right)\sum_1^7 e_ia_i\right\}$$

$$e_ob_o - \left\{\left(\sum_0^7 e_ib_i \sum_0^7 e_ia_i\right)e_oa_o\right\}\sum_1^7 e_ib_i + \left\{\left(\sum_0^7 e_ib_i \sum_0^7 e_ia_i\right)\right.$$

$$\left.\sum_1^7 e_ia_i\right\}\sum_1^7 e_ib_i$$

$$= \left(\sum_0^7 e_ib_i \sum_0^7 e_ia_i\right)\left(e_oa_oe_ob_o\right) - \left(\sum_0^7 e_ib_i \sum_0^7 e_ia_i\right)\left(\sum_1^7 e_ia_i\; e_ob_o\right)$$

$$- \left(\sum_0^7 e_ib_i \sum_0^7 e_ia_i\right)\left(e_oa_o \sum_1^7 e_ib_i\right) + \left(\sum_0^7 e_ib_i \sum_0^7 e_ia_i\right)\left(\sum_1^7 e_ia_i \sum_1^7 e_ib_i\right)^{*\,**}$$

$$= \left(\sum_0^7 e_ib_i \sum_0^7 e_ia_i\right)\left[\left(e_oa_o\, e_ob_o\right) - \left(\sum_1^7 e_ia_i\; e_ob_o\right) - \left(e_oa_o \sum_1^7 e_ib_i\right)\right.$$

$$\left.+ \left(\sum_1^7 e_ia_i \sum_1^7 e_ib_i\right)\right\}$$

$$= \left(\sum_0^7 e_ib_i \sum_0^7 e_ia_i\right)\left\{\left(e_oa_o - \sum_1^7 e_ia_i\right)\left(e_ob_o - \sum_1^7 e_ib_i\right)\right\}$$

$$= (Y\,X)\,(\bar{X}\,\bar{Y})$$

$$= (Y\,X)\,(\overline{Y\,X})$$

$$= |Y\,X|^2 = |X\,Y|^2$$

Note:

$*$, $**$, see the following lemma 1 and 2.

$$|X\,Y|^2 = (a_ob_o - a_1b_1 - a_2b_2 - a_3b_3 - a_4b_4 - a_5b_5 - a_6b_6 - a_7b_7)^2$$

$$+ (a_1b_o + a_ob_1 - a_3b_2 + a_2b_3 - a_5b_4 + a_4b_5 - a_7b_6 + a_6b_7)^2$$

$$+ (a_2b_o + a_3b_1 + a_ob_2 - a_1b_3 + a_6b_4 - a_7b_5 - a_4b_6 + a_5b_7)^2$$

$$+ (a_3b_o - a_2b_1 + a_1b_2 + a_ob_3 - a_7b_4 - a_6b_5 + a_5b_6 + a_4b_7)^2$$

$$+ (a_4b_o + a_5b_1 - a_6b_2 + a_7b_3 + a_ob_4 - a_1b_5 + a_2b_6 - a_3b_7)^2$$

$$+ (a_5b_o - a_4b_1 + a_7b_2 + a_6b_3 + a_1b_4 + a_ob_5 - a_3b_6 - a_2b_7)^2$$

$$+ (a_6b_o + a_7b_1 + a_4b_2 - a_5b_3 - a_2b_4 + a_3b_5 + a_ob_6 - a_1b_7)$$

$$+ (a_7b_o - a_6b_1 - a_5b_2 - a_4b_3 + a_3b_4 + a_2b_5 + a_1b_6 + a_ob_7)^2$$

[redacted] ....

I. Lemma 1: $\sum_{0}^{7} e_i b_i \left(\sum_{0}^{7} e_i a_i \sum_{1}^{7} e_i a_i\right) = \left(\sum_{0}^{7} e_i b_i \sum_{0}^{7} e_i a_i\right) \sum_{1}^{7} e_i a_i$

Proof

Left handside $= \sum_{0}^{7} e_i b_i \left(\sum_{0}^{7} e_i a_i \sum_{1}^{7} e_i a_i\right)$

$$= \sum_{0}^{7} e_i b_i \left(-\sum_{1}^{7} a_i^2 + a_0 \sum_{1}^{7} e_i a_i\right)$$

$$= -\sum_{1}^{7} a_i^2 \left(\sum_{0}^{7} e_i b_i\right) + a_0 \left(\sum_{0}^{7} e_i b_i \sum_{1}^{7} e_i a_i\right)$$

$$= -\sum_{1}^{7} a_i^2 \left(\sum_{0}^{7} e_i b_i\right) + a_0 \left\{-\sum_{1}^{7} b_i a_i + b_0 \sum_{1}^{7} e_i a_i + \sum_{1}^{7} e_i A_i\right\}$$

where $A_1 = (-b_3a_2 + b_2a_3 - b_7a_6 + b_6a_7 - b_5a_4 + b_4a_5)$

$A_2 = (b_3a_1 - b_1a_3 - b_7a_5 + b_5a_7 + b_6a_4 - b_4a_6)$

$A_3 = (-b_2a_1 + b_1a_2 - b_7a_4 + b_4a_7 - b_6a_5 + b_5a_6)$

$A_4 = (b_5a_1 - b_1a_5 - b_6a_2 + b_2a_6 + b_7a_3 - b_3a_7)$

$A_5 = (-b_4a_1 + b_1a_4 + b_6a_3 - b_3a_6 + b_7a_2 - b_2a_7)$

$A_6 = (b_4a_2 - b_2a_4 - b_5a_3 + b_3a_5 + b_7a_1 - b_1a_7)$

$A_7 = (-b_4a_3 + b_3a_4 - b_5a_2 + b_2a_5 - b_6a_1 + b_1a_6)$

$$= -\sum_{1}^{7} a_i^2 \left(\sum_{0}^{7} e_i b_i\right) - a_0 \sum_{1}^{7} b_i a_i + a_0 b_0 \left(\sum_{1}^{7} e_i a_i\right) + a_0 \left(\sum_{1}^{7} e_i A_i\right)$$

Right handside $= \left(\sum_{0}^{7} e_i b_i \sum_{0}^{7} e_i a_i\right) \sum_{1}^{7} e_i a_i$

$$= \left\{a_0 b_0 - \sum_{1}^{7} b_i a_i + b_0 \sum_{1}^{7} e_i a_i + \sum_{1}^{7} e_i A_i + a_0 \sum_{1}^{7} e_i b_i\right\} \sum_{1}^{7} e_i a_i$$

$$= a_0 b_0 \left(\sum_{1}^{7} e_i a_i\right) - \sum_{1}^{7} b_i a_i \left(\sum_{1}^{7} e_i a_i\right) - b_0 \sum_{1}^{7} a_i^2 + \sum_{1}^{7} e_i A_i \sum_{1}^{7} e_i a_i$$

$$- a_0 \sum_{1}^{7} b_i a_i + a_0 \sum_{1}^{7} e_i A_i$$

$$= a_0 b_0 \left(\sum_{1}^{7} e_i a_i\right) - a_0 \sum_{1}^{7} b_i a_i + \left\{\left(\sum_{1}^{7} e_i A_i \sum_{1}^{7} e_i a_i + a_0 \sum_{1}^{7} e_i A_i\right.\right.$$

$$\left. - \sum_{1}^{7} b_i a_i \left(\sum_{1}^{7} e_i a_i\right) - b_0 \sum_{1}^{7} a_i^2 \right\}$$

$$= a_o b_o \left(\sum_1^7 e_i a_i\right) - a_o \sum_1^7 b_i a_i + \Big\{ \sum_1^7 e_i A_i \sum_o^7 e_i a_i$$

$$- \sum_1^7 b_i a_i \sum_1^7 e_i a_i - b_o \sum_1^7 a_i^2 \Big\}$$

$$= a_o b_o \left(\sum_1^7 e_i a_i\right) - a_o \sum_1^7 b_i a_i + \Big\{ a_o \sum_1^7 e_i A_i - \sum_1^7 a_i^2 \sum_o^7 e_i b_i \Big\}$$

$$= - \sum_1^7 a_i^2 \left(\sum_o^7 e_i b_i\right) - a_o \sum_1^7 b_i a_i + a_o b_o \left(\sum_1^7 e_i a_i\right) + a_o \left(\sum_1^7 e_i A_i\right)$$

Therefore $\sum_o^7 e_i b_i \left(\sum_o^7 e_i a_i \sum_1^7 e_i a_i\right) = \left(\sum_o^7 e_i b_i \sum_o^7 e_i a_i\right) \sum_1^7 e_i a_i$

Lemma 2: $\Big\{ \left(\sum_o^7 e_i b_i \sum_o^7 e_i a_i\right) \sum_1^7 e_i a_i \Big\} \sum_1^7 e_i b_i$

$$= \left(\sum_o^7 e_i b_i \sum_o^7 e_i a_i\right)\left(\sum_1^7 e_i a_i \sum_1^7 e_i b_i\right)$$

Proof

Left handside $= \Big\{ - \sum_1^7 a_i^2 \left(\sum_o^7 e_i b_i\right) - a_o \sum_1^7 b_i a_i + a_o b_o \left(\sum_1^7 e_i a_i\right) + a_o \left(\sum_1^7 e_i A_i\right) \Big\} \sum_1^7 e_i b_i$

$$= - a_o \sum_1^7 b_i a_i \left(\sum_1^7 e_i b_i\right) - \sum_1^7 a_i^2 \left(\sum_o^7 e_i b_i \sum_1^7 e_i b_i\right) +$$

$$a_o b_o \left(\sum_1^7 e_i a_i \sum_1^7 e_i b_i\right) + a_o \left(\sum_1^7 e_i A_i \sum_1^7 e_i b_i\right)$$

$$= - a_o \sum_1^7 b_i a_i \left(\sum_1^7 e_i b_i\right) - \sum_1^7 a_i^2 \left(b_o \sum_1^7 e_i b_i - \sum_1^7 b_i^2\right) +$$

$$a_o b_o \left(- \sum_1^7 a_i b_i - \sum_1^7 e_i A_i\right) + a_o \left(\sum_1^7 e_i A_i \sum_1^7 e_i b_i\right)$$

Right handside $= \left(a_o b_o + b_o \sum_1^7 e_i a_i + a_o \sum_1^7 e_i b_i - \sum_1^7 b_i a_i + \sum_1^7 e_i A_i\right)$

$$\Big\{ - \sum_1^7 a_i b_i - \sum_1^7 e_i A_i \Big\}$$

$$= a_o b_o \left(- \sum_1^7 a_i b_i - \sum_1^7 e_i A_i\right) - a_o \sum_1^7 b_i a_i \left(\sum_1^7 e_i b_i\right)$$

$$- a_o \sum_1^7 e_i b_i \sum_1^7 e_i A_i + \Big(- b_o \sum_1^7 b_i a_i \sum_1^7 e_i a_i - b_o \sum_1^7 e_i a_i$$

$$\sum_1^7 e_i A_i\Big) + \Big[\left(\sum_1^7 b_i a_i\right)^2 - \left(\sum_1^7 e_i A_i\right)^2 \Big]$$

$$= a_0 b_0 \left(-\sum_1^7 a_i b_i - \sum_1^7 e_i A_i\right) - a_0 \sum_1^7 b_i a_i \left(\sum_1^7 e_i b_i\right)$$

$$- a_0 \sum_1^7 e_i b_i \sum_1^7 e_i A_i + \left(b_0^2 \sum_0^7 a_i^2 - b_0 \sum_1^7 a_i^2 \sum_0^7 e_i b_i\right)$$

$$+ \left\{\left(\sum_1^7 b_i a_i\right)^2 - \sum_1^7 A_i^2\right\}$$

$$= a_0 b_0 \left(-\sum_1^7 a_i b_i - \sum_1^7 e_i A_i\right) - a_0 \sum_1^7 b_i a_i \left(\sum_1^7 e_i b_i\right)$$

$$- a_0 \sum_1^7 e_i b_i \sum_1^7 e_i A_i + \left(b_0 \sum_1^7 a_i^2 \sum_1^7 e_i b_i\right) + \left(\sum_1^7 a_i^2 \sum_1^7 b_i^2\right)$$

$$= - a_0 \sum_1^7 b_i a_i \left(\sum_1^7 e_i b_i\right) - \sum_1^7 a_i^2 \left(b_0 \sum_1^7 e_i b_i - \sum_1^7 b_i^2\right)$$

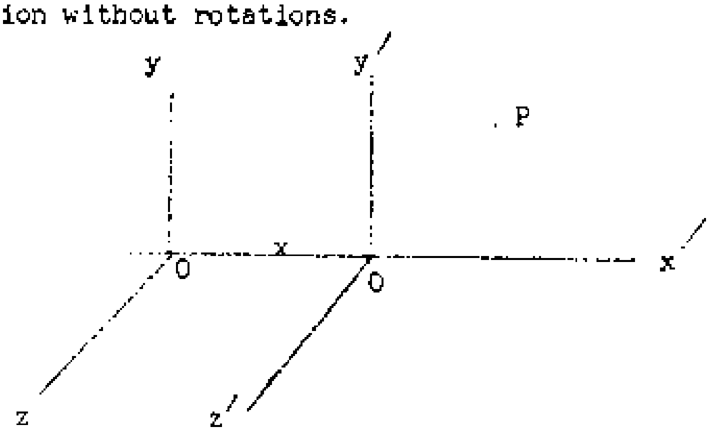
$$+ a_0 b_0 \left(-\sum_1^7 a_i b_i - \sum_1^7 e_i A_i\right) + a_0 \left(\sum_1^7 e_i A_i \sum_1^7 e_i b_i\right)$$

Left handside = Right handside

Note : An identity for $(a_0^2 + a_1^2 + \ldots\ldots + a_{n-1}^2)(b_0^2 + b_1^2 + \ldots + b_{n-1}^2)$ similar to the identities (4.1) , (4.2) and (4.3) cannot be found for every n, but only for n = 1, 2, 4, 8 . (13, p. 100-125).

The Lorentz transformation of Special Relativity in Quaternion Forms.

Let (x, y, z, t) be the cartesian coordinates and the time of an event P in the Galilean reference frame S, and let (x', y', z', t') be the coordinates in another reference frame S', where the x, y, and z axes of S coincide with the x', y', and z' axes of S' and $t = t'$ where $t = 0$, and S' is moving with a velocity v relative to S in the x direction without rotations.



The Galilean transformation is the mapping which maps a point (x, y, z, t) in S onto the point (x', y', z', t') in S' where

$$x' = x - vt$$

$$y' = y$$

$$z' = z$$

$$t' = t$$

This transformation is used in Newtonian mechanics. According to the relativity principle, the transformation equations are

$$x' = \frac{x - vt}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}}$$ where C is the speed of light = 300,000 km/sec.

$$y' = y$$

$$z' = z$$

$$t' = \frac{t - \frac{vx}{C^2}}{\left(1 - \frac{v^2}{C^2}\right)^{\frac{1}{2}}}$$

These transformation equation are called Lorentz equations.

Let $$\frac{1}{\left(1 - \frac{v^2}{C^2}\right)^{\frac{1}{2}}} = \gamma$$

$$x' = \gamma(x - vt)$$

$$y' = y$$

$$z' = z$$

$$t' = \gamma\left(t - \frac{vx}{C^2}\right)$$

Now let $\tau = i\,C\,t$, where i is $\sqrt{-1}$

Then $$x' = \gamma\left(x - \frac{v\tau}{iC}\right) = \gamma\left(x + \frac{iv\tau}{C}\right)$$

$$y' = y$$

$$z' = z$$

$$\tau' = \gamma\left(\frac{\tau}{iC} - \frac{vx}{C^2}\right) iC$$

$$= \gamma\left(\tau - \frac{vxi}{C}\right)$$

This can be written

$$\begin{pmatrix} x' \\ y' \\ z' \\ \tau' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & \frac{\gamma vi}{C} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{\gamma vi}{C} & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ \tau \end{pmatrix} \quad \ldots.(1)$$

The Lorentz transformation can be viewed as a rotation in four dimensional space from the point (x, y, z, τ) to (x', y', z', τ') (12, p.11,12) and rotations in fourdimensional space can be represented by a transformation of quaternions of the form $Q' = A\,Q\,B$, (3, p. 68) where the point represented by the components of Q is transformed into the point represented by the components of Q'. Therefore we shall try to find the quaternions A and B that represent the Lorentz transformation (I).

Consider the transformation of quaternions

$$Q' = A\,Q\,B \quad \text{where}$$

$$A = a + e_1 b + e_2 c + e_3 d$$

$$B = p + e_1 q + e_2 r + e_3 s$$

$$Q = x + e_1 y + e_2 z + e_3 \tau$$

$$Q' = x' + e_1 y' + e_2 z' + e_3 \tau'$$

the transformation is

$$x' + e_1 y' + e_2 z' + e_3 \tau' = (a + e_1 b + e_2 c + e_3 d)(x + e_1 y + e_2 z + e_3 \tau)$$

$(p + e_1 q + e_2 r + e_3 s)$ or

$$\begin{pmatrix} x' \\ y' \\ z' \\ \tau' \end{pmatrix} = \begin{pmatrix} E_1 & E_2 & E_3 & E_4 \\ E_5 & E_6 & E_7 & E_8 \\ E_9 & E_{10} & E_{11} & E_{12} \\ E_{13} & E_{14} & E_{15} & E_{16} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ \tau \end{pmatrix} \quad \ldots\ldots \text{II}$$

Where $E_1, E_2, \ldots\ldots\, E_{16}$ are elements in matrix of transformation in (II) which is similar to (I). Then we have,

$$E_1 = ap - bq - cr - ds = \gamma ,$$

$$E_2 = -bp - aq - dr + cs = 0 ,$$

$$E_3 = -cp + dq - ar - bs = 0 ,$$

$$E_4 = -dp - cq + br - as = \frac{\gamma v i}{c} ,$$

$$E_5 = bp + aq - dr + cs = 0 ,$$

$$E_6 = ap - bq + cr + ds = 1 .$$

$$E_7 = -dp - cq - br + as = 0 ,$$

$$E_8 = cp - dq - ar - bs = 0 ,$$

$$E_9 = cp + dq + ar - bs = 0 ,$$

$$E_{10} = dp - cq - br - as = 0 ,$$

$$E_{11} = ap + bq - cr + ds = 1,$$

$$E_{12} = -bp + aq - dr - cs = 0,$$

$$E_{13} = dp - cq + br + as = \frac{-\gamma vi}{C},$$

$$E_{14} = -cp - dq + ar - bs = 0,$$

$$E_{15} = bp - aq - dr - cs = 0,$$

$$E_{16} = ap + bq + cr - ds = \gamma,$$

From the above sixteen equations, we find $b = c = r = q = 0$, and $\frac{a}{d} = \frac{p}{s} = -\frac{vi\,\gamma}{C(1-\gamma)} = \frac{-(1+\gamma)C}{\gamma vi}$.

$$\text{Therefore} \quad A = \left(\frac{a}{d} + e_3\right) d \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{III}$$

$$B = \left(\frac{p}{s} + e_3\right) s \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{IV}$$

The above transformation $Q' = AQB$ becomes

$$Q' = d\left(\frac{a}{d} + e_3\right)(x + e_1 y + e_2 z + e_3 \tau)\left(\frac{p}{s} + e_3\right)s$$

The right handside is $sd\Big[\Big\{x\left(\frac{a}{d}\cdot\frac{p}{s} - 1\right) - \tau\left(\frac{a}{d} + \frac{p}{s}\right)\Big\} +$
$e_1\Big\{y\left(\frac{a}{d}\cdot\frac{p}{s} + 1\right) + z\left(\frac{a}{d} - \frac{p}{s}\right)\Big\} + e_2\Big\{z\left(\frac{a}{d}\cdot\frac{p}{s} + 1\right) + y\left(\frac{p}{s} - \frac{a}{d}\right)\Big\}$
$+ e_3\Big\{x\left(\frac{a}{d} + \frac{p}{s}\right) + \tau\left(\frac{a}{d}\cdot\frac{p}{s} - 1\right)\Big\}\Big]$.

Comparing with (I) we have,

$$sd\left(\frac{a}{d}\cdot\frac{p}{s} - 1\right) = \gamma = sd\left(\frac{2\gamma}{1-\gamma}\right),$$

$$sd\left(\frac{a}{d} + \frac{p}{s}\right) = -\frac{\gamma vi}{C} = -2\,sd\,\frac{\gamma vi}{C(1-\gamma)},$$

$$sd\left(\frac{a}{d}\cdot\frac{p}{s}\right) + 1 = 1 = sd\left(\frac{2}{1-\gamma}\right).$$

From the above three equations we get $sd = \frac{1-\gamma}{2}$. Therefore we can choose s and d arbitrarily subject to this condition. If s is put equal to d, we have $s = d = \sqrt{\frac{1-\gamma}{2}}$ and the quaternion A is equal to the quaternion B.

We can threrfore represent theLorentz transformation (I) by the quaternion transformation

$$Q' = A\,Q\,B$$

Where $$A = B = \frac{-\,vi\gamma}{C\sqrt{2\,(1-\gamma)}} + e_3\sqrt{\frac{1-\gamma}{2}}$$

Note: The transformation $Q' = AQ$ is not sufficient to represent the general Lorentz Transformation because there are not enough parameters in A to specify a general rotation in 4 - dimensional space.