CHAPTER I

THE FORMS OF MAXWELL'S EQUATIONS

Formerly before the operator nabla ($\nabla = \bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}$) had been introduced to the vector calculus, Maxwell's equations were written as 8 equations :

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = \frac{1}{c}\left(\frac{\partial E_x}{\partial t} + 4\pi\rho v_x\right)$$

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = \frac{1}{c}\left(\frac{\partial E_y}{\partial t} + 4\pi\rho v_y\right)$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = \frac{1}{c}\left(\frac{\partial E_z}{\partial t} + 4\pi\rho v_z\right)$$

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\frac{1}{c}\frac{\partial H_x}{\partial t}$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\frac{1}{c}\frac{\partial H_y}{\partial t}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\frac{1}{c}\frac{\partial H_z}{\partial t}$$

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 4\pi\rho$$

$$\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} = 0.$$

Later, when the operator nabla had been introduced, Maxwell's equations were written as 4 equations :

$$\nabla \times \bar{H} = \frac{1}{c}\left(\frac{\partial \bar{E}}{\partial t} + 4\pi\rho\bar{v}\right)$$

$$\nabla \times \bar{E} = -\frac{1}{c}\frac{\partial \bar{H}}{\partial t}$$

$$\nabla \cdot \bar{E} = 4\pi\rho$$

$$\nabla \cdot \bar{H} = 0$$

Earlier in this thesis, we put $\overline{\Psi} = \overline{E} + i\overline{H}$, and Maxwell's equations in three dimensional form were reduced to 2 equations :

$$\nabla \times \overline{\Psi} = \frac{1}{c}\left(\frac{\partial \overline{\Psi}}{\partial t} + 4\pi\rho\overline{v} \right)$$

$$\nabla \cdot \overline{\Psi} = 4\pi\rho$$

We shall show below that by using four dimensional vectors (space and time) we can write Maxwell's equations as 2 equations in yet another way. Also by putting $\overline{\Psi}^{*} = \overline{E}^{*} + i\overline{H}^{*}$ (star means the field in four dimensions) we shall show that in four dimensional form, Maxwell's equations can be written as 1 equation.

CHAPTER II.

VECTORS IN FOUR DIMENSIONS

To write Maxwell's equations in four dimensional form, we shall use rectangular cartesian coordinates x_1, x_2, x_3, x_4 and take $\bar{i}_1$, $\bar{i}_2$, $\bar{i}_3$, $\bar{i}_4$, to be unit vectors in the four coordinate directions. Set $x_1 = x$, $x_2 = y$, $x_3 = z$, $x_4 = ict$ to produce the set of coordinates that are called the world coordinates.

The vector $\bar{A}$ in four dimensions which has components a_1, a_2, a_3, a_4 can be written

$$\bar{A} = a_1\bar{i}_1 + a_2\bar{i}_2 + a_3\bar{i}_3 + a_4\bar{i}_4 .$$

The dot product between any pair of vectors may be defined with the help of the relations.

$$i_j \cdot i_j = 1 \ , \quad i_j \cdot i_k = 0 \ (j \neq k) \ ,$$

in agreement with the definition for vectors in three dimensions, so that

$$\bar{A} \cdot \bar{B} = a_1 b_1 + a_2 b_2 + a_3 b_3 + a_4 b_4 .$$

this definition may be used in any number of dimensions.

We shall now define the cross product of two vectors in any number of dimensions in such a way that it is an extension of the definition in three dimensions.

Each component of the cross product of two vectors in three dimensions is calculate from expression containing the components of the vectors in the other two dimensions.

For example, the x - component of $\bar{A} \times \bar{B}$ comes from the y and z - component of $\bar{A}$ and $\bar{B}$, ie. $(\bar{A} \times \bar{B})_x = a_2 b_3 - a_3 b_2$ where $\bar{A} = a_1 \bar{i} + a_2 \bar{j} + a_3 \bar{k}$, $\bar{B} = b_1 \bar{i} + b_2 \bar{j} + b_3 \bar{k}$. Since each component of the cross product of two vectors comes from the other two components of both vectors, there are ${}^3c_2 = 3$ components of the cross product in three dimensions. Hence the cross product of two three - dimensional vectors is a vector in three dimensions. If we let the components of the cross product of four dimensional vectors come from pairs of components of the two vectors, then the cross product will have ${}^4c_2 = 6$ components which cannot be represented as a vector in the four dimensional space.

Returning for a moment to the three - dimensional case, we remark that $\bar{A} \times \bar{B}$ has three components which are more clearly described by pairs of suffixes than by single ones. We can write in fact

$$c_{23} = a_2 b_3 - a_3 b_2 = - c_{32} ,$$

$$c_{31} = a_3 b_1 - a_1 b_3 = - c_{13} ,$$

$$c_{12} = a_1 b_2 - a_2 b_1 = - c_{21} ,$$

and arrange the c's as the elements of the anti - symmetric matrix

$$\begin{bmatrix} 0 & c_{12} & c_{13} \\ c_{21} & 0 & c_{23} \\ c_{31} & c_{32} & 0 \end{bmatrix}$$

where the element in the j-th row and k-th column is $c_{jk} = a_j b_k - a_k b_j$. In analogous manner we can define the cross product $\bar{A} \times \bar{B}$ of two four - dimensional vectors as an anti - symmetric matrix of four rows and columns

$$\bar{A} \times \bar{B} = \begin{bmatrix} 0 & c_{12} & c_{13} & c_{14} \\ c_{21} & 0 & c_{23} & c_{24} \\ c_{31} & c_{32} & 0 & c_{34} \\ c_{41} & c_{42} & c_{43} & 0 \end{bmatrix}$$

and it will be observed that six of the c ' s are independent. $\bar{A}$ and $\bar{B}$ is not therefore a four - dimensional vector.

This method can be used for any number of dimensions. In two dimensions the cross product has only one component and is threfore a scalar.

Define the four dimensional vector operater :

$$\Box \equiv \bar{i}_1 \frac{\partial}{\partial x_1} + \bar{i}_2 \frac{\partial}{\partial x_2} + \bar{i}_3 \frac{\partial}{\partial x_3} + \bar{i}_4 \frac{\partial}{\partial x_4}$$

For the world vector we have

$$\Box \equiv \bar{i}_1 \frac{\partial}{\partial x} + \bar{i}_2 \frac{\partial}{\partial y} + \bar{i}_3 \frac{\partial}{\partial z} + \bar{i}_4 \frac{1}{ic} \cdot \frac{\partial}{\partial t} \cdot$$

If Φ be a scalar function of x_1, x_2, x_3, x_4, then

$$\text{grad}\,\Phi \equiv \Box\Phi = \frac{\partial\Phi}{\partial x_1} \bar{i}_1 + \frac{\partial\Phi}{\partial x_2} \bar{i}_2 + \frac{\partial\Phi}{\partial x_3} \bar{i}_3 + \frac{\partial\Phi}{\partial x_4} \bar{i}_4 \cdot$$

Further

$$\text{div } \bar{A} \equiv \Box \cdot \bar{A} = \frac{\partial a_1}{\partial x_1} + \frac{\partial a_2}{\partial x_2} + \frac{\partial a_3}{\partial x_3} + \frac{\partial a_4}{\partial x_4} ,$$

so that

$$\text{div grad } \bar{\Phi} = \Box^2 \bar{\Phi} = \frac{\partial^2 \Phi}{\partial x_1^2} + \frac{\partial^2 \Phi}{\partial x_2^2} + \frac{\partial^2 \Phi}{\partial x_3^2} + \frac{\partial^2 \Phi}{\partial x_4^2} .$$

The expression for curl $\bar{A}$ will, however, be an anti- symmetric matrix and in fact

$$\text{curl } \bar{A} \equiv \Box \times \bar{A} = \begin{bmatrix} 0 & b_{12} & b_{13} & b_{14} \\ b_{21} & 0 & b_{23} & b_{24} \\ b_{31} & b_{32} & 0 & b_{34} \\ b_{41} & b_{42} & b_{43} & 0 \end{bmatrix}$$

where $$b_{jk} = \frac{\partial a_k}{\partial x_j} - \frac{\partial a_j}{\partial x_k} .$$

It follows that all the elements of

$$\text{curl grad } \bar{\Phi} \equiv \Box \times (\Box \bar{\Phi})$$

are identically zero, just as all the components of $\nabla \times (\nabla \phi)$ are zero .

CHAPTER III

MAXWELL'S EQUATIONS IN FOUR DIMENSIONAL FORM

Consider Maxwell's equations in three dimensional form :

$$\left.\begin{aligned} \nabla \times \overline{H} &= \frac{1}{c}\left(\frac{\partial \overline{E}}{\partial t} + 4\pi\rho\overline{v}\right) \\ \nabla \times \overline{E} &= -\frac{1}{c}\frac{\partial \overline{H}}{\partial t} \\ \nabla \cdot \overline{E} &= 4\pi\rho \\ \nabla \cdot \overline{H} &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (1)$$

When we put $\overline{\Psi} = \overline{E} + i\overline{H}$, system (1) reduces to

$$\left.\begin{aligned} \nabla \times \overline{\Psi} &= \frac{i}{c}\left(\frac{\partial \overline{\Psi}}{\partial t} + 4\pi\rho\overline{v}\right) \\ \nabla \cdot \overline{\Psi} &= 4\pi\rho \end{aligned}\right\} \quad \ldots\ldots\ldots (1')$$

The fourth equation of (1) implies that

$$\overline{H} = \nabla \times \overline{A} \quad \ldots\ldots\ldots\ldots (2)$$

where $\overline{A}$ is some vector function. Then equation (2) suggests that we should write the components of the magnetic field ($\overline{H}$) in four dimensions as an anti - symmetric matrix.

Define H^{*}, the magnetic field in four dimensions, as

$$H^{*} = \begin{bmatrix} 0 & H_z & H_y & -iE_x \\ -H_z & 0 & H_x & -iE_y \\ H_y & -H_x & 0 & -iE_z \\ iE_x & iE_y & iE_z & 0 \end{bmatrix}$$

From this, we next define $\bar{H}^*$ as

$$\bar{H}^* = \begin{bmatrix} \bar{H}_1 \\ \bar{H}_2 \\ \bar{H}_3 \\ \bar{H}_4 \end{bmatrix} = \begin{bmatrix} 0 + H_z\bar{i}_2 - H_y\bar{i}_3 - iE_x\bar{i}_4 \\ -E_z\bar{i}_1 + 0 + H_x\bar{i}_3 - iE_y\bar{i}_4 \\ H_y\bar{i}_1 - H_x\bar{i}_2 + 0 - iE_z\bar{i}_4 \\ iE_x\bar{i}_1 + iE_y\bar{i}_2 + iE_z\bar{i}_3 + 0 \end{bmatrix} \quad \ldots\ldots(3)$$

And define v^*, the velocity in four dimensions of charge as

$$v^* = \begin{bmatrix} v_x \\ v_y \\ v_z \\ ic \end{bmatrix}$$

Then

$$\Box \cdot \bar{H}^* = \frac{1}{c} 4\pi \rho v^* \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

corresponding to the first and the third equations of (1)

Now (3) suggests that we should defind the electric field vector, $\bar{E}^*$, as

$$\bar{E}^* = \begin{bmatrix} \bar{E}_1 \\ \bar{E}_2 \\ \bar{E}_3 \\ \bar{E}_4 \end{bmatrix} = \begin{bmatrix} 0 + E_z\bar{i}_2 - E_y\bar{i}_3 + iH_x\bar{i}_4 \\ -E_z\bar{i}_1 + 0 + E_x\bar{i}_3 + iH_y\bar{i}_4 \\ E_y\bar{i}_1 - E_x\bar{i}_2 + 0 + iH_z\bar{i}_4 \\ -iH_x\bar{i}_1 - iH_y\bar{i}_2 - iH_z\bar{i}_3 + 0 \end{bmatrix}$$

Then we have

$$\Box \circ \overline{E}^{*} = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

which corresponds to the second and the fourth equations of (1) .

Now Maxwell's equations in four dimensional form can be written in 2 equations as equations (4) and (5) , ie

$$\left.\begin{aligned} \Box \circ \overline{H}^{*} &= \frac{1}{c} 4\pi \rho v^{*} \\ \Box \circ \overline{E}^{*} &= 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots (6)$$

Next, we put $\overline{\Psi}^{*} = \overline{E}^{*} + i\overline{H}^{*}$. Then system (6) reduces to

$$\Box \circ \overline{\Psi}^{*} = \frac{i}{c} 4\pi \rho v^{*} \qquad \ldots\ldots\ldots\ldots\ldots (6)$$

We shall now show that the equation (6) is equivalent to the system (1). For

$$\overline{\Psi}^{*} = \overline{E}^{*} + i\overline{H}^{*}$$

$$= \begin{pmatrix} 0 + E_z\overline{i}_2 - E_y\overline{i}_3 + iH_x\overline{i}_4 \\ -E_z\overline{i}_1 + 0 + E_x\overline{i}_3 + iH_y\overline{i}_4 \\ E_y\overline{i}_1 - E_x\overline{i}_2 + 0 + iH_z\overline{i}_4 \\ -iH_x\overline{i}_1 - iH_y\overline{i}_2 - iH_z\overline{i}_3 + 0 \end{pmatrix} + \begin{pmatrix} 0 + H_z\overline{i}_2 - H_y\overline{i}_3 - iE_x\overline{i}_4 \\ -H_z\overline{i}_1 + 0 + H_x\overline{i}_3 - iE_y\overline{i}_4 \\ H_y\overline{i}_1 - E_x\overline{i}_2 + 0 - iE_z\overline{i}_4 \\ iE_x\overline{i}_1 + iE_y\overline{i}_2 + iE_z\overline{i}_3 + 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 + (E_z + iH_z)\overline{i}_2 - (E_y + iH_y)\overline{i}_3 + (E_x + iH_x)\overline{i}_4 \\ -(E_z + iH_z)\overline{i}_1 + 0 + (E_x + iH_x)\overline{i}_3 + (E_y + iH_y)\overline{i}_4 \\ (E_y + iH_y)\overline{i}_1 - (E_x + iH_x)\overline{i}_2 + 0 + (E_z + iH_z)\overline{i}_4 \\ -(E_x + iH_x)\overline{i}_1 - (E_y + iH_y)\overline{i}_2 - (E_z + iH_z)\overline{i}_3 + 0 \end{pmatrix}$$

$$\overline{\Psi}^{*} = \begin{bmatrix} 0 + \Psi_z \overline{i}_2 - \Psi_y \overline{i}_3 + \Psi_x \overline{i}_4 \\ -\Psi_z \overline{i}_1 + 0 + \Psi_x \overline{i}_3 + \Psi_y \overline{i}_4 \\ \Psi_y \overline{i}_1 - \Psi_x \overline{i}_2 + 0 + \Psi_z \overline{i}_4 \\ -\Psi_x \overline{i}_1 - \Psi_y \overline{i}_2 - \Psi_z \overline{i}_3 + 0 \end{bmatrix}$$

From this we see that $\square \circ \overline{\Psi}^{*} = \frac{i}{c} 4\pi \rho v^{*}$ is equivalent to the system $\nabla \times \overline{\Psi} = \frac{i}{c} \left(\frac{\partial \overline{\Psi}}{\partial t} + 4\pi \rho \overline{v} \right)$, $\nabla \cdot \overline{\Psi} = 4\pi\rho$.

To complete this chapter, we shall find the meaning of the new vectors which we have defined, namely $\overline{\Psi}$, $\overline{E}^{*}$, $\overline{H}^{*}$ and $\overline{\Psi}^{*}$

Since

$$\overline{\Psi} = \overline{E} + i\overline{H},$$

we have

$$\begin{aligned} |\overline{\Psi}|^2 &= (\overline{E} + i\overline{H}) \circ (\overline{E} - i\overline{H}) \\ &= \overline{E} \cdot \overline{E} + \overline{H} \cdot \overline{H} \\ &= 8\pi \left(\frac{\overline{E} \cdot \overline{E} + \overline{H} \cdot \overline{H}}{8\pi} \right). \end{aligned}$$

That is the square of the modulus of $\overline{\Psi}$ is 8π times the electromagnetic energy density in ether.

For vector $\overline{E}^{*}$, Since

$$\overline{E}^{*} = \begin{bmatrix} 0 + E_z \overline{i}_2 - E_y \overline{i}_3 + iH_x \overline{i}_4 \\ -E_z \overline{i}_1 + 0 + E_x \overline{i}_3 + iH_y \overline{i}_4 \\ E_y \overline{i}_1 - E_x \overline{i}_2 + 0 + iH_z \overline{i}_4 \\ -iH_x \overline{i}_1 - iH_y \overline{i}_2 - iH_z \overline{i}_3 + 0 \end{bmatrix},$$

we have

$$|\overline{E}^*|^{\alpha} = \overline{E}^* . \overline{E}_c^*$$

where $\overline{E}_c^* = \begin{pmatrix} 0 + E_z\overline{i}_2 - E_y\overline{i}_3 - iH_x\overline{i}_4 \\ -E_z\overline{i}_1 + 0 + E_x\overline{i}_3 - iH_y\overline{i}_4 \\ E_y\overline{i}_1 - E_x\overline{i}_2 + 0 - iH_z\overline{i}_4 \\ iH_x\overline{i}_1 + iH_y\overline{i}_2 + iH_z\overline{i}_3 + 0 \end{pmatrix}$: the conjugate of $\overline{E}^*$.

Hence

$$|\overline{E}^*|^{\alpha} = 2\,(E_x^2 + E_y^2 + E_z^2 + H_x^2 + H_y^2 + H_z^2)$$

$= 16\pi$ (electromagnetic energy density in ether)

Similarly, we can show that

$$|\overline{H}^*|^{\alpha} = 2\,(E_x^2 + E_y^2 + E_z^2 + H_x^2 + H_y^2 + H_z^2)$$

16π (electromagnetic energy density in ether)

For vector $\overline{\Psi}^*$, since

$$\overline{\Psi}^* = \overline{E}^* + i\overline{H}^*$$

$$= \begin{pmatrix} 0 + (E_z + iH_z)\,\overline{i}_2 - (E_y + iH_y)\,\overline{i}_3 + (E_x + iH_x)\,\overline{i}_4 \\ -(E_z + iH_z)\,\overline{i}_1 + 0 + (E_x + iH_x)\,\overline{i}_3 + (E_y + iH_y)\,\overline{i}_4 \\ (E_y + iH_y)\,\overline{i}_1 - (E_x + iH_x)\,\overline{i}_2 + 0 + (E_z + iH_z)\,\overline{i}_4 \\ -(E_x + iH_x)\,\overline{i}_1 - (E_y + iH_y)\,\overline{i}_2 - (E_z + iH_z)\,\overline{i}_3 + 0 \end{pmatrix}$$

we have

$$|\overline{\Psi}^*|^{\alpha} = \overline{\Psi}^* . \overline{\Psi}_c^*$$

where $\overline{\Psi}_c^{*} = \begin{bmatrix} 0 + (E_z - iH_z)\,\overline{i}_2 - (E_y - iH_y)\,\overline{i}_3 + (E_x - iH_x)\,\overline{i}_4 \\ -(E_z - iH_z)\,\overline{i}_1 + 0 + (E_x - iH_x)\,\overline{i}_3 + (E_y - iH_y)\,\overline{i}_4 \\ (E_y - iH_y)\,\overline{i}_1 - (E_x - iH_x)\,\overline{i}_2 + 0 + (E_z - iH_z)\,\overline{i}_4 \\ -(E_x - iH_x)\,\overline{i}_1 - (E_y - iH_y)\,\overline{i}_2 - (E_z - iH_z)\,\overline{i}_3 + 0 \end{bmatrix}$,

the conjugate of $\overline{\Psi}^{*}$.

Therefore

$$\left|\overline{\Psi}^{*}\right|^{2} = 4\,(E_x^2 + E_y^2 + E_z^2 + H_x^2 + H_y^2 + H_z^2)$$

$= 32\,\pi$ (electromagnetic energy density in ether).

CHAPTER IV

MAGNETIC CHARGE

In this chapter we shall introduce the interesting idea of magnetic charge.* Magnetic charges may be introduced into Maxwell's equations without difficulty although they have never been found experimentally.

If ρ_e and ρ_m represent electric and magnetic charge densities, $\bar{v}_e$ and $\bar{v}_m$ represent the velocities of electric and magnetic charge respectively, Then, for the symmetry between electric and magnetic quantities, Maxwell's equations for any medium have the form

$$\left.\begin{aligned} \nabla \times \bar{H} &= \frac{1}{c}\left(\frac{\partial \bar{D}}{\partial t} + 4\pi \rho_e \bar{v}_e \right) \\ \nabla \times \bar{E} &= -\frac{1}{c}\left(\frac{\partial \bar{B}}{\partial t} + 4\pi \rho_m \bar{v}_m \right) \\ \nabla \cdot \bar{D} &= 4\pi \rho_e \\ \nabla \cdot \bar{B} &= 4\pi \rho_m \end{aligned}\right\} \quad \ldots\ldots\ldots(1)$$

This is considered as the most general form of Maxwell's equations. The equations in previous chapters are special cases of this in which $\rho_m = 0$.

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* P.G.H. Sanders. Contemp. Phys. Vol. 7. page 419 (1966)

Since $\rho_e \overline{v}_e = \overline{J}_e$ = electric current density, and $\rho_m \overline{v}_m = \overline{J}_m$ = magnetic current density, the system (1) can be written as

$$\left.\begin{aligned} \nabla \times \overline{H} &= \frac{1}{c}\left(\frac{\partial \overline{D}}{\partial t} + 4\pi \overline{J}_e \right) \\ \nabla \times \overline{E} &= -\frac{1}{c}\left(\frac{\partial \overline{B}}{\partial t} + 4\pi \overline{J}_m \right) \\ \nabla \cdot \overline{D} &= 4\pi \rho_e \\ \nabla \cdot \overline{B} &= 4\pi \rho_m \end{aligned}\right\} \quad \cdots\cdots\cdots (1')$$

When the medium is ether, the system (1) becomes

$$\left.\begin{aligned} \nabla \times \overline{H} &= \frac{1}{c}\left(\frac{\partial \overline{E}}{\partial t} + 4\pi \rho_e \overline{v}_e \right) \\ \nabla \times \overline{E} &= -\frac{1}{c}\left(\frac{\partial \overline{H}}{\partial t} + 4\pi \rho_m \overline{v}_m \right) \\ \nabla \cdot \overline{E} &= 4\pi \rho_e \\ \nabla \cdot \overline{H} &= 4\pi \rho_m \end{aligned}\right\} \quad \cdots\cdots\cdots (2)$$

Also, the system (1') becomes

$$\left.\begin{aligned} \nabla \times \overline{H} &= \frac{1}{c}\left(\frac{\partial \overline{E}}{\partial t} + 4\pi \overline{J}_e \right) \\ \nabla \times \overline{E} &= -\frac{1}{c}\left(\frac{\partial \overline{H}}{\partial t} + 4\pi \overline{J}_m \right) \\ \nabla \cdot \overline{E} &= 4\pi \rho_e \\ \nabla \cdot \overline{H} &= 4\pi \rho_m \end{aligned}\right\} \quad \cdots\cdots\cdots (2')$$

Now, we put

$$\sigma = \rho_e + i\rho_m$$

$$\overline{\Gamma} = \overline{J}_e + i\,\overline{J}_m$$

$$\overline{\Psi} = \overline{E} + i\overline{H}$$

where $i = \sqrt{-1}$. Then the system $(2')$ becomes

$$\left.\begin{aligned} \nabla \times \overline{\Psi} &= \frac{i}{c}\left(\frac{\partial \overline{\Psi}}{\partial t} + 4\pi \overline{\Gamma} \right) \\ \nabla \cdot \overline{\Psi} &= 4\pi\sigma \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (3)$$

In four demensional form, the system $(2')$ becomes

$$\left.\begin{aligned} \Box \cdot \overline{H}^{*} &= \frac{1}{c}\, 4\pi\, \overline{J}^{*}_{e} \\ \Box \cdot \overline{E}^{*} &= -\frac{1}{c}\, 4\pi\, \overline{J}^{*}_{m} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (4)$$

where $\overline{H}^{*}$ and $\overline{E}^{*}$ are defined in the previous chapter,

$$J^{*}_{e} = \begin{pmatrix} (J_e)_x \\ (J_e)_y \\ (J_e)_z \\ ic\,\rho_e \end{pmatrix} , \quad \text{and} \quad J^{*}_{m} = \begin{pmatrix} (J_m)_x \\ (J_m)_y \\ (J_m)_z \\ ic\,\rho_m \end{pmatrix}$$

Now we put $\overline{\Psi}^{*} = \overline{E}^{*} + i\overline{H}^{*}$ and $\Gamma^{*} = J^{*}_{e} + iJ^{*}_{m}$. Then the system (4) becomes.

$$\Box \cdot \overline{\Psi}^{*} = \frac{1}{c}\, 4\pi\, \Gamma^{*} \quad \ldots\ldots\ldots\ldots (5)$$

This equation is also equivalent to the system (3)

For

$$\overline{\Psi}^{*} = \overline{E}^{*} + i\overline{H}^{*} = \begin{pmatrix} 0 + \Psi_z \overline{i}_2 - \Psi_y \overline{i}_3 + \Psi_x \overline{i}_4 \\ -\Psi_z \overline{i}_1 + 0 + \Psi_x \overline{i}_3 + \Psi_y \overline{i}_4 \\ \Psi_y \overline{i}_1 - \Psi_x \overline{i}_2 + 0 + \Psi_z \overline{i}_4 \\ -\Psi_x \overline{i}_1 - \Psi_y \overline{i}_2 - \Psi_z \overline{i}_3 + 0 \end{pmatrix}$$

(see previous chapter), and

$$\Gamma^{*} = J_e^{*} + i\, J_m^{*} = \begin{pmatrix} (J_e)_x + i\,(J_m)_x \\ (J_e)_y + i\,(J_m)_y \\ (J_e)_z + i\,(J_m)_z \\ ic\,(\rho_e + \rho_m) \end{pmatrix} = \begin{pmatrix} \Gamma_x \\ \Gamma_y \\ \Gamma_z \\ ic\sigma \end{pmatrix}$$

From this it is easily to verify that equation (5) is equivalent to the system (3).