CHAPTER III

INTEGRATION OVER SIMPLEXES

In this chapter, we shall begin by discussing the integral of a real function over a simplex and its properties. Next, we shall talk about integrals of differential forms over oriented affine simplexes, which will be used in the next chapter.

Throughout this chapter we assume $k, n \in Z^{+}$ with $k \leq n$.

3.1 Simplexes in R^n and Their Volumes

Let $x_0, x_1, \ldots, x_k$ be distinct points of R^n such that the differences $x_1 - x_0, \ldots, x_k - x_0$ are linearly independent. By a *convex combination* of $x_0, x_1, \ldots, x_k$ we mean a linear combination

(1) $$\sum_{j=0}^{k} t_j x_j,$$

where $0 \leq t_j \leq 1$ for all $j \in \{0, 1, \ldots, k\}$ and $\sum_{j=0}^{k} t_j = 1$. The set of all convex combinations of $x_0, x_1, \ldots, x_k$ is called the *k-simplex in R^n with vertices $x_0, x_1, \ldots, x_k$*, and denoted by $\Delta(x_0, x_1, \ldots, x_k)$. Note that a 1-simplex is a line segment, a 2-simplex is a triangle, and a 3-simplex is a tetrahedron, and that each point x of a k-simplex $\Delta(x_0, x_1, \ldots, x_k)$ can be written in a unique way as the convex combination (1); that is,

if $\sum_{j=0}^{k} t_j x_j$ and $\sum_{j=0}^{k} t_j' x_j$ are convex combinations of $x_0, x_1, \ldots, x_k$ such that

$$\sum_{j=0}^{k} t_j x_j = \sum_{j=0}^{k} t_j' x_j,$$

then $t_j = t_j'$ for all $j \in \{0, 1, \ldots, k\}$. The numbers t_j are called the *barycentric coordinates* of x.

If $k = n$, we define the volume of the n-simplex $\Delta(x_0, x_1, \ldots, x_n)$ by

$$\mathrm{Vol}(\Delta(x_0, x_1, \ldots, x_n)) = \frac{1}{n!}\left| \det \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_0^{(1)} & x_1^{(1)} & \cdots & x_n^{(1)} \\ \vdots & \vdots & & \vdots \\ x_0^{(n)} & x_1^{(n)} & \cdots & x_n^{(n)} \end{bmatrix} \right|$$

Let Sim(n) denote the set of all n-simplexes in R^n. If $e_1, \ldots, e_n$ is the standard basis for R^n, we call the simplex $\Delta(0, e_1, \ldots, e_n)$ the *standard n-simplex* in R^n and denote it by $\Delta(n)$. Note that

$$\Delta(n) = \{ x \in R^n \mid 0 \le x^{(i)} \le 1 \text{ for all } i \in \bar{n} \text{ and } \sum_{i=1}^{n} x^{(i)} \le 1 \}.$$

3.2 Integral of a Real Function over a Simplex

Let $\sigma = \Delta(p_0, p_1, \ldots, p_k) \in$ Sim(k). A function δ: $\sigma \to R^+$ is called a *σ-gauge*. A *partition* of σ into k-simplexes is a finite collection of nonoverlapping k-simplexes $\{T_i \mid i \in \bar{m}\}$ such that $\sigma = \bigcup_{i=1}^{m} T_i$. A *tagged* partition τ of σ into k-simplexes (briefly, a *k-partition* of σ) is a finite collection of ordered pairs $\{(t_i, T_i) \mid i \in \bar{m}\}$ such that $\{T_i \mid i \in \bar{m}\}$ is a partition of σ into k-simplexes and $t_i \in T_i$ for all $i \in \bar{m}$. The elements T_i are called *k-subsimplexes* of σ and each t_i is called the *tag* of T_i. Given a σ-gauge δ, a k-partition τ of σ is said to be *δ-fine* iff for each $i \in \bar{m}$, $T_i \subseteq B(t_i, \delta(t_i))$.

For $l \ge 2$, define S_l : $\{(q_1, \ldots, q_l) \mid q_1, \ldots, q_l \in R^l$ and $q_2 - q_1, \ldots, q_l - q_1$ are linearly independent$\} \to R^+$ by

$$S_l((q_1,\ldots,q_l)) = \left(\sum_{r=1}^{l} \left(\det \begin{bmatrix} \pi_r(q_2-q_1) \\ \vdots \\ \pi_r(q_l-q_1) \end{bmatrix} \right)^2 \right)^{\frac{1}{2}}.$$

Convention: Let $(i_0, i_1, \ldots, i_k)$ be any $(k+1)$-tuple of objects. For each $a \in \{0,1,\ldots,k\}$, the notation $(i_0, i_1, \ldots, \hat{i_a}, \ldots, i_k)$ refers to the k-tuple $(i_{j_1}, \ldots, i_{j_k})$, where $(j_1, \ldots, j_k)$ is the unique ascending k-tuple from the set $\{0, 1, \ldots, k\} \setminus \{a\}$. We will also apply this notation to other arrays of objects.

Define the *outer volume* of $\sigma = \Delta(p_0, p_1, \ldots, p_k) \in \mathrm{Sim}(k)$, denoted by $\mathrm{OV}(\sigma)$, by

$$\mathrm{OV}(\sigma) = \begin{cases} \mathrm{diam}(\sigma) & \text{if } k = 1, \\ \dfrac{1}{(k-1)!}\mathrm{diam}(\sigma)\displaystyle\sum_{a=0}^{k} S_k((p_0, p_1, \ldots, \hat{p_a}, \ldots, p_k)) & \text{otherwise.} \end{cases}$$

For each $M \geq \mathrm{OV}(\sigma)$, the k-partition $\tau = \{(t_i, T_i) \mid i \in \overline{m}\}$ of σ is said to be *M-bounded* iff $\sum_{i=1}^{m} \mathrm{OV}(T_i) \leq M$.

Lemma 3.2.1

(i) If p_0 and p_1 are points of R^2, then $S_2(p_0, p_1) = |p_1 - p_0|$.

(ii) If $\sigma = \Delta(p_0, p_1, p_2) \in \mathrm{Sim}(2)$, then $\mathrm{OV}(\sigma) = \mathrm{diam}(\sigma)\mathrm{perimeter}(\sigma)$.

Proof. Clear.

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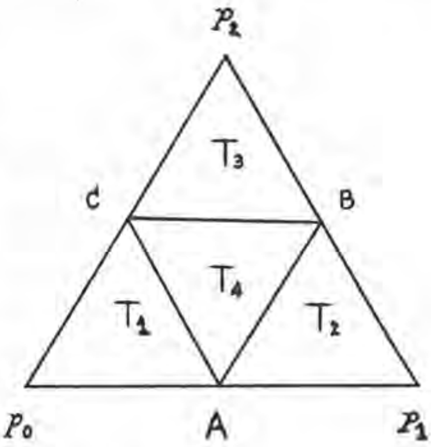
The following lemma ensures the existence of δ-fine M-bounded k-partitions of σ. Although we believe it is true for all $k \in Z^+$, we can only prove the case $k \leq 2$ at present.

Lemma 3.2.2 (*Cousin's Lemma*) Let $\sigma = \Delta(p_0, p_1, \ldots, p_k) \in \text{Sim}(k)$. Suppose that δ is a σ-gauge and $M \geq \text{OV}(\sigma)$. Then there exists a δ-fine M-bounded k-partition of σ.

Proof. First, suppose that $k = 1$. Then p_0 and p_1 are two real numbers. WLOG, we may assume that $p_0 < p_1$, so that σ is the closed interval $[p_0, p_1]$ in R. It is easy to check that any δ-fine tagged partition of $[p_0, p_1]$, as defined in Section 1.3, is a δ-fine 1-partition of $[p_0, p_1]$, and conversely. Furthermore, the sum of the outer volumes of the 1-subsimplexes of any 1-partition of σ equals the outer volume of σ, so every 1-partition of σ is M-bounded. Therefore, the case $k = 1$ of this lemma is equivalent to Lemma 1.3.1.

Finally, suppose that $k = 2$. Then σ is a triangle. Assume to the contrary that no δ-fine M-bounded 2-partition of σ exists. Let A, B, C be the midpoints of the line segments $\overline{p_0p_1}$, $\overline{p_1p_2}$, $\overline{p_2p_0}$, respectively. We then get 4 subtriangles $T_1, \ldots, T_4$ of σ as in the figure. Note that



$$\frac{M}{4} \geq \frac{\text{diam}(\sigma)}{2} \cdot \frac{\text{perimeter}(\sigma)}{2}$$
$$= \text{diam}(T_i)\text{perimeter}(T_i)$$
$$= \text{OV}(T_i)$$

for all $i \in \{1, \ldots, 4\}$. If all of these subtriangles have δ-fine $\frac{M}{4}$-bounded 2-partitions, then the combination of their δ-fine $\frac{M}{4}$-bounded 2-partitions is a δ-fine

M-bounded 2-partition of σ, which is impossible. Hence there is a subtriangle $T^{(1)}$ of σ which has no δ-fine $\frac{M}{4}$-bounded 2-partition such that

$$\text{diam}(T^{(1)}) = \frac{1}{2}\text{diam}(\sigma)$$

and

$$\text{perimeter}(T^{(1)}) = \frac{1}{2}\text{perimeter}(\sigma).$$

Continuing this process inductively, we obtain a decreasing chain of triangles

$$\sigma \supseteq T^{(1)} \supseteq T^{(2)} \supseteq T^{(3)} \supseteq \ldots$$

such that for all $l \in Z^+$,

(i) $T^{(l)}$ has no δ-fine $\frac{M}{4^l}$-bounded 2-partition,

(ii) $\text{diam}(T^{(l)}) = \frac{1}{2^l}\text{diam}(\sigma)$,

and

(iii) $\text{perimeter}(T^{(l)}) = \frac{1}{2^l}\text{perimeter}(\sigma)$.

Since each $T^{(l)}$ is a nonempty compact set, $\bigcap_{l=1}^{\infty} T^{(l)} \neq \varnothing$. Let $t \in \bigcap_{l=1}^{\infty} T^{(l)}$; choose $l \in Z^+$ such that $\text{diam}(T^{(l)}) < \delta(t)$. It is easy to see that $\{(t, T^{(l)})\}$ is a δ-fine $\frac{M}{4^l}$-bounded 2-partition of $T^{(l)}$, which is a contradiction. Hence σ must have a δ-fine M-bounded 2-partition.

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Let $f : \sigma \to R$ be a function, and let $\tau = \{(t_i, T_i) \mid i \in \overline{m}\}$ be a k-partition of σ. Then the sum

$$\sum_{i=1}^{m} f(t_i) Vol(T_i)$$

is called the *Riemann sum* of f given by τ, and it is denoted by $R(f, \tau)$. Let $M \geq$ OV(σ). We call a real number I an *M-integral of f over* σ iff for every $\varepsilon > 0$, there is a σ-gauge δ such that

$$|R(f, \tau) - I| < \varepsilon$$

for every δ-fine M-bounded k-partition τ of σ. In case this integral exists we say that f is *M-integrable over* σ.

Lemma 3.2.3 Let $\sigma \in$ Sim(k), and let $M, M_1, M_2 \geq$ OV(σ). Suppose $f: \sigma \to R$ is a function. Then:

(i) If I_1 and I_2 are M-integrals of f over σ, then $I_1 = I_2$.

(ii) If I_i is an M_i-integral of f over σ ($i = 1, 2$), then $I_1 = I_2$.

Proof. (i) Suppose I_1 and I_2 are M-integrals of f over σ. Let $\varepsilon > 0$ be given. For each $i \in \{1, 2\}$, there exists a σ-gauge δ_i such that for every δ_i-fine M-bounded k-partition τ_i of σ,

$$|R(f, \tau_i) - I_i| < \frac{\varepsilon}{2}.$$

We define $\delta = \min\{\delta_1, \delta_2\}$; thus δ is a σ-gauge. Let τ be a δ-fine M-bounded k-partition of σ, then τ is a δ_i-fine M-bounded k-partition of σ for all $i \in \{1, 2\}$. It follows that

$$|I_1 - I_2| \leq |I_1 - R(f, \tau)| + |R(f, \tau) - I_2| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $I_1 = I_2$. This proves (i).

(ii) WLOG, assume that $M_1 \leq M_2$. Note that for any σ-gauge δ, the set of all δ-fine M_1-bounded k-partitions of σ is a subset of the set of all δ-fine M_2-bounded k-partitions of σ. Thus I_2 is also an M_1-integral of f over σ, and the result follows from part (i).

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This lemma shows that f has at most one M-integral over a given k-simplex σ, and this integral does not depend on M. The function f is said to be *integrable over* σ iff there is an $M \geq \mathrm{OV}(\sigma)$ such that the M-integral of f over σ exists. In this case we denote the M-integral of f over σ by

$$\int_\sigma f$$

or by

$$\int_\sigma f(x)\ dx,$$

and call it the *integral of* f *over* σ. If $k = 1$, one can check that the above integral is the same as the generalized Riemann integral over a closed interval in R.

Proposition 3.2.4 (*Cauchy Criterion*) Let $\sigma \in \mathrm{Sim}(k)$ and $f: \sigma \to R$. Then for all $M \geq \mathrm{OV}(\sigma)$, f is M-integrable over σ iff for every $\varepsilon > 0$, there is a σ-gauge δ such that for all δ-fine M-bounded k-partitions τ_1 and τ_2 of σ,

$$|\mathrm{R}(f,\ \tau_1) - \mathrm{R}(f,\ \tau_2)| < \varepsilon.$$

Proof. Let $M \geq \mathrm{OV}(\sigma)$ be given. The necessity is clear. We now prove the sufficiency. For each $i \in Z^+$, let γ_i be a σ-gauge such that for all γ_i-fine M-bounded k-partitions τ and τ' of σ,

$$|\mathrm{R}(f,\ \tau) - \mathrm{R}(f,\ \tau')| < \frac{1}{i}.$$

For each $i \in Z^+$, define $\delta_i = \min\{\gamma_l \mid l \in \bar{i}\}$, and let τ_i be a δ_i-fine M-bounded k-partition of σ. Clearly,

$$(2) \qquad |\mathrm{R}(f,\ \tau_i) - \mathrm{R}(f,\ \tau_j)| < \frac{1}{i}$$

for all $i, j \in Z^+$ with $i \leq j$. This shows that the sequence

$$\{\mathrm{R}(f,\ \tau_i)\}_{i=1}^{\infty}$$

is Cauchy. Let $A = \lim_{j\to\infty} \mathrm{R}(f, \tau_j)$. It follows from (2) by taking $j \to \infty$ that

$$|\mathrm{R}(f,\ \tau_i) - A| \leq \frac{1}{i}$$

for all $i \in Z^+$.

We claim that A is the M-integral of f, which will be sufficient to show f is M-integrable over σ. Let $\varepsilon > 0$ be given; choose $i_0 \in Z^+$ such that $\frac{1}{i_0} < \frac{\varepsilon}{2}$. Suppose τ is a δ_{i_0}-fine M-bounded k-partition of σ; then

$$\begin{aligned} |\mathrm{R}(f, \tau) - A| &\le |\mathrm{R}(f, \tau) - \mathrm{R}(f, \tau_{i_0})| + |\mathrm{R}(f, \tau_{i_0}) - A| \\ &< \frac{1}{i_0} + \frac{1}{i_0} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

The claim is proved.

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Corollary 3.2.5 Let $\sigma \in \mathrm{Sim}(k)$ and $f: \sigma \to R$. Suppose f is N-integrable over σ for some $N \ge \mathrm{OV}(\sigma)$. Then f is M-integrable over σ for all $M \in [\mathrm{OV}(\sigma), N]$, and these integrals are equal. In particular, all integrable functions are $\mathrm{OV}(\sigma)$-integrable over σ.

Proposition 3.2.6 Let $\sigma \in \mathrm{Sim}(k)$ and $c \in R$. Suppose $f, g: \sigma \to R$ are integrable over σ. Then

(i) $f + g$ is integrable over σ and $\int_\sigma (f + g) = \int_\sigma f + \int_\sigma g$,

and

(ii) cf is integrable over σ and $\int_\sigma (cf) = c\int_\sigma f$.

Proof. To prove (i), let $\varepsilon > 0$ be given. Since both f and g are $\mathrm{OV}(\sigma)$-integrable over σ, we can find a σ-gauge δ such that for all δ-fine $\mathrm{OV}(\sigma)$-bounded k-partitions τ of σ,

$$|\mathrm{R}(f, \tau) - \int_\sigma f| < \frac{\varepsilon}{2} \quad \text{and} \quad |\mathrm{R}(g, \tau) - \int_\sigma g| < \frac{\varepsilon}{2}.$$

Thus, if τ is any δ-fine OV(σ)-bounded k-partition of σ, then

$$\begin{aligned} \left|R(f+g,\tau)-\left(\int_\sigma f+\int_\sigma g\right)\right| &= \left|R(f,\tau)-\int_\sigma f + R(g,\tau)-\int_\sigma g\right| \\ &\le \left|R(f,\tau)-\int_\sigma f\right| + \left|R(g,\tau)-\int_\sigma g\right| \\ &< \frac{\varepsilon}{2}+\frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

This proves (i).

To prove (ii): The result is obvious when c = 0. Suppose c ≠ 0, and let $\varepsilon > 0$ be given. There exists a σ-gauge δ such that for all δ-fine OV(σ)-bounded k-partitions τ of σ,

$$\left|R(f,\tau)-\int_\sigma f\right| < \frac{\varepsilon}{|c|}.$$

Thus, if τ is any δ-fine OV(σ)-bounded k-partition of σ, then

$$\begin{aligned} \left|R(cf,\tau)-c\int_\sigma f\right| &= |c|\left|R(f,\tau)-\int_\sigma f\right| \\ &< \varepsilon. \end{aligned}$$

This proves the case c ≠ 0.

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The remainder of this section is devoted to the proof that a continuous function is always integrable over a k-simplex in R^k, and that the integral is equal to the ordinary Riemann integral.

Proposition 3.2.7 The standard k-simplex $\Delta(k)$ has content.

Proof. We have to show that Bd($\Delta(k)$) has content zero. If $k = 1$ then the result is obvious. Suppose that $k \ge 2$. Let I denote the unit cube $\times_{i=1}^{k} [0, 1]$ in R^k. If we can show that

(i) $\mathrm{Bd}(\Delta(k)) \subseteq \bigcup_{i=1}^{k} \{(x^{(1)},\dots,x^{(k)}) \in I \mid x^{(i)} = 0\} \cup \{(x^{(1)},\dots,x^{(k)}) \in I \mid \sum_{i=1}^{k} x^{(i)} = 1\}$,

(ii) for all $i \in \bar{k}$, $\{(x^{(1)},\dots,x^{(k)}) \in I \mid x^{(i)} = 0\}$ has content zero,

and

(iii) $\{(x^{(1)},\dots,x^{(k)}) \in I \mid \sum_{i=1}^{k} x^{(i)} = 1\}$ has content zero,

then $\mathrm{Bd}(\Delta(k))$ has content zero. Therefore, $\Delta(k)$ has content.

To prove (i), let $y = (y^{(1)},\dots,y^{(k)}) \in \mathrm{Bd}(\Delta(k))$ be arbitrary. If there exists $i \in \bar{k}$ such that $y^{(i)} = 0$, then $y \in \{(x^{(1)},\dots,x^{(k)}) \in I \mid x^{(i)} = 0\}$, so we are done.

Suppose that $y^{(i)} > 0$ for all $i \in \bar{k}$. Note that $0 < \sum_{i=1}^{k} y^{(i)} \le 1$. Assume for a contradiction that $\sum_{i=1}^{k} y^{(i)} < 1$. Put

$$r = \min\{y^{(1)}, 1 - y^{(1)}, \dots, y^{(k)}, 1 - y^{(k)}, \frac{1 - \sum_{i=1}^{k} y^{(i)}}{k}\}.$$

Thus, $0 < r < 1$.

We claim that $B(y, r) \subseteq \Delta(k)$. Let $z = (z^{(1)},\dots,z^{(k)}) \in B(y, r)$. Then $|z^{(i)} - y^{(i)}| < r$ for all $i \in \bar{k}$, so $y^{(i)} - r < z^{(i)} < y^{(i)} + r$ for all $i \in \bar{k}$. Hence $0 < z^{(i)} < 1$ for all $i \in \bar{k}$, and

$$\begin{aligned} \sum_{i=1}^{k} z^{(i)} &< \sum_{i=1}^{k} (y^{(i)} + r) \\ &= kr + \sum_{i=1}^{k} y^{(i)} \end{aligned}$$

$$\leq \; k\left(\frac{1-\sum_{i=1}^{k} y^{(i)}}{k}\right) + \sum_{i=1}^{k} y^{(i)}$$

$$= \; 1.$$

Therefore, $z \in \Delta(k)$, and the claim is proved.

This implies that $y \in \text{Int}(\Delta(k))$. Hence $y \notin \text{Bd}(\Delta(k))$, which is a contradiction.

Thus, $\sum_{i=1}^{k} y^{(i)} = 1$, so $y \in \{(x^{(1)},\ldots,x^{(k)}) \in I \mid \sum_{i=1}^{k} x^{(i)} = 1\}$. This proves (i).

(ii) is obvious.

To prove (iii), define $\varphi: R^k \to R^k$ by

$$\varphi((x^{(1)},\ldots,x^{(k)})) = (x^{(1)},\ldots,x^{(k-1)}, 1-\sum_{i=1}^{k-1} x^{(i)})$$

for all $(x^{(1)},\ldots,x^{(k)}) \in R^k$. Clearly, φ is a C^1-mapping. It follows from (ii) and Proposition 1.2.1(i) that $A = \{(x^{(1)},\ldots,x^{(k-1)},0) \in I \mid \sum_{i=1}^{k-1} x^{(i)} \leq 1\}$ has content zero. Since $\varphi(A) = \{(x^{(1)},\ldots,x^{(k)}) \in I \mid \sum_{i=1}^{k} x^{(i)} = 1\}$, it follows from Theorem 1.2.9 that $\{(x^{(1)},\ldots,x^{(k)}) \in I \mid \sum_{i=1}^{k} x^{(i)} = 1\}$ has content zero.

This completes the proof.

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Corollary 3.2.8 If $\Delta(p_0, p_1,\ldots,p_k) \dot{\in} \text{Sim}(k)$, then $\Delta(p_0, p_1,\ldots,p_k)$ has content.

Proof. Define $\varphi: R^k \to R^k$ by $\varphi(x) = p_0 + \sum_{i=1}^{k} x^{(i)}(p_i - p_0)$ for all $x \in R^k$. It is easy to see that for all $x \in R^k$, the derivative $\varphi'(x) = A$, where $A \in \text{Lin}(R^k, R^k)$ is defined by

$$A(x) = \sum_{i=1}^{k} x^{(i)}(p_i - p_0)$$

for all $x \in R^k$. Note that if $e_1, \ldots, e_k$ is the standard basis for R^k, then $A(e_i) = p_i - p_0$ for all $i \in \bar{k}$. Thus all first-order partial derivatives of all of the component functions of φ are constant functions, from which it follows that all higher-order partial derivatives are zero. It is then clear that all partial derivatives of all of the component functions of φ of all orders exist and are continuous, so $\varphi \in C^\infty(R^k)$.

Note that $\varphi(0) = p_0$, $\varphi(e_i) = p_i$ for all $i \in \bar{k}$, and $\varphi(x) = p_0 + A(x)$ for all $x \in R^k$. It is easy to see that

$$\varphi(\Delta(k)) = \Delta(p_0, p_1, \ldots, p_k).$$

Since $p_1 - p_0, \ldots, p_k - p_0$ are linearly independent, we have $J_\varphi(x) = \det[\varphi'(x)] = \det[A] \neq 0$ for all $x \in R^k$. Because $\Delta(k)$ has content and $\varphi(\Delta(k)) = \Delta(p_0, p_1, \ldots, p_k)$, it follows from Theorem 1.2.10 that $\Delta(p_0, p_1, \ldots, p_k)$ has content.

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We can now show that every real continuous function on $\sigma \in \text{Sim}(k)$ is integrable over σ, with the integral equal to $(R)\int_\sigma f$.

Proposition 3.2.9 Let $\sigma \in \text{Sim}(k)$, and let $f : \sigma \to R$ be continuous. Then f is integrable over σ and

$$\int_\sigma f = (R)\int_\sigma f.$$

Proof. Let $\varepsilon > 0$ be given. Since σ is compact, f is uniformly continuous, so there is a number $\gamma > 0$ such that for all $x, y \in \sigma$, $|x - y| < \gamma$ implies $|f(x) - f(y)| < \varepsilon$. Define a constant σ-gauge δ by $\delta(x) = \gamma$ for all $x \in \sigma$. Let $\tau = \{(t_i, T_i) | i \in \bar{m}\}$ be any δ-fine OV(σ)-bounded k-partition of σ. Since f is continuous, it follows from Theorem 1.2.6 and Theorem 1.2.7(ii) that f is (R)-integrable on σ and on T_i for all $i \in \bar{m}$, and

$$(\mathrm{R})\int_\sigma f = \sum_{i=1}^{m} (\mathrm{R})\int_{T_i} f.$$

For each $i \in \overline{m}$, let $M_i = \max\{f(x) \mid x \in T_i\}$ and $m_i = \min\{f(x) \mid x \in T_i\}$. Thus

$$m_i \mathrm{Vol}(T_i) = (\mathrm{R})\int_{T_i} m_i \le (\mathrm{R})\int_{T_i} f \le (\mathrm{R})\int_{T_i} M_i = M_i \mathrm{Vol}(T_i)$$

for all $i \in \overline{m}$. If we can show that for all $i \in \overline{m}$,

(3) $$\left| f(t_i)\mathrm{Vol}(T_i) - (\mathrm{R})\int_{T_i} f \right| < \varepsilon \mathrm{Vol}(T_i),$$

then

$$\begin{aligned}\left|\mathrm{R}(f, \tau) - (\mathrm{R})\int_\sigma f\right| &= \left|\sum_{i=1}^{m} f(t_i)\mathrm{Vol}(T_i) - \sum_{i=1}^{m} (\mathrm{R})\int_{T_i} f\right| \\ &\le \sum_{i=1}^{m} \left| f(t_i)\mathrm{Vol}(T_i) - (\mathrm{R})\int_{T_i} f \right| \\ &< \varepsilon \sum_{i=1}^{m} \mathrm{Vol}(T_i) \\ &= \varepsilon \mathrm{Vol}(\sigma).\end{aligned}$$

This will show that f is integrable over σ and $\int_\sigma f = (\mathrm{R})\int_\sigma f$, as required.

To prove (3), let $i \in \overline{m}$ be fixed. Note that $m_i = f(x_i)$ for some $x_i \in T_i$,

$$m_i \mathrm{Vol}(T_i) \le f(t_i)\mathrm{Vol}(T_i) \le M_i \mathrm{Vol}(T_i),$$

and

$$m_i \mathrm{Vol}(T_i) \le (\mathrm{R})\int_{T_i} f \le M_i \mathrm{Vol}(T_i).$$

Since $|x_i - t_i| < \delta(t_i) = \gamma$, so $|f(t_i) - m_i| < \varepsilon$. If $(\mathrm{R})\int_{T_i} f \le f(t_i)\mathrm{Vol}(T_i)$, then

$$\begin{aligned}\left| f(t_i)\mathrm{Vol}(T_i) - (\mathrm{R})\int_{T_i} f \right| &= f(t_i)\mathrm{Vol}(T_i) - (\mathrm{R})\int_{T_i} f \\ &\le f(t_i)\mathrm{Vol}(T_i) - m_i \mathrm{Vol}(T_i) \\ &= |f(t_i) - m_i| \mathrm{Vol}(T_i) \\ &< \varepsilon \mathrm{Vol}(T_i).\end{aligned}$$

The other case can be proved similarly.

This completes the proof.

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3.3 Integration of Differential Forms

This section is based on the material in [7] and [10], which can be consulted for further details.

To say that f is a C^1-mapping of a compact set $D \subseteq R^k$ into R^n will mean that there is a C^1-mapping g of an open set $W \subseteq R^k$ into R^n such that $D \subseteq W$ and $g(x) = f(x)$ for all $x \in D$.

Suppose E is a nonempty open subset of R^n. A *k-surface* in E is a C^1-mapping φ from a compact set $D \subseteq R^k$ into E. The compact set D is called the *parameter domain* of φ. For this thesis we shall confine ourselves to the simple situation in which D is the standard k-simplex $\Delta(k)$.

We stress that k-surfaces in E are defined to be mappings into E, not subsets of E. This agrees with the standard definition of curves. In fact, the 1-surfaces are precisely the continuously differentiable curves.

A mapping $f{:}R^k \to R^n$ is said to be *affine* iff $f - f(0)$ is linear. In other words, the requirement is that there exists $A \in \mathrm{Lin}(R^k, R^n)$ such that

$$f(x) = f(0) + A(x)$$

for all $x \in R^k$.

Let $e_1, \ldots, e_k$ be the standard basis for R^k.

Lemma 3.3.1 Let $p_0, p_1, \ldots, p_k \in R^n$. Let $\gamma{:}R^k \to R^n$ be defined by

$$\gamma(x) = p_0 + \sum_{i=1}^{k} x^{(i)}(p_i - p_0)$$

for all $x \in R^k$. Then γ is an affine C^∞-mapping.

Proof. It is clear that $\gamma - \gamma(0) = \gamma - p_0$ is linear, and thus γ is affine. By the same argument given in the proof of Corollary 3.2.8, $\gamma \in C^\infty(R^k)$.

#

Let E be a nonempty open subset of R^n, and let $p_0, p_1, \ldots, p_k$ be elements of E such that the convex hull of the set $\{p_0, p_1, \ldots, p_k\}$, denoted by Conv($\{p_0, p_1, \ldots, p_k\}$), is a subset of E. The *oriented affine k-simplex* $\sigma = [p_0, p_1, \ldots, p_k]$ is the k-surface in E with parameter domain $\Delta(k)$ such that $\sigma(x) = \gamma(x)$ for all $x \in \Delta(k)$, where γ is as in the above lemma. Note that

(i) if $\sigma: \Delta(k) \to E$ is an oriented affine k-simplex, then $\sigma = [p_0, p_1, \ldots, p_k]$, where $p_0 = \sigma(0)$ and $p_i = \sigma(e_i)$ for all $i \in \bar{k}$,

(ii) if $\sigma = [0, p_1, \ldots, p_k]$, then σ is linear and $\sigma(e_i) = p_i$ for all $i \in \bar{k}$,

and

(iii) if $\sigma = [p_0, p_1, \ldots, p_k]$, then $\sigma(x) = p_0 + A(x)$ for all $x \in \Delta(k)$, where $A = [0, p_1 - p_0, \ldots, p_k - p_0]$.

Suppose, in addition, that $p_1 - p_0, \ldots, p_k - p_0$ are linearly independent. Note that the k-simplex $\Delta(p_0, p_1, \ldots, p_k)$ is a subset of E, whereas the oriented affine k-simplex $[p_0, p_1, \ldots, p_k]$ is a function mapping $\Delta(k)$ into E. This can be confusing, but it is standard practice. There is a connection between these two objects, however: The image of the function $[p_0, p_1, \ldots, p_k]$ is the set $\Delta(p_0, p_1, \ldots, p_k)$. (In general, the image of the function $[p_0, p_1, \ldots, p_k]$ is the set Conv($\{p_0, p_1, \ldots, p_k\}$).)

Definition 3.3.2 Let E be a nonempty open subset of R^n. Let $\omega = \sum_I b_I \, dx^{(I)}$ be the standard presentation of a k-form ω on E, and let $\sigma = [p_0, p_1, \ldots, p_k]$ be an oriented affine k-simplex in E. We define the integral of ω over σ by the equation

$$\int_{\sigma} \omega \; = \; \sum_{I=(i_1,\ldots,i_k)} \int_{\Delta(k)} b_I \circ \sigma(z) \frac{\partial(\sigma^{(i_1)},\ldots,\sigma^{(i_k)})}{\partial(x^{(1)},\ldots,x^{(k)})}(z) \; dz$$

provided that all integrals on the right exist.

Any point p of E is also called an oriented affine 0-simplex in E, denoted by $[p]$. If p is a point of E and if f is a real function on E, we define

$$\int_{[p]} f \; = f(p).$$

Although we have just defined an object called an *oriented* affine simplex, in fact in general there is no way to define the orientation of such an object. However, there is one situation where the orientation can be defined in a natural way. This happens when $k = n$ and when the vectors $p_1 - p_0, \ldots, p_k - p_0$ are linearly independent.

Suppose that $p_0, p_1, \ldots, p_k \in R^k$ are such that $p_1 - p_0, \ldots, p_k - p_0$ are linearly independent, and let $\sigma = [p_0, p_1, \ldots, p_k]$ be the corresponding oriented affine k-simplex. Define the orientation of σ as follows: Let $A \in \mathrm{Lin}(R^k, R^k)$ be such that $\sigma(x) = p_0 + A(x)$ for all $x \in \Delta(k)$. Since $p_1 - p_0, \ldots, p_k - p_0$ is a basis for R^k, it follows that A is invertible. Hence $\det[A] \neq 0$. The oriented affine k-simplex σ is said to be *positively* (respectively, *negatively*) oriented iff $\det[A] > 0$ (respectively, $\det[A] < 0$). In particular, the oriented affine k-simplex $[0, e_1, \ldots, e_k]$, given by the identity mapping, has positive orientation.

Let E be a nonempty open subset of R^n. Let $k \in Z^+ \cup \{0\}$, and let $\mathrm{Aff}_E(k)$ denote the set of all oriented affine k-simplexes in E. Let F be the free abelian group on $\mathrm{Aff}_E(k)$ (see [3], § II.1). Thus the elements of F consist of finite formal sums $\sum_{i=1}^{l} m_i \, \sigma_i$, where $m_i \in Z$ and $\sigma_i \in \mathrm{Aff}_E(k)$ for all $i \in \overline{l}$. Let S be the subgroup of F

generated by the set

$$\{[p_0, p_1, \ldots, p_k] - [p_{\sigma(0)}, p_{\sigma(1)}, \ldots, p_{\sigma(k)}] \mid [p_0, p_1, \ldots, p_k] \in \mathrm{Aff}_E(k) \text{ and } \sigma \text{ is an even permutation on the set } \{0, 1, \ldots, k\}\}$$

$$\cup \ \{[p_0, p_1, \ldots, p_k] + [p_{\sigma(0)}, p_{\sigma(1)}, \ldots, p_{\sigma(k)}] \mid [p_0, p_1, \ldots, p_k] \in \mathrm{Aff}_E(k) \text{ and } \sigma \text{ is an odd permutation on the set } \{0, 1, \ldots, k\}\}.$$

The elements of the quotient group F/S are called the *affine k-chains in E.* Technically, an affine k-chain is written as a coset $\sum_{i=1}^{l} m_i \, \sigma_i \ + \ S$, where $m_i \in Z$ and $\sigma_i \in \mathrm{Aff}_E(k)$ for all $i \in \overline{l}$. However, the standard notation is to omit the $+ S$, and just write it as $\sum_{i=1}^{l} m_i \, \sigma_i$. When do this, we then have that for any $[p_0, p_1, \ldots, p_k] \in \mathrm{Aff}_E(k)$ and any permutation σ on $\{0, 1, \ldots, k\}$,

$$[p_{\sigma(0)}, p_{\sigma(1)}, \ldots, p_{\sigma(k)}] \ = \ [p_0, p_1, \ldots, p_k]$$

(as k-chains) if σ is an even permutation and

$$[p_{\sigma(0)}, p_{\sigma(1)}, \ldots, p_{\sigma(k)}] \ = -[p_0, p_1, \ldots, p_k]$$

(as k-chains) if σ is an odd permutation. Finally, if $\Gamma = \sum_{i=1}^{l} m_i \, \sigma_i$ is an affine k-chain and ω is a k-form on E, we define

$$\int_\Gamma \ \omega \ = \ \sum_{i=1}^{l} m_i \int_{\sigma_i} \ \omega$$

provided that the integral $\int_{\sigma_i} \omega$ exists whenever $m_i \neq 0$. It is straightforward, but tedious, to show that $\int_\Gamma \omega$ is well-defined. (Note that the representation $\Gamma = \sum_{i=1}^{l} m_i \sigma_i$ is not unique.)

The boundary of the oriented affine k-simplex $\sigma = [p_0, p_1, \ldots, p_k]$, denoted by $\partial\sigma$, is defined to be the affine $(k-1)$-chain

$$\partial\sigma = \sum_{i=0}^{k} (-1)^i [p_0, \ldots, \hat{p}_i, \ldots, p_k].$$

For example, if $\sigma = [p_0, p_1, p_2]$, then

$$\begin{aligned} \partial\sigma &= [p_1, p_2] - [p_0, p_2] + [p_0, p_1] \\ &= [p_0, p_1] + [p_1, p_2] + [p_2, p_0] \end{aligned}$$

which coincides with the usual notion of the oriented boundary of a triangle.

Suppose that $p_0, p_1, \ldots, p_k \in R^k$ are such that $p_1 - p_0, \ldots, p_k - p_0$ are linearly independent. If $\sigma = [p_0, p_1, \ldots, p_k]$ is a positively oriented affine k-simplex, then it is can be regarded as the k-simplex $\Delta(p_0, p_1, \ldots, p_k)$, and conversely. We will make use of this dual viewpoint in the proof of Stokes' theorem.