CHAPTER II

DIFFERENTIAL FORMS

This chapter is a summary of well-known material on differential forms which will be needed later on. It is based on the material in [7] and [11], and does not include any proofs. The first section builds up the algebraic background needed for the definition of the differential forms, while the second section discusses differential forms themselves.

2.1 Algebraic Preliminaries

If V is a vector space over R, then a function $T{:}V^k \to R$ will be called *multilinear* iff for all $i \in \overline{k}$, for all $v_1, \ldots, v_i, \ldots, v_k, v_i' \in V$, and for all $a \in R$,

$$T(v_1, \ldots, v_i + v_i', \ldots, v_k) = T(v_1, \ldots, v_i, \ldots, v_k) + T(v_1, \ldots, v_i', \ldots, v_k), \text{ and}$$
$$T(v_1, \ldots, av_i, \ldots, v_k) = aT(v_1, \ldots, v_i, \ldots, v_k).$$

A multilinear function $T{:}V^k \to R$ is also called a *k-tensor on V*, and the set of all k-tensors on V, denoted by $\mathrm{L}^k(V)$, becomes a vector space over R if for S, $T \in \mathrm{L}^k(V)$ and $a \in R$, we define

$$(S+T)(v_1, \ldots, v_k) = S(v_1, \ldots, v_k) + T(v_1, \ldots, v_k), \text{ and}$$
$$(aS)(v_1, \ldots, v_k) = aS(v_1, \ldots, v_k)$$

for all $(v_1, \ldots, v_k) \in V^k$. There is an additional operation connecting the various spaces $\mathrm{L}^k(V)$. If $S \in \mathrm{L}^k(V)$, and $T \in \mathrm{L}^l(V)$, we define the *tensor product* $S \otimes T \in \mathrm{L}^{k+l}(V)$ by

$$S \otimes T(v_1, \ldots, v_k, v_{k+1}, \ldots, v_{k+l}) = S(v_1, \ldots, v_k) \cdot T(v_{k+1}, \ldots, v_{k+l}).$$

The following properties are very easy to prove:

(i) $\quad (S_1 + S_2) \otimes T = (S_1 \otimes T) + (S_2 \otimes T),$

(ii) $S \otimes (T_1 + T_2) = (S \otimes T_1) + (S \otimes T_2)$,

(iii) $(aS) \otimes T = S \otimes (aT) = a(S \otimes T)$, and

(iv) $(S \otimes T) \otimes U = S \otimes (T \otimes U)$.

Both $(S \otimes T) \otimes U$ and $S \otimes (T \otimes U)$ are usually denoted simply by $S \otimes T \otimes U$; higher-order products $T_1 \otimes \cdots \otimes T_r$ are defined similarly.

Note that $L^1(V)$ is just the dual space V^*. The operation $\otimes$ allows us to express the other vector spaces $L^k(V)$ in terms of $L^1(V)$.

Theorem 2.1.1 ([11], Theorem 4.1) Let $v_1, \ldots, v_n$ be a basis for V, and let $\phi_1, \ldots, \phi_n$ be the dual basis, i.e., for all $i, j \in \bar{n}$, $\phi_i(v_j) = \delta_{ij}$. Then

$$B = \{ \phi_{i_1} \otimes \cdots \otimes \phi_{i_k} \mid i_1, \ldots, i_k \in \bar{n} \}$$

is a basis for $L^k(V)$, which therefore has dimension n^k.

Note: To simplify the notation, if $I = (i_1, \ldots, i_k)$ is an element of $(\bar{n})^k$, we will write ϕ_I for $\phi_{i_1} \otimes \cdots \otimes \phi_{i_k}$. Thus, the basis B above can be written as $\{ \phi_I \mid I \in (\bar{n})^k \}$.

One important construction, familiar for the case of dual spaces, can also be made for tensors. If $f: V \to W$ is a linear transformation, then the linear transformation $f^*: L^k(W) \to L^k(V)$ is defined by

$$f^*(T)(v_1, \ldots, v_k) = T(f(v_1), \ldots, f(v_k))$$

for $T \in L^k(W)$ and $(v_1, \ldots, v_k) \in V^k$. It is easy to verify that

$$f^*(S \otimes T) = f^*(S) \otimes f^*(T).$$

A k-tensor $\omega \in L^k(V)$ is called *alternating* iff

$$\omega(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = -\omega(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k)$$

for all $v_1, \ldots, v_k \in V$ and for all $i, j \in \bar{k}$ with $i \neq j$. (In this equation v_i and v_j are interchanged and all other v_m's are left fixed.) The set of all alternating k-tensors on V, denoted by $A^k(V)$, is clearly a subspace of $L^k(V)$.

Let Perm($\bar{k}$) be the set of all permutations on $\bar{k}$. Recall that the sign of $\sigma \in$ Perm($\bar{k}$), denoted by $sgn(\sigma)$, is +1 iff σ is even and −1 iff σ is odd. If $T \in \mathrm{L}^k(V)$, we define $Alt(T): V^k \to R$ by

$$Alt(T)(v_1,\dots,v_k) \;=\; \frac{1}{k!}\sum_{\sigma\in \mathrm{Perm}(\bar{k})}\mathrm{sgn}(\sigma)T(v_{\sigma(1)},\dots,v_{\sigma(k)})$$

for all $(v_1,\dots,v_k) \in V^k$.

Theorem 2.1.2 ([11], Theorem 4.3)

(i) If $T \in \mathrm{L}^k(V)$, then $Alt(T) \in \mathrm{A}^k(V)$.

(ii) If $\omega \in \mathrm{A}^k(V)$, then $Alt(\omega) = \omega$.

(iii) If $T \in \mathrm{L}^k(V)$, then $Alt(Alt(T)) = Alt(T)$.

Proposition 2.1.3 If $k \in Z^+$, then Alt: $\mathrm{L}^k(V) \to \mathrm{A}^k(V)$ is linear.

To determine the dimension of $\mathrm{A}^k(V)$, we would like a theorem analogous to Theorem 2.1.1. There is one small problem which must be solved before we can prove such a theorem, however: If $\omega \in \mathrm{A}^k(V)$ and $\eta \in \mathrm{A}^l(V)$, then $\omega \otimes \eta$ is usually not in $\mathrm{A}^{k+l}(V)$. We will therefore define a new product, the *wedge product* $\omega \wedge \eta \in \mathrm{A}^{k+l}(V)$ by

$$\omega \wedge \eta = \frac{(k+l)!}{k!\,l!}\,Alt(\omega \otimes \eta).$$

Proposition 2.1.4 (see [11], p.79) Let $\omega, \omega_1, \omega_2 \in \mathrm{A}^k(V)$, let $\eta, \eta_1, \eta_2 \in \mathrm{A}^l(V)$, and let $a \in R$. Then:

(i) $(\omega_1 + \omega_2) \wedge \eta = (\omega_1 \wedge \eta) + (\omega_2 \wedge \eta)$.

(ii) $\omega \wedge (\eta_1 + \eta_2) = (\omega \wedge \eta_1) + (\omega \wedge \eta_2)$.

(iii) $(a\omega) \wedge \eta = \omega \wedge (a\eta) = a(\omega \wedge \eta)$.

(iv) $\omega \wedge \eta = (-1)^{kl}\, \eta \wedge \omega$.

Proposition 2.1.5 (see [11], p.79) Let $f: V \to W$ be linear, $\omega \in A^k(W)$, and $\eta \in A^l(W)$. Then $f^*(\omega) \in A^k(V)$, $f^*(\eta) \in A^l(V)$, and

$$f^*(\omega \wedge \eta) = f^*(\omega) \wedge f^*(\eta).$$

Proposition 2.1.6 ([11], Theorem 4.4)

(i) If $S \in L^k(V)$, $T \in L^l(V)$, and $Alt(S) = 0$, then $Alt(S \otimes T) = Alt(T \otimes S) = 0$.

(ii) If $\omega \in L^k(V)$, $\eta \in L^l(V)$, and $\theta \in L^m(V)$, then

$$Alt(Alt(\omega \otimes \eta) \otimes \theta) = Alt(\omega \otimes \eta \otimes \theta) = Alt(\omega \otimes Alt(\eta \otimes \theta)).$$

(iii) If $\omega \in A^k(V)$, $\eta \in A^l(V)$, and $\theta \in A^m(V)$, then

$$(\omega \wedge \eta) \wedge \theta = \omega \wedge (\eta \wedge \theta) = \frac{(k+l+m)!}{k!\,l!\,m!} Alt(\omega \otimes \eta \otimes \theta).$$

Naturally, $(\omega \wedge \eta) \wedge \theta$ and $\omega \wedge (\eta \wedge \theta)$ are both denoted simply by $\omega \wedge \eta \wedge \theta$. More generally, if $\omega_i \in A^{k_i}(V)$ $(i = 1, \ldots, r)$, we define

$$\omega_1 \wedge \ldots \wedge \omega_r \quad = \quad \frac{(k_1 + \cdots + k_r)!}{k_1! \cdots k_r!} Alt(\omega_1 \otimes \cdots \otimes \omega_r).$$

Lemma 2.1.7 If $\omega \in A^k(V)$, $\eta \in A^l(V)$, and $\omega = 0$, then $\omega \wedge \eta = 0 = \eta \wedge \omega$.

Lemma 2.1.8 Let $k, n \in Z^+$ be such that $2 \le k \le n$, and let V be a vector space over R with $\dim(V) = n$. If $v_1, \ldots, v_n$ is a basis for V, and if $\phi_1, \ldots, \phi_n$ is the dual basis, then

$$\phi_{i_1} \wedge \ldots \wedge \phi_{i_k} = 0$$

whenever $i_1, \ldots, i_k \in \bar{n}$ are not all distinct.

Theorem 2.1.9 ([11], Theorem 4.5) Let $k, n \in Z^+$ be such that $k \le n$, and let V be a vector space over R with $\dim(V) = n$. If $v_1, \ldots, v_n$ is a basis for V, and if $\phi_1, \ldots, \phi_n$ is the dual basis, then

$$\{ \phi_{i_1} \wedge \ldots \wedge \phi_{i_k} \mid 1 \le i_1 < i_2 < \ldots < i_k \le n \}$$

is a basis for $A^k(V)$, which therefore has dimension $\dfrac{n!}{k!(n-k)!}$.

As we did after Theorem 2.1.1, we will introduce some simplifying notation. If $I = (i_1, ..., i_k) \in (\bar{n})^k$, let $\psi_I = \phi_{i_1} \wedge ... \wedge \phi_{i_k}$. Also we will call I an *ascending* k-tuple iff $i_1 < i_2 < ... < i_k$. Thus, the above basis can be written as

$$\{ \psi_I \mid I \text{ is an ascending } k\text{-tuple from the set } \bar{n} \}$$

The ψ_I, where I ranges over all ascending k-tuples from the set $\bar{n}$, are called the *elementary alternating k-tensors on V.*

Proposition 2.1.10 (see [11], p.84) Let $k, n \in Z^+$ be such that $k \leq n$, and let $I = (i_1, ..., i_k)$ and $J = (j_1, ..., j_k)$ be ascending k-tuples from the set $\bar{n}$. Let $e_1, ..., e_n$ be the standard basis for R^n, and let $\phi_1, ..., \phi_n$ be the dual basis. Then:

(i) $$\phi_{i_1} \wedge ... \wedge \phi_{i_k} (e_{j_1}, ..., e_{j_k}) = \begin{cases} 1 & \text{if } I = J, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) For all $v_1, ..., v_k \in R^n$, $\phi_{i_1} \wedge ... \wedge \phi_{i_k} (v_1, ..., v_k)$ is the determinant of the $k \times k$ minor of the matrix

$$\begin{bmatrix} v_1^{(1)} & \cdots & v_k^{(1)} \\ \vdots & & \vdots \\ v_1^{(n)} & \cdots & v_k^{(n)} \end{bmatrix}$$

obtained by selecting rows $i_1, ..., i_k$. That is,

$$\phi_{i_1} \wedge ... \wedge \phi_{i_k} (v_1, ..., v_k) = \det \begin{bmatrix} v_1^{(i_1)} & \cdots & v_k^{(i_1)} \\ \vdots & & \vdots \\ v_1^{(i_k)} & \cdots & v_k^{(i_k)} \end{bmatrix}.$$

2.2 Tangent Spaces and Differential Forms

Now we introduce the concept of a tensor field. We are primarily interested in the special case of an alternating tensor field, which is called a *differential form*. Next, we shall introduce a certain operator on differential forms, called the *differential* *operator* d, which is a unifying generalization of the operators *grad*, *curl*, and *div*. This operator is crucial in the formulation of the basic theorems concerning integrals of differential forms, which we shall discuss in the next two chapters.

Let n be in Z^+. Given $x \in R^n$, we define a *tangent vector to* R^n *at* x to be a pair $(x; v)$, where $v \in R^n$. The set of all tangent vectors to R^n at x forms a vector space over R if we define

$$(x; v) + (x; w) = (x; v+w),$$
$$c(x; v) = (x; cv),$$

where $(x; v)$ and $(x; w)$ are tangent vectors and $c \in R$. It is called the *tangent space to* R^n *at* x, and is denoted by $\mathrm{T}_x(R^n)$.

Let E be a nonempty open subset of R^n, and let k be in Z^+ such that $k \leq n$. A *k-tensor field in* E is a function ω assigning, to each $x \in E$, a k-tensor on the vector space $\mathrm{T}_x(R^n)$. That is,

$$\omega(x) \in \mathrm{L}^k(\mathrm{T}_x(R^n))$$

for each $x \in E$. Thus $\omega(x)$ is a multilinear function mapping k-tuples of tangent vectors to R^n at x into R. If $\omega(x) \in \mathrm{A}^k(\mathrm{T}_x(R^n))$ for each $x \in E$, then ω is called a *differential form of order k on E* (or simply, a *k-form on E*).

Let $e_1, \ldots, e_n$ be the standard basis for R^n. Then $(x; e_1), \ldots, (x; e_n)$ is called the *standard basis* for $\mathrm{T}_x(R^n)$. We define a 1-form $\overline{\phi}_i$ on R^n by the equation

$$\overline{\phi}_i(x)((x; e_j)) = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases}$$

The forms $\overline{\phi}_1, \ldots, \overline{\phi}_n$ are called the *elementary 1-forms* on R^n. Similarly, given an ascending k-tuple $I = (i_1, \ldots, i_k)$ from the set $\overline{n}$, we define a k-form $\overline{\psi}_I$ on R^n by the equation

$$\overline{\psi}_I(x) = \overline{\phi}_{i_1}(x) \wedge \ldots \wedge \overline{\phi}_{i_k}(x).$$

The forms $\overline{\psi}_I$ are called the *elementary k-forms on* R^n.

Note that for each $x \in R^n$, the 1-tensors $\overline{\phi}_1(x), \ldots, \overline{\phi}_n(x)$ constitute the basis for $L^1(T_x(R^n))$ dual to the standard basis for $T_x(R^n)$, the k-tensor $\overline{\psi}_I(x)$ is the corresponding elementary alternating k-tensor on $T_x(R^n)$, and that for all $v, v_1, \ldots, v_n \in R^n$,

$$\overline{\phi}_i(x)((x; v)) = v^{(i)},$$

and

$$\overline{\psi}_I(x)((x; v_1), \ldots, (x; v_k)) = \det \begin{bmatrix} v_1^{(i_1)} & \cdots & v_k^{(i_1)} \\ \vdots & & \vdots \\ v_1^{(i_k)} & \cdots & v_k^{(i_k)} \end{bmatrix},$$

where $I = (i_1, \ldots, i_k)$ is an ascending k-tuple from the set $\overline{n}$.

If ω is a k-form on an open subset E of R^n, then for each $x \in E$, the k-tensor $\omega(x)$ can be written uniquely in the form

$$\omega(x) = \sum_I b_I(x)\, \overline{\psi}_I(x), \tag{1}$$

for some scalars $b_I(x)$. The summation in (1) extends over all ascending k-tuples from the set $\overline{n}$. The functions b_I are called the *components* of ω relative to the elementary k-forms on R^n. We call ω *differentiable* iff all of the functions b_I are differentiable.

Let $f : E \to R$ be a function; then f is called a *scalar field* in E. We also call f a *differential form of order* 0 (briefly, a *0-form*) on E. If f is differentiable on E, then it is said to be a *differentiable 0-form.*

We define the wedge product of two 0-forms f and g by the rule $f \wedge g = f \cdot g$, which is just the usual product of real functions. More generally, we define the wedge product of the 0-form f and the k-form ω by the rule

$$(\omega \wedge f)(x) = (f \wedge \omega)(x) = f(x) \cdot \omega(x);$$

this is just the usual product of the tensor $\omega(x)$ and the scalar $f(x)$.

We now introduce an operator d on differentiable forms. In general, the operator d, when applied to a k-form, gives a $(k+1)$-form. We begin by defining d for differentiable 0-forms.

Definition 2.2.1 Let E be a nonempty open subset of R^n, let $f: E \to R$ be differentiable on E. We define a 1-form df on E by the formula

$$df(x)((x; v)) = f'(x) \cdot v = \sum_{i=1}^{n} D_i f(x) v^{(i)}$$

for all $x \in A$ and for all $v \in R^n$, where f' denotes the derivative (gradient) of f, and $D_i f$ denotes the partial derivative of f with respect to its i^{th} argument. The 1-form df is called the *differential of f*.

Using the operator d, we can obtain a new way of expressing the elementary 1-forms $\bar{\phi}_i$ on R^n.

Lemma 2.2.2 ([7], Lemma 30.2) Let $\bar{\phi}_1, \ldots, \bar{\phi}_n$ be the elementary 1-forms on R^n. Then $d\pi^{(i)} = \bar{\phi}_i$ for all $i \in \bar{n}$. (See Section 1.1 for the definition of $\pi^{(i)}$.)

It is common in this subject to abuse notation slightly, denoting the i^{th} projection function not by $\pi^{(i)}$ but by $x^{(i)}$. Then in this notation, $\bar{\phi}_i$ is equal to $dx^{(i)}$. We shall use this notation henceforth:

Convention: If x denotes a general point of R^n, we denote the i^{th} projection function mapping R^n into R by the symbol $x^{(i)}$. Then $dx^{(i)}$ equals the elementary 1-form $\bar{\phi}_i$ on R^n. If $I = (i_1, \ldots, i_k)$ is an ascending k-tuple from the set $\bar{n}$, then we introduce the notation

$$dx^{(I)} = dx^{(i_1)} \wedge \ldots \wedge dx^{(i_k)}$$

for the elementary k-form $\overline{\psi}_I$ on R^n. A general k-form ω can then be written uniquely in the form

$$(2) \qquad \omega = \sum_I b_I\, dx^{(I)},$$

for some scalar functions b_I, where I ranges over all ascending k-tuples from the set $\bar{n}$. This is called the *standard presentation* of ω.

The forms $dx^{(i)}$ and $dx^{(I)}$ are of course characterized by the equations

$$dx^{(i)}(x)((x; v)) = v^{(i)}$$

and

$$dx^{(I)}(x)((x; v_1), \ldots, (x; v_k)) = \det \begin{bmatrix} v_1^{(i_1)} & \cdots & v_k^{(i_1)} \\ \vdots & & \vdots \\ v_1^{(i_k)} & \cdots & v_k^{(i_k)} \end{bmatrix},$$

where $I = (i_1, \ldots, i_k)$ is an ascending k-tuple from the set $\bar{n}$.

For convenience, we extend this notation to an arbitrary k-tuple $J = (j_1, \ldots, j_k)$ from the set $\bar{n}$, putting

$$dx^{(J)} = dx^{(j_1)} \wedge \ldots \wedge dx^{(j_k)}.$$

Note that whereas $dx^{(j)}$ is the differential of a 0-form, $dx^{(J)}$ does not denote the differential of a form, but rather a wedge product of elementary 1-forms.

Theorem 2.2.3 ([7], Theorem 30.3) Let E be a nonempty open subset of R^n; let $f: A \to R$ be differentiable on E. Then

$$df = (D_1 f)dx^{(1)} + \cdots + (D_n f)dx^{(n)}.$$

In particular, $df = 0$ if f is constant.

Definition 2.2.4 Let $k, n \in Z^+$ be such that $k < n$, and let E be a nonempty open subset of R^n. Let ω be a differentiable k-form on E, with the standard presentation

$$\omega = \sum_{I} b_I \, dx^{(I)},$$

and define

$$d\omega = \sum_{I} (db_I) \wedge dx^{(I)}$$

$$= \sum_{I} \sum_{j=1}^{n} (D_j b_I) \, dx^{(j)} \wedge dx^{(I)}.$$