CHAPTER 5

THE EULER – LAGRANGE DIFFERENTIAL EQUATION.

The problem discussed in chapters 2, 3 and 4 is that of finding a curve $y = y(x)$ satisfying the conditions $y(x_o) = y_o$, $y(x_n) = y_n$ which make the integral

$$I = \int_{x_o}^{x_n} F\,(x, y, y')\;dx \qquad (5.1)$$

a minimum.

In the direct method for solving this problem we consider the functional I for argument functions which are polygonal curves, consisting of line segments whose vertices have the fixed abscissae $x_o, x_1, \ldots\ldots x_n$. Along such polygonal curves the functional I is a function of $y_1, y_2, \ldots\ldots y_{n-1}$, which are the ordinates of the vertices of the polygonal curves, that is

$$I = \int_{x_o}^{x_n} F(x, y, y')\,dx$$

$$= \psi\,(y_1, y_2, \ldots\ldots y_{n-1}) .$$

We approximate the value of I along each segment of the polygonal curve by putting

$$X_i = \frac{x_i + x_{i+1}}{2}$$

$$Y_i = \frac{y_i + y_{i+1}}{2}$$

$$Y_i' = \frac{y_{i+1} - y_i}{\Delta x}$$

$$dx = \Delta x = x_{i+1} - x_i .$$

The functional I then becomes

$$I = \sum_{i=0}^{n-1} F\left(\frac{x_i + x_{i+1}}{2}, \frac{y_i + y_{i+1}}{2}, \frac{y_{i+1} - y_i}{\Delta x} \right) \Delta x \,.. \quad (5.2)$$

or $$I = \sum_{i=0}^{n-1} F(X_i, Y_i, Y_i') \Delta x.$$

By constructing a sequence of polygons P^k using the method of chapter 3 and 4 we obtain a monotomic decreasing sequence of values of I. Then the required polygon that makes the value of I a minimum is the polygon P^m whose vertices have ordinates satisfying the conditions $\frac{\partial \varphi}{\partial y_i} = 0, \quad i = 1, 2, \ldots\ldots \quad n - 1.$

<u>Theorem</u> As $n \longrightarrow \infty$ the polygon P^m approaches the curve which satisfies The Euler - Lagrange differential equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \qquad (5.3)$$

provided that $y_i \longrightarrow y_{i+1}$ for all i.

<u>Proof</u> Since

$$X_i = \frac{x_i + x_{i+1}}{2},$$

$$Y_i = \frac{y_i + y_{i+1}}{2},$$

$$Y_i' = \frac{y_{i+1} - y_i}{\Delta x},$$

we have

$$\psi(y_1, y_2, \ldots, y_{n-1}) = \sum_{i=0}^{n-1} F(X_i, Y_i, Y_i')\,\Delta x,$$

$$\frac{\partial \psi}{\partial y_i} = \sum_{i=0}^{n-1}\left[\frac{\partial F}{\partial X_i}\cdot\frac{\partial X_i}{\partial y_i} + \frac{\partial F}{\partial Y_i}\cdot\frac{\partial Y_i}{\partial y_i} + \frac{\partial F}{\partial Y_i'}\cdot\frac{\partial Y_i'}{\partial y_i}\right]\Delta x$$

$$= \left[\frac{\partial F}{\partial Y_{i-1}}\cdot\frac{1}{2} + \frac{\partial F}{\partial Y_i}\cdot\frac{1}{2} + \frac{\partial F}{\partial Y_{i-1}'}\left(\frac{1}{\Delta x}\right) + \frac{\partial F}{\partial Y_i'}\left(-\frac{1}{\Delta x}\right)\right]\Delta x$$

$$= 0 .$$

Therefore

$$\frac{1}{2}\left[\frac{\partial F}{\partial Y_{i-1}} + \frac{\partial F}{\partial Y_i}\right] - \frac{1}{\Delta x}\left[\frac{\partial F}{\partial Y_i'} - \frac{\partial F}{\partial Y_{i-1}'}\right] = 0 \qquad (5.4)$$

If $y_{i+1} \longrightarrow y_i$ for all i as $\Delta x \longrightarrow 0$

then since $$Y_i = \frac{y_{i+1} + y_i}{2}$$

we have $$Y_i \longrightarrow y_i$$

and $$\frac{\partial Y_i}{\partial y_i} \longrightarrow 1 , \quad i = 1, 2, \ldots\ldots n - 1 .$$

Then $$\frac{1}{2} \left[\frac{\partial F}{\partial Y_{i-1}} + \frac{\partial F}{\partial Y_i} \right] \longrightarrow \frac{1}{2} \left[\frac{\partial F}{\partial y_{i-1}} + \frac{\partial F}{\partial y_i} \right] \longrightarrow \frac{\partial F}{\partial y_i} .$$

Also $Y_i' = \dfrac{y_{i+1} - y_i}{\Delta x}$ approaches the slope at any point along the limitting curve so we may write

$$Y_i' = \frac{y_{i+1} - y_i}{\Delta x} \longrightarrow y_i' .$$

Then in the limit as $\Delta x \longrightarrow 0$ the equation (5.4) has the form (5.3)

Example 4.

a) In case of the shortest arc joining two given points,

$$I = \int_{x_0}^{x_1} \sqrt{1 + (y')^2} \, dx ,$$

where $F = F(y') = \sqrt{1 + (y')^2}$.

The solution of this problem is the solution of the Euler - Lagrange equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0,$$

which is $y = c_1 x + c_2$ ......

In example 1, chapter 2

$$y(0) = 0, \quad y(7) = 0.$$

Then the solution is

$$y = 0$$

which is the same result as that obtained by the direct method in example 1.

b) In the case of the surface of revolution of minimum area

$$I = 2\pi \int_{x_0}^{x_1} y\sqrt{1 + (y')^2} \, dx ,$$

where $F = F(y, y') = y\sqrt{1 + (y')^2}$.

The solution of this problem is the solution of the Euler - Lagrange equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 ,$$

which is

$$y = b \cosh \left(\frac{x - a}{b} \right) .$$

In example 2, chapter 3 the polygons obtained in the direct method converge to the curve $y = 2 \cosh \left(\frac{x - 1}{2} \right)$.

c) In the case of the Brachistochrone problem,

$$I = \frac{1}{\sqrt{2g}} \int_{x_0}^{x_{n+1}} \frac{\sqrt{1 + (y')^2}}{\sqrt{y}} \, dx$$

and $$F = \sqrt{\frac{1 + (y')^2}{y}} .$$

The solution of this problem is the solution of the Euler - Larganage equation.

$$\frac{\partial F}{\partial y} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) = 0 ,$$

which is

$$x = a\,(t - \sin t)$$

$$y = a\,(1 - \cos t) .$$

This is a cycloid.

In example 5, chapter 4. The polygons obtained by the direct method converge to the curve

$$x = 3\,(t - \sin t)$$

$$y = 3\,(1 - \cos t) .$$

APPENDIX (see page 15 and page 36)

(a) To make all $y_1^{k+1'}$ less than E we may choose $\mathcal{E}_1 = E + \frac{E}{2^n}$,

where n is the number of vertices of the polygon.

$$\text{Then} \quad y_o^{k+1'} = 0, \quad (\text{since } y_o^k = 0 \text{ for all } k)$$

$$y_1^{k+1'} = \frac{1}{2} (y_o^{k+1'} + y_2^{k'}) = \frac{1}{2} (y_2^{k'})$$

$$\leq \frac{1}{2} \mathcal{E}_1$$

$$\leq \frac{E}{2} + \frac{E}{2^n . 2}$$

$$< E \quad , \text{since } n > 1.$$

$$y_2^{k+1'} = \frac{1}{2} (y_1^{k+1'} + y_3^{k'})$$

$$=< \frac{1}{2} (\frac{1}{2} E + \frac{1}{2^n . 2} E + E + \frac{E}{2^n})$$

$$< \frac{E}{2^2} + \frac{E}{2^n . 2^2} + \frac{E}{2} + \frac{E}{2^n . 2} < E, \quad \text{since } n > 2$$

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$$\text{Similary} \quad y_{n-1}^{k+1'} = \frac{1}{2} (y_{n-2}^{k+1} + y_n^{k'})$$

$$y_{n-1}^{k+1'} < \frac{E}{2^{n-1}} + \frac{E}{2^n . 2^{n-1}} + \frac{E}{2^{n-2}} + \frac{E}{2.2^n 2^{n-2}} + \dots\dots + \frac{E}{2} + \frac{E}{2^n . 2}$$

$$= E (\frac{1}{2} + \frac{1}{2^2} + \dots\dots \frac{1}{2^{n-1}}) + \frac{E}{2^n} (\frac{1}{2} + \frac{1}{2^2} + \dots \frac{1}{2^{n-1}})$$

$$= E\left(\frac{\frac{1}{2}\left(1-\left(\frac{1}{2}\right)^{n-1}\right)}{1-\frac{1}{2}}\right) + \frac{E}{2^n}\left(\frac{\frac{1}{2}\left(1-\left(\frac{1}{2}\right)^{n-1}\right)}{1-\frac{1}{2}}\right)$$

$$= E\left(1-\left(\frac{1}{2}\right)^{n-1}\right) + \frac{E}{2^n}\left(1-\left(\frac{1}{2}\right)^{n-1}\right)$$

$$= E - \frac{E}{2^{n-1}} + \frac{E}{2^n} - \frac{E}{2^{2n-1}}$$

$$= E + \left(\frac{E}{2\cdot 2^{n-1}} - \frac{E}{2^{n-1}}\right) - \frac{E}{2^{2n-1}}$$

$$= E + \left(\frac{1}{2}-1\right)\frac{E}{2^{n-1}} - \frac{E}{2^{2n-1}}$$

$$= E - \frac{E}{2\cdot 2^{n-1}} - \frac{E}{2^{2n-1}}$$

$$< E$$

(b)

$$I = \frac{1}{\sqrt{2g}} \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \sqrt{\frac{1+\left(\frac{y_{i+1}-y_i}{\Delta x}\right)^2}{y}}\, dx \quad \text{---(4.3)}$$

$$= \frac{1}{\sqrt{2g}} \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \frac{\sqrt{1+\left(\frac{y_{i+1}-y_i}{\Delta x}\right)^2}}{\sqrt{\frac{y_{i+1}-y_i}{\Delta x}(x-x_i)+y_i}}\, dx$$

$$I = \frac{1}{\sqrt{2g}} \sum_{i=0}^{n-1} \int_0^{\Delta x} \frac{\sqrt{1 + \left(\frac{y_{i+1}-y_i}{\Delta x}\right)^2}}{\sqrt{\left(\frac{y_{i+1}-y_i}{\Delta x}\right) t + y_i}} \, dt$$

$$= \frac{1}{\sqrt{2g}} \sum_{i=0}^{n-1} \sqrt{1 + \left(\frac{y_{i+1}-y_i}{\Delta x}\right)^2} \left[\left(\frac{\Delta x}{y_{i+1}-y_i}\right) 2 \sqrt{\left(\frac{y_{i+1}-y_i}{\Delta x}\right) t + y_i} \right]_0^{\Delta x}$$

$$= \frac{2}{\sqrt{2g}} \sum_{i=0}^{n-1} \frac{\sqrt{1 + \left(\frac{y_{i+1}-y_i}{\Delta x}\right)^2}}{(y_{i+1} - y_i)} \left[\sqrt{y_{i+1}} - \sqrt{y_i} \right] \Delta x$$

$$= \frac{2}{\sqrt{2g}} \sum_{i=0}^{n-1} \frac{\sqrt{1 + \left(\frac{y_{i+1}-y_i}{\Delta x}\right)^2}}{\sqrt{y_{i+1}} + \sqrt{y_i}} \Delta x \qquad \text{-----}(4.4)$$