CHAPTER 4

BRACHISTOCHRONE PROBLEM.

Let $A(x_o, y_o)$ and $B(x_n, y_n)$ be two given points in the xy - plane. We want to find a curve $y = y(x)$ joining them so that a particle falls under gravity along this curve from A to B in the shortest time, friction being neglected.

Take the origin of the coordinate system at the point A, the y - axis directed vertically downward and the x- axis horizontal so that the passage from A to B is marked by an increase in x.

Let the particle start from A with the initial velocity zero, then the velocity of the particle at any point along the curve is given by $v = \frac{ds}{dt}$,. The total time of descent is

$$I = \int_{x_o}^{x_n} \frac{ds}{v} = \int_{x_o}^{x_n} \frac{(1 + (y')^2)^{\frac{1}{2}}}{v} dx \text{ ....(4.1)}$$

where the velocity v at any point distance y below the point A is computed by

$$\frac{1}{2} m v^2 = m g y$$

or

$$v = \sqrt{2 gy} .$$

It should be noted that y cannot be negative anywhere on the curve, otherwise the particle will fail to reach B. Then (4.1) becomes

$$I = \frac{1}{\sqrt{2g}} \int_{x_o}^{x_n} \sqrt{\frac{1 + (y')^2}{y}} dx. \text{ .... (4.2)}$$

We want to find the curve $y = y(x)$ that makes I a minimum.

In the direct method for solving this problem we consider argument functions which are polygonal curves as defined in chapters 2 and 3.

Then the equation (4.2) may be written

$$I = \frac{1}{\sqrt{2g}} \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \sqrt{\frac{1 + \left(\frac{y_{i+1} - y_i}{\Delta x}\right)^2}{y}} \, dx \;\ldots\ldots\ldots (4.3)$$

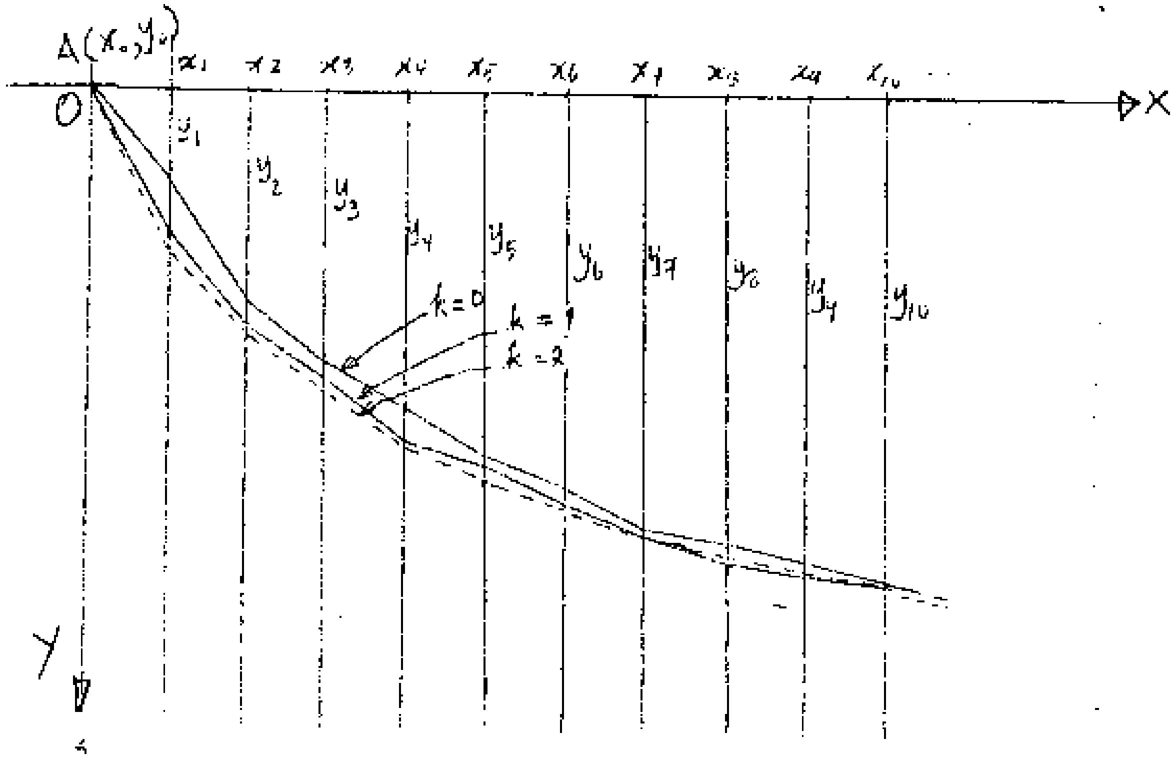

Fig. 10. Polygons constructed to solve the brachistochrone problem.

In order to compute the value of I along the segments of polygonal curves, the equation (4. 3) becomes

$$I = \frac{1}{\sqrt{2g}} \, 2 \sum_{i=0}^{n-1} \frac{\sqrt{1 + \left(\frac{y_{i+1} - y_i}{\Delta x}\right)^2}}{\sqrt{y_{i+1}} + \sqrt{y_i}} \cdot \Delta x \qquad \ldots\ldots(4.4) \; *$$

$$= \psi \, (y_1, y_2, \ldots\ldots \, y_{n-1})$$

Let $Y_i = \dfrac{y_i + y_{i+1}}{2}$,

$$Y_i' = \frac{y_{i+1} - y_i}{\Delta x} \quad , \text{ for } i = 0, 1, \ldots\ldots n-1.$$

Then $\psi \, (y_1, y_2, \ldots\ldots \, y_{n-1})$

$$= \frac{2}{\sqrt{2g}} \sum_{i=0}^{n-1} F \, (x_i, Y_i, Y_i') \, . \, \Delta x \, .$$

where $F \, (x_i, Y_i, Y_i') = \dfrac{\sqrt{1 + (Y_i')^2}}{\sqrt{Y_i + \frac{(\Delta x) Y_i'}{2}} + \sqrt{Y_i - \frac{(\Delta x) Y_i'}{2}}}$

(As in chapter 3 we include x_i in F)

Initially we choose an arbitrary polygon P^o whose vertices have the ordinates y_0^o , y_1^o ,..... y_n^o , none of which are negative.

Equation (4.4) then gives

$$I_o = \frac{2}{\sqrt{2g}} \sum_{i=0}^{n-1} \frac{\sqrt{1 + \left(\frac{y_{i+1}^o - y_i^o}{\Delta x}\right)^2}}{\sqrt{y_{i+1}^o} + \sqrt{y_i^o}} \cdot \Delta x,$$

where I_o is the time the particle falls from A to B along P^o.

---

* See Appendix (b).

Now by letting y_0^o and y_2^o remain fixed and by varying y_1^1 on $x = x_1$ the time of fall

$$\frac{2}{\sqrt{2g}}\left(\frac{\sqrt{(\Delta x)^2 + (y_1^1 - y_0^o)^2}}{\sqrt{y_1^1} + \sqrt{y_0^o}} + \frac{\sqrt{(\Delta x)^2 + (y_2^o - y_1^1)^2}}{\sqrt{y_2^o} + \sqrt{y_1^1}}\right)$$

along the new polygonal curve from x_0 to x_2 is made a minimum; this must be less than or equal to its previous value

$$\frac{2}{\sqrt{2g}}\left(\frac{\sqrt{(\Delta x)^2 + (y_1^o - y_0^o)^2}}{\sqrt{y_0^o} + \sqrt{y_1^o}} + \frac{\sqrt{(\Delta x)^2 + (y_2^o - y_1^o)^2}}{\sqrt{y_2^o} + \sqrt{y_1^o}}\right)$$

Again by letting y_1^1 and y_3^o remain fixed and by varying y_2^1 on $x = x_2$ the time of fall

$$\frac{2}{\sqrt{2g}}\left(\frac{\sqrt{(\Delta x)^2 + (y_2^1 - y_1^1)^2}}{\sqrt{y_2^1} + \sqrt{y_1^1}} + \frac{\sqrt{(\Delta x)^2 + (y_3^o - y_2^1)^2}}{\sqrt{y_3^o} + \sqrt{y_2^1}}\right)$$

along the new polygonal curve from $x = x_1$ to $x = x_3$ is made a minimum; this must be less than or equal to its previous value

$$\frac{2}{\sqrt{2g}}\left(\frac{\sqrt{(\Delta x)^2 + (y_2^o - y_1^1)^2}}{\sqrt{y_2^o} + \sqrt{y_1^1}} + \frac{\sqrt{(\Delta x)^2 + (y_3^o - y_2^o)^2}}{\sqrt{y_3^o} + \sqrt{y_2^o}}\right)$$

and so on.

Repeating the same process for the remaining interval to construct y_3^1, y_4^1 .... y_{n-1}^1, we obtain a new polygon P^1 the ordinates of the vertices of which are $y_0^1 = y_0^o$, y_1^1, y_2^1,.... $y_n^1 = y_n^o$. The value of the integral I along P^1 is

$$I_1 = \frac{2}{\sqrt{2g}} \sum_{i=0}^{n-1} \frac{\sqrt{(\Delta x)^2 + (y^1_{i+1} - y^1_i)^2}}{\sqrt{y^1_{i+1}} + \sqrt{y^1_i}}$$

Which must be less than or equal to I_o.

Next applying the same process to P^1 we get the polygon P^2, and by repeating the process again and again we obtain a sequence of polygons P^3, P^4,....... P^k,..... In the c se of the polygon P^k the ordinates of the vertices are $y^k_o = y^o_o$, y^k_1, y^k_2,...... y^k_{n-1}, $y^k_n = y^o_n$, and the value of the integral I is

$$I_k = \frac{2}{\sqrt{2g}} \sum_{i=0}^{n-1} \frac{\sqrt{(\Delta x)^2 + (y^k_{i+1} - y^k_i)^2}}{\sqrt{y^k_{i+1}} + \sqrt{y^k_i}}$$

which must be less than I_{k-1}.

By an argument similar to that in the proof of lemma 2, I_o, I_1, I_2 I_2,...... I_k ,..... as defined in the preceeding section is a monotonic decreasing sequence, moreover it is bounded below by zero. Therefore this sequence converges to I_m, the greatest lower bound of the sequence.

In other words the polygonal curves converge to the polygon P^m that makes the value of the integral I a minimum,

or we may say that the ordinates of the vertices y^k_i of the polygon P^k converge. in such a way that in the limit y_i satisfies the equation $\frac{\partial \varphi}{\partial y_i} = 0$, $i = 1, 2, \ldots n - 1$.

Example 3

To find the curve $y(x)$, where $y(0) = 0$, and $y\ (9.4247) = 6$, that makes the integral

$$I = \frac{1}{\sqrt{2g}} \int_0^{9.4247} \frac{\sqrt{1 + (y')^2}}{\sqrt{y}}\, dx \quad \text{a minimum.}$$

Choose the initial arbitrary polygonal curve P^0 with vertices (0, 0), (1, 0.5), (2, 2), (3, 2.5), (4, 4), (5, 4.25), (6, 4.75), (7, 5.0), (8, 5.25), (9.4247, 6.00).

Then construct the polygon P^k, $k = 1, 2, 3, \ldots\ldots$ by the method of chapter 4. To construct y_1^1 we must minimize the expression.

$$\frac{\sqrt{(\Delta x)^2 + (y_1^1 - y_0^0)^2}}{\sqrt{y_1^1} + \sqrt{y_0^0}} + \frac{\sqrt{(\Delta x)^2 + (y_2^0 - y_1^1)^2}}{\sqrt{y_2^0} + \sqrt{y_1^1}} = T.$$

where y_0^0 and y_2^0 are fixed.

When $y_1^0 = 0.5$

$$T = \frac{\sqrt{1 + (0.5)^2}}{\sqrt{0.5} + \sqrt{0}} + \frac{\sqrt{1 + (1,5)^2}}{\sqrt{1.5} + \sqrt{2}}$$

$$= 2.4309$$

When $y_1^1 = 1.0$,

$$T = 1.923,$$

When $y_1^1 = 1.5$, then

$$T = 1.8957,$$

which is less than 1.923 and less than 2.4309. When $y_1^1 = 2.0$, then $T = 1.9445$ which is greater than 1.8957.
Therefore we select $y_1^1 = 2.5$,

Continuing this process we obtain the values of y_i^k as shown in table 3 and the graphs of P^k as shown in figure 9. The table also shows that $\frac{I_k}{\sqrt{g}}$ decreases as k increases, and the graphs show that the sequence of polygons approaches the curves calculated analytically in chapter 5.

Table 3. Example 3 Values of y_i^k.

$k \backslash i$	0	1	2	3	4	5	6	7	8	9.4247	$\frac{I_k}{\sqrt{\pi}}$
0	0	0.5	2.0	2.5	4.0	4.25	4.75	5.0	5.25	6.00	6.5416
1	0	1.5	2.1	3.5	4.0	4.5	4.75	5.1	5.6	6.0	5.6555
2	0	1.6	2.5	3.4	4.1	4.5	4.9	5.2	5.7	6.0	5.5941
3	0	1.7	2.7	3.5	4.2	4.8	4.9	5.4	5.8	6.0	5.5641
4	0	1.7	2.7	3.5	4.2	4.6	5.1	5.5	5.8	6.0	5.5551
5	0	1.7	2.7	3.5	4.2	4.7	5.1	5.5	5.8	6.0	5.5528
-	-	-	-	-	-	-	-	-	-	-	-
-	-	-	-	-	-	-	-	-	-	-	-
-	-	-	-	-	-	-	-	-	-	-	-
Analytical solution	0	2.10	3.10	3.95	4.55	5.05	5.45	5.70	5.85	6.00	5.4412

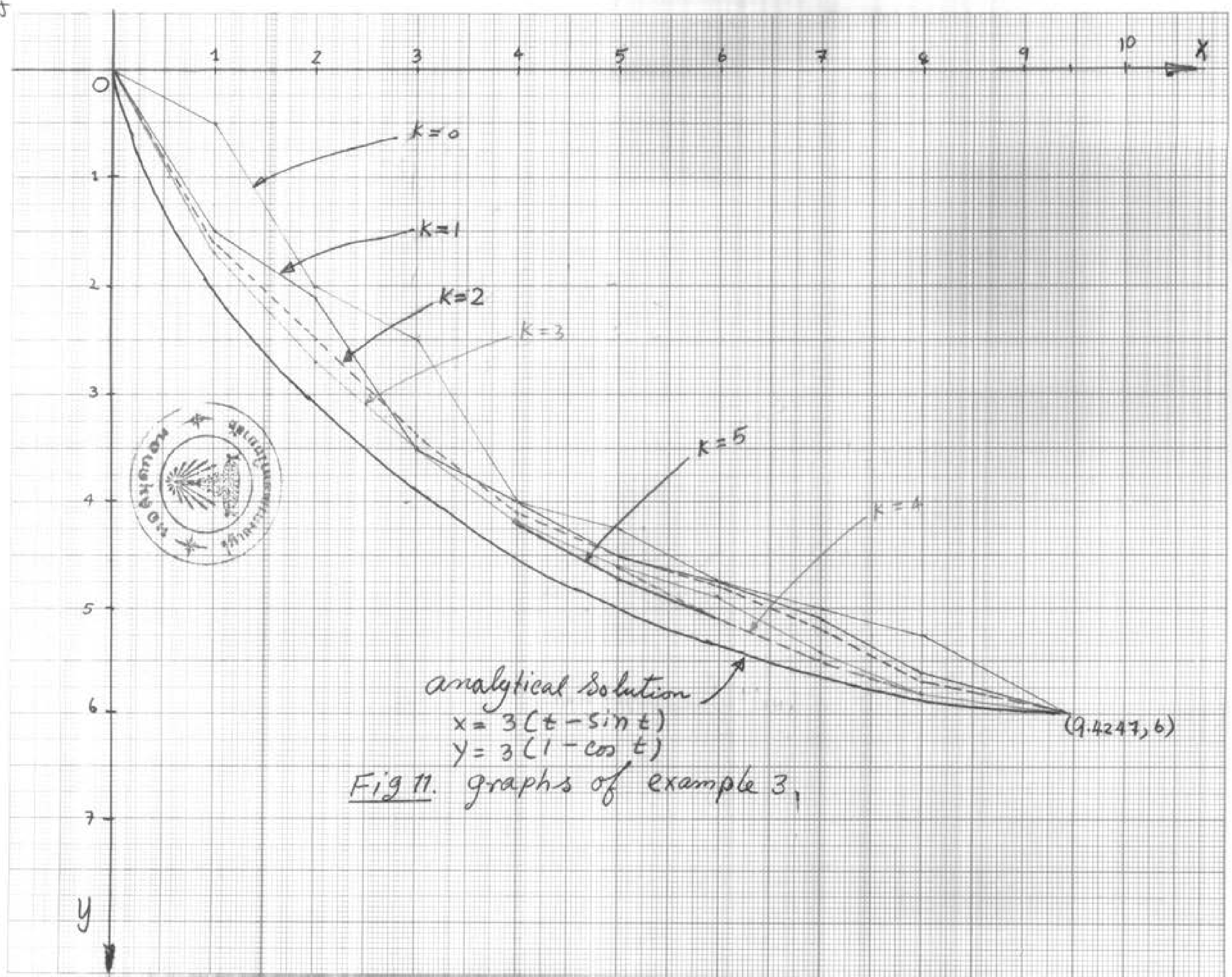

Fig 11. graphs of example 3,