CHAPTER 3

SURFACE OF REVOLUTION OF MINIMUM AREA.

If (x_0, y_0) and (x_n, y_n) are two given points, we want to find the curve $y = y(x)$ joining them such that $y(x) \geqslant 0$ in the interval x_0 to x_n, and

$$I = 2\pi \int_{x_0}^{x_n} y \sqrt{1 + (y')^2}\, dx \quad \text{..........(3.1)}$$

is a minimum.

The functional I given in (3.1) varies according to the given argument function $y = y(x)$. But in the direct method for solving this problem we consider the functional I along argument functions which are polygonal curves of a prescribed number of line segments with vertices on the lines $x = x_0$, $x = x_1$, $x = x_2$ .... $x = x_n$, where $x_i = x_0 + i\Delta x$ for $i = 1, 2, \ldots, n - 1$, and $\Delta x = \frac{x_n - x_0}{n}$. So the functional I depends on the ordinates of vertices $y_1, y_2, \ldots\ldots y_{n-1}$ of the polygons, or in other words the functional I is a function of $y_1, y_2, \ldots\ldots y_{n-1}$, say $I = \varphi(y_1, y_2, \ldots\ldots y_{n-1})$.

For the polygonal curves we obtain from (3.1)

$$I = 2\pi \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} y\sqrt{1 + (y')^2}\, dx$$

$$= 2\pi \sum_{i=0}^{n-1} \left(\frac{y_{i+1} + y_i}{2}\right)\sqrt{1 + \left(\frac{y_{i+1} - y_i}{\Delta x}\right)^2}\,\Delta x$$

$$= \varphi(y_1, y_2, \ldots\ldots y_{n-1}).$$

Let $$Y_i = \frac{y_i + y_{i+1}}{2}$$

and $$Y_i' = \frac{y_{i+1} - y_i}{\Delta x}, \quad \text{for } i = 0, 1, 2, \ldots\ldots n - 1.$$

Then $\varphi(y_1, y_2, \ldots\ldots y_{n-1}) = 2\pi \sum_{i=0}^{n-1} F(x_i, Y_i, Y_i') \Delta x$,

where $F(x_i, Y_i, Y_i') = Y_i\sqrt{1 + (Y_i')^2}$. Although x_i does not appear on the right hand side, it has been included in F so that the general result obtained later in chapter 5 may be applied here.

Let the initial polygonal curve joining (x_o, y_o) and (x_n, y_n) be P^o with vertices $P_i^o\ (x_i, y_i^o)$, $i = 0, 1, 2, \ldots\ldots n$, and let I_o be the value of the integral I along P^o. Then

$$I_o = 2\pi \sum_{i=0}^{n-1} \left(\frac{y_i^o + y_{i+1}^o}{2}\right)\sqrt{1 + \left(\frac{y_{i+1}^o - y_i^o}{\Delta x}\right)^2}\ \Delta x$$

$$= 2\pi \sum_{i=0}^{n-1} P_i^o P_{i+1}^o,$$

Where $P_i^o P_{i+1}^o$ is the surface area generated by revolving the segment joining the consecutive vertices of the polygon P^o about the x - axis.

Now by fixing y_o^o and y_2^o we can choose y_1^1 on $x = x_1$ to make the surface area

$$2\pi\left[\left(\frac{y_o^o + y_1^1}{2}\right)\sqrt{1 + \left(\frac{y_1^1 - y_o^o}{\Delta x}\right)^2}\cdot \Delta x + \left(\frac{y_1^1 + y_2^o}{2}\right)\sqrt{1 + \left(\frac{y_2^o - y_1^1}{\Delta x}\right)^2}\Delta x\right]$$

generated by the new polygonal curve from x_o to x_2 a minimum. This must be less than or equal to its previous value

$$2\pi\left[\left(\frac{y_o^o + y_1^o}{2}\right)\sqrt{1 + \left(\frac{y_1^o - y_o^o}{\Delta x}\right)^2}\Delta x + \left(\frac{y_1^o + y_2^o}{2}\right)\sqrt{1 + \left(\frac{y_2^o - y_1^o}{\Delta x}\right)^2}\Delta x\right].$$

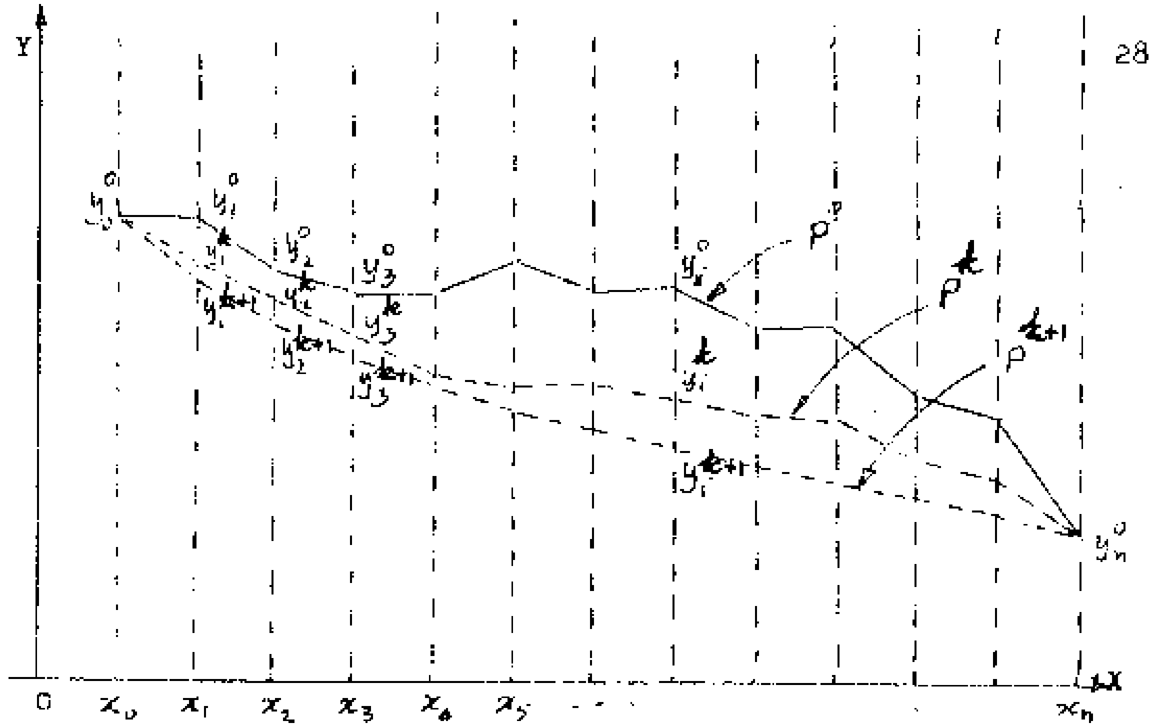

Fig. 8. Polygons constructed to find the minimum surface of revolution.

Then again by fixing y_1^1 and y_3^o we can choose y_2^1 on $x = x_2$ to make the surface area

$$2\pi\left[\left(\frac{y_1^1+y_2^1}{2}\right)\sqrt{1+\left(\frac{y_2^1-y_1^1}{\Delta x}\right)^2}\cdot\Delta x+\left(\frac{y_2^1+y_3^o}{2}\right)\sqrt{1+\left(\frac{y_3^o-y_2^1}{\Delta x}\right)^2}\cdot\Delta x\right]$$

generated by the new polygonal curve from x_1 to x_3 a minimum.. This must be less than or equal to its previous value

$$2\pi\left[\left(\frac{y_1^1+y_2^o}{2}\right)\sqrt{1+\left(\frac{y_2^o-y_1^1}{\Delta x}\right)^2}\cdot\Delta x+\left(\frac{y_2^o+y_3^o}{2}\right)\sqrt{1+\left(\frac{y_3^o-y_2^o}{\Delta x}\right)^2}\cdot\Delta x\right]$$

and so on. (see Fig.8)

Repeating the same process for the remaining intervals to get y_3^1, y_4^1, ... y_{n-1}^1. we obtain a new polygon P^1 the ordinates of the vertices of which are $y_o^1 = y_o^o$, y_1^1, y_2^1, ... y_{n-1}^1, $y_n^1 = y_n^o$.

Let the value of the integral I along P^1 be I_1. Then

$$I_1 = 2\pi \sum_{i=0}^{n-1} \left(\frac{y_i^1 + y_{i+1}^1}{2}\right) \sqrt{1 + \left(\frac{y_{i+1}^1 - y_i^1}{\Delta x}\right)^2} \cdot \Delta x.$$

Next applying the same process to P^1 we get the polygon P^2, and by repeat the process again and again we obtain a sequence of polygons P^3, P^4,...... P^k,...... For the polygon P^k the ordinates of the vertices are $y_0^k = y_0^o$, y_1^k, y_2^k,...... y_{n-1}^k, $y_n^k = y_n^o$ and the value of the integral I is .

$$I_k = 2\pi \sum_{i=0}^{n-1} \left(\frac{y_i^k + y_{i+1}^k}{2}\right) \sqrt{1 + \left(\frac{y_{i+1}^k - y_i^k}{\Delta x}\right)^2} \cdot \Delta x$$

By an argument similar to that in the proof of lemma 2, the sequence I_0, I_1, I_2,...... I_k,.... as defined in the preceeding section is monotonic decreasing, moreover it is bounded below by zero.

Therefore this sequence converges to a limit I_m, which is the greatest lower bound of the sequence .

In other words the polygonal curves converge to the polygon P^m that makes the value of the integral I minimum, or we may say that the ordinates of the vertices y_i^k of the polygon P^k converge in such a way that in the limit as $k \to \infty$ the equation

$$\frac{\partial \psi}{\partial y_i} = 0 \quad , i = 1, 2, \text{.........} n - 1,$$

is satisfied.

Example 2.

Find a curve $y(x)$, where $y(0) = 2.2552$, $y(5) = 7.5244$ that makes the integral

$$I = 2\pi \int_0^5 y \sqrt{1 + (y')^2} \, dx$$

a minimum. This particular end points are so chosen as to simplify comparison between the numerical approximation found below and the known analytical solution (see chapter 5).

Choose the initial arbitrary polygonal curve P^0 with vertices (0,2.2552), (1, 4), (2, 4.5), (3, 5.7), (4, 6.5), (5, 7.5244). Construct the polygons P^k, $k = 1, 2, \ldots\ldots$ by the method in chapter 3.

Thus to construct y_1^1 we first note that the surface area generated between x_0 and x_2 by P^0 is

$$\pi \left[(2.2552 + 4) \sqrt{1 + \left(\frac{4 - 2.2552}{1}\right)^2} \cdot 1 + (4 + 4.5) \sqrt{1 + \left(\frac{4.5 - 4}{1}\right)^2} \cdot 1 \right]$$

$$= 22.0822\ (\pi)$$

Next - choose $y_1^1 = 3.00$ the surface area is

$$\pi \left[(2.2552 + 3) \sqrt{1 + \left(\frac{3.00 - 2.2552}{1}\right)^2} \cdot 1 + (3.00 + 4.5) \sqrt{1 + \left(\frac{4.5 - 3.00}{1}\right)^2} \cdot 1 \right]$$

$$= 20.0721\ (\pi)$$

which is less than $22.0822\ (\pi)$

Next choose $y_1^1 = 2.0$ then the surface area is 21.8932 (π). which is greater than 20.0721 (π). Therefore we select $y_1^1 = 3.00$.

The results are given in table 2 together with the value of I_k, and are illustrated in figure 8. It should be observed that to save labour y_i^k is found only to a few significant figures of accuracy for small value of k, but for larger values of k the accuracy is increased. Figure 8 also shows the analytical solution (see chapter 5).

Table 2. Example 2. Values of y_i^k.

k \ i	0	1	2	3	4	5	$I_k(\pi)$
0	2.2552	4.0	4.5	5.70	6.5	7.5244	73.8460
1	2.2552	3.0	4.0	5.0	6.0	7.5244	69.3921
2	2.2552	2.8	3.5	4.5	5.7	7.5244	68.3061
3	2.2552	2.6	3.3	4.3	5.6	7.5244	67.7850
4	2.2552	2.5	3.1	4.0	5.4	7.5244	67.4969
5	2.2552	2.4	2.9	3.8	5.2	7.5244	67.3247
6	2.2552	2.3	2.7	3.6	5.1	7.5244	67.2091
7	2.2552	2.2	2.6	3.5	5.0	7.5244	67.1685
8	2.2552	2.19	2.55	3.4	4.95	7.5244	67.1543
9	2.2552	2.15	2.5	3.35	4.85	7.5244	67.1484
10	2.2552	2.14	2.45	3.3	4.84	7.5244	67.1419
11	2.2552	2.13	2.44	3.28	4.83	7.5244	67.1399
12	2.2552	2.12	2.42	3.27	4.83	7.5244	67.1381
13	2.2552	2.10	2.41	3.26	4.83	7.5244	67.1357
14	2.2552	2.09	2.40	3.24	4.82	7.5244	67.1349
15	2.2552	2.085	2.38	3.23	4.81	7.5244	67.13441
..	....	....	....	.....	....	....	....
..	....	....	....	.....	....	....	....
..	....	....	....	.....	....	....	....
Analytical solution	2.2552	2.0000	2.2552	3.0862	4.7048	7.5244	66.9304

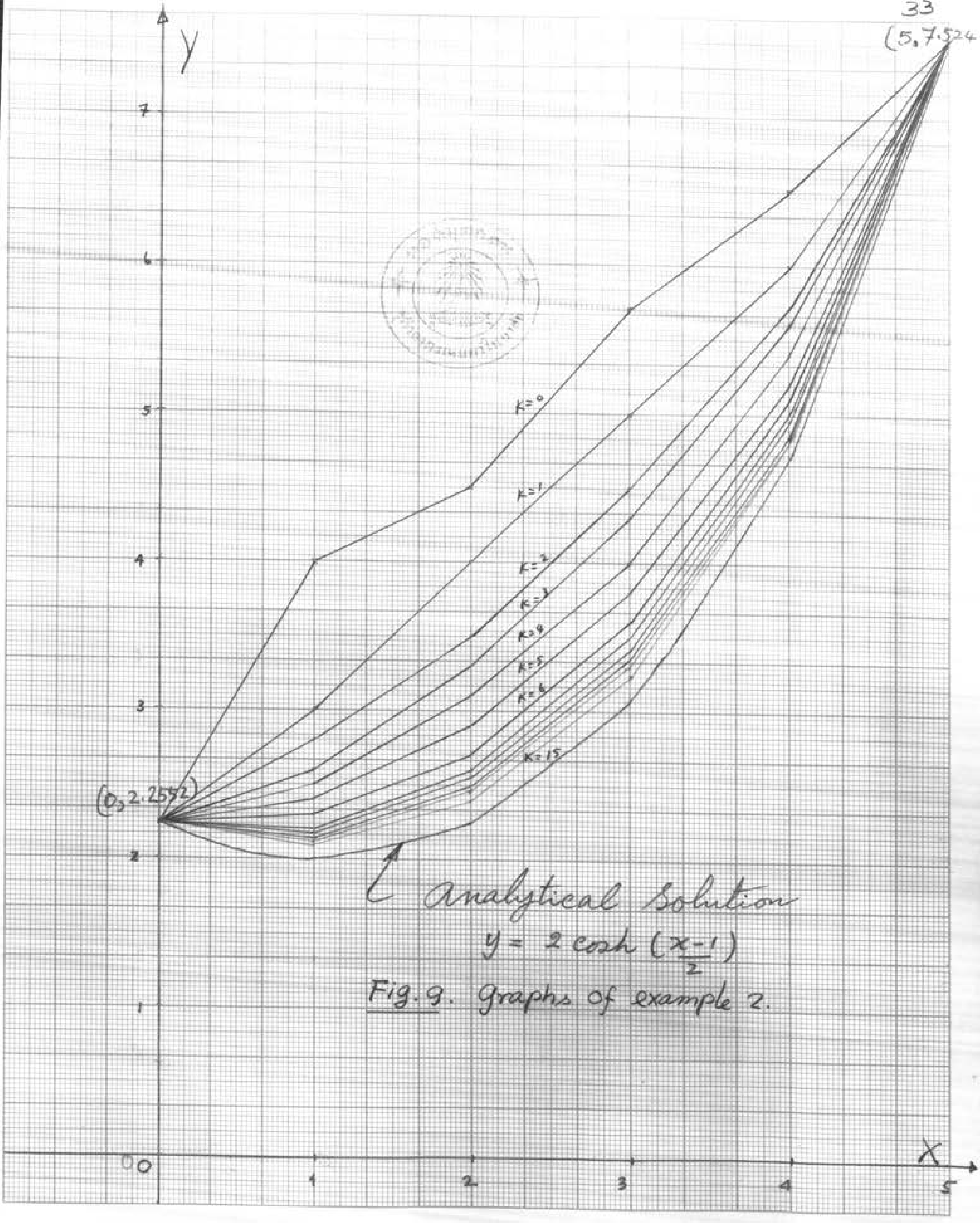

Fig. 9. Graphs of example 2.