CHAPTER I

PATH ALGEBRAS AND PATH PROBLEMS

This chapter first present the definition of a path algebra and some examples of path algebras. We then consider digraphs over the path algebras and illustrate some path problems.

1.1 Path Algebras

A **path algebra** is a semiring $(P, \oplus, \otimes)$ which has the following properties.

(i) The operation $\oplus$ is idempotent and commutative

$a \oplus a = a$ for all $a \in P$,

$a \oplus b = b \oplus a$ for all $a, b \in P$.

(ii) The set P contains a **zero** θ and a **unit** e such that for each $a \in P$,

$\theta \oplus a = a$,

$e \otimes a = a = a \otimes e$ and

$a \otimes \theta = \theta = \theta \otimes a$.

For convenience, the operations $\oplus$ and $\otimes$ are called the **addition** and **multiplication** of the path algebra.

From now on, the set of real numbers is denoted by $\mathbb{R}$, the set of positive integers is denoted by $\mathbb{N}$ and the set of non-negative integers is denoted by $\mathbb{N}_0$.

Ten concrete examples of path algebras [3] are given as follows.

Example 1.1.1 The two-element Boolean algebra is clearly a path algebra.

Example 1.1.2 Let P_2 be $\mathbb{R} \cup \{\infty\}$. The addition is defined by

$x \oplus y = \min\{x, y\}$ for all $x, y \in \mathbb{R}$,

$x \oplus \infty = x = \infty \oplus x$ for all $x \in P_2$,

and the multiplication is defined by

$x \otimes y = x + y$ for all $x, y \in \mathbb{R}$,

$x \otimes \infty = \infty = \infty \otimes x$ for all $x \in P_2$.

Thus, $(P_2, \oplus, \otimes)$ forms a path algebra with zero ∞ and unit 0.

Example 1.1.3 Let P_3 be $\mathbf{R} \cup \{-\infty\}$. The addition is defined by

$$x \oplus y = \max\{x, y\} \qquad \text{for all } x, y \in \mathbf{R},$$
$$x \oplus (-\infty) = x = (-\infty) \oplus x \qquad \text{for all } x \in P_3,$$

and the multiplication is defined by

$$x \otimes y = x + y \qquad \text{for all } x, y \in \mathbf{R},$$
$$x \otimes (-\infty) = -\infty = (-\infty) \otimes x \qquad \text{for all } x \in P_3.$$

Thus, $(P_3, \oplus, \otimes)$ forms a path algebra with zero $-\infty$ and unit 0.

Example 1.1.4 Let P_4 be $\{x \in \mathbf{R} \,/\, 0 \le x \le 1\}$. The addition is defined by

$$x \oplus y = \max\{x, y\} \qquad \text{for all } x, y \in P_4,$$

and the multiplication is defined by

$$x \otimes y = x \times y \qquad \text{for all } x, y \in P_4.$$

Thus, $(P_4, \oplus, \otimes)$ forms a path algebra with zero 0 and unit 1.

Example 1.1.5 Let P_5 be $\{x \in \mathbf{R} \,/\, x \ge 0\} \cup \{\infty\}$. The addition is defined by

$$x \oplus y = \max\{x, y\} \qquad \text{for all } x, y \in \{x \in \mathbf{R} \,/\, x \ge 0\},$$
$$x \oplus \infty = \infty = \infty \oplus x \qquad \text{for all } x \in P_5,$$

and the multiplication is defined by

$$x \otimes y = \min\{x, y\} \qquad \text{for all } x, y \in \{x \in \mathbf{R} \,/\, x \ge 0\},$$
$$x \otimes \infty = x = \infty \otimes x \qquad \text{for all } x \in P_5.$$

Thus, $(P_5, \oplus, \otimes)$ forms a path algebra with zero 0 and unit ∞.

The following algebras are derived from the linguistic concepts.

Example 1.1.6 An **alphabet** is a finite set Σ of symbols and the elements of Σ are called **letters**. A **word over an alphabet** Σ is a finite sequence of zero or more letters from Σ. The sequence of zero letters is called the **empty word**, denoted by λ. The set of all words over an alphabet Σ is denoted by Σ^*, and the subsets of Σ^* are called **languages over the alphabet** Σ.

Let s and t be any words in Σ^*. The **concatenation** $\bullet$ of s and t is defined as follows.

$$s \bullet t = s_1s_2\ldots s_mt_1t_2\ldots t_n \quad \text{if } s = s_1s_2\ldots s_m,\ t = t_1t_2\ldots t_n,$$
$$t \bullet \lambda = \lambda \bullet t = t.$$

Then the concatenation $\bullet$ is an associative binary operation on Σ^*.

Let P_6 be the power set of Σ^*. The addition is defined by

$$X \oplus Y = X \cup Y \qquad \text{for all } X, Y \in P_6,$$

and the multiplication is defined by

$$X \otimes Y = \{\alpha \bullet \beta \,/\, \alpha \in X \text{ and } \beta \in Y\} \qquad \text{for all } X, Y \in P_6,$$

where $\alpha \bullet \beta$ is the concatenation of the words α and β. Using the associativity of the concatenation $\bullet$ on Σ^*, hence $(P_6, \otimes)$ is a semigroup and $(P_6, \oplus, \otimes)$ forms a path algebra with zero $\varnothing$, the null language, and unit, the language $\{\lambda\}$.

Example 1.1.7 Let Σ be any alphabet. A word w is said to be **simple** if no letter of Σ appears in w more than once. Let S be the set of all simple words over Σ and let P_7 be the power set of S. The addition is defined by

$$X \oplus Y = X \cup Y \qquad \text{for all } X, Y \in P_7,$$

and the multiplication is defined by

$$X \otimes Y = \{\alpha \bullet \beta \in S \,/\, \alpha \in X \text{ and } \beta \in Y\} \qquad \text{for all } X, Y \in P_7.$$

where $\alpha \bullet \beta$ is the concatenation of the words α and β. Thus, $(P_7, \oplus, \otimes)$ forms a path algebra with zero $\varnothing$, the null set, and unit, the language $\{\lambda\}$.

Example 1.1.8 Let Σ be any alphabet. An **abbreviation** of a word w over Σ is any word which can be obtained by removing at least one (and possibly all) of the letters of w (note that every word with at least one letter has the abbreviation).

Let $\mathscr{P}(\Sigma^*)$ be the power set of Σ^* and let $X \in \mathscr{P}(\Sigma^*)$. A word w in the language X is **basic** to X if X does not contain any abbreviation of w. The **basis** b(X) of X is the set of all basic words of X. If b(X) = X then X is called a **basic language**; in particular, the null set $\varnothing$ and the set $\{\lambda\}$ are both basic languages.

Let P_8 be the set of all basic languages over Σ. The addition is defined by

$$X \oplus Y = b(X \cup Y) \qquad \text{for all } X, Y \in P_8,$$

and the multiplication is defined by

$$X \otimes Y = \{\alpha \bullet \beta \,/\, \alpha \in X \text{ and } \beta \in Y\} \qquad \text{for all } X, Y \in P_8,$$

where $\alpha \bullet \beta$ is the concatenation of the words α and β. Thus, $(P_8, \oplus, \otimes)$ forms a path algebra with zero $\varnothing$, the null set, and unit, the language $\{\lambda\}$.

Example 1.1.9 Let Σ be any alphabet and let S be any set of subsets of Σ. A member M of S is a **minimal member** of S if S does not contain any proper subsets of M. The **reduction** r(S) of S is the set of all minimal members of S. If r(S) = S then S is called a **reduced set** of sets; in particular, the null set $\varnothing$ and the set $\{\varnothing\}$ are both reduced sets of sets.

Let P_9 be the set of all reduced sets of subsets of Σ. The addition is defined by

$$X \oplus Y = r(\{\alpha \cup \beta \,/\, \alpha \in X \text{ and } \beta \in Y\}) \quad \text{for all } X, Y \in P_9,$$

and the multiplication is defined by

$$X \otimes Y = r(X \cup Y) \quad \text{for all } X, Y \in P_9.$$

Thus, $(P_9, \oplus, \otimes)$ forms a path algebra with zero $\{\varnothing\}$ and unit $\varnothing$.

Example 1.1.10 Let Σ be any alphabet and let w be any symbol which does not belong to Σ. Let $\Omega = \{w\}$, and let $\mathscr{P}(\Sigma)$ be the power set of Σ. Let P_{10} be $\mathscr{P}(\Sigma) \cup \{\Omega\}$. The addition is defined by

$$X \oplus Y = X \cap Y \quad \text{for all } X, Y \in \mathscr{P}(\Sigma),$$

$$X \oplus \Omega = X = \Omega \oplus X \quad \text{for all } X \in P_{10},$$

and the multiplication is defined by

$$X \otimes Y = X \cup Y \quad \text{for all } X, Y \in \mathscr{P}(\Sigma),$$

$$X \otimes \Omega = \Omega = \Omega \otimes X \quad \text{for all } X \in P_{10}.$$

Thus, $(P_{10}, \oplus, \otimes)$ forms a path algebra with zero Ω and unit $\varnothing$, the null set.

1.2 Path Problems

A **directed graph** or a **digraph** G is an ordered pair (X, U) such that X = {1, 2, . . ., n} is a finite set of elements, called **nodes** and U is a set of ordered pairs of nodes, called **arcs**.

A digraph has a pictorial representation in which nodes is represented by dots and each arc (i, j) by an arrow drawn from node i to node j. For convenience, we here represent dots by small squares. A given pictorial representation uniquely determines a digraph.

As an illustration, Figure 1.2.1 represents the digraph G = (X, U) where

X = {1, 2, . . ., 10} and

U = {(1, 1), (1, 2), (2, 8), (3, 2), (3, 6), (4, 3), (4, 5), (5, 6), (5, 10), (6, 4), (7, 6), (7, 8), (7, 10),(8, 3), (9, 1), (9, 7), (9, 8), (10, 9)}.

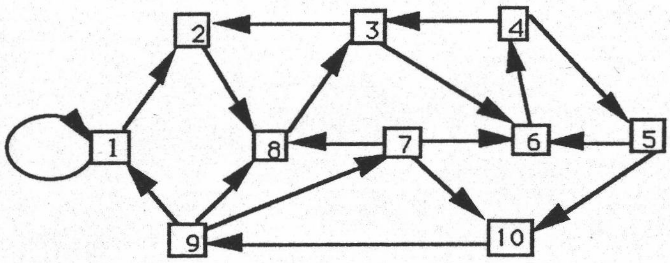

Figure 1.2.1

For an arc (i, j), the nodes i and j are called the **initial** and **terminal endnodes** respectively.

An arc whose endnodes are coincident is called a **loop**.

In a digraph G, a **directed path** or a **dipath of order r from node i_0 to node i_r** is a sequence of r consecutive arcs of the form

$$\mu = (i_0, i_1)(i_1, i_2) \ldots (i_{r-1}, i_r).$$

The nodes i_0 and i_r are called the **initial** and **terminal nodes** of the dipath respectively.

A **null dipath** is a dipath of order zero which the initial and terminal nodes are the same.

A digraph G = (X, U) is **connected** if for any two distinct nodes i and j in X, there is a dipath from node i to node j.

A dipath is **simple** if all the arcs on the dipath are distinct.

A dipath is **elementary** if all the nodes on the dipath are distinct.

A **cycle** is a dipath such that the initial and terminal nodes of the dipath coincide.

An **acyclic digraph** is a digraph which does not contain any cycles.

Let i and j be any nodes of a digraph $G = (X, U)$. An **(i, j)-separating arc set** is a subset W of U such that every dipath from node i to node j contains at least one arc of W. A **proper (i, j)-separating arc set** is an (i, j)-separating arc set W such that no proper subsets of W have this property.

An **(i, j)-separating node sets** is a set V of nodes, not containing i and j, such that every dipath from node i to node j contains at least one node of V. A **proper (i, j)-separating node set** is an (i, j)-separating node set V such that no proper subsets of V have this property.

An arc $u = (i, j)$ is called a **bridge** if in the digraph obtained from G by removing u, there is not a dipath from node i to node j.

An **(i, j)-separating node** is a node k such that every dipath from node i to node j contains k.

A **digraph over a path algebra** $(P, \oplus, \otimes)$ is a triple $G = (X, U, v)$ such that (X, U) is a digraph and $v: U \to P$ is a function.

Let S be the set of all dipaths in the digraph G. Then the function v can be extended to S by for each $\mu \in S$,

$$v(\mu) = \begin{cases} e & \text{if } \mu \text{ is a null dipath,} \\ v(i_0, i_1) \otimes v(i_1, i_2) \otimes \cdots \otimes v(i_{r-1}, i_r) & \text{if } \mu = (i_0, i_1)(i_1, i_2) \cdots (i_{r-1}, i_r) \\ & \text{is a non-null dipath,} \end{cases}$$

where e is the unit of P.

To establish the connection between path algebras and path problems, we introduce the following notation [1].

Let I be any finite index set. Then a formal sum $\oplus\sum_{i \in I} x_i$ in a path algebra $(P, \oplus, \otimes)$ is a well defined element of P which satisfies the following properties.

(i) If $I = \varnothing$ then $\oplus\sum_{i\in I} x_i = \theta$, where θ is the zero of P.

(ii) If $I = \{i\}$ then $\oplus\sum_{j\in I} x_j = x_i$;

(iii) if $I = \{1, 2, \ldots, n\}$ then $\oplus\sum_{i\in I} x_i = x_1 \oplus x_2 \oplus \ldots \oplus x_n$. (1.2.1)

(iv) If $I = \bigcup_{j\in J} I_j$ is a disjoint partition of I, then $\oplus\sum_{i\in I} x_i = \oplus\sum_{j\in J} (\oplus\sum_{i\in I_j} x_i)$.

(v) $z(\oplus\sum_{i\in I} x_i) = \oplus\sum_{i\in I} zx_i$ and $(\oplus\sum_{i\in I} x_i)z = \oplus\sum_{i\in I} x_i z$ for all $z \in P$.

Let $G = (X, U, v)$ be a digraph over a path algebra $(P, \oplus, \otimes)$ and for any $i, j \in X$, $h \in \mathbf{N}_0$ and $k \in \mathbf{N}$, let $T^h_{ij} = \{\mu \,/\, \mu$ is a dipath of order r, $0 \le r \le h$, from node i to node j in G$\}$ and $W^k_{ij} = \{\mu \,/\, \mu$ is a dipath of order r, $1 \le r \le k$, from node i to node j in G$\}$. Then by a path problem we mean either the determination of $\oplus\sum_{\mu\in T^h_{ij}} v(\mu)$ or $\oplus\sum_{\mu\in W^k_{ij}} v(\mu)$ for some $i, j \in X$, $h \in \mathbf{N}_0$ and $k \in \mathbf{N}$.

Some path problems [3] are given as follows.

In a digraph $G = (X, U, v)$ over a path algebra $(P, \oplus, \otimes)$, let i and j be any nodes in X. Then the formal sum $\oplus\sum_{\mu\in T^h_{ij}} v(\mu)$ for some $h \in \mathbf{N}_0$ and the formal sum $\oplus\sum_{\mu\in W^k_{ij}} v(\mu)$ for some $k \in \mathbf{N}$, can be interpreted as follows.

(1) The determination of existence of dipaths from node i to node j if $(P, \oplus, \otimes)$ is a path algebra in Example 1.1.1.

(2) The determination of shortest dipaths from node i to node j if $(P, \oplus, \otimes)$ is a path algebra in Example 1.1.2.

(3) The determination of longest dipaths from node i to node j if the digraph G is acyclic and $(P, \oplus, \otimes)$ is a path algebra in Example 1.1.3.

(4) The determination of most reliable dipaths from node i to node j if $(P, \oplus, \otimes)$ is a path algebra in Example 1.1.4.

(5) The determination of maximal capacity dipaths from node i to node j if $(P, \oplus, \otimes)$ is a path algebra in Example 1.1.5.

(6) The enumeration of dipaths from node i to node j if the digraph G is acyclic and $(P, \oplus, \otimes)$ is a path algebra in Example 1.1.6.

(7) The enumeration of simple dipaths from node i to node j if $(P, \oplus, \otimes)$ is a path algebra in Example 1.1.7.

(8) The enumeration of elementary dipaths from node i to node j if $(P, \oplus, \otimes)$ is a path algebra in Example 1.1.8.

(9) The enumeration of proper (i, j)-separating arc (node) sets if $(P, \oplus, \otimes)$ is a path algebra in Example 1.1.9.

(10) The enumeration of bridges ((i, j)-separating nodes) if $(P, \oplus, \otimes)$ is a path algebra in Example 1.1.10.

To illustrate some path problems in details, the following examples are selected from the above.

Example 1.2.1 Determination of shortest dipaths between two given nodes.

Let $G = (X, U, v)$ be a digraph over the path algebra $(P_2, \oplus, \otimes)$ in Example 1.1.2 and for each arc (i, j), we represent v(i, j) as the arc (i, j) length. To determine the shortest dipaths from node i to node j, we mean that the shortest length of dipaths from node i to node j is sought and it is given by the formal sum

$$\oplus\sum_{\mu \in T_{ij}^q} v(\mu) = \begin{cases} \min\{v(\mu) \,/\, \mu \in T_{ij}^q\} & \text{if } T_{ij}^q \neq \emptyset \\ \infty & \text{if } T_{ij}^q = \emptyset, \end{cases}$$

where

$$v(\mu) = \begin{cases} 0 & \text{if } \mu \text{ is a null dipath,} \\ v(i_0, i_1) + v(i_1, i_2) + \ldots + v(i_{r-1}, i_r) & \text{if } \mu = (i_0, i_1)(i_1, i_2)\ldots(i_{r-1}, i_r) \\ & \text{is a non-null dipath,} \end{cases}$$

$v(\mu)$ is represented as the length of μ, q is the maximum order of simple dipaths from node i to node j in G, and $T^q_{ij} = \{\mu \,/\, \mu$ is a dipath of order r, $0 \le r \le q$, from node i to node j in G$\}$.

Let us consider a digraph $G = (X, U, v)$ over P_2 of Figure 1.2.1. By inspection, all simple dipaths from node 9 to node 3 and their lengths are

$\mu_1 = (9, 7)(7, 6)(6, 2)(2, 3)$, $\quad v(\mu_1) = 25.5$,

$\mu_2 = (9, 10)(10, 6)(6, 2)(2, 3)$, $\quad v(\mu_2) = 27.1$,

$\mu_3 = (9, 7)(7, 11)(11, 10)(10, 6)(6, 2)(2, 3)$, $\quad v(\mu_3) = 44.3$.

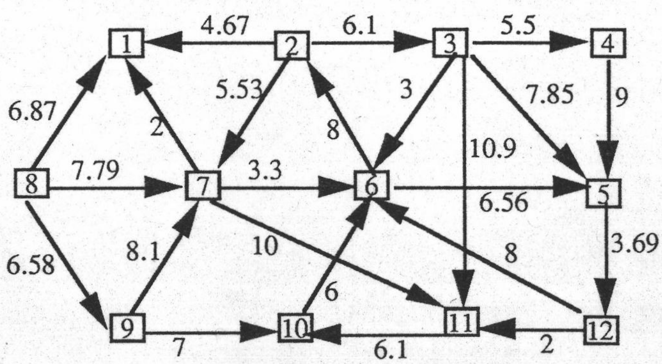

Figure 1.2.1

We note that the maximum order of simple dipaths from node 9 to node 3 is 6. Thus, the shortest length of dipaths from node 9 to node 3 is

$$\underset{\mu \in T^6_{93}}{\oplus\sum} v(\mu) = \min\{v(\mu) \,/\, \mu \in T^6_{93}\}$$

$$= \min\{v(\mu_1), v(\mu_2), v(\mu_3)\}$$

$$= \min\{25.5, 27.1, 44.3\}$$

$$= 25.5.$$

Clearly, there is only one null dipath from node 8 to node 8, then the shortest length of dipaths from node 8 to node 8 is $\oplus\sum_{\mu \in T^0_{88}} v(\mu) = \min\{v(\mu) \,/\, \mu \in T^0_{88}\} = \min\{0\} = 0.$

Also, a dipath from node 5 to node 8 does not exist for all order $q \in \mathbf{N}_0$, then the shortest length of dipaths from node 5 to node 8 is $\oplus\sum_{\mu \in T^q_{58}} v(\mu) = \infty$ for all $q \in \mathbf{N}_0$.

Example 1.2.2 Enumeration of simple dipaths between two given nodes.

Let $G = (X, U)$ be a digraph and let i and j be any nodes in X. The algebraic structure which turns out to be appropriate for describing the simple dipaths from node i to node j, is a path algebra $(P_7, \oplus, \otimes)$ in Example 1.1.7. The reasoning here is based on the fact that a simple dipath from node i to node j is a dipath from node i to node j, which contains distinct arcs. This implicit description of simple dipaths from node i to node j leads naturally to an algebraic formulation in terms of the simple dipaths. Let Σ be the set of distinct names of arcs of G. We define a function $v: U \rightarrow P_7$ by for each arc $(i', j') \in U$,

$$v(i', j') = \{n_{i'j'}\}$$

where $n_{i'j'}$ is the name of the arc (i', j'). Let μ be any simple dipath for node i_0 to node i_r such that $i_0 = i$ and $i_r = j$. Then the label of μ is

$$v(\mu) = \begin{cases} \{\lambda\} & \text{if } \mu \text{ is a null dipath,} \\ v(i_0, i_1) \otimes v(i_1, i_2) \otimes \ldots \otimes v(i_{r-1}, i_r) & \text{if } \mu = (i_0, i_1)(i_1, i_2) \ldots (i_{r-1}, i_r) \\ & \text{is a non-null dipath,} \end{cases}$$

where λ is an empty word and

$$v(i_m, i_{m+1}) \otimes v(i_{m+1}, i_{m+2}) = \{\alpha \bullet \beta \in S \,/\, \alpha \in v(i_m, i_{m+1}) \text{ and } \beta \in v(i_{m+1}, i_{m+2})\}$$

$\alpha \bullet \beta$ is the concatenation of α and β, and $m = 0, 1, 2, \ldots, r - 2$. Therefore, $v(\mu)$ is a set of the name corresponding to the simple dipath μ and so the set of names of all non-null simple dipaths from node i to node j is given by the formal sum

$$\oplus \sum_{\mu \in W_{ij}^q} v(\mu) = \begin{cases} \bigcup_{\mu \in W_{ij}^q} v(\mu) & \text{if } W_{ij}^q \neq \emptyset, \\ \emptyset & \text{if } W_{ij}^q = \emptyset, \end{cases}$$

where q is the maximum order of simple dipaths from node i to node j in G, and W_{ij}^q = $\{\mu \,/\, \mu$ is a dipath of order r, $1 \le r \le q$, from node i to node j in G$\}$.

Consider a digraph G = (X, U) of Figure 1.2.2(a), whose arcs have distinct names from Σ = {a, b, . . . ,w}, Figure 1.2.2(b) shows a digraph G = (X, U, v) over P_7.

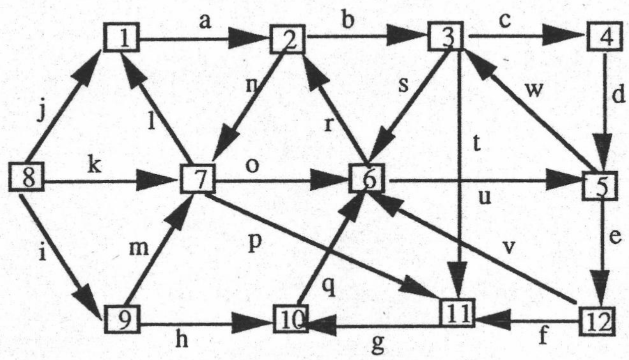

Figure 1.2.2(a)

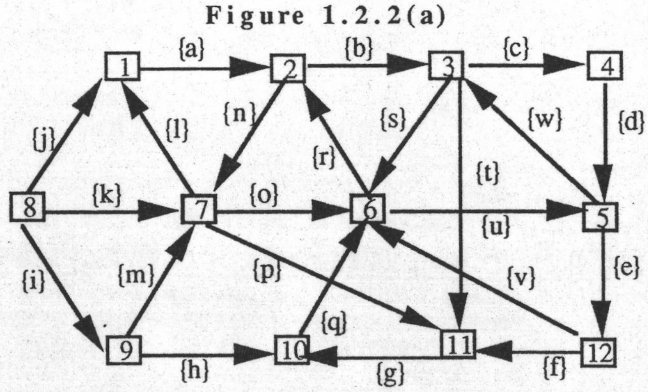

Figure 1.2.2(b)

By inspection, all simple dipaths from node 1 to node 3 and their labels are

μ_1 = (1, 2)(2, 3), $v(\mu_1)$ = {ab},

μ_2 = (1, 2)(2, 3)(3, 4)(4, 5)(5, 3), $v(\mu_2)$ = {abcdw},

μ_3 = (1, 2)(2, 7)(7, 11)(11, 10)(10, 6)(6, 2)(2, 3), $v(\mu_3)$ = {anpgqrb},

μ_4 = (1, 2)(2, 7)(7, 6)(6, 2)(2, 3), $v(\mu_4)$ = {anorb}.

We note that the maximum order of simple dipaths from node 1 to node 3 is 7. Thus, the set of names of all non-null simple dipaths from node 1 to node 3 is

$$\underset{\mu\in W_{13}^{7}}{\oplus\sum} v(\mu) = \bigcup_{\mu\in W_{13}^{7}} v(\mu)$$

$$= \{ab\} \cup \{abcdw\} \cup \{anpgqrb\} \cup \{anorb\}$$

$$= \{ab, abcdw, anpgqrb, anorb\}.$$

Clearly, a dipath from node 3 to node 8 does not exist for all order $q \in \mathbb{N}_0$. Therefore, the set of names of all non-null simple dipaths from node 3 to node 8 is $\underset{\mu\in W_{38}^{q}}{\oplus\sum} v(\mu) = \emptyset$ for all $q \in \mathbb{N}$.

Example 1.2.3 Determination of proper (i, j)-separating arc sets.

Let G = (X, U) be a digraph and let i and j be any nodes in X. The algebraic structure which turns out to be appropriate for describing the proper (i, j)-separating arc sets of G, is a path algebra $(P_9, \oplus, \otimes)$ in Example 1.1.9. The reasoning here is based on the fact that a proper (i, j)-separating arc set is a minimal set of arcs such that every dipath from node i to node j contains at least one arc in it. This implicit description of proper (i, j)-separating arc sets leads naturally to an algebraic formulation in terms of the simple dipaths. Let Σ be the set of distinct names of arcs of G. We define a function $v: U \to P_9$ by for each arc $(i', j') \in U$,

$$v(i', j') = \{\{n_{i'j'}\}\}$$

where $n_{i'j'}$ is the name of the arc (i', j'). Let μ be any simple dipath for node i_0 to node i_r such that $i_0 = i$ and $i_r = j$. Then the label of μ is

$$v(\mu) = \begin{cases} \emptyset & \text{if } \mu \text{ is a null dipath,} \\ \\ v(i_0, i_1) \otimes v(i_1, i_2) \otimes \ldots \otimes v(i_{r-1}, i_r) & \text{if } \mu = (i_0, i_1)(i_1, i_2) \ldots (i_{r-1}, i_r) \\ & \text{is a non-null dipath,} \end{cases}$$

where

$$v(i_m, i_{m+1}) \otimes v(i_{m+1}, i_{m+2}) = r(v(i_m, i_{m+1}) \cup v(i_{m+1}, i_{m+2}))$$

for $m = 0, 1, 2, \ldots, r - 2$. Therefore, $v(\mu)$ is simply the collection of sets corresponding to the arcs of μ. A non-null dipath from node i to node j must contain at least one arc in a proper (i, j)-separating arc set, so each subset presented in $v(\mu)$ indicates a possible choice of a proper (i, j)-separating arc set. Since all non-null dipaths from node i to node j contain at least one arc in a proper (i, j)-separating arc set, thus the set of all proper (i, j)-separating arc sets is given by the formal sum

$$\oplus \sum_{\mu \in W_{ij}^q} v(\mu) \tag{1.2.2}$$

where for any two distinct dipaths μ and μ' in W_{ij}^q, $v(\mu) \oplus v(\mu') = r(\{\alpha \cup \beta \mid \alpha \in v(\mu) \text{ and } \beta \in v(\mu')\})$, q is the maximum order of simple dipaths from node i to node j in G, and $W_{ij}^q = \{\mu \mid \mu$ is a dipath of order r, $1 \le r \le q$, from node i to node j in G$\}$, if $W_{ij}^q \neq \emptyset$. Since there is only one proper (i, j)-separating arc set $\emptyset$ when $W_{ij}^q = \emptyset$, then in case of $W_{ij}^q = \emptyset$, we denote (1.2.2) by $\{\emptyset\}$.

Consider a digraph G = (X, U) of Figure 1.2.3(a) whose arcs have distinct names from $\Sigma = \{a, b, \ldots, n\}$, Figure 1.2.3(b) shows a digraph G = (X, U, v) over P_9.

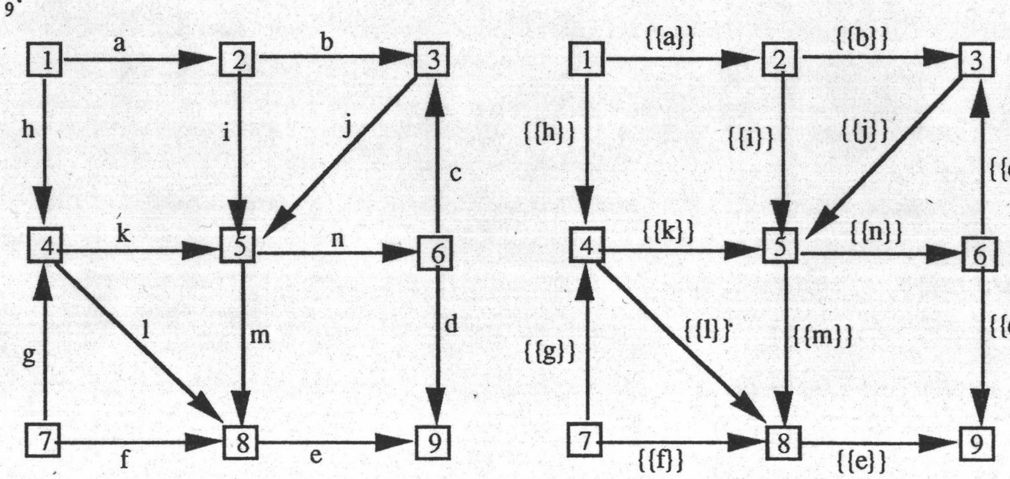

Figure 1.2.3(a) **Figure 1.2.3(b)**

By inspection, all simple dipaths from node 4 to node 9 and their labels are

$$\mu_1 = (4, 8)(8, 9), \qquad v(\mu_1) = \{\{l\}, \{e\}\},$$

$$\mu_2 = (4, 5)(5, 6)(6, 9), \qquad v(\mu_2) = \{\{k\}, \{n\}, \{d\}\},$$

$$\mu_3 = (4, 5)(5, 8)(8, 9), \qquad v(\mu_3) = \{\{k\}, \{m\}, \{e\}\}.$$

We note that the maximum order of simple dipaths from node 4 to node 9 is 3. Thus, the set of all proper (4, 9)-separating arc sets is given by the formal sum

$$\begin{aligned}
\oplus\sum_{\mu\in W_{49}^3} v(\mu) &= v(\mu_1) \oplus v(\mu_2) \oplus v(\mu_3) \\
&= \{\{l\}, \{e\}\} \oplus \{\{k\}, \{n\}, \{d\}\} \oplus \{\{k\}, \{m\}, \{e\}\} \\
&= r(\{\{l\} \cup \{k\} \cup \{k\}, \{l\} \cup \{k\} \cup \{m\}, \{l\} \cup \{k\} \cup \{e\}, \\
&\qquad \{l\} \cup \{n\} \cup \{k\}, \{l\} \cup \{n\} \cup \{m\}, \{l\} \cup \{n\} \cup \{e\}, \\
&\qquad \{l\} \cup \{d\} \cup \{k\}, \{l\} \cup \{d\} \cup \{m\}, \{l\} \cup \{d\} \cup \{e\}, \\
&\qquad \{e\} \cup \{k\} \cup \{k\}, \{e\} \cup \{k\} \cup \{m\}, \{e\} \cup \{k\} \cup \{e\}, \\
&\qquad \{e\} \cup \{n\} \cup \{k\}, \{e\} \cup \{n\} \cup \{m\}, \{e\} \cup \{n\} \cup \{e\}, \\
&\qquad \{e\} \cup \{d\} \cup \{k\}, \{e\} \cup \{d\} \cup \{m\}, \{e\} \cup \{d\} \cup \{e\}\} \\
&= r(\{\{l, k\}, \{l, k, m\}, \{l, k, e\}, \\
&\qquad \{l, n, k\}, \{l, n, m\}, \{l, n, e\}, \\
&\qquad \{l, d, k\}, \{l, d, m\}, \{l, d, e\}, \\
&\qquad \{e, k\}, \{e, k, m\}, \{e, k\}, \\
&\qquad \{e, n, k\}, \{e, n, m\}, \{e, n\}, \\
&\qquad \{e, d, k\}, \{e, d, m\}, \{e, d\}\} \\
&= \{\{l, k\}, \{e, k\}, \{e,n\}, \{e,d\}, \{l, n, m\}, \{l, d, m\}\}.
\end{aligned}$$

Clearly, a dipath from node 9 to node 1 does not exist for all order $k \in \mathbf{N}_0$. Therefore, the set of all proper (9,1)-separating arc sets is $\oplus\sum_{\mu\in W_{91}^k} v(\mu) = \{\emptyset\}$

for all $k \in \mathbf{N}$.