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SYMPLECTIC GRAPHS OVER FINITE COMMUTATIVE RINGS

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ธรรมนูญ ผุยรอด : กราฟซิมเพล็กติกเหนือริงสลับที่จำกัด (SYMPLECTIC GRAPHS OVER FINITE COMMUTATIVE RINGS) อ. ที่ปรึกษาวิทยานิพนธ์หลัก : รศ.ดร. ยศนันต์ มีมาก, 43 หน้า.

งานวิจัยนี้ อาศัยนิยามของกราฟ ซิมเพล็กติก $\mathcal{G}_{\mathrm{Sp}_R(V)}$ [12] เมื่อ V เป็นปริภูมิซิมเพล็กติกเหนือริงสลับที่จำกัด R โดยสำหรับ $R = \mathbb{Z}_{p^n}$ และ $V = R^{2v}$ มีผู้แสดงไว้ว่ากราฟซิมเพล็กติกเป็นกราฟปกติอย่างเข้ม เมื่อ $v = 1$ และเป็นกราฟเดซาโดยแท้ เมื่อ $v \geq 2$

ในวิทยานิพนธ์ฉบับนี้ เราศึกษากราฟ ซิมเพล็กติกเหนือริงสลับที่จำกัด โดยเราได้เงื่อนไขในการจำแนกกราฟออกเป็นกราฟปกติอย่างเข้มและกราฟเดซา และเรายังพิสูจน์ว่ากราฟซิมเพล็กติกมีสมบัติถ่ายทอดบนอาร์ก คำนวณ รงคเลขและกรุปอัตสัณฐาน และยิ่งกว่านั้น เรา ประยุกต์วิธีเชิงการนับเดียวกันนี้เพื่อพิสูจน์สมบัติบนกราฟย่อยของกราฟซิมเพล็กติกเหนือริงจำกัดเฉพาะที่อีกด้วย

ภาควิชา คณิตศาสตร์และวิทยาการคอมพิวเตอร์ ลายมือชื่อนิสิต

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This work is based on ideas of Meemark and Prinyasart [12] who introduced the symplectic graph $\mathcal{G}_{\mathrm{Sp}_R(V)}$, where V is a symplectic space over a finite commutative ring R. When $R = \mathbb{Z}_{p^n}$ and $V = R^{2\nu}$, they proved that $\mathcal{G}_{\mathrm{Sp}_R(V)}$ is a strongly regular graph when $\nu = 1$ and Li, Wang and Guo [10] showed that it is strictly Deza graph when $\nu \geq 2$. In this dissertation, we study symplectic graphs over finite commutative rings. We can classify if our graph is a strongly regular graph or a Deza graph. We also show that it is arc transitive, and determine chromatic numbers and automorphism groups. Moreover, we apply the combinatorial technique presented in [12] to prove similar results on subconstituents of symplectic graphs over finite local rings.

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CONTENTS

CHAPTER I
PRELIMINARIES

1.1 Introduction

The general symplectic graph associated with nonsingular alternate matrices over a field is introduced by Tang and Wan [13] as a new family of strongly regular graphs. This graph was firstly defined for a symplectic space over a commutative ring by Meemark and Prinyasart [12]. They showed that their symplectic graph is vertex transitive and arc transitive when $R = \mathbb{Z}_{p^n}$, p is an odd prime and $n \geq 1$. There are many articles influenced by this definition such as [10], [11], [6] and [5]. Mostly, the work was on strong regularity, automorphism groups, vertex and arc transitivities, chromatic numbers and subconstituents of symplectic graphs over a finite field, modulo p^n, and modulo pq, where p and q are primes and $n \geq 1$.

In what follows, we study those topics over finite local rings and obtain results parallel to [13], [12], [10], [11], [6] and [5]. We use combinatorial approach similar to [12]. In addition, we present some results over finite commutative rings in the final chapter.

1.2 Symplectic graphs

Let R be a commutative ring and let V be a free R-module of R-dimension 2ν, where $\nu \geq 1$. Assume that we have a function β: $V \times V \to R$ which is R-bilinear, $\beta(\vec{x}, \vec{x}) = 0$ for all $\vec{x}$ in V and the R-module morphism from V to $V^* = \mathrm{Hom}_R(V, R)$ given by $\vec{x} \mapsto \beta(\cdot, \vec{x})$ is an isomorphism. We call the pair (V, β) a **symplectic space**. A vector $\vec{x}$ in V is said to be **unimodular** if there is an f in V^*

with $f(\vec{x})$ = 1; equivalently, if $\vec{x} = \alpha_1\vec{b}_1 + \cdots + \alpha_{2\nu}\vec{b}_{2\nu}$, where $\{\vec{b}_1, \ldots, \vec{b}_{2\nu}\}$ is a basis for V, then the ideal $(\alpha_1, \ldots, \alpha_{2\nu}) = R$. If $\vec{x}$ is unimodular, then the **line** $R\vec{x}$ is a free R-direct summand of dimension one.

A **hyperbolic pair** $\{\vec{x}, \vec{y}\}$ is a pair of unimodular vectors in V with the property that $\beta(\vec{x}, \vec{y}) = 1$. The module $H = R\vec{x} \oplus R\vec{y}$ is called a **hyperbolic plane**. Let (V, β) be a symplectic space. An R-module automorphism σ on V is an **isometry** on V if $\beta(\sigma(\vec{x}), \sigma(\vec{y})) = \beta(\vec{x}, \vec{y})$ for all $\vec{x}, \vec{y} \in V$. The group of isometries on V is called the **symplectic group of** (V, β) **over** R and denoted by $\mathrm{Sp}_R(V)$.

Define the graph $\mathcal{G}_{\mathrm{Sp}_R(V)}$ with the vertex set is the set of lines $\{R\vec{x} : \vec{x}$ is a unimodular vector in $V\}$ and with adjacency given by

$$R\vec{x} \text{ is adjacent to } R\vec{y} \text{ if and only if } \beta(\vec{x}, \vec{y}) \in R^{\times}.$$

Here, $R^{\times}$ denotes the group of invertible elements in R. We call $\mathcal{G}_{\mathrm{Sp}_R(V)}$, the **symplectic graph of** (V, β) **over** R.

Example 1.2.1. Let p be a prime number and let R be the ring of integers modulo p^n, $\mathbb{Z}_{p^n}$, or the field of p^n elements, $\mathbb{F}_{p^n}$, where $n \in \mathbb{N}$. For $\nu \geq 1$, let V denote the set of 2ν-tuples $(a_1, \ldots, a_{2\nu})$ of elements in R. Define $\beta : V \times V \to R$ by

$$\beta\big((a_1, \ldots, a_{2\nu}), (b_1, \ldots, b_{2\nu})\big) = (a_1, \ldots, a_{2\nu}) \begin{pmatrix} 0 & I_\nu \\ -I_\nu & 0 \end{pmatrix}_{2\nu \times 2\nu} (b_1, \ldots, b_{2\nu})^t,$$

where I_ν is the $\nu \times \nu$ identity matrix, for all vectors $(a_1, \ldots, a_{2\nu})$, $(b_1, \ldots, b_{2\nu})$ in V. Then (V, β) is a symplectic space, and unimodular vectors in V are those $(a_1, \ldots, a_{2\nu})$ of elements in R such that $a_i \in R^{\times}$ for some $i \in \{1, 2, \ldots, 2\nu\}$. We generalize this result in Theorem 2.1.2. We write $\mathrm{Sp}^{(2\nu)}(R)$ for this symplectic graph.

Theorem 1.2.2. *[12, 13] Let p be a prime number and let n and ν be positive integers. For $R = \mathbb{F}_{p^n}$ or $R = \mathbb{Z}_{p^n}$, the graph $\mathrm{Sp}^{(2\nu)}(R)$ is $(p^n)^{2\nu-1}$-regular and every two adjacent vertices of $\mathrm{Sp}^{(2\nu)}(R)$ has $(p^n)^{2\nu-2}(p^n - p^{n-1})$ common neighbors.*

1.3 Terminologies and literature reviews

A **strongly regular graph** with parameters (v, k, λ, μ) is a k-regular graph on v vertices such that for every pair of adjacent vertices there are λ vertices adjacent to both, and for every pair of non-adjacent vertices there are μ vertices adjacent to both.

Theorem 1.3.1. *Let p be a prime number and let n and ν be positive integers.*

(1) *[13] The symplectic graph $\mathrm{Sp}^{(2\nu)}(\mathbb{F}_{p^n})$ is a strongly regular graph with parameters*

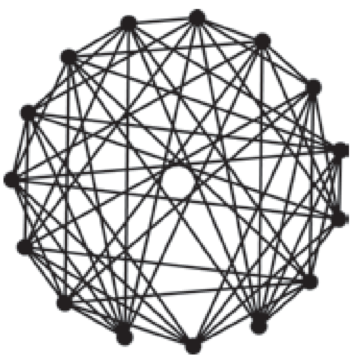
$$\left(\frac{(p^n)^{2\nu}-1}{p^n-1}, (p^n)^{2\nu-1}, (p^n)^{2\nu-2}(p^n-1), (p^n)^{2\nu-2}(p^n-1)\right).$$

(2) *[12] The symplectic graph $\mathrm{Sp}^{(2)}(\mathbb{Z}_{p^n})$ is a strongly regular graph with parameters*

$$\left(p^n + p^{n-1}, p^n, p^n - p^{n-1}, p^n\right)$$

and $\mathrm{Sp}^{(2\nu)}(\mathbb{Z}_{p^n})$ is not strongly regular when $\nu \geq 2$ and $n \geq 2$.

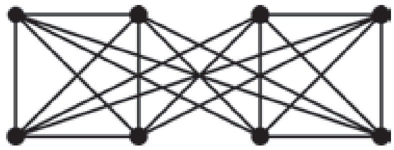
Example 1.3.2. The following figure shows the symplcetic graph $\mathrm{Sp}^{(4)}(\mathbb{F}_2)$. This graph is strongly regular with parameters (15, 8, 4, 4).



As a generalization of strongly regular graphs, Erickson and Fernando [3] introduced Deza graphs, which were firstly introduced in a slightly more restricted form by Deza and Deza [2]. A regular graph with degree k on v vertices is said to be a (v, k, λ, μ)-**Deza graph** if any two distinct vertices x and y have λ

or μ common adjacent vertices. A Deza graph of diameter two is called a **strictly Deza graph** if it is not strongly regular.

Example 1.3.3. The following figure shows a (8, 5, 4 2)-Deza graph. It is also a strictly Deza graph.



Li, Wang and Guo [10] showed that:

Theorem 1.3.4. *[10] The symplectic graph* $\mathrm{Sp}^{(2\nu)}(\mathbb{Z}_{p^n})$ *is a strictly Deza graph when* $\nu \geq 2$ *and* $n \geq 2$ *with parameters*

$$\left(\frac{(p^n)^{2\nu}-1}{p^n-1}, (p^n)^{2\nu-1}, (p^n)^{2\nu-2}(p^n-p^{n-1}), (p^n)^{2\nu-1}\right).$$

For a graph G, we write $\mathcal{V}(G)$ for its vertex set and $\mathcal{E}(G)$ for its edge set. Let G and H be graphs. A function f from $\mathcal{V}(G)$ to $\mathcal{V}(H)$ is a **homomorphism** from G to H if $f(g_1)$ and $f(g_2)$ are adjacent in H whenever g_1 and g_2 are adjacent in G. It is called an **isomorphism** if it is a bijection and f^{-1} is a homomorphism from H onto G. Moreover, an isomorphism on G is called an **automorphism**. The set of all automorphisms of a graph G is denoted by $\mathrm{Aut}(G)$. It is a group under composition, called the **automorphism group of** G.

Theorem 1.3.5. *Let* R *be a commutative ring and* (V, β) *a symplectic space over* R. *For each* $\sigma \in \mathrm{Sp}_R(V)$, σ *can be considered as an automorphism of* $\mathcal{G}_{\mathrm{Sp}_R(V)}$. *That is, we have the imbedding* $\mathrm{Sp}_R(V) \hookrightarrow \mathrm{Aut}(\mathcal{G}_{\mathrm{Sp}_R(V)})$.

Proof. Let $\sigma \in \mathrm{Sp}_R(V)$. Define the map $\bar{\sigma}$ on $\mathcal{G}_{\mathrm{Sp}_R(V)}$ by

$$\bar{\sigma} : R\vec{x} \mapsto R\sigma(\vec{x})$$

for all unimodular vectors $\vec{x} \in V$. Since σ is an isometry, $\bar{\sigma}$ is a bijection and

$\beta(\vec{x},\vec{y}) = \beta(\sigma(\vec{x}),\sigma(\vec{y}))$ for all unimodular vectors $\vec{x},\vec{y} \in V$. Thus,

$$\beta(\vec{x},\vec{y}) \in R^{\times} \Leftrightarrow \beta(\sigma(\vec{x}),\sigma(\vec{y})) \in R^{\times}$$

for all unimodular vectors $\vec{x},\vec{y} \in V$. Hence, $\bar{\sigma} \in \mathrm{Aut}(\mathcal{G}_{\mathrm{Sp}_R(V)})$. □

We give more reviews and study the automorphism group of symplectic graphs and their subconstituents in Sections 2.4 and 3.3, respectively.

A graph G is **vertex transitive** if its automorphism group acts transitively on the vertex set. That is, for any two vertices of G, there is an automorphism carrying one to the other. An **arc** in G is an ordered pair of adjacent vertices, and G is **arc transitive** if its automorphism group acts transitively on its arcs. Note that an arc transitive graph is necessarily vertex and edge transitive. More on transitive graphs can be found in Chapter 3 of Godsil's book [4]. We have the next results.

Theorem 1.3.6. *[10, 12, 13] Let p be a prime number and let n be a positive integer. For $R = \mathbb{F}_{p^n}$ or $R = \mathbb{Z}_{p^n}$, the symplectic graph* $\mathrm{Sp}^{(2\nu)}(R)$ *is arc transitive.*

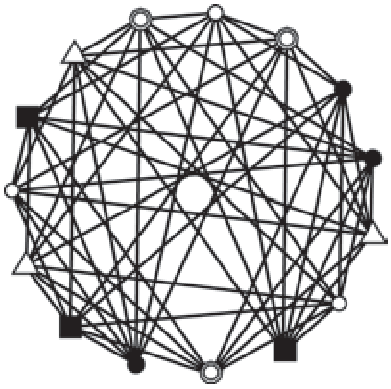
Theorem 1.3.7. *[11] Let $m \geq 2$ be an integer. For $R = \mathbb{Z}_m$, the symplectic graph* $\mathrm{Sp}^{(2\nu)}(R)$ *is arc transitive.*

The **chromatic number** of a graph G is the smallest number of colors needed to color the vertices of G so that no two adjacent vertices share the same color. The chromatic number of a graph G is commonly denoted by $\chi(G)$. Some results on chromatic numbers are as follows. We refer the reader to Sections 2.3, 3.2 and 4.2 for our work on chromatic numbers.

Theorem 1.3.8.

1. *[13] If k is the field of q elements and V is the symplectic graph of dimension 2ν, $\nu \geq 1$, then $\chi(\mathcal{G}_{\mathrm{Sp}_k(V)}) = q^{\nu} + 1$.*
2. *[12] $\chi(\mathrm{Sp}^{(2)}(\mathbb{Z}_{p^n})) = p + 1$ for all $n \geq 1$.*

Example 1.3.9. The symplectic graph $\mathrm{Sp}^{(4)}(\mathbb{F}_2)$ has the chromatic number equal to $2^2 + 1 = 5$. We can assign a coloring as shown below.



CHAPTER II

SYMPLECTIC GRAPHS OVER FINITE LOCAL RINGS

A **local ring** is a commutative ring which has a unique maximal ideal. Note that for a local ring R, its unique maximal ideal is given by $M = R \setminus R^{\times}$ and we call the field R/M, the **residue field of** R. For example, if p is a prime, then $\mathbb{Z}_{p^n}$, $n \in \mathbb{N}$, is a local ring with maximal ideal $p\mathbb{Z}_{p^n}$ and residue field $\mathbb{Z}_{p^n}/p\mathbb{Z}_{p^n}$ isomorphic to $\mathbb{Z}_p$. Moreover, every field is a local ring with maximal ideal $\{0\}$.

In this chapter, we study the symplectic graph $\mathcal{G}_{\mathrm{Sp}_R(V)}$ when R is a finite local ring. We obtain a classification for our graph to be strongly regular or to be a strictly Deza graph. Moreover, we prove that this graph is vertex and arc transitive and study the chromatic number. In the final section, we work on the automorphism group.

2.1 Strong regularity

Let R be a local ring with unique maximal ideal M and let (V, β) be a symplectic space of R-dimension 2ν, where $\nu \geq 1$. By Theorem 1 of [8], V possesses a canonical basis $\{\vec{e}_1, \ldots, \vec{e}_{2\nu}\}$ such that $\{\vec{e}_j, \vec{e}_{\nu+j}\}$ is a hyperbolic pair for all $1 \leq j \leq \nu$ and V is an orthogonal direct sum $V = H_1 \perp H_2 \perp \ldots \perp H_\nu$, where $H_j = R\vec{e}_j \oplus R\vec{e}_{\nu+j}$ is a hyperbolic plane for all $1 \leq j \leq \nu$. Recall a common theorem about local rings that:

Theorem 2.1.1. *Let R be a local ring with unique maximal ideal M. Then $1 + m$ is a unit in R for all $m \in M$.*

Proof. Let $m \in M$. Assume that $1 + m$ is not invertible. Then $1 + m \in M$ because R is a local ring. Thus, $(1 + m) - m = 1 \in M$, a contradiction. Hence, $1 + m$ is a unit of R. □

We have a criterion to determine whether a vector in V is unimodular as follows.

Theorem 2.1.2. *A vector $\vec{x} = a_1\vec{e}_1 + \cdots + a_{2\nu}\vec{e}_{2\nu}$ in V is unimodular if and only if a_i is a unit of R for some $i \in \{1, \ldots, 2\nu\}$.*

Proof. If some a_i is a unit in R, then $(a_1, \ldots, a_{2\nu}) = R$, so $\vec{x}$ is unimodular. Conversely, assume that $\vec{x}$ is unimodular. Then there exists an $f \in V^*$ such that $1 = f(\vec{x}) = a_1 f(\vec{e}_1) + \cdots + a_{2\nu} f(\vec{e}_{2\nu})$. Suppose that a_i is not a unit in R for all i. Since R is a local ring, $a_i \in M$ for all i, and thus $a_1 f(\vec{e}_1) + \cdots + a_{2\nu} f(\vec{e}_{2\nu}) \in M$. By Theorem 2.1.1, $0 = 1 - (a_1 f(\vec{e}_1) + \cdots + a_{2\nu} f(\vec{e}_{2\nu}))$ is a unit in R, which is a contradiction. Therefore, a_i is a unit of R for some $i \in \{1, \ldots, 2\nu\}$. $\square$

If R is finite, the above theorem gives the number of vertices of $\mathcal{G}_{\mathrm{Sp}_R(V)}$, namely,

$$\left|\mathcal{V}(\mathcal{G}_{\mathrm{Sp}_R(V)})\right| = |\{R\vec{x} : \vec{x} \text{ is a unimodular vector in } V\}| = \frac{|R|^{2\nu} - |M|^{2\nu}}{|R^\times|}.$$

Write unimodular vectors $\vec{a} = a_1\vec{e}_1 + \cdots + a_{2\nu}\vec{e}_{2\nu}$ and $\vec{b} = b_1\vec{e}_1 + \cdots + b_{2\nu}\vec{e}_{2\nu}$ for some $a_i, b_i \in R$. Then

$$\begin{aligned}\beta(\vec{a}, \vec{b}) &= \beta(a_1\vec{e}_1 + \ldots + a_{2\nu}\vec{e}_{2\nu}, b_1\vec{e}_1 + \ldots + b_{2\nu}\vec{e}_{2\nu}) \\ &= \sum_{i=1}^{2\nu}\sum_{j=1}^{2\nu} a_i b_j \beta(\vec{e}_i, \vec{e}_j) = \sum_{i=1}^{\nu}(a_i b_{\nu+i} - a_{\nu+i} b_i)\end{aligned}$$

because $\beta(\vec{e}_i, \vec{e}_i) = 0, \beta(\vec{e}_i, \vec{e}_{\nu+i}) = 1$ and $\beta(\vec{e}_i, \vec{e}_j) = -\beta(\vec{e}_j, \vec{e}_i)$ for all $i, j \in \{1, \ldots, 2\nu\}$. Hence, the adjacency condition becomes

$$R\vec{a} \text{ is adjacent to } R\vec{b} \quad \text{if and only if} \quad \sum_{i=1}^{\nu}(a_i b_{\nu+i} - a_{\nu+i} b_i) \in R^\times. \tag{2.1.1}$$

We shall use it in the next lemma.

Lemma 2.1.3. *Let $\vec{a} = a_1\vec{e}_1 + \cdots + a_{2\nu}\vec{e}_{2\nu}$ and $\vec{b} = b_1\vec{e}_1 + \cdots + b_{2\nu}\vec{e}_{2\nu}$ be unimodular vectors in V and assume that $a_i \in R^\times$ for some $i \in \{1, \ldots, 2\nu\}$. If $R\vec{a}$ is adjacent to $R\vec{b}$, then $a_i b_l - a_l b_i$ is a unit for some $l \in \{1, \ldots, 2\nu\}$ and $l \neq i$.*

Proof. Assume that $a_i b_l - a_l b_i \in M$ for all $l \in \{1, \ldots, 2\nu\}$. Then

$$\sum_{j=1}^{\nu}(a_i a_j b_{\nu+j} - a_{\nu+j} a_j b_i) + \sum_{j=1}^{\nu}(a_i a_{\nu+j} b_j - a_j a_{\nu+j} b_i) = -\sum_{j=1}^{\nu}(a_i a_j b_{\nu+j} - a_i a_{\nu+j} b_i) \in M,$$

which implies $\sum_{j=1}^{\nu}(a_i b_{\nu+j} - a_{\nu+j} b_j) \in M$. Thus, $R\vec{a}$ is not adjacent to $R\vec{b}$. □

Theorem 2.1.4. *Let R be a finite local ring and let (V, β) be a symplectic space of dimension 2ν, where $\nu \geq 1$.*

(1) The symplectic graph $\mathcal{G}_{\mathrm{Sp}_R(V)}$ is $|R|^{2\nu-1}$-regular with $\dfrac{|R|^{2\nu} - |M|^{2\nu}}{|R^\times|}$ vertices.

(2) Every two adjacent vertices of $\mathcal{G}_{\mathrm{Sp}_R(V)}$ has $|R|^{2\nu-2}\,|R^\times|$ common neighbors.

(3) Every two non-adjacent vertices of $\mathcal{G}_{\mathrm{Sp}_R(V)}$ has $|R|^{2\nu-2}|R^\times|$ or $|R|^{2\nu-1}$ common neighbors.

Proof. Let $\vec{a} = a_1\vec{e}_1 + \cdots + a_{2\nu}\vec{e}_{2\nu}$ and $\vec{b} = b_1\vec{e}_1 + \cdots + b_{2\nu}\vec{e}_{2\nu}$ be unimodular vectors in V and assume that $R\vec{a}$ is adjacent to $R\vec{b}$. Since $\vec{a}$ is unimodular, there exists an $i \in \{1, \ldots, 2\nu\}$ such that $a_i \in R^\times$. If $i \leq \nu$, then

$$\begin{aligned} b_{\nu+i} = a_i^{-1}\big(&r + (a_{\nu+1}b_1 - a_1 b_{\nu+1}) + (a_{\nu+2}b_2 - a_2 b_{\nu+2}) + \cdots \\ &+ (a_{\nu+i-1}b_{i-1} - a_{i-1}b_{\nu+i-1}) + a_{\nu+i}b_i + (a_{\nu+i+1}b_{i+1} - a_{i+1}b_{\nu+i+1}) + \cdots \\ &+ (a_{2\nu}b_\nu - a_\nu b_{2\nu})\big) \end{aligned}$$

for some $r \in R^\times$ and if $i > \nu$, then

$$\begin{aligned} b_{i-\nu} = a_i^{-1}\big(&(a_1 b_{\nu+1} - a_{\nu+1}b_1) + (a_2 b_{\nu+2} - a_{\nu+2}b_2) + \cdots + (a_{i-1-\nu}b_{i-1} - a_{i-1}b_{i-1-\nu}) \\ &+ a_{i-\nu}b_i + (a_{i+1-\nu}b_{i+1} - a_{i+1}b_{i+1-\nu}) + \cdots + (a_\nu b_{2\nu} - a_{2\nu}b_\nu) - s\big) \end{aligned}$$

for some $s \in R^{\times}$. Therefore, there are $|R|^{2\nu-1}$ classes adjacent to the vertex $R\vec{a}$, and hence $\mathcal{G}_{\mathrm{Sp}_R(V)}$ is $|R|^{2\nu-1}$-regular. This proves (1).

Next, we let $\vec{x} = x_1\vec{e}_1 + \cdots + x_{2\nu}\vec{e}_{2\nu}$ be a unimodular vector in V such that $R\vec{x}$ is a common neighbor of $R\vec{a}$ and $R\vec{b}$. Then

$$(a_1x_{\nu+1} - a_{\nu+1}x_1) + (a_2x_{\nu+2} - a_{\nu+2}x_2) + \cdots + (a_\nu x_{2\nu} - a_{2\nu}x_\nu) = r' \qquad (2.1.2)$$

and

$$(b_1x_{\nu+1} - b_{\nu+1}x_1) + (b_2x_{\nu+2} - b_{\nu+2}x_2) + \cdots + (b_\nu x_{2\nu} - b_{2\nu}x_\nu) = s' \qquad (2.1.3)$$

for some $r', s' \in R^{\times}$. Since $a_i \in R^{\times}$ and we may assume without loss of generality that $i \leq \nu$, from Eq. (2.1.2) we have

$$\begin{aligned} x_{\nu+i} = a_i^{-1}\big(r' &+ (a_{\nu+1}x_1 - a_1x_{\nu+1}) + (a_{\nu+2}x_2 - a_2x_{\nu+2}) + \cdots \\ &+ (a_{\nu+i-1}x_{i-1} - a_{i-1}x_{\nu+i-1}) + a_{\nu+i}x_i + (a_{\nu+i+1}x_{i+1} - a_{i+1}x_{\nu+i+1}) + \cdots \\ &+ (a_{2\nu}x_\nu - a_\nu x_{2\nu})\big). \end{aligned}$$

Subtracting $b_i\times$ (2.1.2) from $a_i\times$(2.1.3) gives

$$-\sum_{j=1}^{\nu}(a_ib_{\nu+j} - a_{\nu+j}b_i)x_j + \sum_{\substack{j=1 \\ j\neq i}}^{\nu}(a_ib_j - a_jb_i)x_{\nu+j} = a_is' - b_ir'. \qquad (2.1.4)$$

Assume that $R\vec{a}$ is adjacent to $R\vec{b}$. By Lemma 2.1.3, we have $a_ib_l - a_lb_i$ is a unit for some $l \in \{1, \ldots, 2\nu\}$ and $l \neq i$. If $l \leq \nu$, then

$$x_{\nu+l} = (a_ib_l - a_lb_i)^{-1}\Big(a_is' - b_ir' + \sum_{j=1}^{\nu}(a_ib_{\nu+j} - a_{\nu+j}b_i)x_j - \sum_{\substack{j=1 \\ j\neq i,l}}^{\nu}(a_ib_j - a_jb_i)x_{\nu+j}\Big)$$

and if $l \geq \nu + 1$, then

$$x_{l-\nu} = (a_i b_l - a_l b_i)^{-1}\Big(a_i s' - b_i r' + \sum_{\substack{j=1 \\ j\neq l-\nu}}^{\nu} (a_i b_{\nu+j} - a_{\nu+j} b_i) x_j - \sum_{\substack{j=1 \\ j\neq i}}^{\nu} (a_i b_j - a_j b_i) x_{\nu+j}\Big).$$

Hence, there are $\frac{|R|^{2\nu-2}|R^\times||R^\times|}{|R^\times|} = |R|^{2\nu-2}|R^\times|$ classes of common neighbors of adjacent vertices $R\vec{a}$ and $R\vec{b}$, and so we have (2).

Finally, suppose that $R\vec{a}$ is not adjacent to $R\vec{b}$. If $a_i b_l - a_l b_i$ is a unit for some $l \in \{1, \ldots, 2\nu\}$ and $l \neq i$, then Eq. (2.1.4) implies that $x_{\nu+l}$ or x_l depends on other $2\nu - 2$ variables similar to the previous paragraph, so that there are $\frac{|R|^{2\nu-2}|R^\times||R^\times|}{|R^\times|} = |R|^{2\nu-2}|R^\times|$ classes of common neighbors. Assume that $a_i b_l - a_l b_i \in M$ for all $l \in \{1, \ldots, 2\nu\} \setminus \{i\}$. Then b_i is a unit, so

$$\begin{aligned} x_{\nu+i} = b_i^{-1}\big(s' &+ (b_{\nu+1}x_1 - b_1 x_{\nu+1}) + (b_{\nu+2}x_2 - b_2 x_{\nu+2}) + \cdots \\ &+ (b_{\nu+i-1}x_{i-1} - b_{i-1}x_{\nu+i-1}) + b_{\nu+i}x_i + (b_{\nu+i+1}x_{i+1} - b_{i+1}x_{\nu+i+1}) + \cdots \\ &+ (b_{2\nu}x_\nu - b_\nu x_{2\nu})\big). \end{aligned}$$

Clearly, if $x_k \in M$ for all $k \in \{1, \ldots, 2\nu\} \setminus \{\nu + i\}$, then $x_{\nu+i} \in R^\times$. Hence, there are $|R|^{2\nu-1}$ classes of common neighbors. This completes the proof of (3). □

Furthermore, the above proof gives the following result.

Theorem 2.1.5. *Let $\vec{a} = a_1\vec{e}_1 + \cdots + a_{2\nu}\vec{e}_{2\nu}$ and $\vec{b} = b_1\vec{e}_1 + \cdots + b_{2\nu}\vec{e}_{2\nu}$ be unimodular vectors in V and assume that $a_i \in R^\times$ for some $i \in \{1, \ldots, 2\nu\}$. If $R\vec{a}$ and $R\vec{b}$ are non-adjacent vectices of $\mathcal{G}_{\mathrm{Sp}_R(V)}$, then the number of common neighbors are*

$$\begin{cases} |R|^{2\nu-2}|R^\times|, & \text{if } a_i b_l - a_l b_i \in R^\times \text{ for some } l \in \{1, \ldots, 2\nu\} \setminus \{i\}, \\ |R|^{2\nu-1}, & \text{if } a_i b_l - a_l b_i \in M \text{ for all } l \in \{1, \ldots, 2\nu\} \setminus \{i\}. \end{cases}$$

We conclude some direct consequences of Theorems 2.1.4 and 2.1.5 in our main classification theorem.

Theorem 2.1.6. *Let R be a finite local ring with unique maximal ideal M and let (V, β) be a symplectic space of dimension 2ν.*

(1) If $\nu = 1$, then $\mathcal{G}_{\mathrm{Sp}_R(V)}$ is a strongly regular graph with parameters

$$(|R| + |M|, |R|, |R^\times|, |R|).$$

Moreover, if $\nu = 1$ and R is a field, then $\mathcal{G}_{\mathrm{Sp}_R(V)}$ is a complete graph.

(2) If $\nu \geq 2$ and R is a field, then $\mathcal{G}_{\mathrm{Sp}_R(V)}$ is a strongly regular graph with parameters

$$\left(\frac{|R|^{2\nu} - 1}{|R| - 1}, |R|^{2\nu-1}, |R|^{2\nu-2}(|R| - 1), |R|^{2\nu-2}(|R| - 1)\right).$$

(3) If $\nu \geq 2$ and R is not a field, then $\mathcal{G}_{\mathrm{Sp}_R(V)}$ is a strictly Deza graph with parameters

$$\left(\frac{|R|^{2\nu} - |M|^{2\nu}}{|R^\times|}, |R|^{2\nu-1}, |R|^{2\nu-2}|R^\times|, |R|^{2\nu-1}\right).$$

2.2 Vertex and arc transitivities

In this section, we show that our symplectic graph $\mathcal{G}_{\mathrm{Sp}_R(V)}$ is arc transitive. We recall Proposition 2.3 of [8] as follows.

Proposition 2.2.1. *[8] Let R be a local ring. If $\{\vec{x}, \vec{a}\}$ and $\{\vec{x}, \vec{b}\}$ are hyperbolic pairs of unimodular vectors in V, then there exists an isometry σ in $\mathrm{Sp}_R(V)$ which leaves $\vec{x}$ invariant and carries $\vec{a}$ to $\vec{b}$.*

Lemma 2.2.2. *Let R be a finite local ring and let (V, β) be a symplectic space of dimension 2ν. Then symplectic group $\mathrm{Sp}_R(V)$ acts transitively on unimodular vectors and on hyperbolic planes.*

Proof. Let $\vec{a}$ and $\vec{b}$ be unimodular vectors in V such that $R\vec{a} \neq R\vec{b}$. By Theorem 2.1.4 (2) and (3), for every two distinct vertices of $\mathcal{G}_{\mathrm{Sp}_R(V)}$, there exists a unimodular vector $\vec{x}$ such that $\{\vec{x}, \vec{a}\}$ and $\{\vec{x}, \vec{b}\}$ are hyperbolic pairs. Then Proposition

2.2.1 gives an isometry σ in $\mathrm{Sp}_R(V)$ which leaves $\vec{x}$ invariant and carries $\vec{a}$ to $\vec{b}$. Hence, $\mathrm{Sp}_R(V)$ acts transitively on unimodular vectors.

Next, let $\{\vec{a}, \vec{b}\}$ and $\{\vec{c}, \vec{d}\}$ be two distinct hyperbolic pairs of unimodular vectors in V. Then there exists an isometry ρ in $\mathrm{Sp}_R(V)$ carries $\vec{a}$ to $\vec{c}$. Since $\{\vec{a}, \vec{b}\}$ is hyperbolic pair, so is the pair $\{\rho(\vec{x}), \rho(\vec{a})\} = \{\vec{c}, \rho(\vec{b})\}$. Again, Proposition 2.2.1 implies an isometry τ in $\mathrm{Sp}_R(V)$ which leaves $\vec{c}$ invariant and carries $\rho(\vec{b})$ to $\vec{d}$. It follows that $\tau \circ \rho \in \mathrm{Sp}_R(V)$ maps $\{\vec{a}, \vec{b}\}$ to $\{\vec{c}, \vec{d}\}$ as desired. □

Theorem 2.2.3. *Let R be a finite local ring and let (V, β) be a symplectic space of dimension 2ν. Then symplectic graph $\mathcal{G}_{\mathrm{Sp}_R(V)}$ is vertex transitive and arc transitive.*

Proof. The previous lemma proves that our graph is vertex transitive. Observe that for any automorphism σ of V, we have the induced automorphism T_σ on the vertex set of the symplectic graph $\mathcal{G}_{\mathrm{Sp}(V)}$ given by

$$T_\sigma : R\vec{a} \mapsto R\sigma(\vec{a})$$

for all unimodular vectors $\vec{a} \in V$. Let $\vec{a}$ and $\vec{b}$ be unimodular vectors in V. By Lemma 2.2.2, there is an isometry $\sigma \in \mathrm{Sp}_R(V)$ such that $\sigma(\vec{a}) = \vec{b}$. Thus, we have $T_\sigma \in \mathrm{Aut}\,\mathcal{G}_{\mathrm{Sp}_R(V)}$ and $T_\sigma : R\vec{a} \mapsto R\sigma(\vec{a}) = R\vec{b}$.

For edge transitivity, we let $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ be unimodular vectors in V such that $\{\vec{a}, \vec{b}\}$ and $\{\vec{c}, \vec{d}\}$ are hyperbolic pairs. Again, by Lemma 2.2.2, there exists an isometry $\sigma \in \mathrm{Sp}_R(V)$ such that $\sigma(\vec{a}) = \vec{c}$ and $\sigma(\vec{b}) = \vec{d}$. Hence, $T_\sigma \in \mathrm{Aut}\,\mathcal{G}_{\mathrm{Sp}_R(V)}$ sends $R\vec{a}$ to $R\vec{c}$ and $R\vec{b}$ to $R\vec{d}$. This proof also shows that the symplectic graph $\mathcal{G}_{\mathrm{Sp}_R(V)}$ is arc transitive. □

2.3 Chromatic numbers

Let R be a finite local ring with unique maximal ideal M and the residue field $k = R/M$. Let V be a free R-module of R-dimension 2ν, $\nu \geq 1$, and let V' be the 2ν-dimensional vector space over k induced from V via the canonical map $\pi : R \to k$ given by

$$\pi : r \mapsto r + M.$$

Moreover, if (V, β) is a symplectic space, then (V', β') is a symplectic space, where β' is given by

$$\beta'(\pi(\vec{a}), \pi(\vec{b})) = \pi(\beta(\vec{a}, \vec{b}))$$

for all $\vec{a}, \vec{b} \in V$. Here, we write $\pi(\vec{a}) = (\pi(a_1), \pi(a_2), \dots, \pi(a_{2\nu}))$ for all $\vec{a} = (a_1, a_2, \dots, a_{2\nu}) \in V$. Note that the relation

$$R\vec{x} \sim R\vec{y} \Leftrightarrow k\pi(\vec{x}) = k\pi(\vec{y}) \tag{2.3.1}$$

is an equivalence relation on the vertex set of the graph $\mathcal{G}_{\mathrm{Sp}_R(V)}$. Since R is a local ring, it follows that

$$\beta(\vec{a}, \vec{b}) \in R^{\times} \Leftrightarrow \pi(\beta(\vec{a}, \vec{b})) \neq M \Leftrightarrow \beta'(\pi(\vec{a}), \pi(\vec{b})) \in k^{\times}.$$

This gives (3) of the next theorem.

Theorem 2.3.1. *Let* $\kappa = \frac{|k|^{2\nu}-1}{|k|-1}$ *and* $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_\kappa$ *be unimodular vectors in* V *such that the vertex set*

$$\mathcal{V}(\mathcal{G}_{\mathrm{Sp}_k(V')}) = \{k\pi(\vec{x}_i) : i = 1, 2, \dots, \kappa\}.$$

(1) The set $\Pi = \{R(\vec{x}_1 + M^{2\nu}), R(\vec{x}_2 + M^{2\nu}), \dots, R(\vec{x}_\kappa + M^{2\nu})\}$ *is a partition of* $\mathcal{V}(\mathcal{G}_{\mathrm{Sp}_R(V)})$*, where* $R(\vec{x}_i + M^{2\nu}) = \{R(\vec{x}_i + \vec{m}) : \vec{m} \in M^{2\nu}\}$ *for all* $\in \{1, 2, \dots, \kappa\}$*. Moreover, for each* $i \in \{1, 2, \dots, \kappa\}$*, any two distinct vertices in* $R(\vec{x}_i + M^{2\nu})$ *are non-adjacent vertices.*

(2) $|R(\vec{x}_i + M^{2\nu})| = |M|^{2\nu-1}$ *for all* $i \in \{1, \dots, \kappa\}$*.*

(3) For unimodular vectors $\vec{a}, \vec{b} \in V$*, we have* $R\vec{a}$ *and* $R\vec{b}$ *are adjacent vertices in* $\mathcal{V}(\mathcal{G}_{\mathrm{Sp}_R(V)})$ *if and only if* $k\pi(\vec{a})$ *and* $k\pi(\vec{b})$ *are adjacent vertices in* $\mathcal{V}(\mathcal{G}_{\mathrm{Sp}_k(V')})$*.*

(4) For $i, j \in \{1, 2, \ldots, \kappa\}$, if $k\pi(\vec{x}_i)$ and $k\pi(\vec{x}_j)$ are adjacent vertices, then $R(\vec{x}_i + \vec{m}_1)$ and $R(\vec{x}_j + \vec{m}_2)$ are adjacent vertices in $\mathcal{V}(\mathcal{G}_{\mathrm{Sp}_R(V)})$ for all $\vec{m}_1, \vec{m}_2 \in M^{2\nu}$.

Proof. The first part of (1) follows from (2.3.1) and (4) is an immediate consequence of (3). Note that

$$\beta(\vec{x}_i + \vec{m}_1, \vec{x}_i + \vec{m}_2) = \beta(\vec{x}_i, \vec{m}_1) + \beta(\vec{m}_2, \vec{x}_i) + \beta(\vec{m}_1, \vec{m}_2) \in M$$

for all $i \in \{1, 2, \ldots, \kappa\}$ and $\vec{m}_1, \vec{m}_2 \in M^{2\nu}$. This proves the second part of (1).

Next, let $\vec{m}_1, \vec{m}_2 \in M^{2\nu}$ and assume that $R(\vec{x}_i + \vec{m}_1) = R(\vec{x}_i + \vec{m}_2)$. Then $\vec{x}_i + \vec{m}_1 = \lambda(\vec{x}_i + \vec{m}_2)$ for some $\lambda \in R^\times$. Thus

$$(1 - \lambda)\vec{x}_i = \lambda\vec{m}_2 - \vec{m}_1 \in M^{2\nu}.$$

Since $\vec{x}_i$ is unimodular, $1 - \lambda \in M$, so $\lambda = 1 + \mu$ for some $\mu \in M$. Hence, $\vec{x}_i + \vec{m}_1 = (1 + \mu)(\vec{x}_i + \vec{m}_2)$. Finally, we show that $R(1+\mu)(\vec{x}+\vec{m}) = R(\vec{x}+\vec{m})$ for all $\mu \in M$, $\vec{x} \in V$ unimodular, and $\vec{m} \in M^{2\nu}$ and we therefore have (2). Clearly, $R(1 + \mu)(\vec{x} + \vec{m}) \subseteq R(\vec{x} + \vec{m})$. Since $\mu \in M$, $1 + \mu \in R^\times$. Then $r(\vec{x} + \vec{m}) = (r(1 + \mu)^{-1})(1 + \mu)(\vec{x} + \vec{m})$ for all $r \in R$, which gives another inclusion. □

Recall that the chromatic number of a graph G is the smallest number of colors needed to color the vertices of G so that no two adjacent vertices share the same color. It follows from Proposition 2.3 of [13] that $\mathcal{G}_{\mathrm{Sp}_k(V')}$ is $|k|^\nu + 1$-partite with partite sets $Y_1, Y_2, \ldots, Y_{|k|^\nu+1}$, where $Y_i \cap Y_j = \varnothing$ for all $i \neq j$ and there is no edge of $\mathcal{G}_{\mathrm{Sp}_k(V')}$ joining two vertices of the same subset. Moreover, the subsets $Y_1, Y_2, \ldots, Y_{|k|^\nu+1}$ can be chosen so that for any distinct indices i and j, every $y \in Y_i$ is adjacent to exactly $|k|^{\nu-1}$ vertices in Y_j. In addition, the chromatic number of $\mathcal{G}_{\mathrm{Sp}_k(V')}$ is $|k|^\nu + 1$ (Theorem 1.3.8). The canonical map $\pi : R \to k$ and Theorem 2.3.1 give the following theorem.

Theorem 2.3.2. *The symplectic graph $\mathcal{G}_{\mathrm{Sp}_R(V)}$ is $|k|^\nu + 1$-partite with partite sets $\pi^{-1}(Y_1), \pi^{-1}(Y_2), \ldots, \pi^{-1}(Y_{|k|^\nu+1})$, where Y_j, $j = 1, 2, \ldots, |k|^\nu + 1$, are subsets of $\mathcal{G}_{\mathrm{Sp}_k(V')}$ discussed above. Moreover, for any distinct indices i and j, every $a \in \pi^{-1}(Y_i)$*

is adjacent to exactly $|M|^{2\nu-1}|k|^{\nu-1}$ vertices in $\pi^{-1}(Y_j)$. As a result, the chromatic number $\chi(\mathcal{G}_{\mathrm{Sp}_R(V)})$ is $|k|^\nu + 1$.

Proof. It remains to derive the chromatic number of $\mathcal{G}_{\mathrm{Sp}_R(V)}$. Since our graph is $|k|^\nu+1$-partite, $\chi(\mathcal{G}_{\mathrm{Sp}_R(V)}) \leq |k|^\nu+1$. To prove the reverse inequality, we consider the induced subgraph of $\mathcal{G}_{\mathrm{Sp}_R(V)}$ whose vertex set is $\{Rx_1, Rx_2, \ldots, Rx_\kappa\}$. By Theorem 2.3.1 (3), this subgraph is isomorphic to the symplectic graph $\mathcal{G}_{\mathrm{Sp}_k(V')}$. Thus, it has chromatic number $|k|^\nu + 1$. Hence, the chromatic number $\chi(\mathcal{G}_{\mathrm{Sp}_R(V)})$ is $|k|^\nu + 1$ as desired. □

2.4 Automorphisms

In this section, we study the automorphism group of our symplectic graph. We begin by recalling the results of Tang and Wan [13] for symplectic graphs over finite fields.

Let k be a field and write $\mathrm{Aut}(k)$ for the group of automorphisms of k. Let φ be the natural action of $\mathrm{Aut}(k)$ on the group $(k^\times)^\nu = k^\times \times \cdots \times k^\times$ (ν copies) defined by

$$\varphi(\phi)((a_1, \ldots, a_\nu)) = (\phi(a_1), \ldots, \phi(a_\nu)),$$

for all $\phi \in$ Aut(k) and $a_1, \ldots, a_\nu \in k^\times$. The **semidirect product** of $(k^\times)^\nu$ by Aut(k) corresponding to φ, denoted by $(k^\times)^\nu \rtimes_\varphi \mathrm{Aut}(k)$, is the group consisting of all elements of the form $((a_1, \ldots, a_\nu), \phi)$, where $a_1, \ldots, a_\nu \in k^\times$ and $\phi \in \mathrm{Aut}(k)$ with multiplication defined by

$$\begin{aligned} ((a_1, \ldots, a_\nu), \phi)((a'_1, \ldots, a'_\nu), \phi') &= ((a_1, \ldots, a_\nu)(\varphi(\phi)((a'_1, \ldots, a'_\nu))), \phi \circ \phi') \\ &= ((a_1\phi(a'_1), \ldots, a_\nu\phi(a'_\nu)), \phi \circ \phi'). \end{aligned}$$

The set of all permutations of a set S is denoted by $\mathrm{Sym}(S)$ or just $\mathrm{Sym}(n)$ if $|S| = n$. Note that $|\,\mathrm{Sym}(n)| = n!$. The automorphism group of the symplectic graph over a finite field can be described as follows.

Theorem 2.4.1. (Theorem 3.4 and Corollary 3.6 of [13]) *Let k be a field and V be a symplectic space over k of dimension 2ν, $\nu \geq 1$. Regard the symplectic group $\mathrm{Sp}_k(V)/\{\pm I_{2\nu}\}$ as a subgroup of $\mathrm{Aut}(\mathcal{G}_{\mathrm{Sp}_k(V)})$ (as shown in Theorem 1.3.5) and let E be the subgroup of $\mathrm{Aut}(\mathcal{G}_{\mathrm{Sp}_k(V)})$ defined as follows:*

$$E = \{\sigma \in \mathrm{Aut}(\mathcal{G}_{\mathrm{Sp}_k(V)}) : \sigma(k\vec{e_i}) = k\vec{e_i} \text{ for all } i = 1, \ldots, 2\nu\},$$

where $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_{2\nu}\}$ is the standard basis of V. Then

$$\mathrm{Aut}(\mathcal{G}_{\mathrm{Sp}_k(V)}) = (\mathrm{Sp}_k(V)/\{\pm I_{2\nu}\}) \cdot E,$$

where

$$E \cong \begin{cases} \mathrm{Sym}(k^\times) = \mathrm{Sym}(|k|-1), & \text{if } \nu = 1, \\ (k^\times)^\nu \rtimes_\varphi \mathrm{Aut}(k), & \text{if } \nu \geq 2. \end{cases}$$

Consequently, the number of automorphisms of the symplectic graph is given by

$$|\mathrm{Aut}(\mathcal{G}_{\mathrm{Sp}_k(V)})| = \begin{cases} |k|(|k|^2-1)\cdot(|k|-2)!, & \text{if } \nu = 1, \\ |k|^{\nu^2} \prod_{i=1}^{\nu} (|k|^{2i}-1)\cdot[k:\mathbb{F}_p], & \text{if } \nu \geq 2, \end{cases}$$

where k is of characteristic p and $[k : \mathbb{F}_p]$ denotes the degree of extension of k over $\mathbb{F}_p$.

For a finite local ring R, we have the following result.

Theorem 2.4.2. *Let R be a finite local ring with unique maximal ideal M and residue field $k = R/M$ and let (V, β) be a symplectic space of dimension 2ν, $\nu \geq 1$. Then*

$$\mathrm{Aut}(\mathcal{G}_{\mathrm{Sp}_R(V)}) \cong \mathrm{Aut}(\mathcal{G}_{\mathrm{Sp}_k(V')}) \times (\mathrm{Sym}(|M|^{2\nu-1}))^\kappa,$$

where $\kappa = \frac{|k|^{2\nu-1}}{|k|-1}$ and V' is the symplectic space over k induced from V.

Proof. Let $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_\kappa$ be unimodular vectors in V such that

$$\mathcal{V}(\mathcal{G}_{\mathrm{Sp}_k(V')}) = \{k\pi(\vec{x}_i) : i \in \{1, 2, \ldots, \kappa\}\}.$$

Theorem 2.3.1 shows that the subgraph of $\mathcal{G}_{\mathrm{Sp}_R(V)}$ induced from the vertex set $\{R\vec{x}_i : i \in \{1, 2, \ldots, \kappa\}\}$ is isomorphic to the symplectic graph $\mathcal{G}_{\mathrm{Sp}_k(V')}$. Moreover, each automorphism of $\mathcal{G}_{\mathrm{Sp}_R(V)}$ corresponds with an automorphism of the graph $\mathcal{G}_{\mathrm{Sp}_k(V')}$ and a permutation of vertices in the set $R(\vec{x}_i + M^{2\nu})$ for all $i \in \{1, 2, \ldots, \kappa\}$. Thus,

$$\begin{aligned}\mathrm{Aut}(\mathcal{G}_{\mathrm{Sp}_R(V)}) &\cong \mathrm{Aut}(\mathcal{G}_{\mathrm{Sp}_k(V')}) \times \prod_{i=1}^{\kappa} \mathrm{Sym}(|R(\vec{x}_i + M^{2\nu})|) \\ &= \mathrm{Aut}(\mathcal{G}_{\mathrm{Sp}_k(V')}) \times (\mathrm{Sym}(|M|^{2\nu-1}))^{\kappa}\end{aligned}$$

because $|R(\vec{x}_i + M^{2\nu})| = |M|^{2\nu-1}$ for all $i \in \{1, 2, \ldots, \kappa\}$. □

CHAPTER III

SUBCONSTITUENTS OF SYMPLECTIC GRAPHS OVER FINITE LOCAL RINGS

In this chapter, we study the subconstituents $\mathcal{G}^{(i)}_{\mathrm{Sp}_R(V)}$, $i = 1, 2$, of symplectic graphs over finite local rings. The first section presents the definition and results on strong regularity. Sections 3.2 and 3.3 include the work on chromatic numbers and automorphism groups, respectively. This chapter generalizes the work in [5], [6] and [9].

3.1 Subconstituents of symplectic graphs

Let R be a finite local ring with unique maximal ideal M. Another aspect in studying the symplectic graph $\mathcal{G}_{\mathrm{Sp}_R(V)}$ is to work on the **subconstituents** $\mathcal{G}^{(i)}_{\mathrm{Sp}_R(V)}$, $i = 1, 2$, defined to be the induced subgraphs of $\mathcal{G}_{\mathrm{Sp}_R(V)}$ on the vertex sets

$$\mathcal{V}_i = \{R\vec{x} : \vec{x} \text{ is a unimodular vector in } V \text{ and } d(R\vec{x}, R\vec{e}_{\nu+1}) = i\}$$

$i = 1, 2$, respectively. Recall from Theorem 2.1.6 that our symplectic graph is a strongly regular or strictly Deza graph, so for each modular vector $\vec{x} \in V$, we have $d(R\vec{x}, R\vec{e}_{\nu+1}) = 1$ or 2, if $R\vec{x} \neq R\vec{e}_{\nu+1}$. Thus, $\mathcal{V}_1$ consists of adjacent vertices of $R\vec{e}_{\nu+1}$ and $\mathcal{V}_2$ consists of non-adjacent vertices of $R\vec{e}_{\nu+1}$, which are not $R\vec{e}_{\nu+1}$. Hence, $\mathcal{G}^{(1)}_{\mathrm{Sp}_R(V)}$ is the induced subgraph of $\mathcal{G}_{\mathrm{Sp}_R(V)}$ on the vertex set

$$\mathcal{V}_1 = \{R\vec{x} : \vec{x} = \vec{e}_1 + a_2\vec{e}_2 + \cdots + a_{2\nu}\vec{e}_{2\nu}, \text{ where } a_i \in R \text{ for all } i \in \{2, \ldots, 2\nu\}\}$$

and $\mathcal{G}^{(2)}_{\mathrm{Sp}_R(V)}$ is the induced subgraph of $\mathcal{G}_{\mathrm{Sp}_R(V)}$ on the vertex set

$$\mathcal{V}_2 = \big\{R\vec{x} : \vec{x} = m\vec{e}_1 + a_2\vec{e}_2 + \cdots + a_{2\nu}\vec{e}_{2\nu} \text{ is unimodular in } V, \\ \text{where } a_i \in R \text{ for all } i \in \{2, \dots, 2\nu\} \text{ and } m \in M\big\}.$$

This allows us to find that

$$|\mathcal{V}_1| = |R|^{2\nu-1} \text{ and } |\mathcal{V}_2| = |\mathcal{V}(\mathcal{G}_{\mathrm{Sp}_R(V)})| - |\mathcal{V}_1| - |\{\vec{e}_{\nu+1}\}| = \tfrac{(|R|^{2\nu-1}-|M|^{2\nu-1})|M|}{|R^\times|} - 1.$$

Remark. Observe that we can define the subconstituents associated with other vertices. However, since the symplectic graph $\mathcal{G}_{\mathrm{Sp}_R(V)}$ is vertex and arc transitive, it suffices to consider only the ones associated with $R\vec{e}_{\nu+1}$.

Let $\vec{a} = \vec{e}_1 + a_2\vec{e}_2 + \cdots + a_{2\nu}\vec{e}_{2\nu}$ and $\vec{b} = \vec{e}_1 + b_2\vec{e}_2 + \cdots + b_{2\nu}\vec{e}_{2\nu}$ be vectors in V. Assume that $R\vec{a}$ is adjacent to $R\vec{b}$. Thus,

$$b_{\nu+1} = r + a_{\nu+1} + (a_{\nu+2}b_2 - a_2b_{\nu+2}) + \cdots + (a_{2\nu}b_\nu - a_\nu b_{2\nu}).$$

for some $r \in R^\times$. Therefore, there are $|R|^{2\nu-2}|R^\times|$ classes adjacent to the vertex $R\vec{a}$ in $\mathcal{G}^{(1)}_{\mathrm{Sp}_R(V)}$, and hence $\mathcal{G}^{(1)}_{\mathrm{Sp}_R(V)}$ is $|R|^{2\nu-2}|R^\times|$-regular. Next, we proceed to prove the following theorem.

Theorem 3.1.1. *Let R be a finite local ring and let (V, β) be a symplectic space of dimension 2ν, where $\nu \geq 1$.*

(1) The subconstituent graph $\mathcal{G}^{(1)}_{\mathrm{Sp}_R(V)}$ is $|R|^{2\nu-2}|R^\times|$-regular with $|R|^{2\nu-1}$ vertices.

(2) Every two adjacent vertices of $\mathcal{G}^{(1)}_{\mathrm{Sp}_R(V)}$ has $|R|^{2\nu-3}|R^\times|^2$ or $|R|^{2\nu-2}(|R^\times| - |M|)$ common neighbors.

(3) Every two non-adjacent vertices of $\mathcal{G}^{(1)}_{\mathrm{Sp}_R(V)}$ has $|R|^{2\nu-3}|R^\times|^2$ or $|R|^{2\nu-2}|R^\times|$ common neighbors.

Proof. We have proved (1) in the above discussion. Let $\vec{x} = \vec{e}_1 + x_2\vec{e}_2 + \cdots + x_{2\nu}\vec{e}_{2\nu}$

be a vector in V such that $R\vec{x}$ is a common neighbor of $R\vec{a}$ and $R\vec{b}$. Then

$$(x_{\nu+1} - a_{\nu+1}) + (a_2 x_{\nu+2} - a_{\nu+2} x_2) + \cdots + (a_\nu x_{2\nu} - a_{2\nu} x_\nu) = r' \tag{3.1.1}$$

and

$$(x_{\nu+1} - b_{\nu+1}) + (b_2 x_{\nu+2} - b_{\nu+2} x_2) + \cdots + (b_\nu x_{2\nu} - b_{2\nu} x_\nu) = s' \tag{3.1.2}$$

for some $r', s' \in R^\times$. From Eq. (3.1.1), we have

$$x_{\nu+1} = r' + a_{\nu+1} + (a_{\nu+2} x_2 - a_2 x_{\nu+2}) + \cdots + (a_{2\nu} x_\nu - a_\nu x_{2\nu}). \tag{3.1.3}$$

Subtracting (3.1.1) from (3.1.2) gives

$$-(b_{\nu+1} - a_{\nu+1}) - \sum_{j=2}^{\nu} (b_{\nu+j} - a_{\nu+j}) x_j + \sum_{j=2}^{\nu} (b_j - a_j) x_{\nu+j} = s' - r'. \tag{3.1.4}$$

Assume that $R\vec{a}$ is adjacent to $R\vec{b}$. By Lemma 2.1.3, we have $b_l - a_l$ is a unit for some $l \in \{2, \dots, 2\nu\}$. If $l \le \nu$, then

$$x_{\nu+l} = (b_l - a_l)^{-1} \Big(s' - r' + b_{\nu+1} - a_{\nu+1} + \sum_{j=2}^{\nu} (b_{\nu+j} - a_{\nu+j}) x_j - \sum_{\substack{j=2 \\ j \ne l}}^{\nu} (b_j - a_j) x_{\nu+j} \Big)$$

and if $l \ge \nu + 1$, then

$$x_{l-\nu} = (b_l - a_l)^{-1} \Big(s' - r' + b_{\nu+1} - a_{\nu+1} + \sum_{\substack{j=2 \\ j \ne l-\nu}}^{\nu} (b_{\nu+j} - a_{\nu+j}) x_j - \sum_{j=2}^{\nu} (b_j - a_j) x_{\nu+j} \Big). \tag{3.1.5}$$

If $l \ne \nu + 1$, then there are $|R|^{2\nu-3}|R^\times|^2$ classes of common neighbors of adjacent vertices $R\vec{a}$ and $R\vec{b}$. Now, we assume that $l = \nu + 1$ and $b_j - a_j \in M$ for all $j \in \{2, \dots, 2\nu\} \smallsetminus \{l\}$. Then, by Eq. (3.1.4), we have

$$r' - s' = (b_{\nu+1} - a_{\nu+1}) + \sum_{j=2}^{\nu} (b_{\nu+j} - a_{\nu+j}) x_j - \sum_{j=2}^{\nu} (b_j - a_j) x_{\nu+j},$$

which implies $r' - s' \in R^\times$. There are

$$\frac{|R|^{2\nu-2}(|R^\times|^2 - |R^\times||M|)}{|R^\times|} = |R|^{2\nu-2}(|R^\times| - |M|) = |R|^{2\nu-2}(|R| - 2|M|)$$

classes of common neighbors. This completes the proof of (2).

Finally, we suppose that $R\vec{a}$ is not adjacent to $R\vec{b}$. If $b_l - a_l$ is a unit for some $l \in \{2, \dots, 2\nu\} \smallsetminus \{\nu+1\}$, then Eq. (3.1.3) and (3.1.4) imply that $x_{\nu+1}$ and $(x_{\nu+l}$ or $x_l)$ depend on other $2\nu - 3$ variables, so that there are $|R|^{2\nu-3}|R^\times|^2$ classes of common neighbors. Assume that $b_l - a_l \in M$ for all $l \in \{2, \dots, 2\nu\} \smallsetminus \{\nu+1\}$. If $b_{\nu+1} - a_{\nu+1} \in R^\times$, then

$$\begin{aligned}(b_{\nu+1} - a_{\nu+1}) + &\sum_{i=2}^{\nu}(a_i b_{\nu+i} - a_{\nu+i} b_i) \\ &= (b_{\nu+1} - a_{\nu+1}) + \sum_{i=2}^{\nu}\big(b_{\nu+i}(a_i - b_i) + b_i(b_{\nu+i} - a_{\nu+i})\big) \in R^\times,\end{aligned}$$

which contradicts the fact that $R\vec{a}$ is not adjacent to $R\vec{b}$. Thus, $b_{\nu+1} - a_{\nu+1} \in M$. Eq. (3.1.4) implies that

$$s' = r' - (b_{\nu+1} - a_{\nu+1}) - \sum_{j=2}^{\nu}(b_{\nu+j} - a_{\nu+j})x_j + \sum_{j=2}^{\nu}(b_j - a_j)x_{\nu+j}$$

is a unit. Hence, there are $|R|^{2\nu-2}|R^\times|$ classes of common neighbors. $\square$

Moreover, the proof of Theorem 3.1.1 gives the following results.

Theorem 3.1.2. *Let $\vec{a} = \vec{e}_1 + a_2\vec{e}_2 + \cdots + a_{2\nu}\vec{e}_{2\nu}$ and $\vec{b} = \vec{e}_1 + b_2\vec{e}_2 + \cdots + b_{2\nu}\vec{e}_{2\nu}$ be unimodular vectors in V.*

(1) If $R\vec{a}$ and $R\vec{b}$ are adjacent vectices of $\mathcal{G}^{(1)}_{\mathrm{Sp}_R(V)}$, then the number of common neighbors are

$$\begin{cases} |R|^{2\nu-3}|R^\times|^2, & \text{if } b_l - a_l \in R^\times \text{ for some } l \in \{2, \dots, 2\nu\} \smallsetminus \{\nu+1\}, \\ |R|^{2\nu-2}(|R^\times| - |M|), & \text{if } b_l - a_l \in M \text{ for all } l \in \{2, \dots, 2\nu\} \smallsetminus \{\nu+1\}. \end{cases}$$

(2) If $R\vec{a}$ and $R\vec{b}$ are non-adjacent vectices of $\mathcal{G}^{(1)}_{\mathrm{Sp}_R(V)}$, then the number of common neighbors are

$$\begin{cases} |R|^{2\nu-3}|R^\times|^2, & \text{if } b_l - a_l \in R^\times \text{ for some } l \in \{2,\ldots,2\nu\} \smallsetminus \{\nu+1\}, \\ |R|^{2\nu-2}|R^\times|, & \text{if } b_l - a_l \in M \text{ for all } l \in \{2,\ldots,2\nu\} \smallsetminus \{\nu+1\}. \end{cases}$$

Results for the subconstituent graph $\mathcal{G}^{(2)}_{\mathrm{Sp}_R(V)}$ are similar as we shall see in the next theorem. If $\nu = 1$, then the graph $\mathcal{G}^{(2)}_{\mathrm{Sp}_R(V)}$ consists of $|M| - 1$ vertices with no edges, so it is an empty graph. Thus, we may assume that $\nu \geq 2$.

Theorem 3.1.3. *Let R be a finite local ring and let (V, β) be a symplectic space of dimension 2ν, where $\nu \geq 2$.*

(1) The subconstituent graph $\mathcal{G}^{(2)}_{\mathrm{Sp}_R(V)}$ is $|R|^{2\nu-2}|M|$-regular.

(2) Every two adjacent vertices of $\mathcal{G}^{(2)}_{\mathrm{Sp}_R(V)}$ has $|R|^{2\nu-3}|R^\times||M|$ common neighbors.

(3) Every two non-adjacent vertices of $\mathcal{G}^{(2)}_{\mathrm{Sp}_R(V)}$ has $|R|^{2\nu-3}|R^\times||M|$ or $|R|^{2\nu-2}|M|$ common neighbors.

Proof. Let $\vec{a} = m\vec{e}_1 + a_2\vec{e}_2 + \cdots + a_{2\nu}\vec{e}_{2\nu}$ and $\vec{b} = m'\vec{e}_1 + b_2\vec{e}_2 + \cdots + b_{2\nu}\vec{e}_{2\nu}$ be unimodular vectors in V and assume that $R\vec{a}$ is adjacent to $R\vec{b}$ in $\mathcal{G}^{(2)}_{\mathrm{Sp}_R(V)}$. Since $\vec{a}$ is unimodular, there exists an $i \in \{2, \ldots, 2\nu\}$ such that $a_i \in R^\times$. If $i \leq \nu$, then

$$\begin{aligned} b_{\nu+i} = a_i^{-1}\big(&r + (a_{\nu+1}m' - mb_{\nu+1}) + (a_{\nu+2}b_2 - a_2b_{\nu+2}) + \cdots \\ &+ (a_{\nu+i-1}b_{i-1} - a_{i-1}b_{\nu+i-1}) + a_{\nu+i}b_i + (a_{\nu+i+1}b_{i+1} - a_{i+1}b_{\nu+i+1}) + \cdots \\ &+ (a_{2\nu}b_\nu - a_\nu b_{2\nu})\big) \end{aligned}$$

for some $r \in R^\times$ and if $i \geq \nu + 1$, then

$$\begin{aligned} b_{i-\nu} = a_i^{-1}\big(&(mb_{\nu+1} - a_{\nu+1}m') + (a_2b_{\nu+2} - a_{\nu+2}b_2) + \cdots + (a_{i-1-\nu}b_{i-1} - a_{i-1}b_{i-1-\nu}) \\ &+ a_{i-\nu}b_i + (a_{i+1-\nu}b_{i+1} - a_{i+1}b_{i+1-\nu}) + \cdots + (a_\nu b_{2\nu} - a_{2\nu}b_\nu) - s\big) \end{aligned}$$

for some $s \in R^\times$. Hence, there are $|R|^{2\nu-2}|M|$ classes adjacent to the vertex $R\vec{a}$, and thus $\mathcal{G}^{(2)}_{\mathrm{Sp}_R(V)}$ is $|R|^{2\nu-2}|M|$-regular. Therefore, we have (1).

Next, let $\vec{x} = m''\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_{2\nu}\vec{e}_{2\nu}$ be a unimodular vector in V such that $R\vec{x}$ is a common neighbor of $R\vec{a}$ and $R\vec{b}$. Then

$$(mx_{\nu+1} - a_{\nu+1}m'') + (a_2x_{\nu+2} - a_{\nu+2}x_2) + \cdots + (a_\nu x_{2\nu} - a_{2\nu}x_\nu) = r' \tag{3.1.6}$$

and

$$(m'x_{\nu+1} - b_{\nu+1}m'') + (b_2x_{\nu+2} - b_{\nu+2}x_2) + \cdots + (b_\nu x_{2\nu} - b_{2\nu}x_\nu) = s' \tag{3.1.7}$$

for some $r', s' \in R^\times$. Since $a_i \in R^\times$, if $i \le \nu$, we have

$$\begin{aligned} x_{\nu+i} = a_i^{-1}\big(&r' + (a_{\nu+1}m'' - mx_{\nu+1}) + (a_{\nu+2}x_2 - a_2x_{\nu+2}) + \cdots \\ &+ (a_{\nu+i-1}x_{i-1} - a_{i-1}x_{\nu+i-1}) + a_{\nu+i}x_i + (a_{\nu+i+1}x_{i+1} - a_{i+1}x_{\nu+i+1}) + \cdots \\ &+ (a_{2\nu}x_\nu - a_\nu x_{2\nu})\big) \end{aligned}$$

and if $i \ge \nu + 1$, we have

$$\begin{aligned} x_{i-\nu} = a_i^{-1}\big(&(mx_{\nu+1} - a_{\nu+1}m'') + (a_2x_{\nu+2} - a_{\nu+2}x_2) + \cdots \\ &+ (a_{i-1-\nu}x_{i-1} - a_{i-1}x_{i-1-\nu}) + a_{i-\nu}x_i + (a_{i+1-\nu}x_{i+1} - a_{i+1}x_{i+1-\nu}) + \cdots \\ &+ (a_\nu x_{2\nu} - a_{2\nu}x_\nu) - s\big). \end{aligned}$$

In what follows, we shall prove (2) and (3) when $i \le \nu$. The proof for the case $i \ge \nu + 1$ is obtained in the same way. Subtracting $b_i\times$ (3.1.6) from $a_i\times$(3.1.7) gives

$$a_i s' - b_i r' = -(a_i b_{\nu+1} - a_{\nu+1} b_i) m'' - \sum_{j=2}^{\nu} (a_i b_{\nu+j} - a_{\nu+j} b_i) x_j + (a_i m' - m b_i) x_{\nu+1} + \sum_{\substack{j=2 \\ j \neq i}}^{\nu} (a_i b_j - a_j b_i) x_{\nu+j}. \tag{3.1.8}$$

Assume that $R\vec{a}$ is adjacent to $R\vec{b}$. By Lemma 2.1.3, we have $a_i b_l - a_l b_i$ is a unit for some $l \in \{2, \ldots, 2\nu\}$ and $l \neq i$. If $l \leq \nu$, then

$$x_{\nu+l} = (a_i b_l - a_l b_i)^{-1} \Big(a_i s' - b_i r' + (a_i b_{\nu+1} - a_{\nu+1} b_i) m'' + \sum_{j=2}^{\nu} (a_i b_{\nu+j} - a_{\nu+j} b_i) x_j - (a_i m' - m b_i) x_{\nu+1} - \sum_{\substack{j=2 \\ j \neq i,l}}^{\nu} (a_i b_j - a_j b_i) x_{\nu+j} \Big) \tag{3.1.9}$$

and if $l \geq \nu + 1$, then

$$x_{l-\nu} = (a_i b_l - a_l b_i)^{-1} \Big(a_i s' - b_i r' + (a_i b_{\nu+1} - a_{\nu+1} b_i) m'' + \sum_{\substack{j=2 \\ j \neq l-\nu}}^{\nu} (a_i b_{\nu+j} - a_{\nu+j} b_i) x_j - (a_i m' - m b_i) x_{\nu+1} - \sum_{\substack{j=2 \\ j \neq i}}^{\nu} (a_i b_j - a_j b_i) x_{\nu+j} \Big). \tag{3.1.10}$$

Thus, there are $\frac{|R|^{2\nu-3}|R^\times|^2|M|}{|R^\times|} = |R|^{2\nu-3}|R^\times||M|$ classes of common neighbors of adjacent vertices $R\vec{a}$ and $R\vec{b}$. This proves (2).

Finally, we suppose that $R\vec{a}$ is not adjacent to $R\vec{b}$. If $a_i b_l - a_l b_i$ is a unit for some $l \in \{2, \ldots, 2\nu\}$ and $l \neq i$, then Eq. (3.1.8) implies that x_i and ($x_{\nu+l}$ or x_l) depend on other $2\nu - 3$ variables similar to the previous paragraph, so that there are $|R|^{2\nu-3}|R^\times||M|$ classes of common neighbors.

Assume that $a_i b_l - a_l b_i \in M$ for all $l \in \{2, \ldots, 2\nu\} \smallsetminus \{\nu + i\}$. If $a_i b_{\nu+i} - a_{\nu+i} b_i \in$

$R^\times$, then

$$\begin{aligned}(mb_{\nu+1}-a_{\nu+1}m') &+ \sum_{j=2}^{\nu}(a_jb_{\nu+j}-a_{\nu+j}b_j)\\ &= (a_ib_{\nu+i}-a_{\nu+i}b_i)+(mb_{\nu+1}-a_{\nu+1}m')+a_i^{-1}\sum_{\substack{j=2\\ j\neq i}}^{\nu}a_i(a_jb_{\nu+j}-a_{\nu+j}b_j)\\ &= (mb_{\nu+1}-a_{\nu+1}m')++a_i^{-1}\sum_{\substack{j=2\\ j\neq i}}^{\nu}\big(a_j(a_ib_{\nu+j}-a_{\nu+j}b_i)+a_{\nu+j}(a_jb_i-a_ib_j)\big)\\ &\quad + (a_ib_{\nu+i}-a_{\nu+i}b_i)\end{aligned}$$

is a unit, which contradicts the fact that $R\vec{a}$ is not adjacent to $R\vec{b}$. Thus, $a_ib_{\nu+i}-a_{\nu+i}b_i \in M$. Again, Eq. (3.1.8) implies that

$$\begin{aligned}s' = a_i^{-1}(b_ir' - (a_ib_{\nu+1}-a_{\nu+1}b_i)m'' &- \sum_{j=2}^{\nu}(a_ib_{\nu+j}-a_{\nu+j}b_i)x_j\\ + (a_im'-mb_i)x_{\nu+1} &+ \sum_{\substack{j=2\\ j\neq i}}^{\nu}(a_ib_j-a_jb_i)x_{\nu+j})\end{aligned}$$

is a unit. Hence, there are $|R|^{2\nu-2}|M|$ classes of common neighbors. The proof completes. □

We may conclude some consequences from the proof of Theorem 3.1.3 in the next theorem.

Theorem 3.1.4. *Let $\vec{a} = m\vec{e}_1 + a_2\vec{e}_2 + \cdots + a_{2\nu}\vec{e}_{2\nu}$ and $\vec{b} = m'\vec{e}_1 + b_2\vec{e}_2 + \cdots + b_{2\nu}\vec{e}_{2\nu}$ be unimodular vectors in V, $m, m' \in M$ and assume that $a_i \in R^\times$ for some $i \in \{2, \ldots, 2\nu\}$.*

(1) If $R\vec{a}$ and $R\vec{b}$ are adjacent vectices of $\mathcal{G}^{(2)}_{\mathrm{Sp}_R(V)}$, then the number of common neighbors is $|R|^{2\nu-3}|R^\times||M|$.

(2) If $R\vec{a}$ and $R\vec{b}$ are non-adjacent vectices of $\mathcal{G}^{(2)}_{\mathrm{Sp}_R(V)}$, then for $i \leq \nu$, the number of

common neighbors are

$$\begin{cases} |R|^{2\nu-3}|R^\times||M|, & \text{if } a_ib_l - a_lb_i \in R^\times \text{ for some } l \in \{2,\dots,2\nu\} \setminus \{\nu+i\}, \\ |R|^{2\nu-2}|M|, & \text{if } a_ib_l - a_lb_i \in M \text{ for all } l \in \{2,\dots,2\nu\} \setminus \{\nu+i\}, \end{cases}$$

and for $i \geq \nu+1$, the number of common neighbors are

$$\begin{cases} |R|^{2\nu-3}|R^\times||M|, & \text{if } a_ib_l - a_lb_i \in R^\times \text{ for some } l \in \{2,\dots,2\nu\} \setminus \{i-\nu\}, \\ |R|^{2\nu-2}|M|, & \text{if } a_ib_l - a_lb_i \in M \text{ for all } l \in \{2,\dots,2\nu\} \setminus \{i-\nu\}. \end{cases}$$

For $d \geq 2$, a k-regular graph G on v vertices is called a *d-Deza graph* with parameters $(v, k, \{c_1, \dots, c_d\})$ if every two distinct vertices of G have $c_1, c_2, \dots, c_d$ common adjacent vertices. In particular, a 2-Deza graph is just an ordinary Deza graph.

Therefore, we can summarize the work on subconstituents of symplectic graphs in this section as follows.

Theorem 3.1.5. *Let R be a finite local ring with unique maximal ideal M and let (V, β) be a symplectic space of dimension 2ν.*

(1) If $\nu = 1$, then $\mathcal{G}^{(1)}_{\mathrm{Sp}_R(V)}$ is a strongly regular graph with parameters

$$\left(|R|, \left|R^\times\right|, |R^\times| - |M|, |R^\times|\right).$$

(2) If $\nu \geq 2$ and R is a field, then $\mathcal{G}^{(1)}_{\mathrm{Sp}_R(V)}$ is a strictly Deza graph with parameters

$$\left(|R|^{2\nu-1}, |R|^{2\nu-2}\left(|R|-1\right), |R|^{2\nu-3}(|R|-1)^2, |R|^{2\nu-2}(|R|-2)\right).$$

(3) If $\nu \geq 2$ and R is not a field, then $\mathcal{G}^{(1)}_{\mathrm{Sp}_R(V)}$ is a 3-Deza graph with parameters

$$\left(|R|^{2\nu-1}, |R|^{2\nu-2}\left|R^\times\right|, \{|R|^{2\nu-3}|R^\times|^2, |R|^{2\nu-2}(|R^\times| - |M|), |R|^{2\nu-2}|R^\times|\}\right).$$

Theorem 3.1.6. *Let R be a finite local ring with unique maximal ideal M and let (V,β) be a symplectic space of dimension 2ν, where $\nu \geq 2$. Then $\mathcal{G}^{(2)}_{\mathrm{Sp}_R(V)}$ is a strictly Deza graph with parameters*

$$\left(\frac{|R|^{2\nu-1}|M| - |M|^{2\nu}}{|R^\times|} - 1, |R|^{2\nu-2}|M|, \{|R|^{2\nu-2}|M|, |R|^{2\nu-3}|R^\times||M|\}\right).$$

3.2 Chromatic Numbers

Let R be a finite local ring with unique maximal ideal M. Let V be a free R-module of R-dimension 2ν, $\nu \geq 1$, and let V' be the 2ν-dimensional row vector space over k induced from V via the canonical map as we have discussed earlier in Section 2.3.

Recall that the subconstituents $\mathcal{G}^{(1)}_{\mathrm{Sp}_R(V)}$ is the induced subgraph of $\mathcal{G}_{\mathrm{Sp}_R(V)}$ on the vertex set

$$\mathcal{V}_1 = \{R\vec{x} : \vec{x} = \vec{e}_1 + a_2\vec{e}_2 + \cdots + a_{2\nu}\vec{e}_{2\nu}, \text{ where } a_i \in R \text{ for all } i \in \{2,\ldots,2\nu\}\}$$

and

$$\mathcal{V}_2 = \big\{R\vec{x} : \vec{x} = m\vec{e}_1 + a_2\vec{e}_2 + \cdots + a_{2\nu}\vec{e}_{2\nu} \text{ is unimodular in } V, \\ \text{where } a_i \in R \text{ for all } i \in \{2,\ldots,2\nu\} \text{ and } m \in M\big\}.$$

Thus, the canonical map π gives that

$$\mathcal{V}_1' = \pi(\mathcal{V}_1) = \{k\vec{x}' : \vec{x}' = \vec{e}_1 + b_2\vec{e}_2 + \cdots + b_{2\nu}\vec{e}_{2\nu}, \text{ where } b_i \in k \text{ for all } i \in \{2,\ldots,2\nu\}\}$$

and

$$\mathcal{V}_2' = \pi(\mathcal{V}_2) = \big\{k\vec{x}' : \vec{x}' = b_2\vec{e}_2 + \cdots + b_{2\nu}\vec{e}_{2\nu} \text{ is unimodular in } V', \\ \text{where } b_i \in k \text{ for all } i \in \{2,\ldots,2\nu\}\big\}$$

are the vertex sets of the subconstituents $\mathcal{G}^{(1)}_{\mathrm{Sp}_k(V')}$ and $\mathcal{G}^{(2)}_{\mathrm{Sp}_k(V')}$, respectively. Let

$$\kappa_1 = |\mathcal{V}_1'| = |k|^{2\nu-1} \quad \text{and} \quad \kappa_2 = |\mathcal{V}_2'| = \frac{|k|^{2\nu-1}-1}{|k|-1} - 1,$$

and let $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_{\kappa_1}$ and $\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_{\kappa_2}$ be unimodular vectors in V such that

$$\mathcal{V}(\mathcal{G}^{(1)}_{\mathrm{Sp}_k(V')}) = \{k\pi(\vec{x}_1), k\pi(\vec{x}_2), \ldots\ k\pi(\vec{x}_{\kappa_1})\}$$

and

$$\mathcal{V}(\mathcal{G}^{(2)}_{\mathrm{Sp}_k(V')}) = \{k\pi(\vec{y}_1), k\pi(\vec{y}_2), \ldots\ k\pi(\vec{y}_{\kappa_2})\}.$$

Similar to Theorem 2.3.1, the above observation yields the next theorem.

Theorem 3.2.1. *Under the above set up, we have the following statements.*

(1) *(i) The set* $\Pi_1 = \{R(\vec{x}_1 + M^{2\nu}), R(\vec{x}_2 + M^{2\nu}), \ldots, R(\vec{x}_{\kappa_1} + M^{2\nu})\}$ *is a partition of* $\mathcal{V}(\mathcal{G}^{(1)}_{\mathrm{Sp}_R(V)})$*. Moreover, for each* $i \in \{1, 2, \ldots, \kappa_1\}$*, any two distinct vertices in* $R(\vec{x}_i + M^{2\nu})$ *are non-adjacent vertices.*

(ii) The set $\Pi_2 = \{R(\vec{y}_1 + M^{2\nu}), R(\vec{y}_2 + M^{2\nu}), \ldots, R(\vec{y}_{\kappa_2} + M^{2\nu})\}$ *is a partition of* $\mathcal{V}(\mathcal{G}^{(2)}_{\mathrm{Sp}_R(V_2)})$*. Moreover, for each* $j \in \{1, 2, \ldots, \kappa_2\}$*, any two distinct vertices in* $R(\vec{y}_j + M^{2\nu})$ *are non-adjacent vertices.*

Here, $R(\vec{x} + M^{2\nu}) = \{R(\vec{x} + \vec{m}) : \vec{m} \in M^{2\nu}\}$ *for all unimodular vectors* $\vec{x}$ *in* V*.*

(2) $|R(\vec{x} + M^{2\nu})| = |M|^{2\nu-1}$ *for all unimodular vectors* $\vec{x}$ *in* V*.*

(3) Let $\vec{a}$ *and* $\vec{b}$ *be unimodular vectors in* V*. For each* $i \in \{1, 2\}$*, we have* $R\vec{a}$ *and* $R\vec{b}$ *are adjacent vertices in* $\mathcal{V}(\mathcal{G}^{(i)}_{\mathrm{Sp}_R(V_i)})$ *if and only if* $k\pi(\vec{a})$ *and* $k\pi(\vec{b})$ *are adjacent vertices in* $\mathcal{V}(\mathcal{G}^{(i)}_{\mathrm{Sp}_k(V_i')})$*.*

(4) For $i \in \{1, 2\}$*, if* $k\pi(\vec{z})$ *and* $k\pi(\vec{w})$ *are adjacent vertices in the subconstituent* $\mathcal{G}^{(i)}_{\mathrm{Sp}_R(V)}$*, then* $R(\vec{z} + \vec{m}_1)$ *and* $R(\vec{w} + \vec{m}_2)$ *are adjacent vertices in* $\mathcal{V}(\mathcal{G}^{(i)}_{\mathrm{Sp}_R(V)})$ *for all* $\vec{m}_1, \vec{m}_2 \in M^{2\nu}$*.*

Proof. The proof is analogous to the proof of Theorem 2.3.1. □

It follows from Theorem 3.5 of [9] that the subconstituent $\mathcal{G}^{(1)}_{\mathrm{Sp}_k(V')}$ is $|k|^\nu$-partite with partite sets $Z_1, Z_2, \ldots, Z_{|k|^\nu}$, where $Z_i \cap Z_j \neq \varnothing$ for all $i \neq j$ and there is no edge of $\mathcal{G}^{(1)}_{\mathrm{Sp}_k(V')}$ joining two vertices of the same subset. Moreover, the subsets $Z_1, Z_2, \ldots, Z_{|k|^\nu}$ can be chosen so that for any distinct indices i and j, every $z \in Z_i$ is adjacent to exactly $|k|^{\nu-1}$ vertices in Z_j. Moreover, the chromatic number of $\mathcal{G}^{(1)}_{\mathrm{Sp}_k(V')}$ is $|k|^\nu$ (Theorem 3.7 of [9]). The canonical map $\pi : R \to k$ and Theorem 3.2.1 give the following theorem.

Theorem 3.2.2. *The subconstituent $\mathcal{G}^{(1)}_{\mathrm{Sp}_R(V)}$ is $|k|^\nu$-partite with partite sets $\pi^{-1}(Z_1)$, $\pi^{-1}(Z_2), \ldots, \pi^{-1}(Z_{|k|^\nu})$, where Z_j, $j = 1, 2, \ldots, |k|^\nu$, are subsets of $\mathcal{G}^{(1)}_{\mathrm{Sp}_k(V')}$ discussed above. Moreover, the subsets $\pi^{-1}(Z_1), \pi^{-1}(Z_2), \ldots, \pi^{-1}(Z_{|k|^\nu})$ can be chosen so that for any distinct indices i and j, every $a \in \pi^{-1}(Z_i)$ is adjacent to exactly $|M|^{2\nu-1}|k|^{\nu-1}$ vertices in $\pi^{-1}(Z_j)$. Consequently, the chromatic number $\chi(\mathcal{G}^{(1)}_{\mathrm{Sp}_R(V)})$ is $|k|^\nu$.*

Proof. Since the subconstituent $\mathcal{G}^{(1)}_{\mathrm{Sp}_R(V)}$ is $|k|^\nu$-partite, its chromatic number is at most $|k|^\nu$. The reverse inequality follows from the fact that the induced subgraph of $\mathcal{G}^{(1)}_{\mathrm{Sp}_R(V)}$ whose vertex set is $\{R\vec{x}_1, \ldots, R\vec{x}_{\kappa_1}\}$ is isomorphic to the graph $\mathcal{G}^{(1)}_{\mathrm{Sp}_k(V')}$ by Theorem 3.2.1 (3). Hence, $\chi(\mathcal{G}^{(1)}_{\mathrm{Sp}_R(V_1)}) = |k|^\nu$. □

The proof of Theorem 4.8 of [9] shows that the vertex set of subconstituent $\mathcal{G}^{(2)}_{\mathrm{Sp}_k(V')}$ can be partitioned into pairwise disjoint sets $W_1, W_2, \ldots, W_{|k|^{\nu-1}+1}$, and there is no edge of $\mathcal{G}^{(2)}_{\mathrm{Sp}_k(V')}$ joining two vertices of the same subset. Moreover, $|W_j| = \frac{|k|^\nu - |k|}{|k|-1}$ for all $1 \leq j \leq |k|^{\nu-1} + 1$. In addition, Theorem 4.6 of [9] says that $\chi(\mathcal{G}^{(2)}_{\mathrm{Sp}_k(V')}) = |k|^{\nu-1} + 1$. Again, the canonical map $\pi : R \to k$ and Theorem 3.2.1 give the following theorem.

Theorem 3.2.3. *The subconstituent $\mathcal{G}^{(2)}_{\mathrm{Sp}_R(V)}$ is $|k|^{\nu-1} + 1$-partite with partite sets $\pi^{-1}(W_1), \pi^{-1}(W_2), \ldots, \pi^{-1}(W_{|k|^{\nu-1}+1})$, where W_j, $j = 1, 2, \ldots, |k|^{\nu-1}+1$, are subsets of $\mathcal{V}(\mathcal{G}^{(2)}_{\mathrm{Sp}_k(V')})$ discussed above. Moreover, there is no edge of $\mathcal{G}^{(2)}_{\mathrm{Sp}_R(V)}$ joining two vertices of the same subset $\pi^{-1}(W_j)$ and $|\pi^{-1}(W_j)| = \frac{|k|^\nu - |k|}{|k|-1}|M|^{2\nu-1}$ for all $1 \leq j \leq |k|^{\nu-1}+1$. Consequently, the chromatic number $\chi(\mathcal{G}^{(2)}_{\mathrm{Sp}_R(V)})$ is $|k|^{\nu-1} + 1$.*

Proof. Since the subconstituent $\mathcal{G}^{(2)}_{\mathrm{Sp}_R(V)}$ is $|k|^{\nu-1}+1$-partite, its chromatic number is at most $|k|^{\nu-1}+1$. The reverse inequality follows from the fact that the induced subgraph of $\mathcal{G}^{(2)}_{\mathrm{Sp}_R(V)}$ whose vertex set is $\{R\vec{y}_1, \ldots, R\vec{y}_{\kappa_2}\}$ is isomorphic to the graph $\mathcal{G}^{(2)}_{\mathrm{Sp}_k(V')}$ by Theorem 3.2.1 (3). Therefore, $\chi(\mathcal{G}^{(2)}_{\mathrm{Sp}_R(V)}) = |k|^{\nu-1} + 1$. $\square$

3.3 Automorphisms

Let R be a commutative ring and let $(V = R^{2\nu}, \beta)$ be a symplectic space with the standard basis $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_{2\nu}\}$. Recall from Section 1.2 that an R-module automorphism σ on V is an isometry if $\beta(\sigma(\vec{x}), \sigma(\vec{y})) = \beta(\vec{x}, \vec{y})$ for all $\vec{x}, \vec{y} \in V$ and the group of isometries $\mathrm{Sp}_R(V)$ is called the symplectic group. Define

$$\mathrm{Sp}^{(1)}_R(V) = \{\sigma \in \mathrm{Sp}_R(V) \ : \sigma(\vec{e}_{\nu+1}) = \vec{e}_{\nu+1}\}.$$

Clearly, it is a subgroup of $\mathrm{Sp}_R(V)$. Moreover, similar to Theorem 1.3.5, we have the imbedding $\mathrm{Sp}^{(1)}_R(V) \hookrightarrow \mathrm{Aut}(\mathcal{G}^{(1)}_{\mathrm{Sp}_R(V)})$ by considering the automorphisms fixing $\vec{e}_{\nu+1}$. Gu and Wan [6] showed that:

Proposition 3.3.1. (Theorem 2.2 and Corollary 2.13 of [6]) *Let k be a field and V a symplectic space over k of dimension 2ν, $\nu \geq 2$. Let E_1 be the subgroup of $\mathrm{Aut}(\mathcal{G}^{(1)}_{\mathrm{Sp}_k(V)})$ defined as follows:*

$$\begin{aligned} E_1 = \big\{\sigma \in \mathrm{Aut}(\mathcal{G}^{(1)}_{\mathrm{Sp}_k(V)}) : \sigma(k\vec{e_1}) = k\vec{e}_1, \sigma(k(\vec{e_1} + \vec{e_{\nu+i}})) = k(\vec{e}_1 + \vec{e_{\nu+i}}),\\ \sigma(k(\vec{e}_1 + \vec{e_i})) = k(\vec{e}_1 + c\vec{e_i}), \textit{ for all } i = 2, 3, \ldots, \nu \textit{ and } c \in k^{\times}\big\}, \end{aligned}$$

where $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_{2\nu}\}$ is the standard basis of V. Then

$$\mathrm{Aut}(\mathcal{G}^{(1)}_{\mathrm{Sp}_k(V)}) = (\mathrm{Sp}^{(1)}_k(V)) \cdot E_1.$$

Moreover, the number of automorphisms of the subconstituent $\mathcal{G}^{(1)}_{\mathrm{Sp}_k(V)}$ is

$$|k|^{\nu^2}(|k| - 1)\prod_{i=1}^{\nu-1}(|k|^{2i} - 1) \cdot [k : \mathbb{F}_p],$$

where k is of characteristic p and $[k:\mathbb{F}_p]$ denotes the degree of extension of k over $\mathbb{F}_p$.

Gu and Wan also studied the automorphism group of the subconstituent $\mathcal{G}^{(2)}_{\mathrm{Sp}_k(V)}$. Their result is as follows.

Proposition 3.3.2. (Theorem 3.1 and Corollary 3.2 of [6]) *Let k be a field and V a symplectic space over k of dimension 2ν, $\nu \geq 2$. The automorphism group of the subconstituent $\mathcal{G}^{(2)}_{\mathrm{Sp}_k(V)}$ is isomorphic to*

$$\mathrm{Aut}(\mathcal{G}_{\mathrm{Sp}_k(W)}) \times (\mathrm{Sym}(k))^{\kappa_2'},$$

where W is the subspace of V generated by the set $\{\vec{e}_2, \ldots, \vec{e}_\nu, \vec{e}_{\nu+2}, \ldots, \vec{e}_{2\nu}\}$ and $\kappa_2' = \frac{|k|^{2(\nu-1)-1}-1}{|k|-1} - 1$ is the number of vertices in the graph $\mathcal{G}^{(2)}_{\mathrm{Sp}_k(W)}$. In addition,

$$|\mathrm{Aut}(\mathcal{G}^{(2)}_{\mathrm{Sp}_k(V)})| = \begin{cases} (|k|+1)!(|k|!)^{|k|+1}, & \text{if } \nu = 2, \\ |k|^{(\nu-1)^2}(|k|!)^{\kappa_2'} \displaystyle\prod_{i=1}^{\nu-1}(|k|^{2i}-1)\cdot[k:\mathbb{F}_p], & \text{if } \nu \geq 3, \end{cases}$$

where k is of characteristic p and $[k:\mathbb{F}_p]$ denotes the degree of extension of k over $\mathbb{F}_p$.

For our results, we let R be a finite local ring with unique maximal ideal M and residue field $k = R/M$ and let (V, β) be a symplectic space of dimension 2ν and V' the 2ν-dimensional row vector space over k induced from V via the canonical map. By Theorem 3.2.1 (1), for $i \in \{1, 2\}$, the set Π_i is a partition of $\mathcal{V}(\mathcal{G}^{(i)}_{\mathrm{Sp}_R(V)})$ and any two distinct vertices in each partite set are non-adjacent. Moreover, the induced subgraph of $\mathcal{G}^{(1)}_{\mathrm{Sp}_R(V)}$ [resp. $\mathcal{G}^{(2)}_{\mathrm{Sp}_R(V)}$], whose vertex set is $\{R\vec{x}_1, \ldots, R\vec{x}_{\kappa_1}\}$ [resp. $\{R\vec{y}_1, \ldots, R\vec{y}_{\kappa_2}\}$], is isomorphic to graph $\mathcal{G}^{(1)}_{\mathrm{Sp}_k(V')}$ [resp. $\mathcal{G}^{(2)}_{\mathrm{Sp}_k(V')}$] (by Theorem 3.2.1 (3) and (4)). Thus, for $i \in \{1, 2\}$, an automorphism of $\mathcal{G}^{(i)}_{\mathrm{Sp}_R(V)}$ corresponds with an automorphism of $\mathcal{G}^{(i)}_{\mathrm{Sp}_k(V')}$ (studied in the previous two propositions) and a permutation of vertices of the subconstituent $\mathcal{G}^{(i)}_{\mathrm{Sp}_R(V)}$ in each partite set of Π_i. Recall also that each partite set is of cardinality $|M|^{2\nu-1}$ by Theorem 3.2.1 (2). Hence, we have the following theorem on the automorphism groups.

Theorem 3.3.3. *Let R be a finite local ring with unique maximal ideal M and residue field $k = R/M$ and let (V, β) be a symplectic space of dimension 2ν, $\nu \geq 2$. Then for $i \in \{1, 2\}$,*

$$\mathrm{Aut}(\mathcal{G}^{(i)}_{\mathrm{Sp}_R(V)}) = \mathrm{Aut}(\mathcal{G}^{(i)}_{\mathrm{Sp}_k(V')}) \times \mathrm{Sym}(|M|^{2\nu-1})^{\kappa_i},$$

where V' is the 2ν-dimensional row vector space over k induced from V via the canonical map, $\kappa_1 = |k|^{2\nu-1}$ and $\kappa_2 = \dfrac{|k|^{2\nu-1} - 1}{|k| - 1} - 1$.

CHAPTER IV

SOME RESULTS OVER FINITE COMMUTATIVE RINGS

It is well known that any finite commutative ring is a product of finite local rings (Theorem 8.7 of [1]) and we completely study our graphs over finite local rings in the previous chapters. In this chapter, we show how to use the decomposition of finite commutative rings into local rings and the work on symplectic graphs in Chapter II to obtain some analogous results. We also include an example with $R = \mathbb{Z}_m$, $m > 1$, to illustrate the theorems. Moreover, it generalizes [11].

4.1 Strong regularity

Let R be a finite commutative ring. Write

$$R = R_1 \times R_2 \times \cdots \times R_t$$

as a direct product of finite local rings R_i, $i = 1, 2, \ldots, t$. Consider $V = R^{2\nu}$, a free R-module of R-dimension 2ν, where $\nu \geq 1$. We have the canonical 1-1 correspondence

$$\vec{x} = (x_1, x_2, \ldots, x_{2\nu}) \overset{\varphi}{\mapsto} ((x_1^{(j)})_{j=1}^t, (x_2^{(j)})_{j=1}^t, \ldots, (x_{2\nu}^{(j)})_{j=1}^t).$$

Note that if $\vec{x},\vec{y}\in V$, then this correspondence induces the symplectic map β on V by

$$\begin{aligned}
\beta(\vec{x},\vec{y}) &= \beta(((x_1^{(j)})_{j=1}^t,(x_2^{(j)})_{j=1}^t,\ldots,(x_{2\nu}^{(j)})_{j=1}^t),((y_1^{(j)})_{j=1}^t,(y_2^{(j)})_{j=1}^t,\ldots,(y_{2\nu}^{(j)})_{j=1}^t))\\
&= (\beta_1(\vec{x}^{(1)},\vec{y}^{(1)}),\beta_2(\vec{x}^{(2)},\vec{y}^{(2)}),\ldots,\beta_t(\vec{x}^{(t)},\vec{y}^{(t)}))\\
&= \Big(\sum_{i=1}^{\nu}(x_i^{(1)}y_{\nu+i}^{(1)}-x_{\nu+i}^{(1)}y_i^{(1)}),\sum_{i=1}^{\nu}(x_i^{(2)}y_{\nu+i}^{(2)}-x_{\nu+i}^{(2)}y_i^{(2)}),\ldots,\sum_{i=1}^{\nu}(x_i^{(t)}y_{\nu+i}^{(t)}-x_{\nu+i}^{(t)}y_i^{(t)})\Big),
\end{aligned}$$

where $\vec{x}^{(j)}=(x_1^{(j)},\vec{x}_2^{(j)},\ldots,\vec{x}_{2\nu}^{(j)})\in V^{(j)}:=R_j^{2\nu}$ and $(V^{(j)},\beta_j)$ is a 2ν-dimensional symplectic space of R_j, for all $j=1,2,\ldots,t$. Since $R^\times=R_1^\times\times R_2^\times\times\cdots\times R_t^\times$, we have

$$\beta(\vec{x},\vec{y})\in R^\times \Leftrightarrow \sum_{i=1}^{\nu}(x_i^{(j)}y_{\nu+i}^{(j)}-x_{\nu+i}^{(j)}y_i^{(j)})\in R_j^\times \text{ for all } j\in\{1,2,\ldots,t\}, \tag{4.1.1}$$

it follows from Eq. (2.1.1) that

$$\mathcal{G}_{\mathrm{Sp}_R(V)}\cong\mathcal{G}_{\mathrm{Sp}_{R_1}(V^{(1)})}\otimes\mathcal{G}_{\mathrm{Sp}_{R_2}(V^{(2)})}\otimes\cdots\otimes\mathcal{G}_{\mathrm{Sp}_{R_t}(V^{(t)})}, \tag{4.1.2}$$

as a graph isomorphism. Here, for two graphs G and H, we define their **tensor product** $G\otimes H$ to be the graph with vertex set $\mathcal{V}(G)\times\mathcal{V}(H)$, where (u,v) is adjacent to (u',v') if and only if u is adjacent to u' and v is adjacent to v'.

From Theorem 2.1.4 (1) and the above discussion, we have the number of vertices of $\mathcal{G}_{\mathrm{Sp}_R(V)}$ is equal to

$$|\mathcal{V}(\mathcal{G}_{\mathrm{Sp}_R(V)})|=\prod_{j=1}^{t}|\mathcal{V}(\mathcal{G}_{\mathrm{Sp}_{R_j}(V^{(j)})})|=\prod_{j=1}^{t}\frac{|R_j|^{2\nu}-|M_j|^{2\nu}}{|R_j^\times|}.$$

and $\mathcal{G}_{\mathrm{Sp}_R(V)}$ is regular of degree $|R_1|^{2\nu-1}|R_2|^{2\nu-1}\ldots|R_t|^{2\nu-1}=|R|^{2\nu-1}$. Moreover, every two adjacent vertices of $\mathcal{G}_{\mathrm{Sp}_R(V)}$ has $|R|^{2\nu-2}|R^\times|$ common neighbors by Theorem 2.1.4 (2). We record these results in the next theorem.

Theorem 4.1.1. *Let R be a finite commutative ring and (V,β) be the induced symplectic space of dimension 2ν, $\nu\geq 1$, discussed above.*

(1) *The symplectic graph* $\mathcal{G}_{\mathrm{Sp}_R(V)}$ *is a* $|R|^{2\nu-1}$*-regular and isomorphic to the graph*

$$\mathcal{G}_{\mathrm{Sp}_{R_1}(V^{(1)})} \otimes \mathcal{G}_{\mathrm{Sp}_{R_2}(V^{(2)})} \otimes \cdots \otimes \mathcal{G}_{\mathrm{Sp}_{R_t}(V^{(t)})}.$$

(2) *Every two adjacent vertices of* $\mathcal{G}_{\mathrm{Sp}_R(V)}$ *has* $|R|^{2\nu-2}|R^\times|$ *common neighbors.*

Example 4.1.2. If $m > 1$ and $m = p_1^{n_1} p_2^{n_2} \dots p_t^{n_t}$, where $n_i \in \mathbb{N}$ and p_i are distinct primes for all $i \in \{1, 2, \dots, t\}$, then by Chinese remainder theorem we have

$$R = \mathbb{Z}_m \cong \mathbb{Z}_{{p_1}^{n_1}} \times \mathbb{Z}_{{p_2}^{n_2}} \times \cdots \times \mathbb{Z}_{{p_t}^{n_t}}.$$

Consider $V = R^{2\nu}$, the induced symplectic space of dimension 2ν, $\nu \geq 1$. By Theorem 4.1.1, we have

(1) the symplectic graph $\mathcal{G}_{\mathrm{Sp}_R(V)}$ is a $m^{2\nu-1}$-regular and isomorphic to the graph product

$$\mathrm{Sp}^{(2\nu)}(\mathbb{Z}_{{p_1}^{n_1}}) \otimes \mathrm{Sp}^{(2\nu)}(\mathbb{Z}_{{p_2}^{n_2}}) \otimes \cdots \otimes \mathrm{Sp}^{(2\nu)}(\mathbb{Z}_{{p_t}^{n_t}}),$$

(see Example 1.2.1), and

(2) every two adjacent vertices of $\mathcal{G}_{\mathrm{Sp}_R(V)}$ has $m^{2\nu-2}\phi(m)$ common neighbors, where ϕ is the Euler ϕ-function.

The numbers of common neighbors for two non-adjacent vertices are studied in the following theorem, where we apply Theorem 2.1.5 on each factor.

Theorem 4.1.3. *Let* R *and* V *be as in Theorem 4.1.1, and let*

$$\vec{a} = \left((a_1^{(j)})_{j=1}^t, (a_2^{(j)})_{j=1}^t, \dots, (a_{2\nu}^{(j)})_{j=1}^t\right) \text{ and } \vec{b} = \left((b_1^{(j)})_{j=1}^t, (b_2^{(j)})_{j=1}^t, \dots, (b_{2\nu}^{(j)})_{j=1}^t\right)$$

be unimodular vectors in V *and assume that* $(a_i^{(j)})_{j=1}^t \in R^\times$ *for some* $i \in \{1, 2, \dots, 2\nu\}$. *Assume that* $R\vec{a}$ *and* $R\vec{b}$ *are non-adjacent vertices of* $\mathcal{G}_{\mathrm{Sp}_R(V)}$. *Let* $\{j_1, j_2, \dots, j_s\} \subseteq \{1, 2, \dots, t\}$ *be such that* $R_{j_k}\left(a_1^{(j_k)}, a_2^{(j_k)}, \dots, a_{2\nu}^{(j_k)}\right)$ *and* $R_{j_k}\left(b_1^{(j_k)}, b_2^{(j_k)}, \dots, b_{2\nu}^{(j_k)}\right)$ *are non-adjacent vertices for all* $k \in \{1, 2, \dots, s\}$. *Then the number of common neighbors* *of $R\vec{a}$ and $R\vec{b}$ are*

$$\left(\prod_{j \in \{1,2,\ldots,t\} \setminus \{j_1, j_2, \ldots, j_s\}} |R_j|^{2\nu-2} |R_j^\times| \right) \prod_{k \in \{1,2,\ldots,s\}} C_k,$$

where

$$C_k = \begin{cases} |R_{j_k}|^{2\nu-2} |R_{j_k}^\times|, & \text{if } a_i^{(j_k)} b_l^{(j_k)} - a_l^{(j_k)} b_i^{(j_k)} \in R_{j_k}^\times \text{ for some } l \in \{1,2,\ldots,2\nu\} \setminus \{i\}, \\ |R_{j_k}|^{2\nu-1}, & \text{if } a_i^{(j_k)} b_l^{(j_k)} - a_l^{(j_k)} b_i^{(j_k)} \in M_{j_k} \text{ for all } l \in \{1,\ldots,2\nu\} \setminus \{i\}, \end{cases}$$

for all $k \in \{1, 2, \ldots, s\}$.

Proof. It follows directly from the isomorphism (4.1.2) and Theorem 2.1.5. □

Remark. Theorem 4.1.3 tells us that, in general, the symplectic graphs over finite commutative rings are neither strongly regular nor strictly Deza. Thus, we shall not talk about their subconstituents.

Let G and H be two graphs. Let σ and τ be automorphisms of G and H, respectively. It is easy to see that the map

$$\rho : (g, h) \mapsto (\sigma(g), \tau(h)) \text{ for all } g \in \mathcal{V}(G),\, h \in \mathcal{V}(H),$$

is an automorphism of $G \otimes H$. Thus, we have showed that:

Theorem 4.1.4. *For graphs G and H, $\operatorname{Aut}(G) \times \operatorname{Aut}(H) \subseteq \operatorname{Aut}(G \otimes H)$.*

Remark. Unfortunately, another inclusion is false in general. Hence, the isomorphism (4.1.2) does not imply the automorphism group of $\mathcal{G}_{\mathrm{Sp}_R(V)}$, which is usually larger than the product $\operatorname{Aut}(\mathcal{G}_{\mathrm{Sp}_{R_1}(V^{(1)})}) \times \cdots \times \operatorname{Aut}(\mathcal{G}_{\mathrm{Sp}_{R_t}(V^{(t)})})$. However, the fact in the above theorem can be use to prove results on transitivities in the next section.

4.2 Vertex and arc transitivities and chromatic numbers

If a finite commutative ring R is decomposed as $R = R_1 \times R_2 \times \cdots \times R_t$, where R_i is a local ring for all $i = 1, 2, \ldots, t$, then

$$\mathcal{G}_{\mathrm{Sp}_R(V)} \cong \mathcal{G}_{\mathrm{Sp}_{R_1}(V^{(1)})} \otimes \mathcal{G}_{\mathrm{Sp}_{R_2}(V^{(2)})} \otimes \cdots \otimes \mathcal{G}_{\mathrm{Sp}_{R_t}(V^{(t)})}$$

as we have seen in Subsection 4.1. Recall from Theorem 2.2.3 that for each i, we have $\mathcal{G}_{\mathrm{Sp}_{R_i}(V^{(i)})}$ is vertex transitive and arc transitive. By Theorem 4.1.4,

$$\mathrm{Aut}(\mathcal{G}_{\mathrm{Sp}_{R_1}(V^{(1)})}) \times \mathrm{Aut}(\mathcal{G}_{\mathrm{Sp}_{R_2}(V^{(2)})}) \times \cdots \times \mathrm{Aut}(\mathcal{G}_{\mathrm{Sp}_{R_t}(V^{(t)})}) \subseteq \mathrm{Aut}(\mathcal{G}_{\mathrm{Sp}_R(V)}),$$

it follows that $\mathcal{G}_{\mathrm{Sp}_R(V)}$ is also vertex transitive and arc transitive. Hence, we have proved:

Theorem 4.2.1. *If (V, β) is a symplectic space over a finite commutative ring R, then the symplectic graph $\mathcal{G}_{\mathrm{Sp}_R(V)}$ is vertex transitive and arc transitive.*

A set I of vertices of a graph G is called an **independent set** if no two distinct vertices of I are adjacent. Write $\alpha(G)$ for the size of largest independent set of G. For example, if R is a local ring, Theorem 2.3.1 implies that the sets $R(\vec{x}_i + M^{2\nu})$, $i \in \{1, 2, \ldots, \kappa\}$, are independent sets in the symplectic graph $\mathcal{G}_{\mathrm{Sp}_R(V)}$. Since the symplectic graph is regular, it follows from Theorem 2.3.2 that:

Theorem 4.2.2. *Let R be a finite local ring with unique maximal ideal M and residue field $k = R/M$ and let (V, β) be a symplectic space of dimension 2ν, $\nu \geq 1$. Then*

$$\alpha(\mathcal{G}_{\mathrm{Sp}_R(V)}) = \Big(\frac{|k|^\nu - 1}{|k| - 1}\Big)|M|^{2\nu-1}.$$

A **fractional coloring of a graph** G is a mapping f which assigns to each independent set I of G a real number $f(I) \in [0, 1]$ such that for any vertex v, $\sum_{v \in I} f(I) = 1$. The **total weight** $w(f)$ of a fractional coloring f of G is the sum of $f(I)$ over all the independent sets I of G. The **fractional chromatic number**

of G, denoted by $\chi^*(G)$, is the minimum total weight of a fractional coloring of G.

The color classes of a proper l-coloring of G form a collection of l pairwise disjoint independent sets $I_1, I_2, \ldots, I_l$ whose union is $\mathcal{V}(G)$. The function f such that $f(I_j) = 1$ for all $j \in \{1, 2, \ldots, l\}$ and $f(S) = 0$ for all other independent sets S is a fractional coloring of weight l. Therefore, $\chi^*(G) \leq \chi(G)$. Moreover, when G is vertex transitive, we have the following proposition.

Proposition 4.2.3. (Corollary 7.5.2 of [4]) *If G is a vertex transitive graph, then*

$$\chi^*(G) = \frac{|\mathcal{V}(G)|}{\alpha(G)}.$$

Let R be a finite local ring with unique maximal ideal M and residue field $k = R/M$ and let (V, β) be a symplectic space of dimension 2ν, $\nu \geq 1$. By Theorems 2.1.4 (1) and 4.2.2, we have

$$|\mathcal{V}(\mathcal{G}_{\mathrm{Sp}_R(V)})| = \frac{|R|^{2\nu} - |M|^{2\nu}}{|R^\times|} \quad \text{and} \quad \alpha(\mathcal{G}_{\mathrm{Sp}_R(V)}) = \Big(\frac{|k|^\nu - 1}{|k| - 1}\Big)|M|^{2\nu-1},$$

respectively. Thus, it follows from Proposition 4.2.3 that

$$\chi^*(\mathcal{G}_{\mathrm{Sp}_R(V)}) = \frac{\dfrac{|R|^{2\nu} - |M|^{2\nu}}{|R^\times|}}{\Big(\dfrac{|k|^\nu - 1}{|k| - 1}\Big)|M|^{2\nu-1}} = \frac{|R|^{2\nu} - |M|^{2\nu}}{|M|^{2\nu}} \frac{|k| - 1}{|k|^\nu - 1} \frac{|M|}{|R| - |M|} = |k|^\nu + 1,$$

which is equal to the chromatic number of $\mathcal{G}_{\mathrm{Sp}_R(V)}$. We record this result in the next theorem.

Theorem 4.2.4. *Let R be a finite local ring with unique maximal ideal M and residue field $k = R/M$ and let (V, β) be a symplectic space of dimension 2ν, $\nu \geq 1$. Then*

$$\chi^*(\mathcal{G}_{\mathrm{Sp}_R(V)}) = |k|^\nu + 1 = \chi(\mathcal{G}_{\mathrm{Sp}_R(V)}).$$

It is easy to see that if there is a homomorphism from a graph X to a graph Y, then $\chi(X) \leq \chi(Y)$. Let G and H be graphs. Since both G and H are homomorphic images of $G \times H$ (using the projection homomorphisms), we have that

$$\chi(G \otimes H) \leq \min\{\chi(G), \chi(H)\}.$$

Hedetniemi [7] has conjectured that for all graphs G and H equality occurs in the above bound. This conjecture is still open. However, Zhu [14] showed that Hedetniemi's conjecture is true for fractional chromatic numbers.

Proposition 4.2.5. (Theorem 2 of [14]) *For any graphs G and H,*

$$\chi^*(G \otimes H) = \min\{\chi^*(G), \chi^*(H)\}.$$

Let R be a finite commutative ring decomposed as $R = R_1 \times R_2 \times \cdots \times R_t$, where R_i is a local ring for all $i = 1, 2, \ldots, t$. Then

$$\mathcal{G}_{\mathrm{Sp}_R(V)} \cong \mathcal{G}_{\mathrm{Sp}_{R_1}(V^{(1)})} \otimes \mathcal{G}_{\mathrm{Sp}_{R_2}(V^{(2)})} \otimes \cdots \otimes \mathcal{G}_{\mathrm{Sp}_{R_t}(V^{(t)})}.$$

as we have seen earlier. By Proposition 4.2.5 and the above discussion,

$$\min_{1\leq i\leq t} \chi^*\big(\mathcal{G}_{\mathrm{Sp}_{R_i}(V^{(i)})}\big) = \chi^*(\mathcal{G}_{\mathrm{Sp}_R(V)}) \leq \chi(\mathcal{G}_{\mathrm{Sp}_R(V)}) \leq \min_{1\leq i\leq t} \chi\big(\mathcal{G}_{\mathrm{Sp}_{R_i}(V^{(i)})}\big).$$

Since $\chi^*(\mathcal{G}_{\mathrm{Sp}_{R_i}(V^{(i)})}) = \chi(\mathcal{G}_{\mathrm{Sp}_{R_i}(V^{(i)})})$ for all $i = 1, 2, \ldots, t$, it forces that

$$\chi^*(\mathcal{G}_{\mathrm{Sp}_R(V)}) = \chi(\mathcal{G}_{\mathrm{Sp}_R(V)}) = \min_{1\leq i\leq t} \chi\big(\mathcal{G}_{\mathrm{Sp}_{R_i}(V^{(i)})}\big).$$

Together with Theorem 2.3.2, we have our final result.

Theorem 4.2.6. *Let R be a finite commutative ring decomposed as $R = R_1 \times R_2 \times \cdots \times R_t$, where R_i is a local ring and k_i is its residue field, for all $i = 1, 2, \ldots, t$. If (V, β) is a symplectic space over R of dimension 2ν, $\nu \geq 1$, then*

$$\chi^*(\mathcal{G}_{\mathrm{Sp}_R(V)}) = \chi(\mathcal{G}_{\mathrm{Sp}_R(V)}) = \min_{1\leq i\leq t} |k_i|^\nu + 1.$$

Corollary 4.2.7. *Let* $m > 1$ *and* $R = \mathbb{Z}_m \cong \mathbb{Z}_{p_1^{n_1}} \times \mathbb{Z}_{p_2^{n_2}} \times \cdots \times \mathbb{Z}_{p_t^{n_t}}$*, where* $n_i \in \mathbb{N}$ *and* p_i *are primes such that* $p_1 < p_2 < \cdots < p_t$*. For the symplectic space* V *over* R *of dimension* 2ν*,* $\nu \geq 1$*, we have the chromatic number of the graph* $\mathcal{G}_{\mathrm{Sp}_R(V)}$ *given by*

$$\chi^*(\mathcal{G}_{\mathrm{Sp}_R(V)}) = \chi(\mathcal{G}_{\mathrm{Sp}_R(V)}) = |p_1|^\nu + 1.$$

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