CHAPTER IV

VIBRATIONS AND MODE ANALYSIS

We shall now use the vibration theory to compute the normal frequencies of the system with 5 masses.

Normal frequency calculations for systems with 5 masses

For the system which belongs to the group D_{4h} in Example 1., we have the following set of Lagrange's equations of motion for the system:

$$m\,\ddot{x}_{1,3} + k\,(x_{1,3} - x_0) = 0 \quad \text{..........(4.1)}$$

$$m\,\ddot{y}_{2,4} + k\,(y_{2,4} - y_0) = 0 \quad \text{..........(4.2)}$$

$$m\,\ddot{z}_{1,3} + k_B(z_{1,3} - z_0 \pm 1\theta) = 0 \quad \text{..........(4.3)}$$

$$m\,\ddot{z}_{2,4} + k_B(z_{2,4} - z_0 \mp 1\phi) = 0 \quad \text{..........(4.4)}$$

$$m\,\ddot{y}_{1,3} + k_B(y_{1,3} - y_0 \pm 1\psi) = 0 \quad \text{..........(4.5)}$$

$$m\,\ddot{x}_{2,4} + k_B(x_{2,4} - x_0 \pm 1\psi) = 0, \quad \text{..........(4.6)}$$

$$M\ddot{x}_0 + m(\ddot{x}_1+\ddot{x}_2+\ddot{x}_3+\ddot{x}_4) = 0, \quad \text{..........(4.6a)}$$

where k, k_B are force constants and ·· denotes a second derivative with respect to time t.

From (4.1),

$$m\,\ddot{x}_1 + k\,(x_1 - x_0) = 0 \quad \text{..........(4.7)}$$

$$m\,\ddot{x}_3 + k\,(x_3 - x_0) = 0. \quad \text{..........(4.8)}$$

Subtracting (4.8) from (4.7), we have

$$m(x_1 - x_3)^{\cdot\cdot} + k\,(x_1 - x_3) = 0 \quad \ldots\ldots\ldots(4.9)$$

Similarly, from (4.2)

$$m(y_2 - y_4)^{\cdot\cdot} + k\,(y_2 - y_4) = 0 \quad \ldots\ldots\ldots(4.10)$$

Let $x_1 = x_{10} \cos \sqrt{\lambda}\, t$

$x_3 = x_{30} \cos \sqrt{\lambda}\, t$

where x_{10} and x_{30} are amplitudes of the displacement and substituting into (4.9), we have

$$\lambda_1 = \frac{k}{m}\,. \quad \ldots\ldots\ldots\ldots(4.11)$$

From (4.1), adding together

$$m\,(x_1 + x_3)^{\cdot\cdot} + k(x_1 + x_3) - 2k\,x_o = 0$$

differentiating two times with respect to t,

$$m\,(x_1 + x_3)^{IV} + k\,(x_1 + x_3)^{\cdot\cdot} - 2k\,\ddot{x}_o = 0 \quad \ldots\ldots(4.12)$$

Since $M\,\ddot{x}_o + m\,(\ddot{x}_1 + \ddot{x}_2 + \ddot{x}_3 + \ddot{x}_4) = 0$,

hence $\ddot{x}_o = -\frac{m}{M}\,(\ddot{x}_1 + \ddot{x}_2 + \ddot{x}_3 + \ddot{x}_4)$.

Setting $\ddot{x}_2 = \ddot{x}_4 = 0$ and substituting into (4.12) which gives

$m(x_1 + x_3)^{IV} + k\,(x_1 + x_3)^{\cdot\cdot}\,(1 + \frac{2m}{M}) = 0$, we obtain

$$\lambda = 0 \quad \text{and} \quad \lambda_2 = \frac{k}{m}(1 + \frac{2m}{M}) \quad \ldots\ldots\ldots\ldots(4.13)$$

where $\lambda = 0$ corresponds to the rigid mode motion of the system.

Similarly, from (4.2), we have the similar result

$$\lambda_3 = \frac{k_B}{m}\left(1 + \frac{2m}{M}\right) \qquad \ldots\ldots\ldots\ldots(4.14)$$

From (4.5) and (4.6), we have

$$m(y_1-y_3)^{\cdot\cdot} + k_B(y_1-y_3) - 2lk_B\psi = 0$$

$$m(x_2-x_4)^{\cdot\cdot} + k_B(x_2-x_4) + 2lk_B\psi = 0$$

dividing the two equations by lk_B,

$$\frac{m}{lk_B}(y_1-y_3)^{\cdot\cdot} + \frac{1}{l}(y_1-y_3) - 2\psi = 0$$

$$\frac{m}{lk_B}(x_2-x_4)^{\cdot\cdot} + \frac{1}{l}(x_2-x_4) + 2\psi = 0.$$

Let $\frac{y_1-y_3}{2l} = \alpha_1$, $\frac{x_2-x_4}{2l} = \alpha_2$, therefore the equations become

$$\frac{2m}{k_B}\ddot{\alpha}_1 + 2\alpha_1 - 2\psi = 0$$

$$\frac{2m}{k_B}\ddot{\alpha}_2 + 2\alpha_2 + 2\psi = 0.$$

Let $\psi = \frac{\alpha_1 - \alpha_2}{2}$,

$$\frac{2m}{k_B}\ddot{\alpha}_1 + \alpha_1 + \alpha_2 = 0,$$

$$\frac{2m}{k_B}\ddot{\alpha}_2 + \alpha_1 + \alpha_2 = 0,$$

the determinant of the coefficients is

$$\begin{vmatrix} 1 - \frac{2m\lambda}{k_B} & 1 \\ 1 & 1 - \frac{2m\lambda}{k_B} \end{vmatrix} = 0,$$

the root of the determinant is

$$\lambda_4 = \frac{k_B}{m} . \qquad \text{..........(4.15)}$$

From (4.3),

$$m(z_1+z_3)^{\cdot\cdot} + k_B\,(z_1+z_3 - 2z_o) = 0$$

or $$m(z_1+z_3)^{IV} + k_B(z_1+z_3)^{\cdot\cdot} - 2k_B\ddot{z}_o = 0 \qquad \text{.......(4.16)}$$

since $$M\,\ddot{z}_o + m(\ddot{z}_1 + \ddot{z}_2 + \ddot{z}_3 + \ddot{z}_4) = 0 ,$$

substituting $\ddot{z}_o = -\frac{m}{M}(\ddot{z}_1 + \ddot{z}_2 + \ddot{z}_3 + \ddot{z}_4)$ into (4.16), and rearranging its terms, we have

$$m(z_1+z_3)^{IV} + k_B\,(1+\frac{2m}{M})\,(z_1+z_3)^{\cdot\cdot} + 2k_B\frac{m}{M}(z_2+z_4)^{\cdot\cdot} = 0$$

............(4.17)

Similarly, from (4.4), we have

$$m(z_2+z_4)^{IV} + k_B\,(1+\frac{2m}{M})(z_2+z_4)^{\cdot\cdot} + 2k_B(z_1+z_3)^{\cdot\cdot} = 0.$$

...........(4.18)

Equations (4.17) and (4.18) can be combined together and expressed in the matrix form as

$$m\begin{Bmatrix} z_1 + z_3 \\ z_2 + z_4 \end{Bmatrix}^{IV} + \begin{bmatrix} k_B(1+\frac{2m}{M}) & 2k_B\frac{m}{M} \\ 2k_B\frac{m}{M} & k_B(1+\frac{2m}{M}) \end{bmatrix}\begin{Bmatrix} z_1 + z_3 \\ z_2 + z_4 \end{Bmatrix}^{\cdot\cdot} = 0$$

There will be a non-trivial solution only if

$$\begin{vmatrix} k_B\,(1+\frac{2m}{M}) - m\lambda & 2k_B\frac{m}{M} \\ 2k_B\frac{m}{M} & k_B(1+\frac{2m}{M}) - m\lambda \end{vmatrix} = 0 ,$$

The roots of this determinant are

$$\lambda_5 = \frac{k_B (M + 4m)}{mM} \quad \ldots\ldots\ldots\ldots(4.19)$$

and $\lambda = \frac{k_B}{m}$ which is the same as λ_4.

From (4.3),

$$m(z_1+z_3)'' + k_B(z_1 + z_3 - 2 z_o) = 0$$

which also yields

$\lambda = \frac{k_B}{m} (1 + \frac{2m}{M})$ which is the same as λ_3. By solving (4.1), and (4.6) simultaneously, we have

$$\lambda_{6,7} = \frac{(k+k_B)(1+\frac{2m}{M}) \pm \sqrt{(k+k_B)^2(1+\frac{2m}{M})^2 - 4kk_B(1+\frac{4m}{M})}}{2m} \quad \ldots(4.20)$$

Therefore, we obtain seven normal frequencies ($\nu_i = \frac{\sqrt{\lambda_i}}{2\pi}$) for the system, of which two are doubly degenerate and five are non-degenerate. **These seven normal frequencies are agreeable with the equation (3.3) in Chapter III.**

For the system which belongs to the group D_{2h} in Example 2, we have the following set of Lagrange's equations of motion:

$$m\,\ddot{x}_{1,3} + k_1(x_{1,3} - x_o) = 0 \quad \ldots\ldots\ldots(4.21)$$

$$m\,\ddot{y}_{2,4} + k_2(y_{2,4} - y_o) = 0 \quad \ldots\ldots\ldots(4.22)$$

$$m\,\ddot{z}_{1,3} + k_1'(z_{1,3} - z_o \pm l_1\theta) = 0 \quad \ldots\ldots(4.23)$$

$$m\,\ddot{z}_{2,4} + k_2'(z_{2,4} - z_o \pm l_2\phi) = 0 \quad \ldots\ldots(4.24)$$

$$m\,\ddot{y}_{1,3} + k_1'(y_{1,3} - y_o \mp l_1\psi) = 0 \quad \ldots\ldots(4.25)$$

$$m\,\ddot{x}_{2,4} + k_2'\,(x_{2,4} - x_0 \pm l_2\psi) = 0, \qquad \text{.......(4.26)}$$

where k_1, k_2, k_1' and k_2' are force constants.

The process used to solve the set of these equations is the same as in the group D_{4h} above. We obtain the following values of λ_i :

$$\lambda_1 = \frac{k_1}{m}\left(1 + \frac{2m}{M}\right) \qquad \text{...........(4.27)}$$

$$\lambda_2 = \frac{k_2}{m}\left(1 + \frac{2m}{M}\right) \qquad \text{...........(4.28)}$$

$$\lambda_3 = \frac{k_1}{m} \qquad \text{...........(4.29)}$$

$$\lambda_4 = \frac{k_2}{m} \qquad \text{...........(4.30)}$$

$$\lambda_5 = \frac{k_1'}{m}\left(1 + \frac{2m}{M}\right) \qquad \text{...........(4.31)}$$

$$\lambda_6 = \frac{k_2'}{m}\left(1 + \frac{2m}{M}\right) \qquad \text{...........(4.32)}$$

$$\lambda_{7,8} = \frac{\left(1+\frac{2m}{M}\right)(k_1'+k_2') \pm \sqrt{\left(1+\frac{2m}{M}\right)^2 (k_1'+k_2')^2 - 4k_1'k_2'\left(1+\frac{4m}{M}\right)}}{2m}$$

$$\text{...........(4.33)}$$

$$\lambda_9 = \frac{k_1' + k_2'}{2m}\,. \qquad \text{...........(4.34)}$$

There are 9 vibrational degrees of freedom for this system. We, therefore, have 9 normal frequencies and all of them are non-degenerate.

Schematic representations

We shall present the schematic representations of possible coordinates having the required symmetry as in Figure 10, and 11.

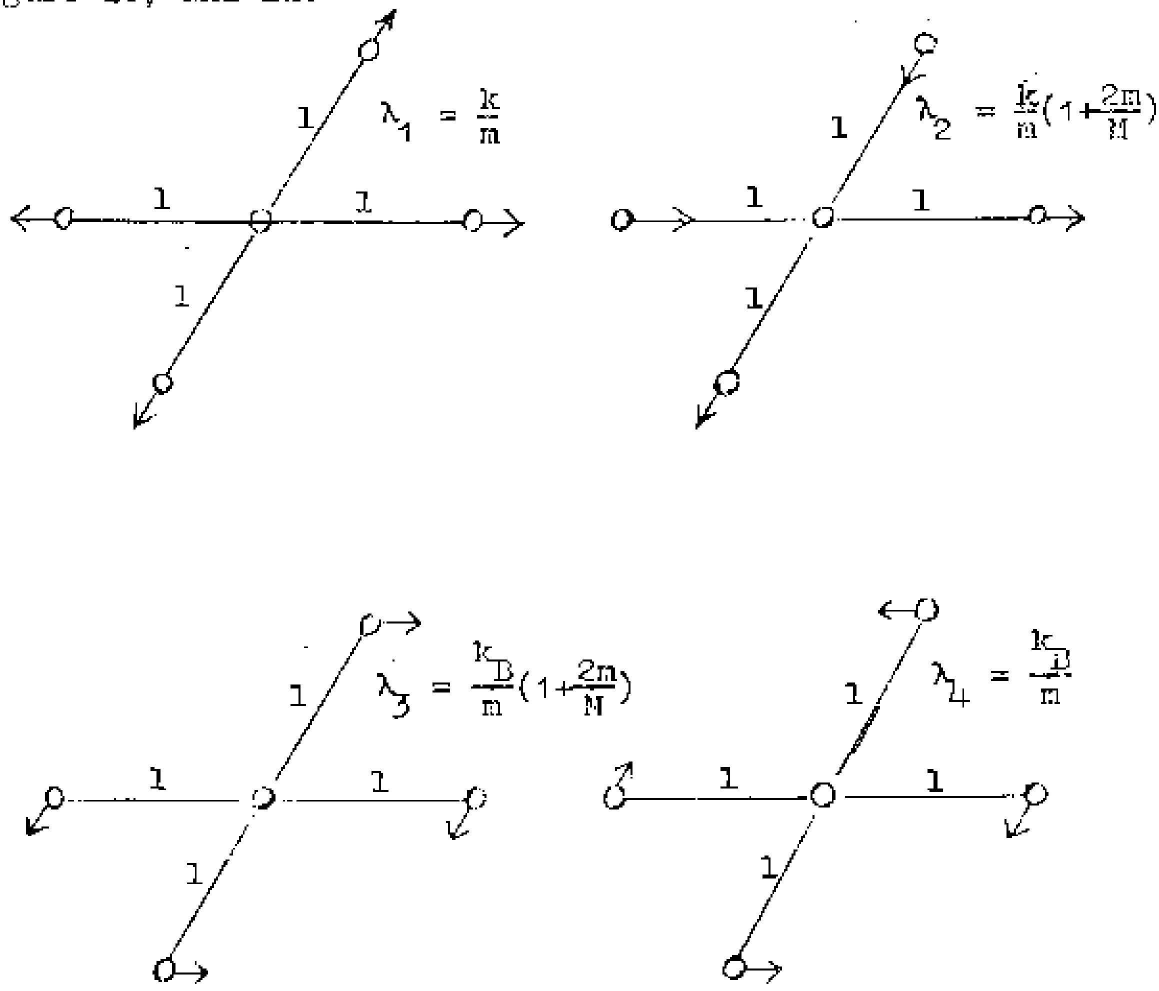


Figure 10(a) Normal modes of vibration of the system belonging to the group D_{4h}.

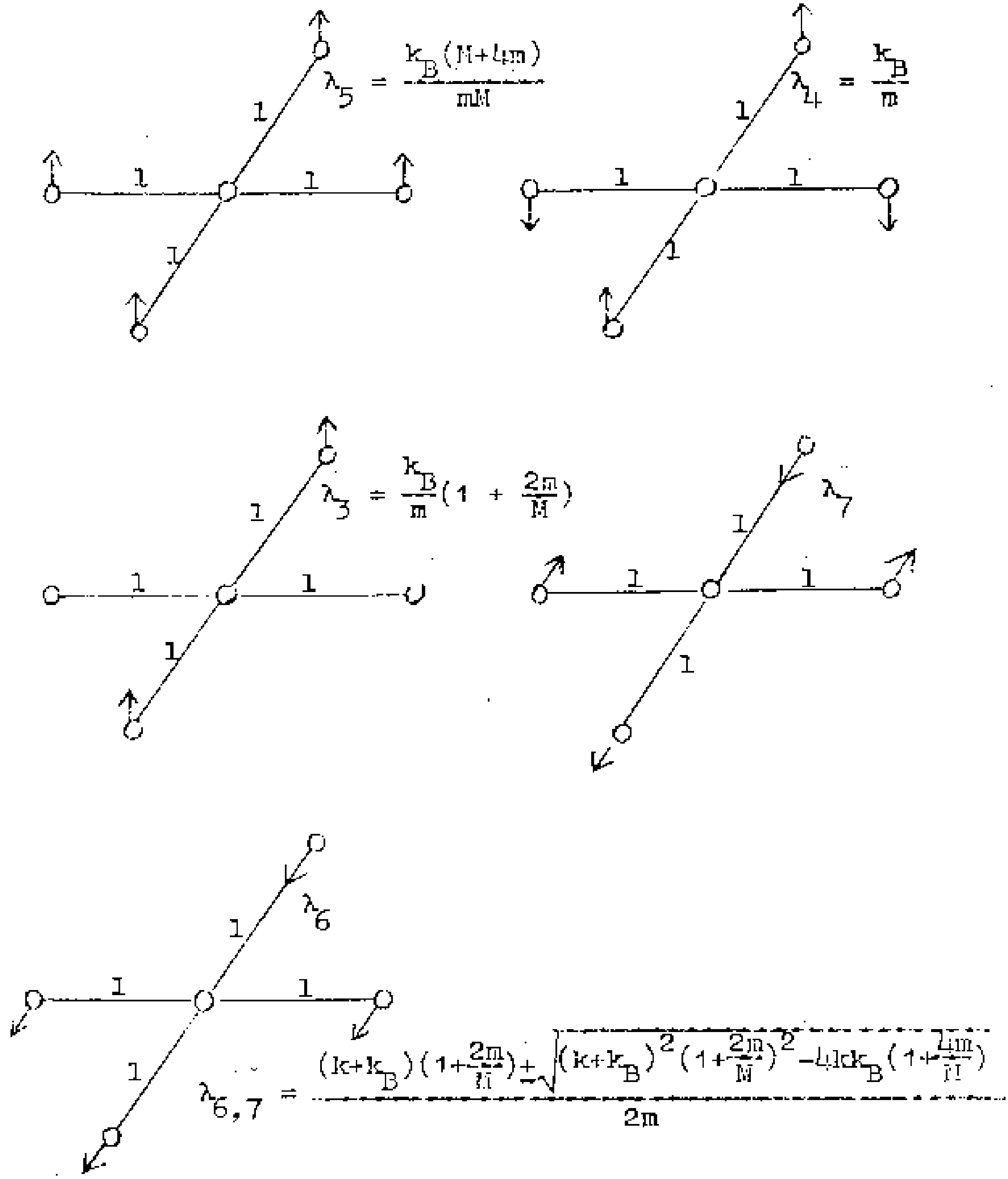


Figure 10(b) Normal modes of vibration of the system belonging to the group D_{4h}.

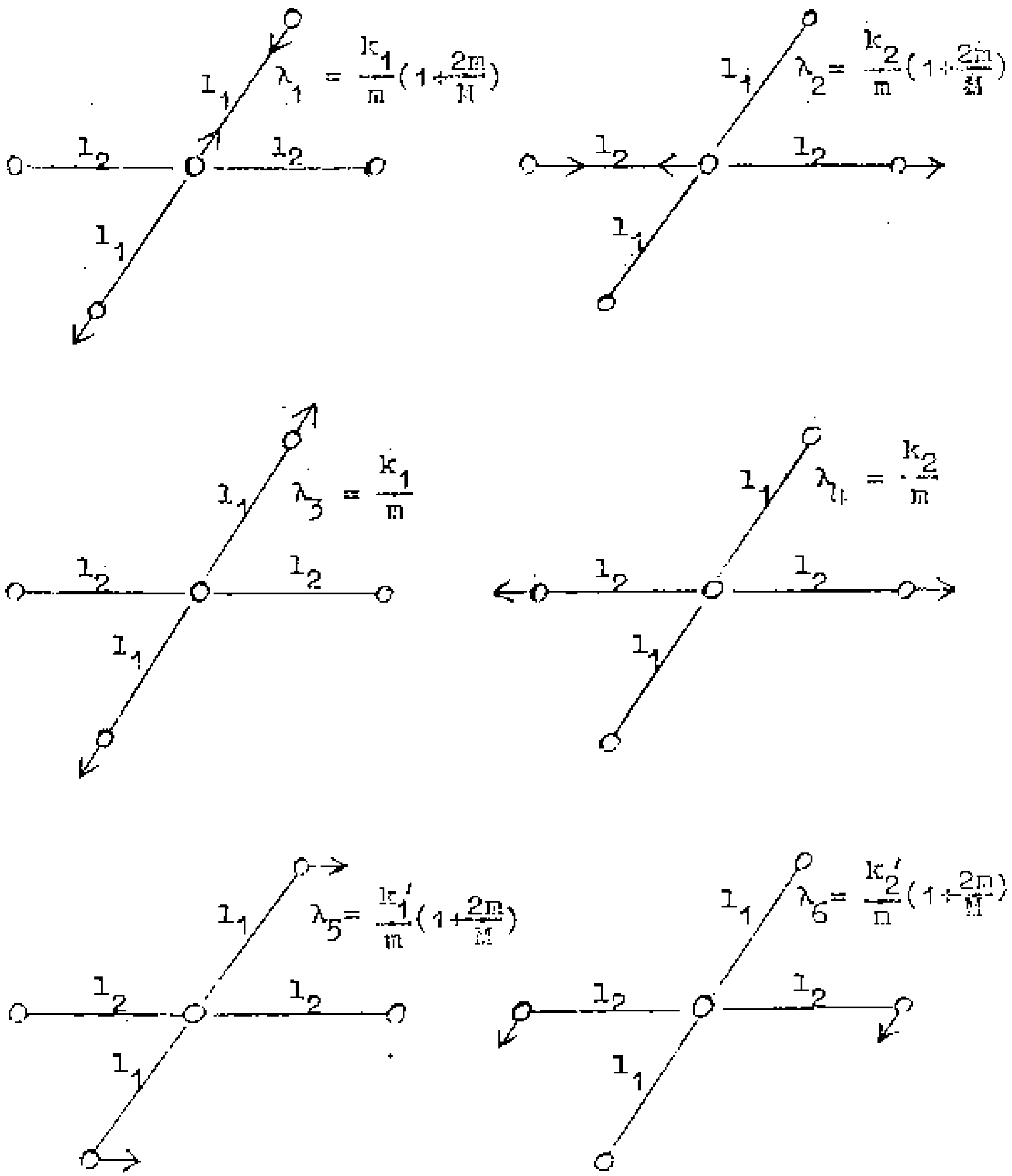
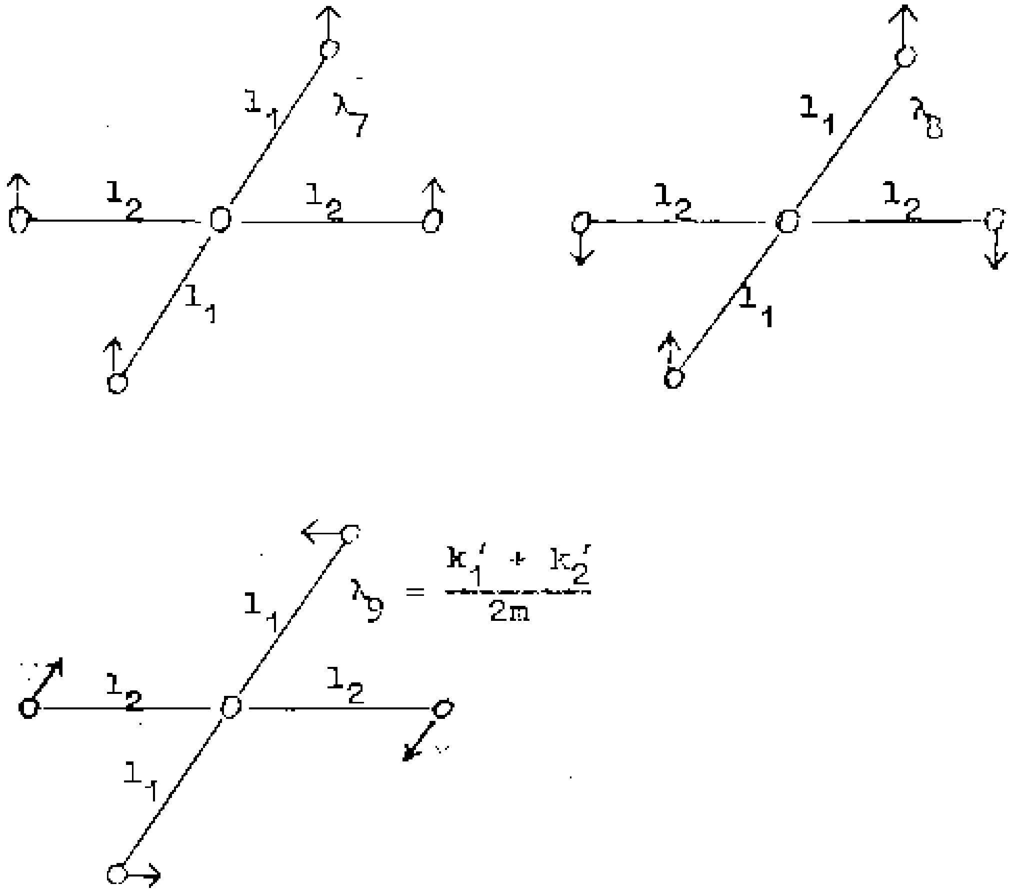


Figure 11(a) Normal modes of vibration of the system belonging to the group D_{2h} .



$$\lambda_9 = \frac{k_1' + k_2'}{2m}$$

$$\lambda_{7,8} = \frac{(1+\frac{2m}{M})(k_1'+k_2') \pm \sqrt{(1+\frac{2m}{M})^2 (k_1'+k_2')^2 - 4k_1'k_2'(1+\frac{4m}{M})}}{2m}$$

Figure 11(b) Normal modes of vibration of the system belonging to the group D_{2h} .