CHAPTER I

INTRODUCTION

M.K. Sen and N.K. Saha [5] introduced the notion of $\Gamma-$semigroups and obtained some properties in semigroup theory analogous to those in $\Gamma-$semigroups. Because the structure of a $\Gamma-$semigroup is similarly defined to semigroup theory, so almost all results should be alike. For example, in 1987, M.K. Sen and N.K. Saha defined relations in $\Gamma-$semigroup analoguous to Green's relations in a semigroup, deduced a condition for an element in a $\Gamma-$semigroup to be regular and proved that if α is a regular element in a $\mathcal{D}-$class D_α containing α, then every element of D_α is regular in the $\Gamma-$semigroup. In fact, every semigroup admits a $\Gamma-$semigroup structure, by choosing $\Gamma = S$. On the other hand, let S be a $\Gamma-$semigroup and α be a fixed element of Γ. We define $a \circ b$ in S by $a \circ b = a\alpha b$ for all $a, b \in S$, then it can be shown that $(S, \circ)$ is a semigroup. Thus, every $\Gamma-$semigroup admits a semigroup structure.

The purpose of this thesis is to conduct research on the structure of some $\Gamma-$semigroups. We characterize different types of $\Gamma-$semigroups in three topics which are real intervals, sets of $m \times n$ matrices over $\mathbb{R}$ and sets of linear operators on a vector space over a division ring. In Chapter II, we investigate real intervals as $\Gamma-$subsemigroups which are motivated by the following propositions:

Proposition 1.1. ([2]) *Let I be a real interval. Then I is a subsemigroup under addition if and only if I is one of the following forms :*

(i) $\mathbb{R}$, *(ii)* $\{0\}$,

(iii) $[a, \infty)$ *where* $a \geq 0$, *(iv)* (a, ∞) *where* $a \geq 0$,

(v) $(-\infty, b]$ *where* $b \leq 0$, *(vi)* $(-\infty, b)$ *where* $b \leq 0$.

Proposition 1.2. ([2]) *Let I be a real interval. Then I is a subsemigroup under multiplication if and only if I is one of the following forms :*

(i) $\mathbb{R}$, *(ii)* $\{0\}$, *(iii)* $\{1\}$,

(iv) $(0, \infty)$, *(v)* $[0, \infty)$,

(vi) (a, ∞) *where* $a \geq 1$ *(vii)* $[a, \infty)$ *where* $a \geq 1$,

(viii) $(0, b)$ *where* $0 < b \leq 1$ *(ix)* $(0, b]$ *where* $0 < b \leq 1$,

(x) $[0, b)$ *where* $0 < b \leq 1$ *(xi)* $[0, b]$ *where* $0 < b \leq 1$,

(xii) (a, b) *where* $-1 \leq a < 0 < a^2 \leq b \leq 1$,

(xiii) $(a, b]$ *where* $-1 \leq a < 0 < a^2 \leq b \leq 1$,

(xiv) $[a, b)$ *where* $-1 < a < 0 < a^2 < b \leq 1$,

(xv) $[a, b]$ *where* $-1 \leq a < 0 < a^2 \leq b \leq 1$.

In Chapters III and IV, we investigate sets of $m \times n$ matrices over $\mathbb{R}$ and sets of linear operators on a vector space over a division ring as $\Gamma-$subsemigroups, respectively.

We first recall some definitions and examples from [3], [4] and [5].

Definition Let S and Γ be two nonempty sets. Then S is called a $\Gamma-$*semigroup* if there exists a mapping $S \times \Gamma \times S \to S$, written the image of (a, γ, b) as $a\gamma b$, satisfying the identity $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b, c \in S$ and $\alpha, \beta \in \Gamma$.

Definition Let S be a $\Gamma-$semigroup. A nonempty subset B of S is said to be a $\Gamma-$*subsemigroup* of S if $B\Gamma B \subseteq B$ where $B\Gamma B = \{a\alpha b \mid a, b \in B \text{ and } \alpha \in \Gamma\}$.

We give some examples of $\Gamma-$semigroups.

Example 1.3. For any nonempty subset Γ of $\mathbb{R}$, $\mathbb{R}$ is a $\Gamma-$semigroup under usual addition and multiplication.

Example 1.4. For any nonempty subset Γ of $M_{nm}(\mathbb{R})$, $M_{mn}(\mathbb{R})$ is a $\Gamma-$semigroup under usual matrix multiplication.

Example 1.5. Let $m \in \mathbb{N}$ with $m > 1$ and $n \in N_{m-1}$. Then $m\mathbb{Z} + n$ is a $\big(m\mathbb{Z} + (m-n)\big)-$semigroup under usual addition since for each $a, b, c \in \mathbb{Z}$

$$(ma + n) + (mb + m - n) + (mc + n) = m(a + b + c + 1) + n \in m\mathbb{Z} + n.$$

Example 1.6. Let $L(V)$ be the set of all linear operators on a vector space V over a division ring. Then $L(V)$ is a $\Gamma-$semigroup under composition for any nonempty subset Γ of $L(V)$.

Example 1.7. In $\mathbb{R}$, let $a, b \in [0,1]$ and $c \in [0,a]$. Then $[0,a]$ is a $[0,b]-$semigroup and $[0,c]$ is a $[0,b]-$subsemigroup of $[0,a]$ under usual multiplication.

Example 1.8. Let $S = 4\mathbb{Z}+3$, $B = \{4n-1 \mid n \in \mathbb{Z}^+\}$ and $\Gamma = \{4n+1 \mid n \in \mathbb{Z}^+\}$. Then it can be shown that S is a $\Gamma-$semigroup and B is a $\Gamma-$subsemigroup of S under usual addition.

Example 1.9. Let $S = M_{32}(\mathbb{R})$, $\Gamma = M_{23}(\mathbb{R})$ and $B = \left\{\left[\begin{smallmatrix} 0 & 0 \\ a & b \\ c & d \end{smallmatrix}\right] \mid a, b, c, d \in \mathbb{R}\right\}$. Then S is a $\Gamma-$semigroup and B is a $\Gamma-$subsemigroup of S under usual matrix multiplication.

Example 1.10. Let X and Y be two nonempty sets. Denote the set of all functions from X into Y by S, $\{\ g \in S \mid g$ is a constant function$\}$ by B and the set of all functions from Y into X by Γ. Then S is a $\Gamma-$semigroup and B is a $\Gamma-$subsemigroup of S under composition.

Proposition 1.11. *A nonempty subset I of $\mathbb{R}$ is a $\mathbb{Q}^c-$semigroup under usual addition if and only if $I = \mathbb{R}$.*

Proof. It is obvious that $\mathbb{R}$ is a $\mathbb{Q}^c-$semigroup. Next, let I be a nonempty subset of $\mathbb{R}$ such that I is a $\mathbb{Q}^c-$semigroup under addition. If $I \cap \mathbb{Q}^c = \phi$, then let $x \in I \cap \mathbb{Q}$, so $x + \sqrt{2} + x \in I \cap \mathbb{Q}^c$ which is a contradiction. Hence $I \cap \mathbb{Q}^c \neq \phi$. Let $y \in I \cap \mathbb{Q}^c$, thus $-2y \in \mathbb{Q}^c$ and $0 = y + (-2y) + y \in I$ which implies that $\mathbb{Q}^c = 0 + \mathbb{Q}^c + 0 \subseteq I$. Next, to show that $\mathbb{Q} \subseteq I$, let $r \in \mathbb{Q}$. Thus for any $y \in I \cap \mathbb{Q}^c$, $r = 0 + (r + y) + (-y) \in I$. Then we conclude that $I = \mathbb{R}$. $\square$