ควอซี-ไอดีลเล็กสุดเฉพาะกลุ่มของเมทริกซ์ริงบางชนิด

นางสาวภัททิตา จันทรขจร

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MINIMAL QUASI-IDEALS OF SOME MATRIX RINGS

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พิมพ์ต้นฉบับบทคัดย่อวิทยานิพนธ์ภายในกรอบสีเขียวนี้เพียงแผ่นเดียว

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ให้ R เป็นริง เราเรียกสับกรุป Q ภายใต้การบวกของ R ว่า *ควอซี-ไอดีล* ของ R ถ้า $RQ \cap QR \subseteq Q$ สำหรับ $a \in R$ ให้ $(a)_q$ เป็นควอซี-ไอดีลของ R ก่อกำเนิดโดย a เรากล่าวว่า ควอซี-ไอดีล Q ของ R *เล็กสุดเฉพาะกลุ่ม* ถ้า $Q \neq \{0\}$ และไม่มีควอซี-ไอดีลของ R ซึ่งไม่ใช่ $\{0\}$ และเป็นสับเซตแท้ของ Q เพราะฉะนั้น ถ้า Q เป็นควอซี-ไอดีลเล็กสุดเฉพาะกลุ่มของ R แล้ว $Q = (a)_q$ สำหรับทุก $a \in Q \setminus \{0\}$

ให้ F เป็นฟีลด์ n เป็นจำนวนเต็มบวก $k \in \{1, 2, \ldots, n\}$ และ

$\mathrm{M}_n(F)$ = เมทริกซ์ริงเต็มมิติ $n \times n$ บน F

$\mathrm{SU}_n(F)$ = ริงของเมทริกซ์สามเหลี่ยมบนโดยแท้มิติ $n \times n$ บน F ทั้งหมด

$\mathrm{C}_{2n+1}(F)$ = ริงของเมทริกซ์ A มิติ $(2n+1) \times (2n+1)$ บน F ทั้งหมด ซึ่ง $A_{ij} = 0$ สำหรับทุก $(i, j) \in \{1, 2, \ldots, 2n+1\} \times \{1, 2, \ldots, 2n+1\} \setminus \{(1, 1), (1, 2n+1), (n+1, n+1), (2n+1, 1), (2n+1, 2n+1)\}$

$\mathrm{R}_n(F, k)$ = ริงของเมทริกซ์ A มิติ $n \times n$ บน F ทั้งหมด ซึ่ง $A_{ij} = 0$ สำหรับทุก $i, j \in \{1, 2, \ldots, n\}$ และ $i \neq k$

ผลสำคัญของการวิจัยมีดังนี้

ทฤษฎีบท 1 สำหรับ $A \in \mathrm{M}_n(F)$, $(A)_q$ เป็นควอซี-ไอดีลเล็กสุดเฉพาะกลุ่มของ $\mathrm{M}_n(F)$ ก็ต่อเมื่อ ค่าลำดับชั้นของ $A = 1$

ทฤษฎีบท 2 ถ้าแคแรกเทอริสติกของ $F = 0$ แล้ว $\mathrm{SU}_n(F)$ ไม่มีควอซี-ไอดีลเล็กสุดเฉพาะกลุ่ม

ทฤษฎีบท 3 ให้แคแรกเทอริสติกของ $F = p > 0$

1) สำหรับ $A \in \mathrm{SU}_n(F)$ ถ้าค่าลำดับชั้นของ $A = 1$ แล้ว $(A)_q$ เป็นควอซี-ไอดีลเล็กสุดเฉพาะกลุ่มของ $\mathrm{SU}_n(F)$

2) บทกลับของ 1) เป็นจริง ก็ต่อเมื่อ $n \leq 3$

ทฤษฎีบท 4 สำหรับ $A \in \mathrm{C}_{2n+1}(F)$, $(A)_q$ เป็นควอซี-ไอดีลเล็กสุดเฉพาะกลุ่มของ $\mathrm{C}_{2n+1}(F)$ ก็ต่อเมื่อ ค่าลำดับชั้นของ $A = 1$

ทฤษฎีบท 5 ให้แคแรกเทอริสติกของ $F = 0$ และ $A \in \mathrm{R}_n(F, k)$ ดังนั้น $(A)_q$ เป็นควอซี-ไอดีลเล็กสุดเฉพาะกลุ่มของ $\mathrm{R}_n(F, k)$ ก็ต่อเมื่อ $A_{kk} \neq 0$

ทฤษฎีบท 6 ถ้าแคแรกเทอริสติกของ $F = p > 0$ แล้วสำหรับ $A \in \mathrm{R}_n(F, k)$ ใดๆ $(A)_q$ เป็นควอซี-ไอดีลเล็กสุดเฉพาะกลุ่มของ $\mathrm{R}_n(F, k)$

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Let R be a ring. An additive subgroup Q of a ring R is said to be a *quasi-ideal* of R if $RQ \cap QR \subseteq Q$. For $a \in R$, let $(a)_q$ denote the quasi-ideal of R generated by a. A quasi-ideal Q of R is said to be *minimal* if $Q \neq \{0\}$ and Q does not properly contain any nonzero quasi-ideal of R. Therefore if Q is a minimal quasi-ideal of R, then $Q = (a)_q$ for every $a \in Q \setminus \{0\}$.

Let F be a field, n a positive integer, $k \in \{1, 2, \ldots, n\}$ and

$\mathrm{M}_n(F)$ = the full $n \times n$ matrix ring over F,

$\mathrm{SU}_n(F)$ = the ring of all strictly upper triangular $n \times n$ matrices over F,

$\mathrm{C}_{2n+1}(F)$ = the ring of all $(2n+1) \times (2n+1)$ matrices A over F with $A_{ij} = 0$ for all $(i, j) \in \{1, 2, \ldots, 2n+1\} \times \{1, 2, \ldots, 2n+1\} \setminus \{(1, 1), (1, 2n+1), (n+1, n+1), (2n+1, 1), (2n+1, 2n+1)\}$ and

$\mathrm{R}_n(F, k)$ = the ring of all $n \times n$ matrices A over F with $A_{ij} = 0$ for all $i, j \in \{1, 2, \ldots, n\}$ and $i \neq k$.

The main results of this research are as follows:

Theorem 1. For $A \in \mathrm{M}_n(F)$, $(A)_q$ is a minimal quasi-ideal of $\mathrm{M}_n(F)$ if and only if $\mathrm{rank}(A) = 1$.

Theorem 2. If $\mathrm{char}(F) = 0$, then $\mathrm{SU}_n(F)$ has no minimal quasi-ideal.

Theorem 3. Let $\mathrm{char}(F) = p > 0$.

1) For $A \in \mathrm{SU}_n(F)$, if $\mathrm{rank}(A) = 1$, then $(A)_q$ is a minimal quasi-ideal of $\mathrm{SU}_n(F)$.

2) The converse of 1) holds if and only if $n \leq 3$.

Theorem 4. For $A \in \mathrm{C}_{2n+1}(F)$, $(A)_q$ is a minimal quasi-ideal of $\mathrm{C}_{2n+1}(F)$ if and only if $\mathrm{rank}(A) = 1$.

Theorem 5. Let $\mathrm{char}(F) = 0$ and $A \in \mathrm{R}_n(F, k)$. Then $(A)_q$ is a minimal quasi-ideal of $\mathrm{R}_n(F, k)$ if and only if $A_{kk} \neq 0$.

Theorem 6. If $\mathrm{char}(F) = p > 0$, then for any $A \in \mathrm{R}_n(F, k)$, $(A)_q$ is a minimal quasi-ideal of $\mathrm{R}_n(F, k)$.

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CONTENTS

INTRODUCTION

Quasi-ideals of rings were first introduced by O. Steinfeid in [3]. They are generizations of left ideals and right ideals. The intersection of a left ideal and a right ideal of a ring is a quasi-ideal. It is well-known that not every quasi-ideal of a ring can be obtained in this way. Up to now, many researches on quasi-ideals in rings have been published. The quasi-ideal of a ring R generated by $X \subseteq R$ in an explicit form was given by H. J. Weinert in [5]. This work is useful for our research. A minimal quasi-ideal of a ring R is defined to be a nonzero quasi-ideal of R not properly containing any nonzero quasi-ideal of R. It follows that a minimal quasi-ideal of a ring R is the quasi-ideal of R generated by any element of its nonzero elements. It is given in [3] that a minimal quasi-ideal of a ring R is either a zero subring or a division subring of R and a quasi-ideal of R which is a division subring of R is a minimal quasi-ideal of R. P. N. Stewart has characterized a minimal quasi-ideal of a ring in terms of left ideals and right ideals in [4] as follows: A quasi-ideal of R is minimal if and only if any two of its nonzero elements generate the same left ideal and the same right ideal of R.

In ring theory, matrix rings are considered to be standard and important rings. We know that any matrix ring over a field F can be considered as a ring of linear transformations of a finite-dimensional vector space over F. Then the knowledge of vector spaces over fields and their linear transformations is sometimes useful to study matrix rings over fields. Characterizations of minimal quasi-ideals of some matrix rings over fields are given in this research in terms of ranks of matrices. However, it is shown that some matrix rings have no minimal quasi-ideal and every nonzero principal quasi-ideal of some matrix rings is minimal.

Preliminaries for this research is given in Chapter I. In Chapter II, minimal quasi-ideals of any full matrix ring over a field are characterized. It is shown that its quasi-ideal generated by A is minimal if and only if rank(A)= 1. We show in Chapter III that for any positive integer n, the ring of all strictly upper triangular $n \times n$ matrices over a field of characteristic 0 has no minimal quasi-ideal. We also study minimal quasi-ideals of this ring where the characteristic of its field is prime. We show that for any element A of this ring, if rank(A) = 1, then the quasi-ideal generated by A is minimal, and the converse is true if and only if $n \leq 3$. In the last chapter, minimal quasi-ideals in some other matrix rings are studied.

CHAPTER I

PRELIMINARIES

Throughout this research, let N and Z denote the set of all positive integers and the set of all integers, respectively.

Let $R = (R, +, .)$ be a ring.

For subsets X and Y of R, let

$$XY = \left\{ \sum_{i=1}^{n} x_i y_i \mid n \in N,\ x_i \in X \text{ and } y_i \in Y \text{ for all } i \in \{1, 2, \ldots, n\} \right\}$$

and

$$ZX = \left\{ \sum_{i=1}^{n} k_i x_i \mid n \in N,\ x_i \in X \text{ and } k_i \in Z \text{ for all } i \in \{1, 2, \ldots, n\} \right\}.$$

For $x \in R$, let Rx, xR and Zx denote $R\{x\}$, $\{x\}R$ and $Z\{x\}$, respectively. Then for $x \in R$,

$$Rx = \{ rx \mid r \in R \},$$

$$xR = \{ xr \mid r \in R \}$$

and

$$Zx = \{ kx \mid k \in Z \}.$$

We have that for any nonempty subset X of R, RX and XR are a left ideal and a right ideal of R, respectively.

An additive subgroup Q of R is said to be a *quasi-ideal* of R if $RQ \cap QR \subseteq Q$. Then every left ideal and every right ideal of R is a quasi-ideal of R. Moreover, the intersection of a left ideal and a right ideal of R is a quasi-ideal of R. However, not every quasi-ideal of R can be obtained in this way. An example was given by M. Sadiq Zia in [2] and an other example was given in [1]. In fact, if R is a commutative ring, then every quasi-ideal of R is an ideal.

If R has an identity and Q is a quasi-ideal of R containing a unit a of R, then for every $x \in R$, $x = (xa^{-1})a = a(a^{-1}x) \in RQ \cap QR \subseteq Q$ and hence $Q = R$.

An arbitrary intersection of quasi-ideals of R is a quasi-ideal of R. For $X \subseteq R$, let $(X)_q$ be the intersection of all quasi-ideals of R containing X and it is called the *quasi-ideal of R generated by X.* For $x \in R$, let $(x)_q$ denote $(\{x\})_q$, and it is called the *principal quasi-ideal of R generated by x.* The following theorem has been known.

Theorem 1.1 ([5]). *For a nonempty subset X of R,*

$$(X)_q = \mathbb{Z}X + (RX \cap XR).$$

From Theorem 1.1, we have directly that *for* $x \in R$,

$$(x)_q = \mathbb{Z}x + (Rx \cap xR).$$

Corollary 1.2. *If R has an identity, then for a nonempty subset X of R,*

$$(X)_q = RX \cap XR,$$

and in particular, for $x \in R$,

$$(x)_q = Rx \cap xR.$$

Proof. Let X be a nonempty subset of R. Then RX and XR are subrings of R which imply that $RX \cap XR$ is a subring of R. Since R has an identity, $X \subseteq RX \cap XR$. But $RX \cap XR$ is an additive subgroup of $(R, +)$, so $\mathbb{Z}X \subseteq RX \cap XR$. Consequently, $\mathbb{Z}X + (RX \cap XR) = RX \cap XR$. Hence $(X)_q = RX \cap XR$. □

A nonzero quasi-ideal Q of R is said to be *minimal* if Q does not properly contain any nonzero quasi-ideal of R. The following statements hold clearly.

1) If Q is a minimal quasi-ideal of R, then $Q = (x)_q$ for every $x \in Q \setminus \{0\}$. Then every minimal quasi-ideal of R is a principal quasi-ideal of R.

2) If $x \in R\backslash\{0\}$ is such that $(x)_q = (y)_q$ for all $y \in (x)_q\backslash\{0\}$, then $(x)_q$ is a minimal quasi-ideal of R.

The necessary conditions of minimal quasi-ideals of R given in [3] are as follows:

Theorem 1.3 ([3]). *If Q is a minimal quasi-ideal of R, then Q is either a zero subring or a division subring of R. In the second case, Q has the form $Q = eRe = eR \cap Re$ where e is the identity of Q.*

Theorem 1.3 has a partial converse as follows:

Theorem 1.4 ([3]). *If a quasi-ideal Q of R is a division subring of R, then Q is a minimal quasi-ideal of R.*

For the case that a quasi-ideal Q of R is a zero subring of R, Q need not be minimal. This can be seen from a zero ring of which its additive structure contains a nonzero proper subgroup.

For $x \in R$, let $(x)_l$ and $(x)_r$ be the left ideal and the right ideal of R generated by x, respectively. Then for every $x \in R$, $(x)_l = Zx + Rx$ and $(x)_r = Zx + xR$. The necessary and sufficient conditions for a quasi-ideal of R to be minimal given by P. N. Stewart in [4] are as follows:

Theorem 1.5 ([4]). *A quasi-ideal Q of R is minimal if and only if for all $x, y \in Q\backslash\{0\}$, $(x)_l = (y)_l$ and $(x)_r = (y)_r$.*

For a field F and $n \in N$, the full $n \times n$ matrix ring is denoted by $M_n(F)$. By a *matrix ring* over a field F, we mean a subring of $M_n(F)$ for some $n \in N$.

It is clear from Theorem 1.1 that if R is a matrix ring over a field F with the property that $aA \in R$ for all $a \in F$ and $A \in R$, then $(aA)_q = a(A)_q$ in R for all $a \in F$ and $A \in R$.

Let F be a field and $n \in N$.

For $A \in \mathrm{M}_n(F)$ and $i\,,\, j \in \{\, 1\,,\, 2\,, \ldots,\, n \,\}$, let A_{ij} denote the entry of A in the $i^{\underline{th}}$ row and the $j^{\underline{th}}$ column.

Let $\mathrm{SU}_n(F)$ be the ring of all strictly upper triangular $n \times n$ matrices over F, that is, $\mathrm{SU}_n(F)$ is the ring of all matrices in $\mathrm{M}_n(F)$ of the form

$$\begin{bmatrix} 0 & a_{12} & a_{13} & \ldots & a_{1,n-1} & a_{1n} \\ 0 & 0 & a_{23} & \ldots & a_{2,n-1} & a_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 0 & a_{n-1,n} \\ 0 & 0 & 0 & \ldots & 0 & 0 \end{bmatrix}.$$

Next, let

$$\mathrm{C}_{2n+1}(F) = \{A \in \mathrm{M}_{2n+1}(F) \,|\, A_{ij} = 0 \text{ for all } (i,j) \in \{1, 2, \ldots, 2n+1\} \times \{1, 2, \ldots, 2n+1\} \setminus \{(1,1), (1, 2n+1), (n+1, n+1), (2n+1, 1), (2n+1, 2n+1)\} \},$$

that is, $\mathrm{C}_{2n+1}(F)$ is the set of all matrices in $\mathrm{M}_{2n+1}(F)$ of the form

$$\begin{array}{cc} & \begin{array}{ccccccccc} & & & & n+1^{\underline{th}} & & & & \end{array} \\ \begin{array}{c} \\ \\ \\ \\ n+1^{\underline{th}} \\ \\ \\ \\ \\ \end{array} & \begin{bmatrix} a & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & b \\ 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & c & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 \\ d & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & e \end{bmatrix} \end{array}.$$

For $k \in \{\, 1\,,\, 2\,, \ldots,\, n \,\}$, let

$$\mathrm{R}_n(F,k) = \{A \in \mathrm{M}_n(F) \mid A_{ij} = 0 \text{ for all } i,j \in \{1, 2, \ldots, n\} \text{ and } i \neq k \},$$

that is, $R_n(F, k)$ is the set of all matrices in $M_n(F)$ of the form

$$\left[\begin{array}{cccc} 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \\ a_{k1} & a_{k2} & \dots & a_{kn} \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{array}\right]\begin{array}{l} \\ \\ \\ k^{\underline{th}} . \\ \\ \\ \\ \end{array}$$

It is clear that $R_n(F, k)$ is a subring of $M_n(F)$ for all $k \in \{1, 2, \dots, n\}$ and $C_{2n+1}(F)$ is a subgroup of $M_{2n+1}(F)$ under matrix addition.

To show that $C_{2n+1}(F)$ is a subring of $M_{2n+1}(F)$, let A, $B \in C_{2n+1}(F)$ and $(i, j) \in \{1, 2, \dots, 2n+1\} \times \{1, 2, \dots, 2n+1\} \setminus \{(1, 1), (1, 2n+1), (n+1, n+1), (2n+1, 1), (2n+1, 2n+1)\}$. Then

$$\begin{aligned} (AB)_{ij} &= \sum_{m=1}^{2n+1} A_{im} B_{mj} \\ &= A_{i1}B_{1j} + A_{i,n+1}B_{n+1,j} + A_{i,2n+1}B_{2n+1,j} \qquad \text{..........}(*) \end{aligned}$$

Case 1 : $i \notin \{1, n+1, 2n+1\}$. Then $A_{im} = 0$ for all $m \in \{1, 2, \dots, 2n+1\}$, so $(AB)_{ij} = 0$.

Case 2 : $i = 1$. Then $j \notin \{1, 2n+1\}$. Thus $B_{1j} = 0$, $A_{i,n+1} = 0$ and $B_{2n+1,j} = 0$, so by $(*)$, $(AB)_{ij} = 0$.

Case 3 : $i = n+1$. Then $j \neq n+1$. Then $A_{i1} = 0$, $B_{n+1,j} = 0$ and $A_{i,2n+1} = 0$, so by $(*)$, $(AB)_{ij} = 0$.

Case 4 : $i = 2n+1$. Then $j \notin \{1, 2n+1\}$. Thus $B_{1j} = 0$, $A_{i,n+1} = 0$ and $B_{2n+1,j} = 0$. By $(*)$, we have that $(AB)_{ij} = 0$.

In fact, the ring $C_{2n+1}(F)$ is clearly isomorphic to the ring $C_3(F)$ by the mapping

$$\begin{array}{cc}
 & \begin{array}{ccccccccc} & & & & n+1^{\underline{th}} & & & & \end{array} \\
\begin{array}{c} \\ \\ \\ \\ n+1^{\underline{th}} \\ \\ \\ \\ \end{array} & \left[\begin{array}{ccccccccc}
a & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & b \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & c & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
d & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & e
\end{array}\right]
\end{array} \mapsto \begin{bmatrix} a & 0 & b \\ 0 & c & 0 \\ d & 0 & e \end{bmatrix}$$

and it also preserves the ranks of matrices.

We know that $M_n(F)$ is a vector space over F under the usual addition and scalar multiplication of matrices.

Let V be a vector space over F and let $\mathrm{Hom}_F(V, V)$ denote the set of all linear transformations of V into itself. Then $\mathrm{Hom}_F(V, V)$ is a ring under the usual addition and composition of functions and it is also a vector space over F under the usual addition and scalar multiplication of functions.

Let $\dim_F V = n$. For any ordered basis B of V and for $T \in \mathrm{Hom}_F(V, V)$, let $[T]_B$ denote the matrix of T relative to the ordered basis B, that is, if $B = \{v_1, v_2, \dots, v_n\}$ and

$$\begin{aligned}
T(v_1) &= a_{11}v_1 + a_{21}v_2 + \dots + a_{n1}v_n \\
T(v_2) &= a_{12}v_1 + a_{22}v_2 + \dots + a_{n2}v_n \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
T(v_n) &= a_{1n}v_1 + a_{2n}v_2 + \dots + a_{nn}v_n
\end{aligned}$$

for $a_{ij} \in F$ and $i, j \in \{1, 2, \dots, n\}$, then

$$[T]_B = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

It is well-known that for any ordered basis B of V, the mapping $T \mapsto [T]_B$ is both a ring isomorphism and a vector space isomorphism of $\mathrm{Hom}_F(V, V)$ onto $M_n(F)$ and for each $T \in \mathrm{Hom}_F(V, V)$, $\mathrm{rank}(T) = \mathrm{rank}([T]_B)$.

CHAPTER II

FULL MATRIX RINGS

The main purpose of this chapter is to show that for a field F, $n \in N$ and $A \in \mathrm{M}_n(F)$, the quasi-ideal of $\mathrm{M}_n(F)$ generated by A, $(A)_q$, is minimal if and only if $\mathrm{rank}(A) = 1$.

We know that if V is a vector space over a field F of dimension n, then $\mathrm{Hom}_F(V, V) \cong \mathrm{M}_n(F)$ as rings and vector spaces by $T \mapsto [T]_B$ for any given ordered basis B of V. To obtain the main result above, we prove that for a finite–dimensional vector space V over a field F and $T \in \mathrm{Hom}_F(V, V)$, $(T)_q$ is a minimal quasi-ideal of $\mathrm{Hom}_F(V, V)$ if and only if $\mathrm{rank}(T) = 1$.

Throughout, let V be a vector space over a field F.

If R is a subring of the ring $\mathrm{Hom}_F(V, V)$ containing $\{aI \mid a \in F\}$ where I is the identity map on V, it is clear that R is a subspace of the vector space $\mathrm{Hom}_F(V, V)$.

Lemma 2.1. *Let R be a subring of $Hom_F(V, V)$ containing the set $\{aI \,/\, a \in F\}$ and T, $U \in R$.*

1) If $U \in (T)_q$, then $ImU \subseteq ImT$ and $KerU \supseteq KerT$.

2) If $(T)_q = (U)_q$, then $ImT = ImU$ and $KerU = KerT$.

Proof. 1) Let $U \in (T)_q$. By Corollary 1.2, $(T)_q = RT \cap TR$. Since $U \in TR$, $U = TT_1$ for some $T_1 \in R$. Hence $\mathrm{Im}U = \mathrm{Im}\,(TT_1) \subseteq \mathrm{Im}T$. Since $U \in RT$, $U = T_2T$ for some $T_2 \in R$. Then $\mathrm{Ker}U = \mathrm{Ker}\,(T_2T) \supseteq \mathrm{Ker}T$.

2) follows directly from 1). □

Lemma 2.2. *Let R be a subring of $Hom_F(V, V)$ containing the set $\{aI \mid a \in F\}$. Then for every $T \in R$, $FT \subseteq (T)_q$ in R where $FT = \{aT \mid a \in F\}$.*

Proof. Let $T \in R$ and $a \in F$. Then $aT = (aI)T = T(aI)$ since T is linear. Because $aI \in R$, $aT \in RT \cap TR = (T)_q$. This proves that $FT \subseteq (T)_q$ in R. $\square$

Lemma 2.3. *Let V be finite-dimensional and R a subring of $Hom_F(V, V)$ containing $\{aI \mid a \in F\}$. Then for every $T \in R$, if $rank(T) = 1$, then $(T)_q = FT$ in R.*

Proof. Assume that $\dim_F V = n$. Let $T \in R$ be such that $\operatorname{rank}(T) = 1$. By Lemma 2.2, $FT \subseteq (T)_q$ in R. Since $\operatorname{rank}(T) = 1$, $\dim_F(\operatorname{Im}T) = 1$. But $\operatorname{rank}(T) + \operatorname{nullity}(T) = \dim_F V = n$, so $\operatorname{nullity}(T) = n\text{-}1$, that is, $\dim_F(\operatorname{Ker}T) = n\text{-}1$. To show that $(T)_q \subseteq FT$, let $U \in (T)_q$ and $U \neq 0$. By Lemma 2.1, $\operatorname{Im}U \subseteq \operatorname{Im}T$ and $\operatorname{Ker}U \supseteq \operatorname{Ker}T$. Then $\dim_F(\operatorname{Im}U) \leq \dim_F(\operatorname{Im}T) = 1$. Since $U \neq 0$ and $\dim_F(\operatorname{Im}U) \leq 1$, it follows that $\dim_F(\operatorname{Im}U) = 1$. Hence $\dim_F(\operatorname{Ker}U) = n\text{-}1$ since $\operatorname{rank}(U) + \operatorname{nullity}(U) = n$. Now we have $\operatorname{Im}U \subseteq \operatorname{Im}T$, $\dim_F(\operatorname{Im}U) = \dim_F(\operatorname{Im}T) = 1$, $\operatorname{Ker}U \supseteq \operatorname{Ker}T$ and $\dim_F(\operatorname{Ker}U) = \dim_F(\operatorname{Ker}T) = n\text{-}1$. Consequently, $\operatorname{Im}U = \operatorname{Im}T$ and $\operatorname{Ker}U = \operatorname{Ker}T$. Let $v \in \operatorname{Im}T \setminus \{0\}$. Since $\dim_F(\operatorname{Im}T) = 1$, $\operatorname{Im}T = Fv$ where $Fv = \{av \mid a \in F\}$. Then $\operatorname{Im}U = \operatorname{Im}T = Fv$.

Next, let $\{v_1, v_2, \dots, v_{n-1}\}$ be a basis of $\operatorname{Ker}T$ ($= \operatorname{Ker}U$) and $\{v_1, v_2, \dots, v_{n-1}, v_n\}$ a basis of V. Then $T(v_i) = U(v_i) = 0$ for all $i \in \{1, 2, \dots, n\text{-}1\}$, $T(v_n) \neq 0$ and $U(v_n) \neq 0$. Since $\operatorname{Im}T = \operatorname{Im}U = Fv$, $T(v_n) = bv$ and $U(v_n) = cv$ for some $b, c \in F \setminus \{0\}$. Then $U(v_i) = ((b^{-1}c)T)(v_i) = 0$ for all $i \in \{1, 2, \dots, n\text{-}1\}$ and $U(v_n) = cv = (b^{-1}c)(bv) = (b^{-1}c)T(v_n) = ((b^{-1}c)T)(v_n)$. This follows that $U = (b^{-1}c)T \in FT$.

Hence we have that $FT = (T)_q$, as required. $\square$

Lemma 2.4. *Let R be a subring of $Hom_F(V, V)$ containing the set $\{aI \mid a \in F\}$. Then every quasi-ideal of R is a subspace of R.*

Proof. Let Q be a quasi-ideal of R. Then Q is an additive subgroup of R and $RQ \cap QR \subseteq Q$. Next, let $T \in Q$ and $a \in F$. Then $aT = (aI)T = T(aI) \in RQ \cap QR \subseteq Q$. Hence Q is a subspace of R. $\square$

From Lemma 2.3 and Lemma 2.4, we have

Corollary 2.5. *Let V be finite-dimensional, R a subring of $Hom_F(V, V)$ containing the set $\{aI \mid a \in F\}$ and $T \in R$. If $rank(T) = 1$, then $(T)_q$ is a subspace of R of dimension 1.*

Lemma 2.6. *Let V be finite-dimensional, R a subring of $Hom_F(V, V)$ containing the set $\{aI \mid a \in F\}$ and $T \in R$. If $rank(T) = 1$, then $(T)_q$ is a minimal quasi-ideal of R.*

Proof. Assume that $\mathrm{rank}(T) = 1$. By Corollary 2.5, $\dim_F(T)_q = 1$. To show that $(T)_q$ is a minimal quasi-ideal of R, let $U \in (T)_q$ and $U \neq 0$. Then $(U)_q \subseteq (T)_q$. From Lemma 2.4, $(U)_q$ is a subspace of R. Thus $\dim_F(U)_q \leq \dim_F(T)_q$. But $\dim_F(T)_q = 1$ and $U \neq 0$, so $\dim_F(U)_q = \dim_F(T)_q$. Consequently, $(U)_q = (T)_q$. Therefore $(T)_q$ is a minimal quasi-ideal of R. $\square$

Lemma 2.7. *Let V be finite-dimensional, $T \in Hom_F(V, V)$ and $T \neq 0$. Then there exists $U \in (T)_q$ such that $rank(U) = 1$.*

Proof. Let $\dim_F V = n$. Since $T \neq 0$, $\mathrm{Im}T \neq \{0\}$.

Case 1: $\mathrm{Im}T = V$. Since V is finite-dimensional, T is an isomorphism on V. Then T is a unit of $\mathrm{Hom}_F(V, V)$. This implies that $(T)_q = \mathrm{Hom}_F(V, V)$. Hence $(T)_q$ contains an element U of $\mathrm{Hom}_F(V, V)$ such that $\mathrm{rank}(U) = 1$.

Case 2: $\mathrm{Im}T \neq V$. Then $\{0\} \subsetneq \mathrm{Im}T \subsetneq V$. Let $\{v_1, v_2, \ldots, v_k\}$ be a basis of $\mathrm{Im}T$ and $\{v_1, v_2, \ldots, v_k, \ldots, v_n\}$ a basis of V. Since $\mathrm{rank}(T) + \mathrm{nullity}(T) = \dim_F V = n$, $\mathrm{nullity}(T) = n - k$. Let $\{w_{k+1}, w_{k+2}, \ldots, w_n\}$ be a basis of $\mathrm{Ker}T$.

Since $v_1, v_2, \ldots, v_k \in \mathrm{Im}T$, there exist $w_1, w_2, \ldots, w_k \in V$ such that $T(w_i) = v_i$

for all $i \in \{1, 2, \ldots, k\}$. To show that $\{w_1, w_2, \ldots, w_n\}$ is a basis of V, let $a_1, a_2, \ldots, a_n \in F$ be such that

$$a_1w_1 + a_2w_2 + \ldots + a_nw_n = 0. \qquad \ldots\ldots\ldots..(*)$$

Then

$$T(a_1w_1 + a_2w_2 + \ldots + a_nw_n) = 0,$$

so

$$a_1T(w_1) + a_2T(w_2) + \ldots + a_nT(w_n) = 0.$$

Since $w_{k+1}, w_{k+2}, \ldots, w_n \in \mathrm{Ker}T$ and $T(w_i) = v_i$ for all $i \in \{1, 2, \ldots, k\}$, we have that

$$a_1v_1 + a_2v_2 + \ldots + a_kv_k = 0.$$

Because of linearly independence of $v_1, v_2, \ldots, v_k$, we have that $a_i = 0$ for all $i \in \{1, 2, \ldots, k\}$. From (*), we have

$$a_{k+1}w_{k+1} + a_{k+2}w_{k+2} + \ldots + a_nw_n = 0.$$

Since $w_{k+1}, w_{k+2}, \ldots, w_n$ are linearly independent, $a_{k+1}, a_{k+2}, \ldots, a_n$ are all zero. This proves that $w_1, w_2, \ldots, w_n$ are linearly independent which implies that $\{w_1, w_2, \ldots, w_n\}$ is a basis of V because $\dim_F V = n$.

Next, let $T_1, T_2 \in \mathrm{Hom}_F(V, V)$ be defined by

$$T_1(v_1) = v_1,\ T_1(v_i) = 0 \text{ for all } i \in \{2, 3, \ldots, n\}$$

and

$$T_2(w_1) = w_1,\ T_2(w_i) = 0 \text{ for all } i \in \{2, 3, \ldots, n\}.$$

Then

$$(T_1T)(w_1) = T_1(T(w_1)) = T_1(v_1) = v_1,$$

$$(T_1T)(w_i) = T_1(T(w_i)) = T_1(v_i) = 0 \text{ for all } i \in \{2, 3, \ldots, k\}$$

and

$$(T_1T)(w_i) = T_1(T(w_i)) = T_1(0) = 0 \text{ for all } i \in \{k+1, k+2, \ldots, n\}.$$

We also have that

$$(TT_2)(w_1) = T(T_2(w_1)) = T(w_1) = v_1$$

and

$$(TT_2)(w_i) = T(T_2(w_i)) = T(0) = 0 \text{ for all } i \in \{2, 3, \ldots, n\}.$$

These imply that $T_1T = TT_2$ and $\mathrm{Im}(T_1T) = \mathrm{Im}(TT_2) = \langle v_1\rangle$. Let $U = T_1T$. Then $\mathrm{rank}(U) = 1$ and $U = T_1T = TT_2 \in \mathrm{Hom}_F(V, V)T \cap T\,\mathrm{Hom}_F(V, V) = (T)_q$.

Hence the lemma is completely proved. $\square$

Lemma 2.8. *Let V be finite-dimensional and $T \in Hom_F(V, V)\setminus\{0\}$. If $(T)_q$ is a minimal quasi-ideal of $Hom_F(V, V)$, then rank(T) = 1.*

Proof. Assume that $(T)_q$ is a minimal quasi-ideal of $\mathrm{Hom}_F(V, V)$. By Lemma 2.7, there exists $U \in (T)_q$ such that $\mathrm{rank}(U) = 1$. Then $U \neq 0$, so $(U)_q \neq \{0\}$. Since $U \in (T)_q$, $(U)_q \subseteq (T)_q$. Then $\{0\} \neq (U)_q \subseteq (T)_q$ which implies that $(U)_q = (T)_q$ since $(T)_q$ is a minimal quasi-ideal of $\mathrm{Hom}_F(V, V)$. By Lemma 2.1 (2), $\mathrm{Im}T = \mathrm{Im}U$. Then $\mathrm{rank}(T) = \mathrm{rank}(U) = 1$. $\square$

From Lemma 2.6 and Lemma 2.8, we have the following theorem.

Lemma 2.9. *Let V be finite-dimensional and $T \in Hom_F(V, V)$. Then $(T)_q$ is a minimal quasi-ideal of $Hom_F(V, V)$ if and only if rank(T) = 1.*

Let V be finite-dimensional, $\dim_F V = n$ and $B = \{v_1, v_2, \ldots, v_n\}$ an ordered basis of V. As we have mentioned in Chapter I on page 9, the map $T \mapsto [T]_B$ is an isomorphism of $\mathrm{Hom}_F(V, V)$ onto $\mathrm{M}_n(F)$ as both rings and vector spaces. Let θ denote the map $T \mapsto [T]_B$.

Let $A \in \mathrm{M}_n(F)$. Then $\theta(T) = A$ for some $T \in \mathrm{Hom}_F(V, V)$. By Corollary 1.2, $(A)_q = \mathrm{M}_n(F)A \cap A\mathrm{M}_n(F)$ in $\mathrm{M}_n(F)$ and $(T)_q = \mathrm{Hom}_F(V, V)T \cap T\mathrm{Hom}_F(V, V)$ in $\mathrm{Hom}_F(V, V)$. Since θ is an isomorphism of $\mathrm{Hom}_F(V, V)$ onto $\mathrm{M}_n(F)$, we have that

$$\begin{aligned}
\theta((T)_q) &= \theta\big(\mathrm{Hom}_F(V, V)T \cap T\mathrm{Hom}_F(V, V)\big) \\
&= \theta\big(\mathrm{Hom}_F(V, V)T\big) \cap \theta\big(T\mathrm{Hom}_F(V, V)\big) \\
&= \theta\big(\mathrm{Hom}_F(V, V)\big)\theta(T) \cap \theta(T)\,\theta\big(\mathrm{Hom}_F(V, V)\big) \\
&= \mathrm{M}_n(F)A \cap A\mathrm{M}_n(F) \\
&= (A)_q.
\end{aligned}$$

Therefore $(A)_q$ is a minimal quasi-ideal of $M_n(F)$ if and only if $(T)_q$ is a minimal quasi-ideal of $\mathrm{Hom}_F(V, V)$. Because $A = [T]_B$ and $\mathrm{rank}([T]_B) = \mathrm{rank}(T)$ (see Chapter I, page 9), by Theorem 2.9, the following theorem is obtained.

Theorem 2.10. *Let F be a field and $n \in N$. Then for $A \in M_n(F)$, $(A)_q$ is a minimal quasi-ideal of $M_n(F)$ if and only if $\mathrm{rank}(A) = 1$.*

We shall give a proof of " if " part of Theorem 2.10 by using the addition, scalar multiplication and multiplication of matrices directly. First, we shall prove that for any $A \in M_n(F)$, if $\mathrm{rank}(A) = 1$, then $(A)_q = FA$ where $FA = \{aA \mid a \in F\}$, that is,

$$(A)_q = \{aA \mid a \in F\}.$$

Let $A \in M_n(F)$ be such that $\mathrm{rank}(A) = 1$. Then A has a nonzero row. Without loss of generality, assume that the first row of A is nonzero. Since $\mathrm{rank}(A) = 1$, there exist $a_1, a_2, \ldots, a_n, x_2, x_3, \ldots, x_n \in F$ such that

$$A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ x_2a_1 & x_2a_2 & \cdots & x_2a_n \\ \cdots & \cdots & \cdots & \cdots \\ x_na_1 & x_na_2 & \cdots & x_na_n \end{bmatrix}.$$

Let $B \in (A)_q$. Since $(A)_q = M_n(F)A \cap AM_n(F)$, $B = CA = AD$ for some C, $D \in M_n(F)$. Let $x_1 = 1$. Then

$$CA = \begin{bmatrix} \sum_{j=1}^{n} C_{1j}x_ja_1 & \sum_{j=1}^{n} C_{1j}x_ja_2 & \cdots & \sum_{j=1}^{n} C_{1j}x_ja_n \\ \sum_{j=1}^{n} C_{2j}x_ja_1 & \sum_{j=1}^{n} C_{2j}x_ja_2 & \cdots & \sum_{j=1}^{n} C_{2j}x_ja_n \\ \cdots & \cdots & \cdots & \cdots \\ \sum_{j=1}^{n} C_{nj}x_ja_1 & \sum_{j=1}^{n} C_{nj}x_ja_2 & \cdots & \sum_{j=1}^{n} C_{nj}x_ja_n \end{bmatrix}$$

$$= \begin{bmatrix} a_1(\sum_{j=1}^{n} C_{1j}x_j) & a_2(\sum_{j=1}^{n} C_{1j}x_j) & \cdots & a_n(\sum_{j=1}^{n} C_{1j}x_j) \\ a_1(\sum_{j=1}^{n} C_{2j}x_j) & a_2(\sum_{j=1}^{n} C_{2j}x_j) & \cdots & a_n(\sum_{j=1}^{n} C_{2j}x_j) \\ \cdots & \cdots & \cdots & \cdots \\ a_1(\sum_{j=1}^{n} C_{nj}x_j) & a_2(\sum_{j=1}^{n} C_{nj}x_j) & \cdots & a_n(\sum_{j=1}^{n} C_{nj}x_j) \end{bmatrix}$$

and

$$AD = \begin{bmatrix} \sum_{j=1}^{n} x_1 a_j D_{j1} & \sum_{j=1}^{n} x_1 a_j D_{j2} & \cdots & \sum_{j=1}^{n} x_1 a_j D_{jn} \\ \sum_{j=1}^{n} x_2 a_j D_{j1} & \sum_{j=1}^{n} x_2 a_j D_{j2} & \cdots & \sum_{j=1}^{n} x_2 a_j D_{jn} \\ \cdots & \cdots & \cdots & \cdots \\ \sum_{j=1}^{n} x_n a_j D_{j1} & \sum_{j=1}^{n} x_n a_j D_{j2} & \cdots & \sum_{j=1}^{n} x_n a_j D_{jn} \end{bmatrix}$$

$$= \begin{bmatrix} x_1(\sum_{j=1}^{n} a_j D_{j1}) & x_1(\sum_{j=1}^{n} a_j D_{j2}) & \cdots & x_1(\sum_{j=1}^{n} a_j D_{jn}) \\ x_2(\sum_{j=1}^{n} a_j D_{j1}) & x_2(\sum_{j=1}^{n} a_j D_{j2}) & \cdots & x_2(\sum_{j=1}^{n} a_j D_{jn}) \\ \cdots & \cdots & \cdots & \cdots \\ x_n(\sum_{j=1}^{n} a_j D_{j1}) & x_n(\sum_{j=1}^{n} a_j D_{j2}) & \cdots & x_n(\sum_{j=1}^{n} a_j D_{jn}) \end{bmatrix}$$

which imply that

$$B = \begin{bmatrix} y_1 a_1 & y_1 a_2 & \cdots & y_1 a_n \\ y_2 a_1 & y_2 a_2 & \cdots & y_2 a_n \\ \cdots & \cdots & \cdots & \cdots \\ y_n a_1 & y_n a_2 & \cdots & y_n a_n \end{bmatrix}$$

$$= \begin{bmatrix} z_1 & z_2 & \cdots & z_n \\ x_2 z_1 & x_2 z_2 & \cdots & x_2 z_n \\ \cdots & \cdots & \cdots & \cdots \\ x_n z_1 & x_n z_2 & \cdots & x_n z_n \end{bmatrix}$$

for some y_1 , y_2 , ... , y_n , z_1 , z_2 , ... , $z_n \in F$. It follows that

$$B = \begin{bmatrix} y_1a_1 & y_1a_2 & \cdots & y_1a_n \\ x_2y_1a_1 & x_2y_1a_2 & \cdots & x_2y_1a_n \\ \cdots & \cdots & \cdots & \cdots \\ x_ny_1a_1 & x_ny_1a_2 & \cdots & x_ny_1a_n \end{bmatrix}$$

$$= y_1 \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ x_2a_1 & x_2a_2 & \cdots & x_2a_n \\ \cdots & \cdots & \cdots & \cdots \\ x_na_1 & x_na_2 & \cdots & x_na_n \end{bmatrix}$$

$$= y_1A.$$

Hence $(A)_q \subseteq FA$. For $a \in F$, $aA = (aI)A = A(aI)$, so $aA \in M_n(F)A \cap AM_n(F)$ $= (A)_q$. This proves that $(A)_q = FA$, as required.

Next, let $A \in M_n(F)$ be such that rank$(A) = 1$. Then $(A)_q = FA$. To show that $(A)_q$ is a minimal quasi-ideal of $M_n(F)$, let $B \in (A)_q$ and $B \neq 0$. Since $(A)_q$ $= FA$ and $B \in (A)_q$, $B = cA$ for some $c \in F$. Since $B \neq 0$, $c \neq 0$, so rank(B) = rank$(A) = 1$. Then $(B)_q = FB$, so $(B)_q = F(cA) = (Fc)A = FA$. Hence $(B)_q = (A)_q$. $\square$

We know from Theorem 1.3 that a minimal quasi-ideal of a ring R is either a zero subring or a division subring of R. We give a remark here that some minimal quasi-ideals of $M_n(F)$ are zero subrings and some are division subrings of $M_n(F)$ for the case that $n > 1$.

Let $n > 1$ and A , $B \in M_n(F)$ defined by

$$A = \begin{bmatrix} 0 & \dots & 0 & 1 \\ 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then $\operatorname{rank}(A) = 1 = \operatorname{rank}(B)$, $A^2 = 0$ and $B^2 = B$. Then $(A)_q = FA$, $(B)_q = FB$ and the quasi-ideals $(A)_q$ and $(B)_q$ are minimal. For c, $d \in F$, $(cA)(dA) = (cd)\,A^2 = 0$. Hence $(A)_q$ is a zero subring. Since $(B)_q = FB$,

$$(B)_q = \left\{ \begin{bmatrix} x & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 \end{bmatrix} \;\middle|\; x \in F \right\}$$

which is isomorphic to F, so $(B)_q$ is a division subring of $M_n(F)$.

CHAPTER III

RINGS OF ALL STRICTLY UPPER TRIANGULAR MATRICES

Throughout this chapter, let F be a field, $n \in N$ and $\mathrm{SU}_n(F)$ the ring of all strictly upper triangular $n \times n$ matrices.

The main purpose of this chapter is to prove the following results.

1) If $\mathrm{char}(F) = 0$, then $\mathrm{SU}_n(F)$ has no minimal quasi-ideal.

2) Let $\mathrm{char}(F) = p > 0$. Then

2.1) for $A \in \mathrm{SU}_n(F)$, if $\mathrm{rank}(A) = 1$, then $(A)_q$ is a minimal quasi-ideal of $\mathrm{SU}_n(F)$ and

2.2) the converse of 2.1) holds if and only if $n \leq 3$.

We note that if $\mathrm{char}(F) = 0$, then $m1_F \neq 0$ for all $m \in Z\setminus\{0\}$ where 1_F is the identity of F which implies that for $m \in Z\setminus\{1, -1\}$ and $k \in Z$ considered as elements of F, $mk \neq 1$. Also, if $\mathrm{char}(F) = p > 0$, then

$$Z1_F = \{ 0, 1_F, 2(1_F), \ldots, (p-1)(1_F) \},$$

and so for $x \in F$,

$$Zx = \{ 0, x, 2x, \ldots, (p-1)x \}$$

and $|Zx| = p$ if $x \neq 0$.

Theorem 3.1. *If $\mathrm{char}(F) = 0$, then $SU_n(F)$ has no minimal quasi-ideal.*

Proof. Let $\mathrm{char}(F) = 0$. To prove that $\mathrm{SU}_n(F)$ has no minimal quasi-ideal, it suffices to prove that for every $A \in \mathrm{SU}_n(F)\setminus\{0\}$, there exists $B \in \mathrm{SU}_n(F)\setminus\{0\}$ such that $(B)_q \subsetneq (A)_q$.

Let $A \in \mathrm{SU}_n(F)$ and $A \neq 0$. Then $2A \in \mathrm{SU}_n(F)$. Since $\mathrm{char}(F) = 0$ and $A \neq 0$, $2A \neq 0$. Since $(A)_q$ is an additive subgroup of $\mathrm{SU}_n(F)$, $2A \in (A)_q$ which implies that $(2A)_q \subseteq (A)_q$. Suppose that $(2A)_q = (A)_q$. By Theorem 1.1,

$$(A)_q = ZA + \big(\mathrm{SU}_n(F)A \cap A\mathrm{SU}_n(F)\big).$$

Since $A \in (A)_q = (2A)_q = 2(A)_q$, there exist $m \in Z$ and $C \in \mathrm{SU}_n(F)$ such that

$$A = 2(mA + CA)$$

which implies that

$$(1 - 2m)A = 2CA. \qquad \text{.............}(*)$$

Since A is strictly upper triangular, $A_{ij} = 0$ for all $i, j \in \{1, 2, \ldots, n\}$ and $i \geq j$. Since $A \neq 0$, $A_{ij} \neq 0$ for some $i \in \{1, 2, \ldots, n-1\}$ and $j \in \{i+1, i+2, \ldots, n\}$. Let

$$k = \max\left\{ i \in \{1, 2, \ldots, n-1\} \middle| A_{ij} \neq 0 \text{ for some } j \in \{i+1, i+2, \ldots, n\} \right\}$$

and let $l \in \{k+1, k+2, \ldots, n\}$ be such that $A_{kl} \neq 0$.
From (*), we have

$$((1-2m)A)_{kl} = (2CA)_{kl},$$

so

$$(1-2m)A_{kl} = 2\left(\sum_{j=1}^{k} C_{kj}A_{jl} + \sum_{j=k+1}^{n} C_{kj}A_{jl}\right).$$

Since C is strictly upper triangular, $C_{k1} = C_{k2} = \ldots = C_{kk} = 0$. Then

$$(1-2m)A_{kl} = 2\sum_{j=k+1}^{n} C_{kj}A_{jl}.$$

By the property of k, we have that $A_{jl} = 0$ for all $j > k$.
Thus

$$(1-2m)A_{kl} = 0.$$

But $A_{kl} \neq 0$, so $1-2m = 0$. Thus $2m = 1$ which is a contradiction since $\operatorname{char}(F) = 0$. Hence $(2A)_q \subsetneq (A)_q$.

This proves that $SU_n(F)$ has no minimal quasi-ideal, as required. □

Lemma 3.2. *For $A \in SU_n(F)$, if $\operatorname{rank}(A) = 1$, then $SU_n(F)A \cap ASU_n(F) = \{0\}$.*

Proof. Let $A \in SU_n(F)$ be such that $\operatorname{rank}(A) = 1$. Then $A \neq 0$ and $A_{ij} = 0$ for all $i, j \in \{1, 2, \ldots, n\}$ with $i \geq j$, so $A_{ij} \neq 0$ for some $i \in \{1, 2, \ldots, n-1\}$ and $j \in \{i+1, i+2, \ldots, n\}$. Let

$$k = \max \left\{ i \in \{ 1, 2, \ldots, n\text{-}1 \} \mid A_{ij} \neq 0 \text{ for some } j \in \{ i+1, i+2, \ldots, n \} \right\}$$

and

$$l = \min \{ j \in \{ k+1, k+2, \ldots, n \} \mid A_{kj} \neq 0 \}.$$

It follows from the properties of k, l and $\mathrm{rank}(A) = 1$ that

$$A = \begin{bmatrix} 0 & \ldots & 0 & x_1 A_{kl} & x_1 A_{k,l+1} & \ldots & x_1 A_{kn} \\ 0 & \ldots & 0 & x_2 A_{kl} & x_2 A_{k,l+1} & \ldots & x_2 A_{kn} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & x_{k-1} A_{kl} & x_{k-1} A_{k,l+1} & \ldots & x_{k-1} A_{kn} \\ 0 & \ldots & 0 & A_{kl} & A_{k,l+1} & \ldots & A_{kn} \\ 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \end{bmatrix}$$

for some $x_1, x_2, \ldots, x_{k-1} \in F$. Let $B \in \mathrm{SU}_n(F)A \cap A\mathrm{SU}_n(F)$. Then $B = CA = AD$ for some C, $D \in \mathrm{SU}_n(F)$. Let $x_k = 1$. Then

$$CA = \begin{bmatrix} 0 & \ldots & 0 & (\sum_{j=2}^{k} C_{1j} x_j) A_{kl} & (\sum_{j=2}^{k} C_{1j} x_j) A_{k,l+1} & \ldots & (\sum_{j=2}^{k} C_{1j} x_j) A_{kn} \\ 0 & \ldots & 0 & (\sum_{j=3}^{k} C_{2j} x_j) A_{kl} & (\sum_{j=3}^{k} C_{2j} x_j) A_{k,l+1} & \ldots & (\sum_{j=3}^{k} C_{2j} x_j) A_{kn} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & (\sum_{j=k-1}^{k} C_{k-2,j} x_j) A_{kl} & (\sum_{j=k-1}^{k} C_{k-2,j} x_j) A_{k,l+1} & \ldots & (\sum_{j=k-1}^{k} C_{k-2,j} x_j) A_{kn} \\ 0 & \ldots & 0 & C_{k-1,k} A_{kl} & C_{k-1,k} A_{k,l+1} & \ldots & C_{k-1,k} A_{kn} \\ 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \end{bmatrix}.$$

.............(*)

Since $A_{i1} = A_{i2} = \ldots = A_{i,l-1} = 0$ for all $i \in \{ 1, 2, \ldots, n \}$, $D_{ll} = D_{l+1,l} = \ldots = D_{nl} = 0$ and $(AD)_{il} = \sum_{j=1}^{l-1} A_{ij} D_{jl} + \sum_{j=l}^{n} A_{ij} D_{jl}$ for all $i \in \{1, 2, \ldots, n\}$, we have that $(AD)_{il} = 0$ for every $i \in \{ 1, 2, \ldots, n \}$. But $CA = AD$, so $(CA)_{il} = (AD)_{il} = 0$ for all $i \in \{ 1, 2, \ldots, n \}$. It follows from (*) that

$$\begin{aligned}
(\sum_{j=2}^{k} C_{1j}x_j)A_{kl} &= 0 \\
(\sum_{j=3}^{k} C_{2j}x_j)A_{kl} &= 0 \\
&\dots\dots \\
(\sum_{j=k-1}^{k} C_{k-2,j}x_j)A_{kl} &= 0 \\
C_{k-1,k}A_{kl} &= 0.
\end{aligned}$$

Since $A_{kl} \neq 0$, $\sum_{j=2}^{k} C_{1j}x_j = \sum_{j=3}^{k} C_{2j}x_j = \dots = \sum_{j=k-1}^{k} C_{k-2,j}x_j = C_{k-1,k} = 0$.
By (*), we have $CA = 0$, so $B = 0$.

Hence $\mathrm{SU}_n(F)A \cap A\mathrm{SU}_n(F) = \{0\}$. $\square$

Corollary 3.3. *For $A \in SU_n(F)$, if $rank(A) = 1$, then $(A)_q = ZA$.*

Proof. Let $A \in \mathrm{SU}_n(F)$ be such that $\mathrm{rank}(A) = 1$. By Lemma 3.2, $\mathrm{SU}_n(F)A \cap A\mathrm{SU}_n(F) = \{0\}$. By Theorem 1.1,

$$(A)_q = ZA + (\,\mathrm{SU}_n(F)A \cap A\mathrm{SU}_n(F)\,).$$

Consequently, $(A)_q = ZA$. $\square$

Lemma 3.4. *Let R be a subring of $M_n(F)$, $A \in R$, $A \neq 0$ and $(A)_q = ZA$ in R.*

1) *If $char(F) = 0$, then $(A)_q$ is not a minimal quasi-ideal of R.*

2) *If $char(F) = p > 0$, then $(A)_q = \{0, A, 2A, \dots, (p\text{-}1)A\}$, $|(A)_q| = p$ and $(A)_q$ is a minimal quasi-ideal of R.*

Proof. Since $A \neq 0$, there exist $i, j \in \{1, 2, \dots, n\}$ such that $A_{ij} \neq 0$.

1) Assume that $\mathrm{char}(F) = 0$. Then $2A \neq 0$. Since $(A)_q$ is an additive subgroup of R, $2A \in (A)_q$ which implies that $\{0\} \neq (2A)_q \subseteq (A)_q$. Since $\mathrm{char}(F) = 0$ and $A_{ij} \neq 0$, $A_{ij} \neq 2mA_{ij}$ for all $m \in Z$ which implies that $A \notin 2ZA$. Since $(2A)_q = 2(A)_q = 2ZA$, $A \notin (2A)_q$. Hence $\{0\} \subsetneq (2A)_q \subsetneq (A)_q$. Thus the quasi-ideal $(A)_q$ is not minimal.

2) Assume that $\text{char}(F) = p > 0$. Then

$$Z1_F = \{ 0 , 1_F , 2(1_F) , \ldots ,(p\text{-}1)(1_F) \} ,$$

and so

$$ZA = (Z1_F)A = \{ 0 , A , 2A , \ldots ,(p\text{-}1)A \}$$

where 1_F is the identity of F. Then $(A)_q = \{ 0 , A , 2A , \ldots ,(p\text{-}1)A \}$.
Since $\text{char}(F) = p$ and $A_{ij} \neq 0$, we have that $0 , A_{ij} , 2A_{ij} , \ldots , (p-1)A_{ij}$ are all distinct in F. Then the matrices $0 , A , 2A , \ldots , (p-1)A$ are all distinct. Hence $|(A)_q| = p$. Let $B \in (A)_q$ and $B \neq 0$. Then $(B)_q$ is an additive subgroup of $(A)_q$ and $(B)_q \neq \{0\}$. Thus $|(B)_q| \mid p$ which implies that $|(B)_q| = p$. Hence $(B)_q = (A)_q$.
Therefore $(A)_q$ is a minimal quasi-ideal of R. $\square$

Theorem 3.5. *Let $\text{char}(F) = p > 0$ and $A \in SU_n(F)$. If $\text{rank}(A) = 1$, then $(A)_q$ is a minimal quasi-ideal of $SU_n(F)$.*

Proof. Assume that $\text{rank}(A) = 1$. By Corollary 3.3, $(A)_q = ZA$. Then by Lemma 3.4(2), $(A)_q$ is a minimal quasi-ideal of $SU_n(F)$. $\square$

Theorem 3.6. *Let $\text{char}(F) = p > 0$. Then the following statements are equivalent.*

1) For $A \in SU_n(F)$, if $(A)_q$ is a minimal quasi-ideal of $SU_n(F)$, then $\text{rank}(A) = 1$.

2) $n \leq 3$.

Proof. To prove (1) implies (2) by contrapositive, assume that $n > 3$.
Let

$$A = \begin{bmatrix} 0 & \ldots & 0 & 0 & 1 \\ 0 & \ldots & 0 & 1 & 0 \\ 0 & \ldots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & 0 & 0 \end{bmatrix} .$$

Then $A \in \mathrm{SU}_n(F)$ and $\mathrm{rank}(A) = 2$. To show that $\mathrm{SU}_n(F)A \cap A\mathrm{SU}_n(F) = \{0\}$, let $B \in \mathrm{SU}_n(F)A \cap A\mathrm{SU}_n(F)$. Then $B = CA = AD$ for some C, $D \in \mathrm{SU}_n(F)$. But

$$CA = \begin{bmatrix} 0 & C_{12} & C_{13} & \dots & C_{1,n-1} & C_{1n} \\ 0 & 0 & C_{23} & \dots & C_{2,n-1} & C_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & C_{n-1,n} \\ 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \dots & 0 & 0 & 1 \\ 0 & \dots & 0 & 1 & 0 \\ 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \dots & 0 & C_{12} & 0 \\ 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & 0 \end{bmatrix}$$

and

$$AD = \begin{bmatrix} 0 & \dots & 0 & 0 & 1 \\ 0 & \dots & 0 & 1 & 0 \\ 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & D_{12} & D_{13} & \dots & D_{1,n-1} & D_{1n} \\ 0 & 0 & D_{23} & \dots & D_{2,n-1} & D_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & D_{n-1,n} \\ 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & D_{n-1,n} \\ 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 \end{bmatrix},$$

so $B = 0$. Thus $\mathrm{SU}_n(F)A \cap A\mathrm{SU}_n(F) = \{0\}$. It follows from Theorem 1.1 that $(A)_q = ZA$. By Lemma 3.4(2), $(A)_q$ is a minimal quasi-ideal of $\mathrm{SU}_n(F)$.

Conversely, to prove (2) implies (1), assume that $n \le 3$ and let $A \in \mathrm{SU}_n(F)$ be such that $(A)_q$ is a minimal quasi-ideal of $\mathrm{SU}_n(F)$. Since $A \neq 0$ and $\mathrm{SU}_1(F) = \{0\}$, $n > 1$. If $n = 2$, then every nonzero matrix in $\mathrm{SU}_n(F)$ has rank 1 and so $\mathrm{rank}(A) = 1$.

Next, assume that $n = 3$. To prove that $\operatorname{rank}(A) = 1$, suppose not. Since

$$A = \begin{bmatrix} 0 & A_{12} & A_{13} \\ 0 & 0 & A_{23} \\ 0 & 0 & 0 \end{bmatrix},$$

we have that $A_{12} \neq 0$ and $A_{23} \neq 0$. Let

$$B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Because

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & A_{23}^{-1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & A_{12} & A_{13} \\ 0 & 0 & A_{23} \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & A_{12} & A_{13} \\ 0 & 0 & A_{23} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & A_{12}^{-1} \\ 0 & 0 & 0 \end{bmatrix},$$

we have that $B \in \mathrm{SU}_3(F)A \cap A\mathrm{SU}_3(F)$. But $(A)_q = ZA + \left(\mathrm{SU}_n(F)A \cap A\mathrm{SU}_n(F)\right)$, so $B \in (A)_q$ and hence $(B)_q \subseteq (A)_q$. Since $\operatorname{rank}(B) = 1$, by Corollary 3.3, $(B)_q = ZB$, so

$$(B)_q = \left\{ \begin{bmatrix} 0 & 0 & m \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mid m \in Z \right\}.$$

Then $A \notin (B)_q$ since $A_{12} \neq 0$. Therefore $\{0\} \subsetneqq (B)_q \subsetneqq (A)_q$. It is a contradiction since $(A)_q$ is a minimal quasi-ideal of $\mathrm{SU}_3(F)$. This proves that $\operatorname{rank}(A) = 1$, as required. $\square$

We shall give a remark that if $\operatorname{char}(F) = p > 0$, then every minimal quasi-ideal in $\mathrm{SU}_n(F)$ is always a zero subring of $\mathrm{SU}_n(F)$ because every nonzero subring of $\mathrm{SU}_n(F)$ has no identity.

CHAPTER IV

SOME OTHER MATRIX RINGS

Let F be a field and $n \in N$. We first recall the following notations.

$C_{2n+1}(F)$ = the ring of all $(2n+1) \times (2n+1)$ matrices over F of the form

$$\begin{array}{cc}
 & \begin{array}{ccccccccc} & & & & n+1^{\underline{th}} & & & & \end{array} \\
\begin{array}{c} \\ \\ \\ \\ n+1^{\underline{th}} \\ \\ \\ \\ \\ \end{array} & \begin{bmatrix}
a & 0 & \dots & 0 & 0 & 0 & \dots & 0 & b \\
0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & \dots & 0 & c & 0 & \dots & 0 & 0 \\
0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\
d & 0 & \dots & 0 & 0 & 0 & \dots & 0 & e
\end{bmatrix}
\end{array}.$$

For $k \in \{1, 2, \dots, n\}$, let

$R_n(F, k)$ = the ring of all $n \times n$ matrices over F of the form

$$\begin{array}{cc}
\begin{array}{c} \\ \\ \\ k^{\underline{th}} \\ \\ \\ \\ \end{array} & \begin{bmatrix}
0 & 0 & \dots & 0 \\
\dots & \dots & \dots & \dots \\
0 & 0 & \dots & 0 \\
a_1 & a_2 & \dots & a_n \\
0 & 0 & \dots & 0 \\
\dots & \dots & \dots & \dots \\
0 & 0 & \dots & 0
\end{bmatrix}
\end{array}.$$

In this chapter we shall give a necessary and sufficient condition for a matrix A of each of these matrix rings such that the quasi-ideal $(A)_q$ is minimal.

We know that if V is a finite-dimensional vector space over F of dimension n, then for any given ordered basis B of V, the map $T \mapsto [T]_B$ is an isomorphism of $\mathrm{Hom}_F(V, V)$ onto $M_n(F)$ as both rings and vector spaces and for any $T \in \mathrm{Hom}_F(V, V)$, $\mathrm{rank}(T) = \mathrm{rank}([T]_B)$. Thus from these facts the following two lemmas are obtained from Lemma 2.3 and Lemma 2.6, respectively.

Lemma 4.1 *Let R be a subring of $M_n(F)$ containing $\{aI \,/\, a \in F\}$ where I is the $n \times n$ identity matrix over F. If $A \in R$ is such that $\mathrm{rank}(A) = 1$, then $(A)_q = FA$ where $FA = \{aA \,/\, a \in F\}$.*

Lemma 4.2 *Let R be a subring of $M_n(F)$ containing $\{aI \,/\, a \in F\}$ where I is the $n \times n$ identity matrix over F. If $A \in R$ is such that $\mathrm{rank}(A) = 1$, then $(A)_q$ is a minimal quasi-ideal of R.*

Lemma 4.3 *For $A \in C_{2n+1}(F)$, $(A)_q$ is a minimal quasi-ideal of $C_{2n+1}(F)$ if and only if $\mathrm{rank}(A) = 1$.*

Proof. We have that the map

$$A \mapsto \begin{bmatrix} A_{11} & 0 & A_{1,2n+1} \\ 0 & A_{n+1,n+1} & 0 \\ A_{2n+1,1} & 0 & A_{2n+1,2n+1} \end{bmatrix}$$

is an isomorphism of $C_{2n+1}(F)$ onto $C_3(F)$ which preserves the ranks of matrices. Then to prove this theorem, it suffices to prove that it is true for $C_3(F)$. Note that $\{aI \mid a \in F\} \subseteq C_3(F)$ where I is the 3×3 identity matrix over F. It follows from Lemma 4.2 that if $A \in C_3(F)$ is such that $\mathrm{rank}(A) = 1$, then $(A)_q$ is a minimal quasi-ideal of $C_3(F)$.

For the converse, let $A \in C_3(F)$ be such that $(A)_q$ is a minimal quasi-ideal of $C_3(F)$. Let

$$A = \begin{bmatrix} a & 0 & b \\ 0 & c & 0 \\ d & 0 & e \end{bmatrix}.$$

To show that rank(A) = 3 or 2 is impossible, first assume that rank(A) = 3. Then A^{-1} exists in $M_3(F)$ and

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A$$

$$= \frac{1}{ace - bcd} \begin{bmatrix} ce & 0 & -bc \\ 0 & ae - bd & 0 \\ -dc & 0 & ac \end{bmatrix},$$

so $A^{-1} \in C_3(F)$. Therefore A is a unit in $C_3(F)$. Consequently, $(A)_q = C_3(F)$. By Lemma 4.1,

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)_q = \left\{ \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mid a \in F \right\}$$

which is properly contained in $(A)_q$. This contradicts the fact that $(A)_q$ is a minimal quasi-ideal of $C_3(F)$, so rank(A) $\neq 3$.

Next, assume that rank(A) = 2.

Case 1 : $c \neq 0$. Then $a \neq 0$ or $b \neq 0$ and there exists $g \in F$ such that

$$A = \begin{bmatrix} a & 0 & b \\ 0 & c & 0 \\ ga & 0 & gb \end{bmatrix}.$$

Claim that the set

$$\left\{ \begin{bmatrix} xa & 0 & xb \\ 0 & y & 0 \\ xga & 0 & xgb \end{bmatrix} \mid x, y \in F \right\}$$

is a subset of $(A)_q$. Let this set be denoted by Q and let $x', y' \in F$. Then

$$\begin{bmatrix} x'a & 0 & x'b \\ 0 & y' & 0 \\ x'ga & 0 & x'gb \end{bmatrix} = \begin{bmatrix} x' & 0 & 0 \\ 0 & c^{-1}y' & 0 \\ 0 & 0 & x' \end{bmatrix} \begin{bmatrix} a & 0 & b \\ 0 & c & 0 \\ ga & 0 & gb \end{bmatrix} = \begin{bmatrix} a & 0 & b \\ 0 & c & 0 \\ ga & 0 & gb \end{bmatrix} \begin{bmatrix} x' & 0 & 0 \\ 0 & c^{-1}y' & 0 \\ 0 & 0 & x' \end{bmatrix},$$

so $\begin{bmatrix} x'a & 0 & x'b \\ 0 & y' & 0 \\ x'ga & 0 & x'gb \end{bmatrix}$ is an element of $C_3(F)A \cap AC_3(F)$. This proves that

$Q \subseteq C_3(F)A \cap AC_3(F)$. Hence $Q \subseteq (A)_q$. Let

$$B = \begin{bmatrix} a & 0 & b \\ 0 & 0 & 0 \\ ga & 0 & gb \end{bmatrix}.$$

Since $a \neq 0$ or $b \neq 0$, $\operatorname{rank}(B) = 1$. By Lemma 4.1 ,

$$(B)_q = \left\{ \begin{bmatrix} xa & 0 & xb \\ 0 & 0 & 0 \\ xga & 0 & xgb \end{bmatrix} \mid x \in F \right\}$$

which is properly contained in Q. But $Q \subseteq (A)_q$, so $\{0\} \neq (B)_q \subseteq (A)_q$ which is a contradiction.

Case 2 : $c = 0$. Then

$$A = \begin{bmatrix} a & 0 & b \\ 0 & 0 & 0 \\ d & 0 & e \end{bmatrix}.$$

If $ae = bd$, then $e(a, 0, b) - b(d, 0, e) = (0, 0, 0)$ which is a contradiction since $\operatorname{rank}(A) = 2$. Then $ae - bd \neq 0$. Let

$$Q' = \left\{ \begin{bmatrix} x & 0 & y \\ 0 & 0 & 0 \\ z & 0 & w \end{bmatrix} \mid x, y, z, w \in F \right\}.$$

Claim that $Q' \subseteq (A)_q$. If x', y', z', $w' \in F$, then

$$\begin{bmatrix} x' & 0 & y' \\ 0 & 0 & 0 \\ z' & 0 & w' \end{bmatrix} = \begin{bmatrix} (ex'-dy')(ae-bd)^{-1} & 0 & (ay'-bx')(ae-bd)^{-1} \\ 0 & 0 & 0 \\ (ez'-dw')(ae-bd)^{-1} & 0 & (aw'-bz')(ae-bd)^{-1} \end{bmatrix} \begin{bmatrix} a & 0 & b \\ 0 & 0 & 0 \\ d & 0 & e \end{bmatrix}$$

$$= \begin{bmatrix} a & 0 & b \\ 0 & 0 & 0 \\ d & 0 & e \end{bmatrix} \begin{bmatrix} (ex'-bz')(ae-bd)^{-1} & 0 & (ey'-bw')(ae-bd)^{-1} \\ 0 & 0 & 0 \\ (az'-dx')(ae-bd)^{-1} & 0 & (aw'-dy')(ae-bd)^{-1} \end{bmatrix},$$

and hence $Q' \subseteq (A)_q$. But

$$(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix})_q = \left\{ \begin{bmatrix} x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \;\middle|\; x \in F \right\}$$

which is properly contained in Q', so $(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix})_q \subsetneq (A)_q$ which is a contradiction.

This completes the proof that rank$(A) = 3$ or 2 is impossible. Hence rank$(A) = 1$. $\square$

Let $k \in \{1, 2, \dots, n\}$. We note that for all A, $B \in \mathrm{R}_n(F, k)$,

$$AB = \begin{bmatrix} 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \\ A_{k1} & A_{k2} & \dots & A_{kn} \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \\ B_{k1} & B_{k2} & \dots & B_{kn} \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 \\ A_{kk}B_{k1} & A_{kk}B_{k2} & \ldots & A_{kk}B_{kn} \\ 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 \end{bmatrix}$$

$$= A_{kk}B.$$

Lemma 4.4. *Let $A \in \mathrm{R}_n(F, k)$.*

1) *If $A_{kk} = 0$, then $(A)_q = ZA$.*
2) *If $A_{kk} \neq 0$, then $(A)_q = FA$.*
3) *If $A_{kk} \neq 0$, then $(A)_q$ is a minimal quasi-ideal of $R_n(F, k)$.*

Proof. 1) Assume that $A_{kk} = 0$. Then $AB = A_{kk}B = 0$ for all $B \in \mathrm{R}_n(F, k)$. Therefore $A\mathrm{R}_n(F, k) = \{0\}$. But $(A)_q = ZA + \left(\mathrm{R}_n(F, k)A \cap A\mathrm{R}_n(F, k)\right)$, so $(A)_q = ZA$.

2) Assume that $A_{kk} \neq 0$. Claim that $(A)_q = FA$. For $a \in F$, we have

$$aA = \begin{bmatrix} 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 \\ a & a & \ldots & a \\ 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 \end{bmatrix} A = A \begin{bmatrix} 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 \\ aA_{k1}A_{kk}^{-1} & aA_{k2}A_{kk}^{-1} & \ldots & aA_{kn}A_{kk}^{-1} \\ 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 \end{bmatrix}$$

which implies that $aA \in \mathrm{R}_n(F, k)A \cap A\mathrm{R}_n(F, k) \subseteq (A)_q$. Hence $FA \subseteq (A)_q$.

Since $(A)_q = ZA + \left(\mathrm{R}_n(F, k)A \cap A\mathrm{R}_n(F, k)\right)$

$$\subseteq ZA + \mathrm{R}_n(F, k)A$$

$$= \{\, mA + BA \mid m \in Z \text{ and } B \in \mathrm{R}_n(F, k) \,\}$$

$$= \{\, mA + B_{kk}A \mid m \in Z \text{ and } B \in \mathrm{R}_n(F, k) \,\}$$

$$= \{\, mA + aA \mid m \in Z \text{ and } a \in F \,\}$$

$= \{ (m + a)A \mid m \in Z \text{ and } a \in F \}$

$= \{ aA \mid a \in F \}$

$= FA$,

we have $(A)_q = FA$.

3) Assume that $A_{kk} \neq 0$. Then $(A)_q = FA \neq \{0\}$. Let $B \in (A)_q$ and $B \neq 0$. Then $B = bA$ for some $b \in F\backslash\{0\}$. Since $A_{kk} \neq 0$ and $b \neq 0$, $B_{kk} \neq 0$. From 2), we get $(B)_q = FB$. Then $(B)_q = F(bA) = (Fb)A = FA = (A)_q$, so $(A)_q$ is a minimal quasi-ideal of $R_n(F, k)$.

Hence the lemma is completely proof. $\square$

Theorem 4.5. *Let char(F) = 0, $k \in \{1, 2, \dots, n\}$ and $A \in R_n(F, k)$. Then $(A)_q$ is a minimal quasi-ideal of $R_n(F, k)$ if and only if $A_{kk} \neq 0$.*

Proof. Assume that $A_{kk} = 0$. By Lemma 4.4(1), then $(A)_q = ZA$. Then by Lemma 3.4, $(A)_q$ is not a minimal quasi-ideal of $R_n(F, k)$. This proves that if $(A)_q$ is a minimal quasi-ideal of $R_n(F; k)$, then $A_{kk} \neq 0$.

Conversely, assume that $A_{kk} \neq 0$. By Lemma 4.4(3), $(A)_q$ is a minimal quasi-ideal of $R_n(F, k)$. $\square$

Theorem 4.6. *Let char(F) = $p > 0$ and $k \in \{1, 2, \dots, n\}$. Then for every $A \in R_n(F, k)\backslash\{0\}$, $(A)_q$ is a minimal quasi-ideal of $R_n(F, k)$.*

Proof. Let $A \in R_n(F, k)$ and $A \neq 0$. If $A_{kk} \neq 0$, then by Lemma 4.4(3), $(A)_q$ is a minimal quasi-ideal of $R_n(F, k)$. If $A_{kk} = 0$, then by Lemma 4.4(1), $(A)_q = ZA$ and hence by Lemma 3.4(2), the quasi-ideal $(A)_q$ is minimal. $\square$

We end this chapter by giving the following remarks.

1) Some minimal quasi-ideals of $C_{2n+1}(F)$ are zero subrings and some are division subrings of $C_{2n+1}(F)$.

2) If char(F) = 0, then every minimal quasi-ideal of $R_n(F, k)$ is a division subring of $R_n(F, k)$.

3) If $\mathrm{char}(F) = p > 0$ and $n > 1$, then $R_n(F, k)$ has both minimal quasi-ideals which are zero subrings of $R_n(F, k)$ and minimal quasi-ideals which are division subrings of $R_n(F, k)$.

To show that (1) holds, let A, $B \in C_{2n+1}(F)$ be defined by

$$A = \begin{bmatrix} 0 & \dots & 0 & 1 \\ 0 & \dots & 0 & 0 \\ \dots & \dots & 0 & 0 \\ 0 & \dots & 0 & 0 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{bmatrix}.$$

Then $\mathrm{rank}(A) = \mathrm{rank}(B) = 1$, so $(A)_q$ and $(B)_q$ are minimal quasi-ideal of $C_{2n+1}(F)$, $(A)_q = FA$ and $(B)_q = FB$. Then

$$(A)_q = \left\{ \begin{bmatrix} 0 & \dots & 0 & x \\ 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 \end{bmatrix} \middle| \; x \in F \right\}$$

which is a zero subring of $C_{2n+1}(F)$ and

$$(B)_q = \left\{ \begin{bmatrix} x & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{bmatrix} \middle| \; x \in F \right\}$$

which is a division subring of $C_{2n+1}(F)$.

To show that (2) holds, let char$(F) = 0$ and let $A \in \mathrm{R}_n(F,\ k)$ be such that the quasi-ideal $(A)_q$ is minimal. Then $A_{kk} \neq 0$ and $(A)_q = FA$, that is,

$$(A)_q = \left\{ \begin{array}{c} \\ \\ \\ k^{\underline{th}} \\ \\ \\ \\ \end{array} \begin{array}{c} \qquad\qquad\quad k^{\underline{th}} \\ \begin{bmatrix} 0 & \dots & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & \dots & 0 \\ xA_{k1} & \dots & xA_{kk} & \dots & xA_{kn} \\ 0 & \dots & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & \dots & 0 \end{bmatrix} \end{array} \;\middle|\; x \in F \right\}.$$

Then the matrix

$$\begin{array}{c} \\ \\ \\ \\ k^{\underline{th}} \\ \\ \\ \\ \end{array} \begin{array}{c} \qquad\qquad k^{\underline{th}} \\ \begin{bmatrix} 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ A_{k1}A_{kk}^{-1} & \dots & A_{k,k-1}A_{kk}^{-1} & 1 & A_{k,k+1}A_{kk}^{-1} & \dots & A_{kn}A_{kk}^{-1} \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{bmatrix} \end{array}$$

is the identity element of $(A)_q$ and for $x \in F\backslash\{0\}$, the matrix

$$\begin{array}{c} \\ \\ \\ \\ k^{\underline{th}} \\ \\ \\ \\ \end{array} \begin{array}{c} \qquad\qquad k^{\underline{th}} \\ \begin{bmatrix} 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ x^{-1}(A_{kk}^{-1})^2 A_{k1} & \dots & x^{-1}(A_{kk}^{-1})^2 A_{k,k-1} & x^{-1}A_{kk}^{-1} & x^{-1}(A_{kk}^{-1})^2 A_{k,k+1} & \dots & x^{-1}(A_{kk}^{-1})^2 A_{kn} \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{bmatrix} \end{array}$$

is the inverse of

$$\begin{array}{cc} & \begin{matrix} \phantom{xA_{k1}} & & k^{th} & & \phantom{xA_{kn}} \end{matrix} \\ \begin{matrix} \\ \\ \\ k^{th} \\ \\ \\ \\ \end{matrix} & \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & \cdots & 0 \\ xA_{k1} & \cdots & xA_{kk} & \cdots & xA_{kn} \\ 0 & \cdots & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix} \end{array}.$$

Thus $(A)_q$ is a division subring of $R_n(F, k)$.

Finally, to show that (3) holds, let $\operatorname{char}(F) = p > 0$, $n > 1$ and let $A, B \in R_n(F, k)$ be defined by

$$\begin{array}{ccc} & \begin{matrix} & & k^{th} & & \end{matrix} & \\ A = & \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & \cdots & 0 \\ 1 & \cdots & 1 & \cdots & 1 \\ 0 & \cdots & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix} & \begin{matrix} \\ \\ \\ k^{th} \\ \\ \\ \\ \end{matrix} \end{array}$$

and

$$\begin{array}{ccc} & \begin{matrix} & & & k^{th} & & & \end{matrix} & \\ B = & \begin{bmatrix} 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 1 & \cdots & 1 & 0 & 1 & \cdots & 1 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} & \begin{matrix} \\ \\ \\ k^{th}. \\ \\ \\ \\ \end{matrix} \end{array}$$

Then $(A)_q$ and $(B)_q$ are both minimal quasi-ideals of $\mathrm{R}_n(F, k)$, $(A)_q = FA$ and $(B)_q = ZB$. As shown in (2), $(A)_q$ is a division subring of $\mathrm{R}_n(F, k)$. Since $B^2 = 0$, it follows that $(B)_q$ is a zero subring of $\mathrm{R}_n(F, k)$.

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