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A STRING-THEORY CALCULATION OF HAWKING RADIATION

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The study of black hole thermodynamics shows that black holes can emit particles due to quantum mechanical effect on the curved spacetime. String theory, which is the promising theory of quantum gravity, should provide a description of the radiation. In this dissertation, we calculate a decay rate of the excited D1D5-brane model and compare with a emission rate of a corresponding nonextremal charged black hole. The comparison shows that both results are exactly the same in low energy approximation.

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CHAPTER I

Introduction

In this chapter, we give an introduction of background materials used in this dissertation. We start with the idea and the motivation for quantum gravity. We then move to the topic of black hole, string theory and D-brane. The aim of this dissertation is to review the works on D1D5 brane decay and Hawking radiation. The calculation and the discussion are mainly follow Sumit R. Das and Samir D. Mathur papers [1] and Akikazu Hashimoto and Igor R. Klebanov papers [2]. The basic of string theory and conformal field theory are reviewed from [3, 4].

1.1 Quantum Gravity and the Obstacles

One of the main goals of theoretical physics is to explain all phenomena in nature, not only to give a precise prediction but also to describe their underline mechanism. What we have done since before is to observe the nature and create mathematical models of it. For a brief example, thermodynamics began with observation, then take the relation of the parameters in mathematics. Later we understand it by the concept and the models in kinetic theory of gas and statistical physics. The modern physics, which aims to study the structure of the fundamental nature, has a slightly different. We make mathematical models based on some principle like the conservation laws, then see if it agrees with the experiments/observation.

However, not all of our attempt to construct mathematical models for every phenomena goes well. There are a lot of obstacles, especially gravitational force, the most common force of four fundamental forces yet we understand the least.

It is necessary to make a quantum mechanical description of a gravity because we cannot consistently couple a classical system to a quantum system. Richard Feynman once tried to recreate the general relativity, the classical theory of gravity, from a non geometrical viewpoint. To reconstruct it in the sense of a field theory, one has to find the spin of the graviton in order to find a suitable representation. First, the Newton's inverse-square law force tells us that the gravitational potential is an inverse variation of the distance from the source. This kind of potential is considered as a long-range interaction. Thus the graviton is a massless particle. Its behaviour as a static force make us able to rule out the possibility of half-integer spin. Also the odd-integer spins, like photons, leads to forces that are both attraction and repulsion depends on charges. We then assume the spin of graviton to be an even-integer spins. First to consider, the spin-0 is not coupled with the spin-1 bosons. Since we know that the gravity couples with the energy content, photon can be affected by gravity. Hence graviton is not a spin-0 particle. This leaves us the next possibility of graviton being a spin-2 particle. Nothing has been found to eliminate this possibility. This goes for the rule ”if it works, don't fix it”. Thus we represent the graviton with a rank-2 tensor field for the spin-2 particle.

There are no less than 16 major approaches in the field of quantum gravity [5]:

1. Canonical quantum gravity
2. Manifestly covariant quantization
3. Euclidean quantum gravity
4. R-squared gravity
5. Supergravity
6. String and brane theory
7. Renormalization group and Weinberg's asymptotic safety

8. Non-commutative geometry
9. Twistor theory
10. Asymptotic quantization
11. Lattice formulation
12. Loop space representation
13. Quantum topology, motivated by Wheeler's quantum geometrodynamics
14. Simplicial quantum gravity and null-strut calculus
15. Condensed-matter view: the universe in a helium droplet
16. Affine quantum gravity.

The first eight approaches are based on the Lagrangian/Hamiltonian framework which use the action principle. String theory is the only one among the first eight approaches that is not a field theory and the spacetime is replaced by extended structures. The later eight approaches use different mathematical structures of conventional pictures.

However, the attempt to quantize the theory of gravity has some problems. We often encounter with the divergence or some inconsistency of quantum gravity theories. For example, field quantization in quantum field theory leaves modes of field that possesses a zero-point energy. The field, however, is made up of an infinite number of modes. The vacuum energy is then infinite and also the gravity which is coupled with energy. In the classical theory of gravity, cosmological constant which is corresponding to the vacuum energy is small. This is a problem. Also in the viewpoint of field theory with the unit $\hbar = c = 1$, the gravitational coupling constant has a unit of $energy^{-2}$. Theories with positive dimensions coupling constant usually turn out to be finite, while theories with dimensionless coupling constant are candidates to be renormalizable. Theories that have negative dimensions, usually have divergences and they are not renormalizable. Quantum general relativity falls into the last category.

1.2 Why Black Holes?

Black holes are one of the most mysterious objects in the universe. A lot of sci-fi novels/films mention about their frightful ability of destruction. However, not only in fiction, but black holes are also interesting topics of studies in physics, especially in particle physics and general relativity.

Black holes were first introduced by John Michell in 1783 as ”dark star”, the bodies so massive that even light cannot escape. This idea, however, makes them still mysterious and difficult to understand. In 1915, Albert Einstein developed theory of general relativity. This theory describes the motion of objects (including light motion) caused by gravity through spacetime curvature. A few weeks later, a german physicist Karl Schwarzschild found a solution to the Einstein’s equation that predicted black hole.

Black holes have often provided insights into various fields of physics. For example, the high densities following the big bang may have formed primordial holes which their singularity similarly to a black hole. Study of black hole may lead us to understand some new physics.

Classical black holes have a well-known property that anything cannot escape from them. In 1972, Jacob Bekenstein suggested that black holes should have a finite temperature and entropy. This means black holes should have a thermal radiation. One year later, James Bardeen, Brandon Carter and Stephen Hawking discovered laws of black hole mechanics which are analogous to the laws of thermodynamics [6]. These analogies turned into identities once Hawking radiation was discovered in 1974.

The black hole radiation, however, leads us into a new problem. It implies that black holes emit its energy to the outside and evaporate. This leads us to the information paradox problem: information fallen into black hole could be permanently lose. To make a brief picture, let consider a quantum state $|\Phi>_{init}$ which contains a quantum structure of the mass falling into a black hole by a gravitational collapse. Normally, we cannot claim that the information could escape once it passed the event horizon of the black hole since it is necessary to exceed the speed of light. The black hole, then slowly radiate away its energy until it evaporate. However, the radiation does not depend on the internal structure, but its temperature which depends on its mass, charge and angular momentum by the no-hair theorem. Such a thermal state radiated outside the black hole is a mixed state which is described quantum mechanically by a density matrix, not the wave function. Since $|\Phi >_{init}$ is a pure quantum state, the transformation between pure state and mixed state throws away information. To be more exact, let the state $|\Phi >_{init}$ described by a set of eigenvalues and coefficients. The transformation leaves only one number, the temperature. There is no way to recover the information loss in the process. This causes a breakdown in quantum mechanics.

The information paradox problem becomes an importance issue in quantum gravity. We hope it will solve the problem by giving the black hole internal structure instead of the classical picture which the object falling into it will hit the central singularity in a finite time. The question is how we conjecture the picture of the internal structure or the microstate. Recently, there is a lot of attempts to answer this question, such as a black hole microstate counting [5, 7–10].

1.3 A Brief Introduction to String Theory

We now introduce string theory, a candidate for a quantum theory of gravity. It has beautiful descriptions of the previous ideas, for examples, standard model and general relativity. There are some elegant features in string theory [3]:

1. String theory always contains a massless spin-2 state, whose interactions reduce to general relativity at low energy. This makes it a consistent theory of quantum gravity, at least in the perturbation theory.

2. The gauge groups in string theory are large enough to include the standard model. Thus, one could say that string theory is the unification of the

standard model.

3. Consistent string theory also requires spacetime supersymmetries, as either a manifest or a spontaneously broken symmetries.
4. String theory allows a chiral symmetry gauge or a parity asymmetry gauge which is required for a gauge interaction in nature.
5. There is no free parameter in string theory.
6. String theory also has no discrete freedom analogous to the choice of gauge groups and representations in field theory. This makes it a unique theory.

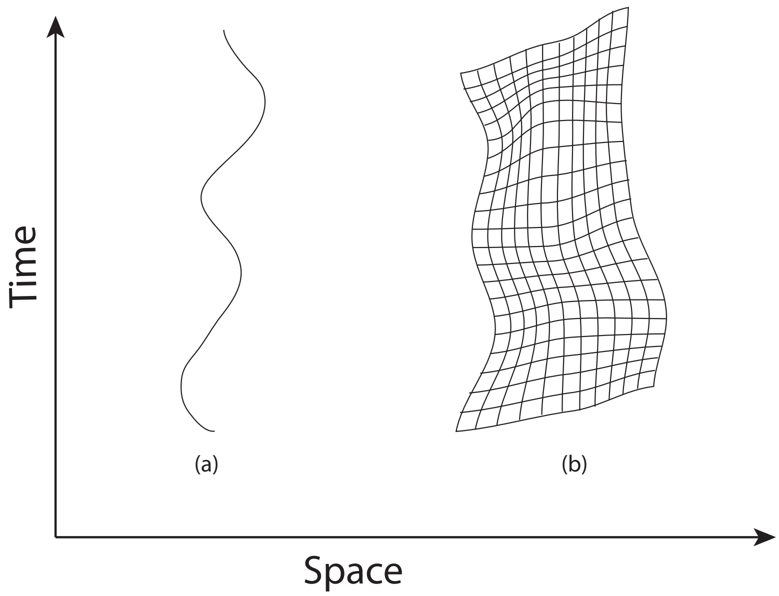


Figure 1.1: A diagram of point-particle worldline (a) and a string worldsheet (b) propagate through the spacetime.

In string theory, point-like particles in the standard particle physics are replaced with 1-dimensional objects or string-like particles. The standard way we study particle physics is the action principle which is proportional to the length of the worldline, the curve of the particle through spacetime. The length of the worldline is also known as the particle's proper time. Similarly, the string action is proportional to the area of the string worldsheet, which is the surface it sweep through spacetime. The action describing the worldsheet is the Numbu-Goto action,

$$S_{NG} = -T(\text{area of worldsheet}) = -T \int d\tau d\sigma \sqrt{-det(\partial_a X^\mu \partial_b X_\mu)} \qquad (1.3.1)$$

where $X^\mu(\tau, \sigma)$ is the classical motion of the string through spacetime. The worldsheet is parameterised with τ and σ, hence a and b are either τ or σ. The constant T is the string tension, which is proportional to the Regge slope α',

$$T = \frac{1}{2\pi\alpha'}. \qquad (1.3.2)$$

The Numbo-Goto action, however, has a square root which is not the appropriate form for studying. One can make the action look more convenient by writing it in the form of the *Polyakov action*,

$$S_P = -\frac{1}{4\pi\alpha'} \int d\tau d\sigma (-\gamma)^{1/2} \gamma^{ab} \partial_a X^\mu \partial_b X_\mu, \qquad (1.3.3)$$

Note that the two-dimensional metric γ_{ab} is an auxiliary variable. We can eliminate γ_{ab} in (1.3.3) by its equation of motion and this restore the Nambu-Goto action (1.3.1).

The worldsheet with boundary also has a surface term in the variation of the action,

$$\delta S_P = \frac{1}{2\pi\alpha'} \int_{-\infty}^{\infty} d\tau \int_0^\ell d\sigma (-\gamma)^{1/2} \delta X^\mu \nabla^2 X_\mu - \frac{1}{2\pi\alpha'} \int_{-\infty}^{\infty} d\tau (-\gamma)^{1/2} \delta X^\mu \, \partial^\sigma X_\mu|_{\sigma=0}^{\sigma=\ell}, \qquad (1.3.4)$$

where we think of τ as a time parameter and σ as spatial, with the coordinate region $-\infty < \tau < \infty$, $0 \leq \sigma \leq \ell$. The boundary term in (1.3.4) vanishes if

$$\partial^\sigma X^\mu(\tau, 0) = \partial^\sigma X^\mu(\tau, \ell) = 0, \qquad (1.3.5)$$

which is the *Neumann boundary condition* on X^μ. This implies that the ends of the open string move freely in spacetime. The surface term also vanishes if we impose periodic boundary conditions

$$X^\mu(\tau, \ell) = X^\mu(\tau, 0), \qquad (1.3.6a)$$

$$\partial^\sigma X^\mu(\tau, \ell) = \partial^\sigma X^\mu(\tau, 0), \qquad (1.3.6b)$$

$$\gamma_{ab}(\tau, \ell) = \gamma_{ab}(\tau, 0). \qquad (1.3.6c)$$

This is the conditions that the string endpoints joined to form a closed loop: *a closed string*.

By using light-cone gauge to eliminate the diff$\times$Weyl redundancy, the Hillbert space of the worldsheet naturally breaks into left-moving and right moving modes, called *holomorphic* and *antiholomorphic* respectively. We can specify the boundary conditions on each separately. An open string is either left-moving or right-moving waves along the string, while a closed string has both.

In superstring theory, when we add a fermions to theory, there are two possible periodic conditions: antiperiodic, called *Neveu-Schwarz* (NS), boundary condition; and periodic, called *Ramond* (R), boundary condition. Since there are two choices for the left-mover and two choices for the right-mover, thus there are four different ways to define a closed superstring; denote by NS-NS, NS-R, R-NS and R-R. The NS-R and R-NS sectors give fermionic degrees of freedom, while the NS-NS and R-R sectors give bosonic degrees of freedom. The massless NS-NS sector of closed string is the same as the previous closed bosonic string we discussed above. The R-R sector gives rise to antisymmetric gauge fields $C^{(n)}$. The superscript index indicates the kind of antisymmetric tensor form of the gauge field. For instant, $C^{(0)}$ is a Lorentz scalar and $C^{(1)}$ is a one-form gauge field.

1.4 D-branes

In string theory, there are also other dynamical objects, called *branes*, which can propagate through spacetime. The Dirichlet-brane, or D-brane for short, is an important class when we consider open strings. As an open string moves through spacetime, Dirichlet boundary conditions imply that its endpoints are required to lie on some other objects or else it breaks Poincare symmetry invariance. Supergravity also has solitonic solutions called Dp-branes, which are $(p+1)$-dimensional extended objects.

D-branes are the sources of Ramond-Ramond gauge fields and carry their charges. In some situations that D-branes are considered as hyperplanes, their charges are related to electric and magnetic charges as Noether charges. The summary of D-branes and their coupled gauge fields are shown in the table 1.4.

	IIB	IIA
NS-NS	g_{MN}, ϕ, B_{MN}	g_{MN}, ϕ, B_{MN}
R-R	$C^{(0)}, C^{(2)}, C^{(4)+}$	$C^{(1)}, C^{(3)}$

IIB	IIA
$D(-1) \to C^{(0)}$	$D0 \to C^{(1)}$
$D(1) \to C^{(2)}$	$D2 \to C^{(3)}$
$D(3) \to C^{(4)+}$	$D4 \to C^{(3)}$ [M]
$D(5) \to C^{(2)}$ [M]	$D6 \to C^{(1)}$ [M]
$D(7) \to C^{(0)}$ [M]	$D8$
$D(9)$	

Table 1.1: The left table shows the bosonic field content of types IIA and IIB string theories in the low-energy limit. The right table is a summary of D-brane content and its coupled Ramond gauge field. Note that the symbol [M] indicates a magnetic type coupling and the plus sign indicates that the gauge field strength is constrained to be self-dual.

One of the important properties of D-branes are Bogomol'nyi-Prasad-Sommerfield (BPS) states. This is a lower bound on the mass with respect to the central charges by the supersymmetry algebra. The states are BPS states when they break half the supersymmetry, have no force between each BPS states, and their mass saturates the bound. Thus, this also the properties of D-branes.

Recently, D-branes are used as black hole models in string theory. With suitable configuration, we can calculate entropy by counting its microstate. Interestingly, the results coincide with Hawking entropies of the corresponding black holes.

it gives an entropy, which is equal to the Hawking entropy of the corresponding black hole. This gives us the idea of the degeneracy of string states corresponding to the black hole.

1.5 Outline

This dissertation focuses on calculating the decay rate of the D1D5 system to understand the radiation property of black holes. In chapter 2, we discuss a semi-classical Hawking radiation using quantum field theory in curved spacetime in detail. The spacetime geometry is taken into the account in the framework of quantum field theory in curved spacetime. This framework leads us to a number density of particles created by Black hole which is match a Bose-Einstein statistic $\frac{1}{e^{\frac{2\pi\omega}{\kappa}}-1}$. This restores the second law of black hole mechanics, the area law, which gives the black hole temperature in a term of surface gravity κ, $T = \frac{\kappa}{2\pi}$. In this chapter, we also discuss examples of the Unruh effect, $1+1$-dimensional Schwarzschild black hole. Finally, we calculate the radiation rate of the 5-dimensional Reissner-Nordstrom black hole from supergravity solution which is corresponded with $D1D5$-brane model. Chapter 3, we describe the conformal field theory which plays a central role in string theory. This chapter mainly focuses on how we defined the vertex operators we use to compute an amplitude of D1D5 brane in chapter 4. We then discuss on a D-brane thermodynamics and compute the decay rate of the system which is exactly the same as the result from chapter 2. Finally, in chapter 5, we present some concluding remarks and some recent attempts related to this dissertation.

CHAPTER II

Black Hole Classical Absorption Rate of Low Energy Scalars

In this chapter we discuss a quantum field theory in curved spacetime using the standard quantum field theory of two observers. The observers in different frames might give the different measure. We give examples of Unruh effect and Schwarzchild radiation. We then use the result to calculate a 5-dimensional Reissner-Nordstrom black hole radiation which gives a final result of energy emission rate. This chapter is reviewed from [1, 11–13]

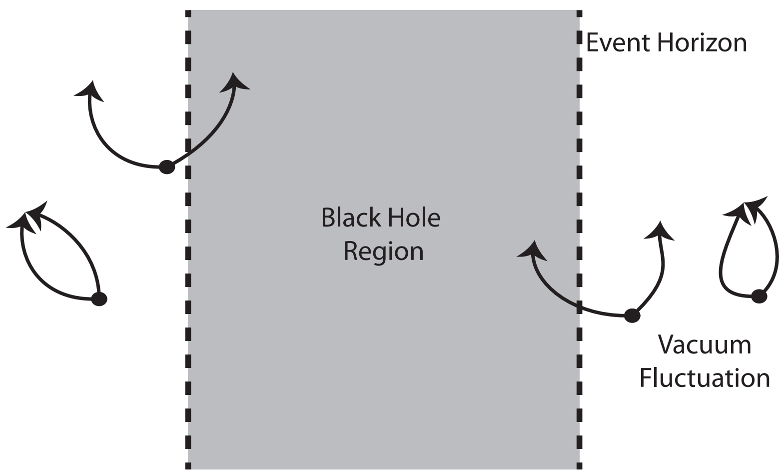


Figure 2.1: The idea to begin with the black hole radiation is about a vacuum fluctuation and a black hole horizon. One might think as a pair of particle fluctuate near the black hole horizon, one is trapped and another is escaping to the infinity.

2.1 Quantum Field Theory in Curved Spacetime

First to start, we need to discuss the basic of quantum field theory in curved spacetime or a semiclassical method to the black hole radiation. The basic idea of semiclassical principle is to treat the matter fields quantum mechanically and the gravity as a background. It means we can consider the mechanism from the standard quantum field theory with generalized metric as it contains the effect of gravity. For our analysis, we will focus on massless scalar field ϕ that satisfy the wave equation,

$$\Box\phi = g^{ab}\nabla_a\nabla_b\phi = 0, \tag{2.1.1}$$

where g_{ab} is the metric from line element $ds^2 = g_{ab}dx^a dx^b$. Note that we work in units with $G = c = \hbar = 1$ as same as the most papers. We treat a scalar field as a quantum operator, by a standard method, it must obey the canonical equal time commutation relations,

$$[\phi(\mathbf{x}, t), \dot{\phi}(\mathbf{x}', t)] = \delta(\mathbf{x} - \mathbf{x}'). \tag{2.1.2}$$

The field can be expressed in a form of mode expansion,

$$\phi = \int d\omega \left(a_\omega f_\omega + a_\omega^\dagger f_\omega^*\right), \tag{2.1.3}$$

where a set $\{f_\omega, f_\omega^*\}$ is a complete orthonormal set of basis functions. The standard choice of basis functions for a scalar field is a plane solution,

$$f_\omega = \frac{1}{\sqrt{2\omega}} e^{-\imath(\omega t - \mathbf{k}\cdot\mathbf{x})}, \tag{2.1.4}$$

where $\omega = +\sqrt{\mathbf{k}\cdot\mathbf{k}}$. The canonical commutation relations for the scalar field imply commutation relations for mode operators a_ω, $a_\omega^\dagger$,

$$[a_\omega, a_{\omega'}^\dagger] = \delta(\omega - \omega'), \quad [a_\omega, a_{\omega'}] = 0 = [a_\omega^\dagger, a_{\omega'}^\dagger]. \tag{2.1.5}$$

Usually, the vacuum state is defined as the state with the lowest possible energy state. On the other hand, it is a state in the absence of particles which is annihilated by all the annihilation operators a_ω,

$$a_\omega \left|0\right\rangle_a = 0 \tag{2.1.6}$$

for all $\omega > 0$. The Fock space of state is constructed by applying creation operators to a vacuum state, for an instant, the state $(a_\omega^\dagger)^n \left|0\right\rangle_a$ contains n particles with energy ω. We can measure the particles in the state ω by defining a number operator $N_\omega = a_\omega^\dagger a_\omega$ for each mode, so that ${}_a\left\langle 0\right| (a_\omega)^n N_\omega (a_\omega^\dagger)^n \left|0\right\rangle_a = n$.

However, measurements do depend on the observers. Any two observers may not observe a coincide result. In order to discuss about this idea, let consider another observer or a different frame. One can define a second expansion on another complete set of basis $\{p_{\omega'}, p_{\omega'}^*\}$,

$$\phi = \int d\omega' \left(b_{\omega'} p_{\omega'} + b_{\omega'}^\dagger p_{\omega'}^* \right), \tag{2.1.7}$$

where mode coefficients $b_{\omega'}$, $b_{\omega'}^\dagger$ also satisfy commutation relations. The annihilation operators $b_{\omega'}$ define another vacuum state, $b_{\omega'} \left|0\right\rangle_b = 0$, for all $\omega' > 0$.The number operator for mode in b-state is $N_{\omega'} = b_{\omega'}^\dagger b_{\omega'}$. The creation operators $b_{\omega'}$ also span another Fock space by applying to $\left|0\right\rangle_b$.

We have seen that the scalar field could be expanded with different sets of basis function. However, since these two observers measure the same system, so there must exist some relation between their measurement. One may have the basis functions f_ω and $p_{\omega'}$ are related to each other through the linear transformation, called *Bogoliubov transformation*,

$$\begin{aligned} p_{\omega'} &= \int d\omega (\alpha_{\omega'\omega} f_\omega + \beta_{\omega'\omega} f_\omega^*) \\ f_\omega &= \int d\omega' (\alpha_{\omega\omega'}^* p_{\omega'} - \beta_{\omega\omega'} p_{\omega'}^*). \end{aligned} \tag{2.1.8}$$

In order to satisfy the orthonormality of the basis functions, Bogoliubov coefficients $\alpha_{\omega\omega'}$, $\beta_{\omega\omega'}$ have to obey

$$\int d\omega'' (|\alpha_{\omega\omega''}|^2 - |\beta_{\omega'\omega''}|^2) = \delta(\omega - \omega'). \tag{2.1.9}$$

Those above leads us to the relation between mode coefficients,

$$b_{\omega'} = \int d\omega(\alpha^*_{\omega'\omega}a_\omega - \beta^*_{\omega'\omega}a^\dagger_\omega). \tag{2.1.10}$$

Now what we want to know is what one observer will see another one state. For an instant, the Unruh effect, one in the accelerating frame could see a bath of particle which is nothing other than a fluctuation in the rest frame. We will discuss about the Unruh effect in the following section. Thus we can evaluate the expression for the measure $N_{\omega'}$ in the a-vacuum state,

$$_a\langle 0|\, N_{\omega'}\, |0\rangle_a =_a \langle 0|\, b^\dagger_{\omega'} b_{\omega'}\, |0\rangle_a = \int d\omega \, |\beta_{\omega'\omega}|^2 \,. \tag{2.1.11}$$

We can easily see in the case that those two are the inertial frames. The coefficient $\beta_{\omega\omega'}$ is equal to zero due to the fact that f_ω and $p_{\omega'}$ should be the same set of basis functions. Therefore each observer sees no particle no matter which state they observe. Generally, however, $|\beta_{\omega\omega'}|^2$ could have a nonzero value which means one might see a bath of particle in another vacuum state. We will discuss how different reference frames give a presence of particle as we study the Unruh effect mechanism.

2.1.1 The Unruh Effect

The Unruh effect is a phenomenon that the accelerating observer will observe a radiation where an inertial observer would not. First, we consider an observer with uniformly accelerated motion. Let the trajectory of the observer be described by a worldline $x^\mu(\tau)$ where is a function of a proper time τ. We can parameterize the proper time to give a condition for its 4-velocity

$$u^\mu u_\mu = 1; \;\; u^\mu \equiv \frac{dx^\mu}{d\tau}. \tag{2.1.12}$$

The 4-acceleration is then defined as

$$a^\mu \equiv \frac{du^\mu}{d\tau}. \tag{2.1.13}$$

One can find that the 4-acceleration is related to 3-acceleration $\mathbf{a}$ via

$$a^\mu a_\mu = -|\mathbf{a}|^2 \tag{2.1.14}$$

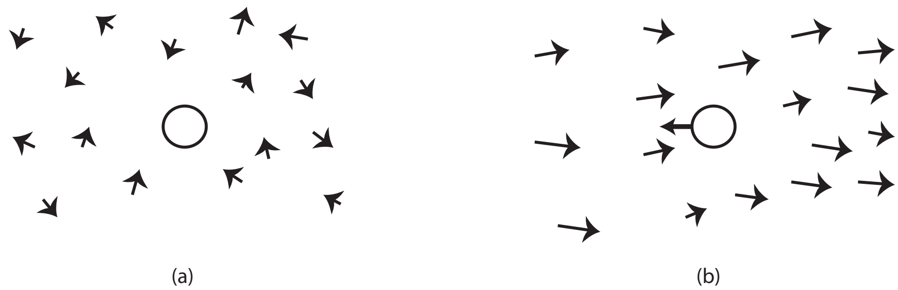


Figure 2.2: Figure shows a concept of the Unruh effect. The rest observer detects no particles due to the fluctuation give the expectation value nothing (a). While the accelerated observer sees every particle accelerated and have higher energy, the observer experiences the presence of the particle from the increased energy (b).

Assume that the acceleration is parallel to x-axis, $\mathbf{a} = (a, 0, 0)$ with $a > 0$, if we give a condition that there is a zero velocity at $\tau = 0$ then we got that

$$x(\tau) = x_0 - \frac{1}{a} + \frac{1}{a}\cosh[a\tau], \quad t(\tau) = t_0 + \frac{1}{a}\sinh[a\tau]. \tag{2.1.15}$$

This trajectory has a simple form if we give an initial conditions $x(0) = a^{-1}, t(0) = 0$, it is a worldline of hyperbola $x^2 - t^2 = a^{-2}$.

However, those are a time and distance measured by the observer at rest. To complete a quantum field theory of the accelerated observer, we have to use the *proper coordinates* (τ, ξ), where the *proper time* τ and the *proper distance* ξ are quantities measured by the accelerated observer. The previous trajectory

$t(\tau), x(\tau)$ should correspond to the line $\xi = 0$. To see the relation between rest frame coordinates and comoving coordinates, let the observer hold a rigid stick of length ξ_0. The stick accelerates together with the observer from $(0, \xi_0$ to (τ, ξ_0) at time τ. Let the stick in the comoving frame represented by the 4-vector $s^{\mu}_{(com)}$ and $s^{\mu}_{(lab)}$ for laboratory frame. At the time τ we can consider the comoving frame as an inertial system with velocity $u^{\mu}(\tau) = dx^{\mu}/d\tau$. Therefore the coordinates s^{μ}_{com} and s^{μ}_{lab} are related by the Lorentz transformation,

$$\begin{bmatrix} s^0_{lab} \\ s^1_{lab} \end{bmatrix} = \frac{1}{\sqrt{1-v^2}} \begin{pmatrix} 1 & v \\ v & 1 \end{pmatrix} \begin{bmatrix} s^0_{com} \\ s^1_{com} \end{bmatrix} = \begin{pmatrix} u^0 & u^1 \\ u^1 & u^0 \end{pmatrix} \begin{bmatrix} s^0_{com} \\ s^1_{com} \end{bmatrix} = \begin{bmatrix} u^1 \xi \\ u^0 \xi \end{bmatrix} \tag{2.1.16}$$

where $v \equiv u^1/u^0$ is the velocity of the stick in the laboratory frame. For the stick attached by the observer moving along $x(\tau)$, the coordinates of the end of the stick are then

$$t(\tau, \xi) = x^0(\tau) + s^0_{lab} = x^0 + \frac{dx^0(\tau)}{d\tau}\xi \tag{2.1.17a}$$

$$x(\tau, \xi) = x^1(\tau) + s^1_{lab} = x^1 + \frac{dx^1(\tau)}{d\tau}\xi. \tag{2.1.17b}$$

Thus, together with (2.1.15) and the initial conditions $x(0) = a^{-1}, t(0) = 0$, the relation between laboratory coordinates and proper coordinates is then

$$t(\tau, \xi) = \frac{1 + a\xi}{a} \sinh[a\tau] \tag{2.1.18a}$$

$$x(\tau, \xi) = \frac{1 + a\xi}{a} \cosh[a\tau] \tag{2.1.18b}$$

and its converse

$$\tau(t, x) = \frac{1}{2a} \ln \frac{x + t}{x - t} \tag{2.1.19a}$$

$$\xi(t, x) = -a^{-1} + \sqrt{x^2 - t^2}. \tag{2.1.19b}$$

The metric in the proper coordinates is

$$ds^2 = dt^2 - dx^2 = (1 + a\xi)^2 d\tau^2 - d\xi^2. \tag{2.1.20}$$

The metric of this form is called *Rindler spacetime.* This can be expressed as a conformal flat metric by replacing the proper coordinate $\tilde{\xi} \equiv \frac{1}{a}\ln(1+a\xi)$, the relation between the laboratory coordinates and the conformal coordinate is then

$$t(\tau,\tilde{\xi}) = a^{-1}e^{a\tilde{\xi}}\sinh[a\tau], \;\; x(\tau,\tilde{\xi}) = a^{-1}e^{a\tilde{\xi}}\cosh[a\tau]. \tag{2.1.21}$$

The metric becomes

$$ds^2 = e^{2a\tilde{\xi}}(d\tau^2 - d\tilde{\xi}^2). \tag{2.1.22}$$

As we discussed in the previous section, the wave equation (2.1.1) is a solution of the action

$$S[\phi] = \frac{1}{2}\int d^2x\sqrt{-g}g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta}. \tag{2.1.23}$$

This action manifests a conformally invariant, by replacing $g_{\alpha\beta} \to \tilde{g}_{\alpha\beta}\Omega^2(t,x)g_{\alpha\beta}$. For our consideration, the actions in laboratory coordinates and conformal coordinates are

$$S[\phi] = \frac{1}{2}\int dtdx[(\partial_t\phi)^2 - (\partial_x\phi)^2] \tag{2.1.24a}$$
$$S[\phi] = \frac{1}{2}\int d\tau d\tilde{\xi}[(\partial_\tau\phi)^2 - (\partial_{\tilde{\xi}}\phi)^2] \tag{2.1.24b}$$

respectively. The solutions to their equations of motion are clearly divided in modes of lightcone coordinates

$$\phi = A(\bar{u}) + B(\bar{v}) = P(u) + Q(v) \tag{2.1.25}$$

where

$$\bar{u} \equiv t - x, \;\; \bar{v} \equiv t + x, \;\; u \equiv \tau - \tilde{\xi}, \;\; v \equiv \tau + \tilde{\xi}. \tag{2.1.26}$$

We now choose the plane wave solution for the scalar field which can be expressed as a $1+1$ dimensional Fourier transformation, in the laboratory coordinates,

$$\begin{aligned}\phi &= \int_{-\infty}^{+\infty} \frac{dk}{(2\pi)^{1/2}} \frac{1}{\sqrt{2|k|}} [e^{-i|k|t+ikx} a_k + e^{i|k|t-ikx} a_k^\dagger] \\ &= \int_{0}^{+\infty} \frac{d\omega}{(2\pi)^{1/2}} \frac{1}{\sqrt{2\omega}} [e^{-i\omega\bar{u}} a_\omega + e^{i\omega\bar{u}} a_\omega^\dagger + e^{-i\omega\bar{v}} a_{-\omega} + e^{i\omega\bar{v}} a_{-\omega}^\dagger] \\ &= A(\bar{u}) + B(\bar{v}) \end{aligned} \tag{2.1.27}$$

where $\omega = |k|$. The same goes for the comoving coordinates,

$$\begin{aligned}\phi &= \int_{0}^{+\infty} \frac{d\Omega}{(2\pi)^{1/2}} \frac{1}{\sqrt{2\Omega}} [e^{-i\Omega u} b_\Omega + e^{i\Omega u} b_\Omega^\dagger + e^{-i\Omega v} b_{-\Omega} + e^{i\Omega v} b_{-\Omega}^\dagger] \\ &= P(u) + Q(v). \end{aligned} \tag{2.1.28}$$

Note that it is not necessary that $\omega = \Omega$ since those coordinates are not the same basis set.

To find out the Bogoliubov coefficients, consider the field expression

$$\phi(u,v) = A(\bar{u}(u)) + B(\bar{v}(v)) = P(u) + Q(v) \tag{2.1.29}$$

so that

$$A(\bar{u}(u)) = P(u), \;\; B(\bar{v}(v)) = Q(v). \tag{2.1.30}$$

We do a Fourier transform on both sides to see the relations, let us consider for $P(u) = A(\bar{u}(u))$,

$$\begin{aligned}\int_{-\infty}^{+\infty} \frac{du}{\sqrt{2\pi}} e^{i\Omega u} P(u) &= \int_{-\infty}^{+\infty} \frac{du}{2\pi} \int_{0}^{+\infty} \frac{d\Omega'}{\sqrt{2\Omega'}} [e^{-i(\Omega'-\Omega)u} b_{\Omega'} + e^{i(\Omega'+\Omega)u} b_{\Omega'}^\dagger] \\ &= \frac{1}{\sqrt{2|\Omega|}} \begin{cases} b_\Omega & ;\Omega > 0 \\ b_{|\Omega|}^\dagger & ;\Omega < 0 \end{cases}, \end{aligned} \tag{2.1.31}$$

and the RHS

$$\begin{aligned}\int_{-\infty}^{+\infty} \frac{du}{\sqrt{2\pi}} e^{i\Omega u} A(\bar{u}) &= \int_{-\infty}^{+\infty} \frac{du}{2\pi} \int_{0}^{+\infty} \frac{d\omega}{\sqrt{2\omega}} [e^{i\Omega u - i\omega\bar{u}} a_\omega + e^{i\Omega u + i\omega\bar{u}} a_\omega^\dagger] \\ &= \int_{0}^{\infty} \frac{d\omega}{\sqrt{2\omega}} [F(\omega,\Omega) a_\omega + F(-\omega,\Omega) a_\omega^\dagger] \end{aligned} \tag{2.1.32}$$

where the function

$$F(\omega,\Omega) \equiv \int_{-\infty}^{+\infty} \frac{du}{2\pi} e^{i\Omega u - i\omega\bar{u}} = \int_{-\infty}^{+\infty} \frac{du}{2\pi} \exp[i\Omega u + i\frac{\omega}{a}e^{-au}]. \tag{2.1.33}$$

It is manifest that the function $F(\omega,\Omega)$ has identities

$$F^*(\omega,\Omega) = F(-\omega,-\Omega) \tag{2.1.34a}$$

$$F(\omega,\Omega) = F(-\omega,\Omega)\exp\left(\frac{\pi\Omega}{a}\right), \quad ;\omega > 0, a > 0. \tag{2.1.34b}$$

By comparing this to (2.1.10), we got the coefficients,

$$\alpha_{\omega\Omega} = \sqrt{\frac{\Omega}{\omega}} F(\omega,\Omega), \;\; \beta_{\omega\Omega} = \sqrt{\frac{\Omega}{\omega}} F(-\omega,\Omega) \tag{2.1.35}$$

for $\omega > 0, \Omega > 0$.

Let us see for the number of particles (2.1.11) the accelerated observer see, to be exact, the number of b-particles in the a-vacuum state. Starting from the identity (2.1.9),

$$\begin{aligned}
\delta(\Omega-\Omega') &= \int_0^{\infty} d\omega(\alpha_{\omega\Omega}\alpha^*_{\omega\Omega'} - \beta_{\omega\Omega}\beta^*_{\omega\Omega'}) \\
&= \int_0^{\infty} \frac{\sqrt{\Omega\Omega'}}{\omega} \left(F(\omega,\Omega)F^*(\omega,\Omega') - F(-\omega,\Omega)F^*(-\omega,\Omega')\right) \\
&= \left(exp\left(\frac{\pi\Omega+\pi\Omega'}{a}\right) - 1\right) \int_0^{\infty} d\omega \frac{\sqrt{\Omega\Omega'}}{\omega} F^*(-\omega,\Omega')F(-\omega,\Omega)
\end{aligned} \tag{2.1.36}$$

so that

$$\int_0^{\infty} d\omega \frac{\sqrt{\Omega\Omega'}}{\omega} F(-\omega,\Omega)F^*(-\omega,\Omega') = \frac{1}{e^{2\pi\Omega/a}-1}\delta(\Omega-\Omega'). \tag{2.1.37}$$

Then this gives us a result of particle number,

$$\begin{aligned}
\langle N_\Omega \rangle &= \int_0^{\infty} d\omega |\beta_{\omega\Omega}|^2 \\
&= \int_0^{\infty} d\omega \frac{\Omega}{\omega} |F(-\omega,\Omega)|^2 \\
&= \frac{1}{e^{2\pi\Omega/a}-1}\delta(0)
\end{aligned} \tag{2.1.38}$$

Note that we expect the number of particles to be divergent from $\delta(0)$ since it is the total of particles in the entire space. We can define a number density by $\int_0^\infty d\omega |\beta_{\omega\Omega}|^2 \equiv n_\Omega \delta(0)$. The density is nicely in the form of Bose-Einstein distribution function

$$n(E) = \frac{1}{e^{E/T} - 1} \tag{2.1.39}$$

One could say that the accelerated observer sees the bath of particles with temperature $T = \frac{a}{2\pi}$ for a given spectrum Ω.

2.1.2 1+1 Dimensional Schwarzschild Black Hole

We now consider 1+1 dimensional Schwarzschild metric which is spherically symmetric and stationary black hole solution to Einstein equation.

$$ds^2 = V(r)dt^2 - V(r)^{-1}dr^2, \quad V(r) \equiv 1 - \frac{2M}{r}. \tag{2.1.40}$$

The horizon in these coordinates is at $r = 2M$. One might find it is convenient to introduce the tortoise coordinates which bring the metric to a conformally flat form,

$$ds^2 = \left(1 - \frac{2M}{r}\right)[dt^2 - dr^{*2}] \tag{2.1.41}$$

where

$$r^* = r - 2M + 2M \ln\left(\frac{r}{2M} - 1\right) \tag{2.1.42a}$$

$$dr^* \equiv V(r)^{-1}dr \tag{2.1.42b}$$

Note that the coordinates (t, r^*) are valid only for $r > 2M$. Again, we express this in the lightcone coordinates $u \equiv t - r^*$ and $v \equiv t + r^*$, the metric becomes

$$ds^2 = \left(1 - \frac{2M}{r}\right) dudv. \tag{2.1.43}$$

Together with the Schwarzschild metric, we have to find a proper coordinates for an observer freely falling into the black hole. The suitable coordinate system is called the *Kruskal coordinates*,

$$\bar{u} = -4M \exp\left(-\frac{u}{4M}\right), \quad \bar{v} = 4M \exp\left(\frac{v}{4M}\right). \tag{2.1.44}$$

The metric (2.1.43) then becomes

$$ds^2 = \frac{2M}{r} \exp\left(1 - \frac{2M}{r}\right) d\bar{u} d\bar{v}. \tag{2.1.45}$$

Note that in the case $r = 2M$, the metric (2.1.45) becomes $ds^2 = d\bar{u}d\bar{v}$ which is same as Minkowski metric, means the flat space. This suggests that the freely falling observer crosses the horizon line without seeing singularity. In the other word, it senses nothing moving into the black hole.

The idea of the particle emission by the black hole is analogous to the Unruh effect. However, there are some misunderstanding about the observers. The observer which analog to the one in the rest frame is not the fixed observer but the freely falling observer. The concept is which observer sense the presence of the acceleration. Let us imagine the classic example of a man in an elevator on the earth. Even the elevator stay still, the man feels his weight due to the gravity. He feels less when the elevator going down and sense nothing when it come to freely falling. Now let the elevator be in the empty space. Of course he should see no gravity when it stays still or even move at a constant velocity. When the elevator is accelerated, the man feels his weight due to the inertia principle. These equivalence principle between gravity and acceleration gives us a concept how to analog the black hole radiation to the Unruh effect.

The calculation to find the number density of particle the fixed observer see emitting from the black hole is the same as what we have done in the Unruh effect section. One can analog the coordinates, the Kruskal coordinates to the laboratory coordinates and the Schwarzschild tortoise coordinates to the comoving coordinates, then the fixed observer is accelerated with $a = 1/2M$. The number of particles created by the black hole is

$$_K \langle 0| N_\omega |0\rangle_K = \frac{1}{e^{4\pi M\omega} - 1}\delta(0). \tag{2.1.46}$$

This makes the black hole has a thermal blackbody radiation with the temperature $T = \frac{1}{4\pi M}$. Note that for any black holes, the temperature is $T = \frac{\kappa}{2\pi}$ where κ is the surface gravity of the black hole. The temperature is known as Hawking temperature. This corresponds with the analogy of black hole mechanics and thermodynamics [6, 14, 15].

2.2 5-dimensional Reissner-Nordstrom Black Hole from Supergravity Solution

In this section we mainly focus on calculating the classical cross section or the emitting rate of the corresponding black hole to the D-brane configuration on the string theory side. We use the fact that a black hole emit scalar particle with the Bose-Einstein distribution where the temperature depends on its surface gravity. However, the previous sections only show in the case of $1 + 1$ dimensional spacetime. For arbitrary dimension, the spacetime geometry affects the result of radiation rate.

For classical radiation of black hole with geometry corresponding to the charges carried by D-branes, we use 5-dimensional extremal black hole metric from SUGRA solution,

$$ds^2 = -f^{-\frac{2}{3}}(r)dt^2 + f^{\frac{1}{3}}(r)dr^2 + f^{\frac{1}{3}}(r)r^2 d\Omega_3^2 \tag{2.2.1}$$

where

$$f(r) = (1 + \frac{Q_1}{r^2})(1 + \frac{Q_2}{r^2})(1 + \frac{Q_3}{r^2}). \tag{2.2.2}$$

A procedure for the solution is provided in appendix A. Consider a spherically symmetric massless minimally coupled scalar wave function with higher angular momentum component (which is not absorbed in low ω limit),

$$\phi(r,t) = R(r)e^{-i\omega t}. \tag{2.2.3}$$

The wave equation is now reduced to

$$\left[\frac{d^2}{dr^2} + \omega^2 f(r) - \frac{3}{4r^2}\right]\psi(r) = 0 \tag{2.2.4}$$

where $\psi(r) = r^{3/2}R(r)$.

The n-dimensional wave function with $n > 2$ can be decomposed into spherical harmonic wave function. In this case, we consider spherically symmetric mode, so the wave equation becomes

$$\Box^{(n)}\phi(r) = (\Box^{(2)} + \Box^{(n-2)})\phi(r) \tag{2.2.5}$$

$$= \left(\frac{d^2}{dr^2} + \omega^2 f(r) + V^{(n-2)}(r)\right)\phi(r) \tag{2.2.6}$$

where $V(r)$ is considered as a barrier-like potential from spacetime geometry. Therefore a particle escaping the black hole needs to tunnel through the potential. This decreases the intensity of the wave by a gray body factor, $\Gamma_{gb}(E) < 1$.

$$\Gamma_\omega = \frac{\text{total outgoing flux at infinity}}{\text{total flux created at horizon}}. \tag{2.2.7}$$

Thus, the number density is in the form of

$$n_E = \Gamma_{gb}(E)[\exp\left(\frac{E}{T}\right) - 1]^{-1} \tag{2.2.8}$$

In order to find a gray body factor, we use matching condition method to find a transmission rate and use low energy limit, $Q\omega^2 << 1$, for approximation.

2.2.1 The Wave Solution

In the calculation, we match the boundaries of the solutions across three regions [1],

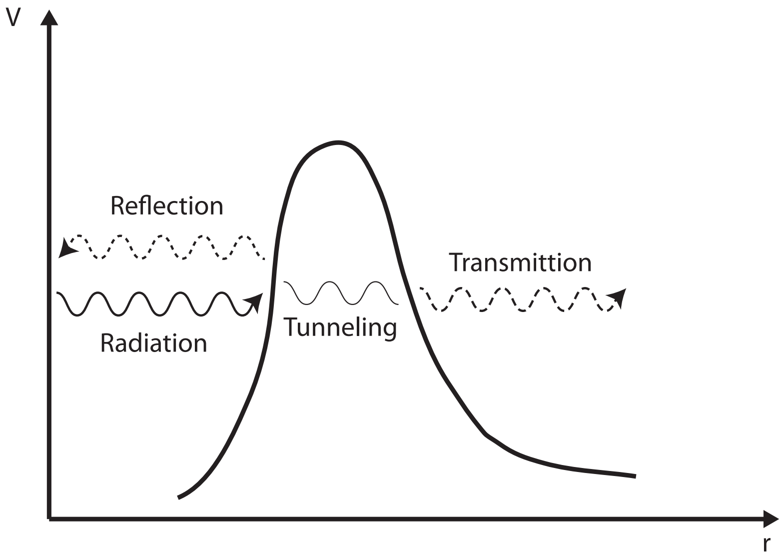


Figure 2.3: Except the $1+1$ dimensional spacetime, a spacetime geometry acts as a barrier blocking the created particles. By the quantum tunneling effect, however, the particles can be transmitted outside while some reflected back. This makes the emission rate decrease with a proportion called *gray body factor*

Outer region $r >> Q_i^{1/2}$.

In this region, the function is approximated to be $f \approx 1 + \frac{Q}{r^2}$ where $Q = Q_1 + Q_2 + Q_3$. We can change the variable by using $r = \frac{\rho}{\omega}$ which make the equation (2.2.4) becomes

$$\left[\frac{d^2}{d\rho^2} + \left(1 + \frac{Q\omega^2 - 3/4}{\rho^2}\right)\right]\psi(r) = 0. \tag{2.2.9}$$

This equation can be easily written in the form of the Bessel equation by change the variable $\psi = \rho^{1/2}\phi$, hence the solution for ψ is

$$\psi(\rho) = \alpha F(\rho) + \beta G(\rho) \tag{2.2.10}$$

where $F(\rho)$ and $G(\rho)$ are in the form of Bessel functions

$$F = \sqrt{\frac{\pi}{2}}\rho J_{+(1-Q\omega^2)^{1/2}}, \quad G = \sqrt{\frac{\pi}{2}}\rho J_{-(1-Q\omega^2)^{1/2}} \tag{2.2.11}$$

In the region that $\rho >> 0$, the functions $F(\rho)$ and $G(\rho)$ become [16]

$$F = \cos(\rho - \frac{\pi}{2}(1 - Q\omega^2)^{1/2} - \frac{\pi}{2}) \tag{2.2.12a}$$

$$G = \cos(\rho + \frac{\pi}{2}(1 - Q\omega^2)^{1/2} - \frac{\pi}{2}). \tag{2.2.12b}$$

Again, the solution ψ can be written in a shifted coordinate $\rho' = \rho - \pi/4$, thus

$$\psi = e^{i\rho'}\frac{i}{2}\left(-\alpha e^{i\pi Q\omega^2/4} + \beta e^{-i\pi Q\omega^2/4}\right) + e^{-i\rho'}\frac{i}{2}\left(\alpha e^{-i\pi Q\omega^2/4} - \beta e^{i\pi Q\omega^2/4}\right). \tag{2.2.13}$$

The equation (2.2.13) is in the form of wave solution with outgoing mode (the left term) and ingoing mode (the right term). Usually, one could find a reflection coefficient $\mathcal{R}$ as

$$\begin{aligned}\mathcal{R} &= \frac{\text{outgoing wave}}{\text{ingoing wave}} \\ &= -e^{i\pi Q\omega^2/2}\frac{1 - \frac{\beta}{\alpha}e^{-i\pi Q\omega^2/2}}{1 - \frac{\beta}{\alpha}e^{i\pi Q\omega^2/2}}.\end{aligned} \tag{2.2.14}$$

which will be useful finding an absorption probability by the relation

$$|\mathcal{A}|^2 = 1 - |\mathcal{R}|^2. \tag{2.2.15}$$

However, the coefficients α and β are needed for the calculation. We will compute them later by matching the solutions from each region. We will need to use the behavior of the Bessel function for $\rho << 1$ where the solution is [1]

$$R(r) \sim \sqrt{\frac{\pi}{2}}\omega^{3/2}\left(\frac{\alpha}{2} + \frac{\beta Q}{r^2}\right). \tag{2.2.16}$$

Intermediate region $r \sim Q_i^{1/2}$.

For what we mainly focus is the low energy case, $Q\omega^2 << 1$, the wave equation (2.2.4) is approximated to be

$$\frac{1}{r^3}\frac{d}{dr}r^3\frac{dR}{dr} = 0 \tag{2.2.17}$$

which has a solution of the form

$$R = C + \frac{D}{r^2}. \tag{2.2.18}$$

Near horizon region $r << Q_i^{1/2}$.

The wave equation (2.2.4) is approximated to be

$$\frac{1}{r^3}\frac{d}{dr}r^3\frac{dR}{dr} + \frac{\omega^2 P}{r^6}(1+\mu r^2)R = 0 \tag{2.2.19}$$

where $P = Q_1 Q_2 Q_3$ and $\mu = \frac{1}{Q_1} + \frac{1}{Q_2} + \frac{1}{Q_3}$. By letting

$$u = -\frac{1}{2r^2}, \;\; \rho = u\omega\sqrt{P}, \;\; \eta = \frac{1}{4}\mu\omega\sqrt{P} \tag{2.2.20}$$

the equation (2.2.19) becomes

$$\frac{d^2R}{d\rho^2} + (1 - \frac{2\eta}{\rho})R = 0 \tag{2.2.21}$$

which is in a form of Coulomb equation [16].

For the near horizon, $\rho \to -\infty$, the solution is in the form of

$$R(\rho) = aF_0 + bG_0 \tag{2.2.22}$$

where

$$F_0 = \sin[\frac{\pi}{4} + \rho + \eta log(\frac{\eta}{2\rho})] \tag{2.2.23a}$$

$$G_0 = \cos[\frac{\pi}{4} + \rho + \eta log(\frac{\eta}{2\rho})]. \tag{2.2.23b}$$

We choose the coefficients a, b for the combination to get an ingoing wave at large negative ρ [1], which is

$$R(r) = G_0(r) - iF_0(r). \tag{2.2.24}$$

For small $|\rho|$, the solution is then

$$R \sim \frac{1}{\sqrt{2}}[(1-i) - (1+i)\rho]. \tag{2.2.25}$$

2.2.2 Matching the Solutions

By matching (2.2.18) and (2.2.25), we got the constants C and D,

$$C = \frac{1}{\sqrt{2}}(1-i), \;\; D = \frac{1}{2\sqrt{2}}(1+i)\omega\sqrt{P}. \tag{2.2.26}$$

We then match (2.2.16) and (2.2.18) using (2.2.26), thus

$$\alpha = \frac{2(1-i)}{\omega^{3/2}\sqrt{\pi}}, \;\; \beta = \frac{(1+i)\sqrt{P}}{2\sqrt{\pi}Q\omega^{1/2}}. \tag{2.2.27}$$

Hence the absorption probability is

$$\begin{aligned} |\mathcal{A}|^2 &= 1 - |\mathcal{R}|^2 \\ &= 1 - \frac{1 + \frac{\omega^2 P}{16Q} + \frac{\omega\sqrt{P}}{4Q} sin[\frac{\pi}{2}Q\omega^2]}{1 + \frac{\omega^2 P}{16Q} - \frac{\omega\sqrt{P}}{4Q} sin[\frac{\pi}{2}Q\omega^2]}. \end{aligned} \tag{2.2.28}$$

For low energy $Q\omega^2 << 1$, we can approximate with the lowest order of ω, then

$$\begin{aligned} |\mathcal{A}|^2 &\approx \frac{1}{2}\pi\omega^3\sqrt{P} \\ &= \frac{1}{4\pi}\omega^3 A_H \end{aligned} \tag{2.2.29}$$

for the area of horizon $A_H = 2\pi^2\sqrt{P}$.

2.3 The Energy Emission Rate

By matching these conditions, one must find an absorption probability (equivalent to transmission probability),

$$|A|^2 = \frac{1}{2}\pi\omega^3\sqrt{Q_1 Q_2 Q_3} = \frac{1}{4\pi}\omega^3 A_H. \tag{2.3.1}$$

Similar to the previous section, the number density of particle emitted by 5-dimensional black hole is in the form of Bose-Einstein distribution but with a gray body factor. Therefore a total energy emission rate in energy range $(\omega, \omega + d\omega)$ should be,

$$\left[\frac{dE(\omega)}{dt}\right]_{SC} = \frac{d\omega}{2\pi}\frac{\omega\Gamma_\omega}{e^{\beta_H\omega} - 1}. \tag{2.3.2}$$

In low energy case, the scalar field emission dominates at low ω [1], the gray body factor is then approximately to the absorption probability, $\Gamma_\omega = |\mathcal{A}|^2(\omega)$. This makes our result become

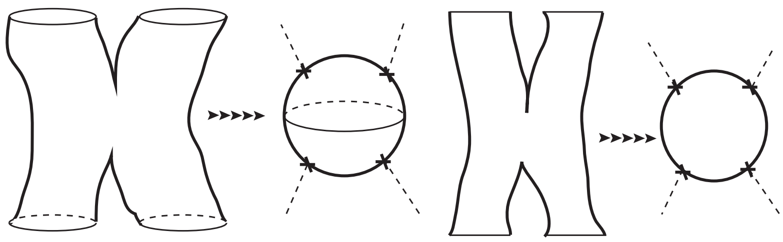
$$\left[\frac{dE(\omega)}{dt}\right]_{SC} = \frac{A_H}{8\pi^2}\frac{\omega^4 d\omega}{e^{\beta_H \omega} - 1}. \tag{2.3.3}$$

This will be compared with the result from the string theory side brane decay model.

CHAPTER III

Conformal Field Theory

In this chapter we introduce some important conformal field theory techniques needed to define vertex operators which we will use to calculate string amplitudes in the next chapter. In string theory, dynamics of a string propagating through spacetime is described by a world sheet theory. A string amplitude is calculated from incoming/outgoing external states of the world sheet. There are a mathematical technique in CFTs allows us to map a world sheet with infinitely long external legs to one with simple geometry, i.e. disk or sphere, by suitable choice of scaling metric $g_{\mu\nu} \to e^{\phi} g_{\mu\nu}$. The in/out external physical states at infinity are mapped to vertex operators applying on the world sheet. This chapter is reviewed from [3, 4]



(a) (b)

Figure 3.1: Conformal mapping maps (a) closed string world sheet to a sphere and maps (b) open string world sheet to a disk, which the external states are mapped to vertex operators on the sphere/disk.

In CFT we use a lot of complex analysis technique, so it is useful to list all the tools we used in our calculations. The complex coordinates are defined by

$$z = \sigma^1 + i\sigma^0, \quad \bar{z} = \sigma^1 - i\sigma^2 \tag{3.0.1}$$

and their derivative components

$$\partial \equiv \partial_z = \frac{1}{2}(\partial_1 - i\partial_2), \qquad \bar{\partial} \equiv \partial_{\bar{z}} = \tfrac{1}{2}(\partial_1 + i\partial_2). \tag{3.0.2}$$

Note that all other vecters change into complex coordinates with the same manners of above

$$v^z = v^1 + iv^2,\ \ v^{\bar{z}} = v^1 - iv^2,\ \ v_z = \frac{1}{2}(v^1 - iv^2),\ \ v_{\bar{z}} = \frac{1}{2}(v^1 + iv^2). \tag{3.0.3}$$

In mapped world sheet, we are using flat Euclidean metric δ_{ab} instead of world sheet metric γ_{ab}. By considering line element $d^2z\,|\det g|^{1/2} = \frac{1}{2}d^2z = d\sigma^1 d\sigma^2$ give us a Euclidean metric in complex coordinates

$$g_{z\bar{z}} = g_{\bar{z}z} = \frac{1}{2},\ \ g_{zz} = g_{\bar{z}\bar{z}} = 0,\ \ g^{z\bar{z}} = g^{\bar{z}z} = 2,\ \ g^{zz} = g^{\bar{z}\bar{z}} = 0 \tag{3.0.4}$$

and delta function in complex coordinates

$$\int d^2z\delta^2(z,\bar{z}) = 1 = \int d\sigma^1 d\sigma^2 \delta(\sigma^1\delta(\sigma^2) \to \delta^2(z,\bar{z}) = \frac{1}{2}\delta(\sigma^1)\delta(\sigma^2) \tag{3.0.5}$$

Divergence theorem in complex coordinates is

$$\int_R d^2z(\partial v^z + \bar{\partial} v^{\bar{z}}) = i\oint_{\partial R}(v^z d\bar{z} - v^{\bar{z}}dz) \tag{3.0.6}$$

where the contour integration in region R is counter clockwise. These are some useful tools to study world sheet theory in complex coordinate system.

Superstring world sheet action on Euclidean metric is given by

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma\left(\partial_a X^\mu \partial^a X_\mu + \alpha' \bar{\boldsymbol{\psi}}^\mu \gamma^a \partial_a \boldsymbol{\psi}_\mu\right) \tag{3.0.7}$$

which is a generalization of the Polyakov action $S = \frac{1}{4\pi\alpha'}\int d^2\sigma \partial_a X^\mu \partial^a X_\mu$. For Majorana spinor, the conjugate fermion field is defined by

$$\bar{\boldsymbol{\psi}} = \boldsymbol{\psi}^\dagger \gamma^0 = \boldsymbol{\psi}^T \gamma^0 \tag{3.0.8}$$

where the Dirac metrics γ^a are satisfied standard anti commutation relation $\{\gamma^a, \gamma^b\} = 2\delta^{ab}$. We are using a convenient representation

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \ \gamma^1 = i\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{3.0.9}$$

The action (3.0.7) is invariant under SUSY transformation,

$$\delta X^\mu = \bar{\epsilon}\boldsymbol{\psi}^\mu, \quad \delta\boldsymbol{\psi}^\mu = \frac{1}{\alpha'}\gamma^a\partial_a X^\mu \epsilon \tag{3.0.10}$$

where ϵ is a Majorana spinor. We are using two component spinor $\boldsymbol{\psi} = \begin{pmatrix} \psi \\ \tilde{\psi} \end{pmatrix}$ so we can write the action in complex coordinates

$$S = \frac{1}{4\pi}\int d^2z \left(\frac{2}{\alpha'}\partial X^\mu \bar{\partial} X_\mu + \psi^\mu \bar{\partial}\psi_\mu + \tilde{\psi}^\mu \partial \tilde{\psi}_\mu\right). \tag{3.0.11}$$

The equations of motion in complex coordinates,

$$\partial\bar{\partial}X^\mu(z,\bar{z}) = 0, \;\; \bar{\partial}\psi^\mu = 0, \;\; \partial\tilde{\psi}^\mu = 0, \tag{3.0.12}$$

imply that we can divide fields into holomorphic (left-moving) fields, $\partial X^\mu(z), \psi^z$, and antiholomorphic (right-moving) fields, $\bar{\partial}X^\mu(\bar{z}), \tilde{\psi}(\bar{z})$.

For two-dimensional theory where the energy momentum tensor is traceless $T_a{}^a = 0$, components of the energy momentum tensor in complex coordinates are $T_{z\bar{z}} = T_{\bar{z}z} = 0$. The conservation laws $\partial^a T_{ab} = 0$ in complex coordinates is $\bar{\partial}T_{zz} = 0, \partial T_{\bar{z}\bar{z}} = 0$. Those above also imply separable of the energy momentum tensor into holomorphic part $T(z) = T_{zz}$ and antiholomorphic part $\tilde{T}(\bar{z}) = T_{\bar{z}\bar{z})}$. For superstring world sheet action, the energy momentum tensors are

$$T_B = -\frac{1}{\alpha'}\partial X^\mu \partial X_\mu - \frac{1}{2}\psi^\mu\partial\psi_\mu, \;\; \tilde{T}_B = -\frac{1}{\alpha'}\bar{\partial}X^\mu\bar{\partial}X_\mu - \frac{1}{2}\tilde{\psi}^\mu\bar{\partial}\tilde{\psi}_\mu \tag{3.0.13a}$$

$$T_F = i\left(\frac{2}{\alpha'}\right)^{1/2}\psi^\mu\partial X_\mu, \;\; \tilde{T}_F = i\left(\frac{2}{\alpha'}\right)^{1/2}\tilde{\psi}^\mu\bar{\partial}X_\mu \tag{3.0.13b}$$

3.1 Operator Product Expansion

In string perturbative theory, similar to a quantum mechanics, any measurement is described by an expectation value of a product of local operators $\langle \mathcal{A}_1(z_1,\bar{z}_1)\mathcal{A}_2(z_2,\bar{z}_2)\ldots\rangle$. The expectation value is defined by path integral

$$\langle \mathcal{F}[X]\rangle = \frac{1}{\mathcal{Z}}\int [dX]\exp(-S)\mathcal{F}[X] \tag{3.1.1}$$

where $\mathcal{F}[X]$ is a functional of field X and $\mathcal{Z}$ is the partition function $\mathcal{Z} = \int [dX]e^{-S}$. In path integral language, choosing the local operators on the mapped world sheet is the way we choose physical states of the external legs. On the other hand it is the vertex operator we want. Our task is to find some suitable vertex operators describing the interesting system.

We are going to study the behavior of local operators via operator product expansion. The main idea is a pair of neighboring operators can be approximated a linear combination of a set of operators within the either point.

$$\mathcal{A}_i(\sigma_1)\mathcal{A}_j(\sigma_2) = \sum_k c_{ij}^k(\sigma_1 - \sigma_2)\mathcal{A}_k(\sigma_2) \tag{3.1.2}$$

Note that all operator equations/statements only hold inside a correlation function. The coefficient functions $c_{ij}^k(\sigma_1 - \sigma_2)$ depend on the distance between the pair and it need to be small compared to any other local operators.

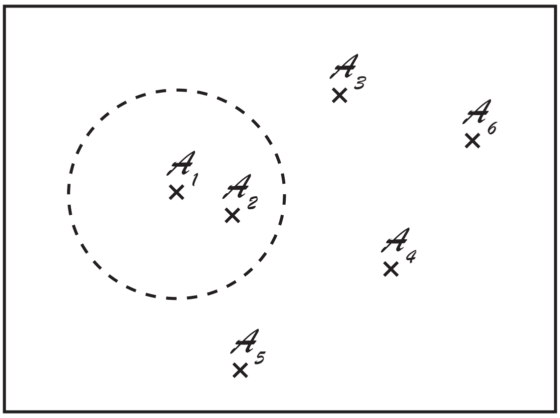


Figure 3.2: The dashed circle shows the radius of convergence which a pair of operators $\mathcal{A}_1, \mathcal{A}_2$ can be replaced by a series of operators at the point of $\mathcal{A}_1$

We start with an example of free scalar field from an action $S = \frac{1}{2\pi\alpha'}\int d^2z\partial X^\mu\bar{\partial}X_\mu$ with the classical equation of motion $\partial\bar{\partial}X^{mu} = 0$. The statement in quantum theory which is analogous with this classical equation of motion can be derived using the fact that path integral of a total derivative must be zero,

$$\begin{aligned}
0 &= \int [dX] \frac{\delta}{\delta X_\mu(z,\bar{z})} e^{-S} \\
&= -\int [dX] e^{-S} \frac{\delta S}{\delta X_{mu}} \\
&= \int [dX] e^{-S} \frac{1}{\pi\alpha'} \partial\bar{\partial} X^\mu \\
&= \frac{1}{\pi\alpha'} \left\langle \partial\bar{\partial} X^\mu \right\rangle ,
\end{aligned} \tag{3.1.3}$$

which can be written in an operator equation as $\partial\bar{\partial}\hat{X}^\mu(z,\bar{z}) = 0$. We can use the same technique to find a propagator by consider

$$\begin{aligned}
0 &= \int [dX] \frac{\delta}{\delta X_\mu(z,\bar{z})} [e^{-S} X^\nu(z',\bar{z}')] \\
&= \int [dX] e^{-S} [\eta^{\mu\nu} \delta^2(z-z',\bar{z}-\bar{z}') + \frac{1}{\pi\alpha'} \partial\bar{\partial} X^\mu(z,\bar{z}) X^\nu(z',\bar{z}')] \\
&= \eta^{\mu\nu} \left\langle \delta^2(z-z',\bar{z}-\bar{z}') \right\rangle + \frac{1}{\pi\alpha'} \partial\bar{\partial} \left\langle X^\mu(z,\bar{z}) X^\nu(z',\bar{z}') \right\rangle
\end{aligned} \tag{3.1.4}$$

so that

$$\frac{1}{\pi\alpha'} \partial\bar{\partial} \left\langle X^\mu(z,\bar{z}) X^\nu(z',\bar{z}') \right\rangle = -\eta^{\mu\nu} \left\langle \delta^2(z-z',\bar{z}-\bar{z}') \right\rangle . \tag{3.1.5}$$

By using $\partial\bar{\partial} \ln |z|^2 = 2\pi\delta^2(z,\bar{z})$, we got the propagator

$$X^\mu(z,\bar{z}) X^\nu(z',\bar{z}') = -\frac{\alpha'}{2} \eta^{\mu\nu} \ln |z-z'|^2 \tag{3.1.6}$$

Normal ordering in the free field theory $:\mathcal{A}:$, or the order of creation and annihilation operators, is used to make sure that the vacuum state has zero energy. For the free scalar field theory, it is defined as

$$: X^\mu(z,\bar{z}) : = X^\mu(z,\bar{z}) \tag{3.1.7a}$$

$$: X^\mu(z_1,\bar{z}_1) X^\nu(z_2,\bar{z}_2) : = X^\mu(z_1,\bar{z}_1) X^\nu(z_2,\bar{z}_2) + \frac{\alpha'}{2} \eta^{\mu\nu} \ln |z_{12}|^2 \tag{3.1.7b}$$

The equation (3.1.7) satisfies an operator equation $\partial_1\bar{\partial}_1 : X^\mu(z_1,\bar{z}_1) X^\nu(z_2,\bar{z}_2) := 0$ which is analogous with the classical equation of motion. We can also write the equation (3.1.7) in terms of Taylor series expansion of $X^\mu(z_1,\bar z_1)$ at $(z_2,\bar z_2)$,

$$X^\mu(z_1,\bar z_1)X^\nu(z_2,\bar z_2) = -\frac{\alpha'}{2}\eta^{\mu\nu}\ln|z_{12}|^2 + :X^\mu X^\nu(z_2,\bar z_2): \\ + \sum_{n=1}^{\infty}\frac{1}{n!}[(z_{12})^n :(\partial^n X^\mu)X^\nu(z_2,\bar z_2): + (\bar z_{12})^n :(\bar\partial^n X^\mu)X^\nu(z_2,\bar z_2):], \tag{3.1.8}$$

so that the expansion is in harmonic functions of holomorphic and antiholomorphicc field. Note that terms in the form of $\partial^i\bar\partial^j$ are absent by the equation of motion.

In a free field theory, the general normal ordering definition for fields is

$$X^{\mu_1}(z_1,\bar z_1)\cdots X^{\mu_n}(z_n,\bar z_n) =: X^{\mu_1}(z_1,\bar z_1)\cdots X^{\mu_n}(z_n,\bar z_n): + \sum \text{contractions}. \tag{3.1.9}$$

For functionals $\mathcal{F}[X]$ and $\mathcal{G}[X]$,

$$:\mathcal{F}::\mathcal{G}:=:\mathcal{F}\mathcal{G}: + \sum \text{cross-contractions} \tag{3.1.10}$$

For further reference, we are using the symbol $\sim$ in equations to indicate that we interested only up to singular terms,

$$X^\mu(z_1,\bar z_1)X^\nu(z_2,\bar z_2) \sim -\frac{\alpha'}{2}\eta^{\mu\nu}\ln|z_{12}|^2 \tag{3.1.11}$$

which is equivalence with

$$X^\mu(z_1,\bar z_1)X^\nu(z_2,\bar z_2) = -\frac{\alpha'}{2}\eta^{\mu\nu}\ln|z_{12}|^2 + \dots \tag{3.1.12}$$

where $\dots$ is nonsingular terms.

To find a quantum version of equation of motion for free fermionic field theory, consider

$$\begin{aligned} 0 &= \int [d\psi]\frac{\delta}{\delta\psi_\mu}e^{-S_{[F}} \\ &= -\int [d\psi]e^{-S_{[F}}\frac{\delta S_{[F}}{\delta\psi_{mu}} \\ &= -\int [d\psi]e^{-S_{[F}}\left(\frac{\eta^{\mu\nu}}{2\pi}\bar\partial\psi_\nu\right), \end{aligned} \tag{3.1.13}$$

so $\langle \bar{\partial}\psi^\mu(z,\bar{z}) \rangle = 0$. For fermionic propagator, consider

$$\begin{aligned} 0 &= \int [d\psi] \frac{\delta}{\delta\psi_\mu(z_1,\bar{z}_1)} (e^{-S_{[F}}\psi^\nu(z_2,\bar{z}_2)) \\ &= \int [d\psi] e^{-S_F} \left(\eta^{\mu\nu}\delta^2(z_{12},\bar{z}_{12}) - \frac{1}{2\pi}\bar{\partial}_1\psi^\mu(z_1,\bar{z}_1)\psi^\nu(z_2,\bar{z}_2) \right), \end{aligned} \tag{3.1.14}$$

so that

$$\bar{\partial}_1 \langle \psi^\mu(z_1,\bar{z}_1)\psi^\nu(z_2,\bar{z}_2) \rangle = 2\pi\eta^{\mu\nu}\delta^2(z_{12},\bar{z}_{12}) = \eta^{\mu\nu}\partial_1\bar{\partial}_1 \ln |z_{12}|^2. \tag{3.1.15}$$

We finally got a fermionic propagator,

$$\begin{aligned} \langle \psi^\mu(z_1,\bar{z}_1)\psi^\nu(z_2,\bar{z}_2) \rangle &= \eta^{\mu\nu}\partial_1 \ln |z_{12}|^2 \\ &= \eta^{\mu\nu}\partial_1(\ln z_{12} + \ln \bar{z}_{12}) \\ &= \frac{\eta^{\mu\nu}}{z_{12}}. \end{aligned} \tag{3.1.16}$$

The same methods go for antiholomorphic field, so we got

$$\psi^\mu(z_1,\bar{z}_1)\psi^\nu(z_2,\bar{z}_2) \sim \frac{\eta^{\mu\nu}}{z_{12}}, \;\; \tilde{\psi}^\mu(z_1,\bar{z}_1)\tilde{\psi}^\nu(z_2,\bar{z}_2) \sim \frac{\eta^{\mu\nu}}{\bar{z}_{12}} \tag{3.1.17}$$

3.2 Ward Identities

In string theory, we focus on studying world sheet symmetries. Ward identities, like Noether's theorem, are the general method to deal with any symmetries in the theory. Later we can consider the symmetries of interest, i.e. translation, rotation, conformal. We start with a general field theory which has an action $S[\phi]$ in d-spacetime dimensions and general field ϕ. We assume there is a symmetry of such a transformation $\phi'(\sigma)_\alpha = \phi_\alpha(\sigma) + \epsilon\delta\phi_\alpha(\sigma)$, where ϵ is an infinitesimal parameter which we will later promote to $\epsilon(\sigma)$ and use the same variation technique as in the classical theory.

Symmetry-ness in quantum theory means the invariance of expectation value under transformation. Thus we only need $[d\phi]e^{-S}$ to be invariant in order to make the correlation function remain the same. The variation is depended with $\partial_a\epsilon$,

$$[d\phi']e^{-S[\phi']} = [d\phi]e^{-S[\phi]}\left[1 - \frac{1}{2\pi}\int d^d\sigma\sqrt{g}j^a(\sigma)\partial_a\epsilon(\sigma) + \mathcal{O}(\epsilon^2)\right]. \tag{3.2.1}$$

The current j^a is contributed form transformation of measure $[d\phi]$ and action $S[\phi]$. Note that the factor $\frac{1}{2\pi}$ is the convention we used for convenience. With a small *epsilon*, invariance of a partition function implies

$$\begin{aligned}0 &= \int [d\phi']e^{-S[\phi]} - \int [d\phi]e^{-S[\phi]} \\ &= -\frac{1}{2\pi}\int d^d\sigma\sqrt{g}\,\langle j^a(\sigma)\partial_a\epsilon(\sigma)\rangle \\ &= \frac{1}{2\pi}\int d^d\sigma\sqrt{g}\epsilon(\sigma)\,\langle\nabla_a j^a(\sigma)\rangle\end{aligned} \tag{3.2.2}$$

This is a quantum version of Noether's theorem, the conservation of currents $\langle\nabla_a j^a(\sigma)\rangle = 0$.

Ward identity has stronger statement. It is not only conserving the partition function but any local operator. We define a time-ordered correlation function as

$$\langle\mathcal{A}_1(\sigma_1)\ldots\mathcal{A}_n(\sigma_n)\rangle = \frac{1}{\mathcal{Z}}\int [d\phi']e^{-S[\phi']}\mathcal{A}_1(\sigma_1)\ldots\mathcal{A}_n(\sigma_n), \tag{3.2.3}$$

where an operator $\mathcal{A}_i$ is a functional which is a general expression of field ϕ. Let consider a symmetry of operator transformation $\mathcal{A}_i \to \mathcal{A}_i + \epsilon\delta\mathcal{A}_i$. The infinitesimal parameter ϵ is now promoted to $\epsilon(\sigma)$ that is only nonzero inside a region R. We now do the same as we have done with Noether's theorem, from

$$\begin{aligned}&\langle\mathcal{A}_1\ldots\mathcal{A}_i'\ldots\mathcal{A}_n\rangle = \frac{1}{\mathcal{Z}}\int [d\phi']e^{-S[\phi']}\mathcal{A}_1\ldots(\mathcal{A}_i + \epsilon(\sigma)\delta\mathcal{A}_i)\ldots\mathcal{A}_n \\ &= \frac{1}{\mathcal{Z}}\int [d\phi]e^{-S[\phi]}\left(1 + \frac{1}{2\pi}\int_R d^d\sigma\sqrt{g}\epsilon(\sigma)\nabla_a j^a(\sigma) + O(\epsilon^2)\right)\mathcal{A}_1\ldots(\mathcal{A}_i + \epsilon(\sigma)\delta\mathcal{A}_i)\ldots\mathcal{A}_n \\ &= \langle\mathcal{A}_1\ldots\mathcal{A}_n\rangle + \frac{1}{2\pi}\int_R d^d\sqrt{g}\epsilon(\sigma)\,\langle\nabla_a j^a(\sigma)\mathcal{A}_1\ldots\mathcal{A}_n\rangle + \epsilon(\sigma)\,\langle\mathcal{A}_1\ldots\delta\mathcal{A}_i\ldots\mathcal{A}_n\rangle + O(\epsilon^2).\end{aligned} \tag{3.2.4}$$

For more convenient, let $\epsilon(\sigma) = \epsilon'\rho(\sigma)$ with $\rho = 1$ inside the region R and zero otherwise. The equation (3.2.3) gives us an operator equation

$$\delta\mathcal{A}_i(\sigma_i) = \frac{1}{2\pi}\int_R d^d\sigma\sqrt{g}\epsilon(\sigma)\nabla_a j^a(\sigma)\mathcal{A}_i(\sigma_i). \tag{3.2.5}$$

In the case that there is no operator insertion in the region R, the absence of field transformation make no operator transformation $\delta\mathcal{A}_i(\sigma_i) = 0$. Therefore the previous Noether's theorem is restored $\langle \nabla_a j^a(\sigma)\mathcal{A}_1(\sigma_1)\ldots\mathcal{A}_n(\sigma_n)\rangle = 0$ for $\sigma \neq \sigma_i$. Note that the Noether's theorem and Ward identity are local properties, they are not depended on any boundary conditions and any symmetries.

We now consider a Ward identity for conformal transformation from (3.2.5) which is equivalent with

$$\nabla_a j^a \mathcal{A}_i(\sigma_i) = g^{-1/2}\delta^d(\sigma - \sigma_i)2\pi\delta\mathcal{A}_i(\sigma_i) + \text{total-derivative}. \tag{3.2.6}$$

Using a divergence theorem, we got $\int_{\partial R} dA n_a j^a \mathcal{A}_i(\sigma_i) = 2\pi\delta\mathcal{A}_i(\sigma_i)$ where dA is an area element and n^a is an outward normal. The flat 2-dimensional version of this equation is

$$i\oint_{\partial R}(jdz - \tilde{j}d\bar{z})\mathcal{A}_i(\sigma_i) = 2\pi\delta\mathcal{A}_i(\sigma_i), \tag{3.2.7}$$

where the holomorphic and antiholomorphic current $j \equiv j_z, \tilde{j} \equiv j_{\bar{z}}$ are separately conserved. We can get residues in (3.2.7) integral to

$$\delta\mathcal{A}(z_0,\bar{z}_0) = -Res_{z\to z_0} j(z)\mathcal{A}(z_0,\bar{z}_0) - Res_{\bar{z}\to\bar{z}_0}\tilde{j}(\bar{z})\mathcal{A}(z_0,\bar{z}_0). \tag{3.2.8}$$

From the classical theory a conformal transformation $\delta z = \epsilon, \delta\bar{z} = \bar{\epsilon}$ implies the currents $j = \epsilon(z)T(z), \tilde{j} = \bar{\epsilon}(\bar{z})\tilde{T}(\bar{z})$, the equation (3.2.8) is now

$$\delta\mathcal{A}(z_0,\bar{z}_0) = -Res_{z\to z_0}(\epsilon(z)T(z)\mathcal{A}(z_0,\bar{z}_0)) - Res_{\bar{z}\to\bar{z}_0}(\bar{\epsilon}(\bar{z})\tilde{T}(\bar{z})\mathcal{A}(z_0,\bar{z}_0)). \tag{3.2.9}$$

Therefore the study of OPEs will mainly focus on operator product with the energy-momentum tensor which leads to the change in operator.

We now consider operator products of $T(z)$ and $\tilde{T}(\bar{z})$ with operator $\mathcal{A}$. Generally the operator products are able to write in the form of Laurent expansion. However the properties of $T(z)$ and $\tilde{T}(\bar{z})$ which are (anti)holomorphic everywhere exclude the insertion point imply that the transformation of operator $\mathcal{A}$ should be determined by singular terms. Therefore we consider a Laurent expansion only in negative powers

$$T(z)\mathcal{A}(\omega,\bar{\omega}) \sim \sum_{n=0}^{\infty}\frac{1}{(z-\omega)^{n+1}}\mathcal{A}^{(n)}(\omega,\bar{\omega}), \tag{3.2.10}$$

where $\mathcal{A}^{(n)}$ is an operator coefficient, and similarly for antiholomorphic. The Ward identity (3.2.9) can be expressed with a Taylor series expansion of the infinitesimal function $\epsilon, \bar{\epsilon}$ around the insertion of operator $\mathcal{A}$. Poles arise when survive cancellation from the Taylor expression and operator products of $T, \tilde{T}$ with $\mathcal{A}$. Consider

$$\begin{aligned} Res_{z\to\omega}\epsilon(z)T(z)\mathcal{A}(\omega,\bar{\omega}) &\sim Res_{z\to\omega}\sum_{m=0}^{\infty}\frac{(z-\omega)^m}{m!}\partial^m\epsilon(\omega)\sum_{n=0}^{\infty}\frac{1}{(z-\omega)^{n+1}}\mathcal{A}^{(n)}(\omega,\bar{\omega}) \\ &\sim Res_{z\to\omega}\sum_{\substack{m=0\\n=0}}^{\infty}\frac{1}{m!}\partial^m\epsilon(\omega)\frac{\mathcal{A}^{(n)}(\omega,\bar{\omega})}{(z-\omega)^{(n-m)+1}} \\ &\sim \sum_{n=0}^{\infty}\frac{1}{n!}\partial^n\epsilon(\omega)\mathcal{A}^{(n)}(\omega,\bar{\omega}) \end{aligned} \tag{3.2.11}$$

and similarly for antiholomorphic, we got an operator transformation for conformal transformation,

$$\delta\mathcal{A}(z,\bar{z}) = -\sum_{n=0}^{\infty}\frac{1}{n!}\left(\partial^n\epsilon(z)\mathcal{A}^{(n)}(z,\bar{z}) + \bar{\partial}^n\bar{\epsilon}(\bar{z})\mathcal{A}^{(n)}(z,\bar{z})\right). \tag{3.2.12}$$

Lets consider a operator transformation of primary field $\mathcal{O}$ with weight $(h, \tilde{h})$ under conformal symmetry. Under general conformal transformation the primary field is changed as

$$\mathcal{O}'(z',\bar{z}') = (\partial_z z')^{-h}(\partial_{\bar{z}}\bar{z}')^{-\tilde{h}}\mathcal{O}(z,\bar{z}). \tag{3.2.13}$$

For $\delta z = \epsilon(z), \delta\bar{z} = \bar{\epsilon}(\bar{z})$, the change in operator is

$$\delta\mathcal{O}(z,\bar{z}) = -\left(h\partial\epsilon + \epsilon\partial + \tilde{h}\bar{\partial}\bar{\epsilon} + \bar{\epsilon}\bar{\partial}\right)\mathcal{O}(z,\bar{z}), \tag{3.2.14}$$

By comparing this with (3.2.12), we got

$$\begin{aligned} T(z)\mathcal{O}(\omega,\bar{\omega}) &\sim \frac{h}{(z-\omega)^2}\mathcal{O}(\omega,\bar{\omega}) + \frac{1}{z-\omega}\partial\mathcal{O}(\omega,\bar{\omega}) \\ \tilde{T}(\bar{z})\mathcal{O}(\omega,\bar{\omega}) &\sim \frac{\tilde{h}}{(\bar{z}-\bar{\omega})^2}\mathcal{O}(\omega,\bar{\omega}) + \frac{1}{\bar{z}-\bar{\omega}}\partial\mathcal{O}(\omega,\bar{\omega}). \end{aligned} \tag{3.2.15}$$

This is the standard method we used to find a weight of fields.

In the canonical quantization an energy-momentum tensor is defined by a normal ordering expression, an order of creation and annihilation operators, to make sure that the vacuum state has zero energy $\langle T\rangle = 0$ by construction. For free scalar field theory the holomorphic and anitiholomorphic energy-momentum tensors are

$$T(z) = -\frac{1}{\alpha'} :\partial X^{\mu}\partial X_{\mu}:, \;\; \tilde{T}(\bar{z}) = -\frac{1}{\alpha'} :\bar{\partial} X^{\mu}\bar{\partial} X_{\mu}: . \tag{3.2.16}$$

One can consider operator products of various forms of scalar fields with these energy-momentum tensors and using (3.2.15) to find their weight. A general operator with form of $:(\prod_i \partial^{m_i} X^{\mu_i})(\prod_j \bar{\partial}^{n_j} X^{\nu_j})e^{ik\cdot X}:$ has weight $(\frac{\alpha' k^2}{4} + \sum_i m_i, \frac{\alpha' k^2}{4} + \sum_j n_j)$.

Field	Weight
X^{μ}	$(0,0)$
∂X^{μ}	$(1,0)$
$\bar{\partial} X^{\mu}$	$(0,1)$
$\partial^2 X^{\mu}$	$(2,0)$
$:e^{ik\cdot X}:$	$(\frac{\alpha' k^2}{4}, \frac{\alpha' k^2}{4})$.

Table 3.1: Scarlar field in various forms and their weight

For further discussion we will encounter a lack of notation in superconformal field theory which has bosonic and fermionic transformations. From now on we will redefine the infinitesimal function $\epsilon(z)$ to be $\epsilon\nu(z)$ for bosonic infinitesimal transformation, where ϵ is now an infinitesimal parameter and $\nu(z)$ is an arbitrary holomorphic function. For fermionic infinitesimal transformation, the $\epsilon(z)$ becomes $\epsilon\eta(z)$. The functions $\nu(z)$ and $\eta(z)$ are commuting and anticommuting respectively.

Lets consider a change in energy-momentum tensor from an operator product,

$$\begin{aligned}
T(z)T(0) &= 4\left(-\frac{1}{\alpha'}\right)\left(-\frac{\alpha'}{2}\partial\partial'\ln|z|^2\right)(:\partial X^\mu(z)\partial' X_\mu(0):) + 2\left(-\frac{1}{\alpha'}\right)\left(-\frac{\alpha'}{2}\partial\partial'\ln|z|^2\right)^2\delta^\mu_\mu \\
&+\left(-\frac{1}{\alpha'}\right):\partial X^\mu\partial X_\mu(z)\partial' X^\nu\partial' X_\nu: \\
&= \frac{\delta^\mu_\mu}{2z^4} - \frac{2}{\alpha' z^2}:\partial X^\mu(z)\partial X_\mu(0): + :T(z)T(0): \\
&= \frac{D}{2z^4} + :T(z)T(0): - \frac{2}{\alpha'}\frac{1}{z^2}\{:\partial' X^\mu(0)\partial' X_{mu}(0): + z:\partial'\partial' X^\mu(0)\partial' X_\mu(0):\} \\
&\sim \frac{D}{2z^4} + \frac{2}{z^2}T(0) + \frac{1}{z}\partial T(0).
\end{aligned} \tag{3.2.17}$$

This implies that the energy-momentum tensor is not a primary field. By comparing this with (3.2.12), we got

$$\epsilon^{-1}\delta T(z) = -\frac{D}{12}\partial_z^3\nu(z) - 2\partial_z\nu(z)T(z) - \nu(z)\partial_z T(z), \tag{3.2.18}$$

which is in the form of general CFT transformation [3],

$$\epsilon^{-1}\delta T(z) = -\frac{c}{12}\partial_z^3\nu(z) - 2\partial_z\nu(z)T(z) - \nu(z)\partial_z T(z), \tag{3.2.19}$$

where the constant c is a central charge. Each scalar field contributes one to central charge, so in the free scalar field theory with D spacetime boson has central charge equal to D.

There are two types of energy-momentum tensor in SCFT by two transformations: conformal transformation and superconformal transformation, which are (3.0.13). Again these are defined under normal ordering even if we drop its notation for convenient. By considering world-sheet supercurrents corresponding to $T_F, \tilde{T}_F$,

$$j^\eta(z) = \eta(z)T_F(z), \;\; \tilde{j}^\eta(\bar{z}) = \eta(z)^*\tilde{T}_F(\bar{z}), \tag{3.2.20}$$

the Ward identity give us superconformal transformations,

$$\epsilon^{-1}\left(2/\alpha'\right)^{1/2}\delta X^\mu = \eta(z)\psi^\mu(z) + \eta(z)^*\tilde{\psi}^\mu(z) \tag{3.2.21a}$$

$$\epsilon^{-1}\left(2/\alpha'\right)^{1/2}\delta\psi^\mu = -\eta(z)\partial X^\mu(z) \tag{3.2.21b}$$

$$\epsilon^{-1}\left(2/\alpha'\right)^{1/2}\delta\tilde{\psi}^\mu = -\eta(z)^*\bar{\partial} X^\mu(\bar{z}). \tag{3.2.21c}$$

Likewise a free bosonic theory, we consider operator products of T_B and T_F combinations, which are

$$T_B(z)T_B(0) \sim \frac{3D}{4z^4} + \frac{2}{z^2}T_B(0) + \frac{1}{z}\partial T_B(0) \tag{3.2.22a}$$

$$T_B(z)T_F(0) \sim \frac{3}{2z^2}T_F(0) + \frac{1}{z}\partial T_F(0) \tag{3.2.22b}$$

$$T_F(z)T_F(0) \sim \frac{D}{z^3} + \frac{2}{z}T_B(0) \tag{3.2.22c}$$

and similar for antiholomorphic currents. The operator product (3.2.22b) implies that T_F is a tensor with weight $(\frac{3}{2}, 0)$. The central charge in superconformal theory comes from scalar fields and fermionic fields which contribute 1 and $\frac{1}{2}$ to the central charge respectively. Therefore it becomes $c = (1 + \frac{1}{2})D = \frac{3}{2}D$ for D-dimensional theory.

3.3 Mode Expansion

In free field theory, fields can be expressed in terms of harmonic oscillators. Starting from free scalar field theory, we consider Laurent expansions of ∂X and $\bar{\partial} X$ which are holomorphic and antiholomorphic respectively,

$$\partial X^\mu(z) = -i\left(\frac{\alpha'}{2}\right)^{1/2} \sum_{m=-\infty}^{\infty} \frac{\alpha_m^\mu}{z^{m+1}}, \quad \bar{\partial} X^\mu(\bar{z}) = -i\left(\frac{\alpha'}{2}\right)^{1/2} \sum_{m=-\infty}^{\infty} \frac{\tilde{\alpha}_m^\mu}{\bar{z}^{m+1}}. \tag{3.3.1}$$

Note that we always write expansions in term of mode m plus the field weight, i.e. $m+1$ for ∂X expansion. Laurent expansion of bosonic closed string is [3]

$$X^\mu(z,\bar{z}) = x^\mu - i\frac{\alpha'}{2}p^\mu \ln|z|^2 + i\left(\frac{\alpha'}{2}\right)^{1/2} \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \frac{1}{m}\left(\frac{\alpha_m^\mu}{z^m} + \frac{\tilde{\alpha}_m^\mu}{\bar{z}^m}\right), \tag{3.3.2}$$

which the spacetime momentum p^μ equal to $p^\mu = (2/\alpha')^{1/2}\alpha_0^\mu = (2/\alpha')^{1/2}\tilde{\alpha}_0^\mu$ for closed string and equal to $(2\alpha')^{-1/2}\alpha_0^\mu$ for open string. The open string case, a Neumann boundary condition $\partial X^\mu = \bar{\partial} X^\mu$ implies that there is only one set of modes, so $\alpha_m^\mu = \tilde{\alpha}_m^\mu$.

The same goes for (anti)holomorphic energy-momentum tensors, we express them in expansions of modes,

$$T(z) = \sum_{m=-\infty}^{\infty} \frac{L_m}{z^{m+2}}, \;\; \tilde{T}(\bar{z}) = \sum_{m=\infty}^{\infty} \frac{\tilde{L}_m}{\bar{z}^{m+2}}, \tag{3.3.3}$$

where $L_m, \tilde{L}_m$ are Virasoro generators which satisfy Virasoro algebra [3],

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n}, \tag{3.3.4}$$

where c is the central charge.

From standard canonical commutation relation $[x^\mu, p^\nu] = i\eta^{\mu\nu}$, we got commutation relation of modes,

$$[\alpha^\mu_m, \alpha^\nu_n] = [\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_n] = m\delta_{m,-n}\eta^{\mu\nu}, \tag{3.3.5}$$

and the otherwise vanish. These mode coefficients are also used as annihilation operators for $n > 0$, and creation operators, for $n < 0$.

We now write a Virasoro generator in terms of mode operators by replace the field X^μ with its Laurent expansion in the energy-momentum tensor, L_m is then

$$\begin{aligned} L_m &= \frac{1}{2\pi i} \oint dz z^{m+1} T(z) \\ &= \frac{1}{2} \sum_{n=-\infty}^{\infty} : \alpha^\mu_{m-n} \alpha_{\mu n} : . \end{aligned} \tag{3.3.6}$$

3.4 State-operator Correspondence

There are two points of view in CFT: states and operators, which can be mapped to each other. In order to understand this we consider a complex coordinate $w = \sigma^1 + i\sigma^0$. We use, for closed string, the spatial coordinate is periodic, $0 \leq Re(w) \leq 2\pi, w \sim w + 2\pi$, and the Euclidean time run from the infinite past to the present, $-\infty \leq Im(w) \leq 0$. These make the w-coordinate a semi-infinite cylinder. The cylinder can be mapped to disk by changing to z-coordinate, $z = e^{-iw}$, which map the infinite past $Im(w) = -\infty$ to the disk origin. Assume there is an initial state $|\mathcal{A}\rangle$ at the infinite past, state-operator mapping implies that there must be a local operator acts at the origin of the corresponding disk. The local operator is the vertex operator we have been talking about since the beginning.

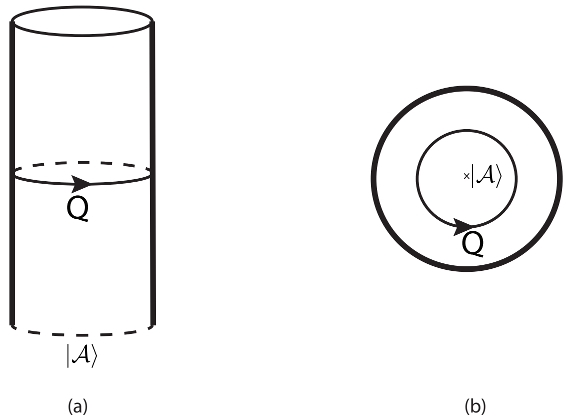


Figure 3.3: Coordinate changing from (a) cylindrical coordinate which has a state $|\mathcal{A}\rangle$ at infinite past and conserved charge Q maps to (b) disk with a contour of conserved charge Q over the state $|\mathcal{A}\rangle$ at the origin.

For a conserved charge Q on the state $|\mathcal{A}\rangle$, by the meaning of space integration, we could find a corresponding operator by using contour integral, as shown in figure 3.3(b). For example, let's consider a state $\alpha^\mu_{-m}\,|1\rangle\,; m \geq 1$ with corresponding charge $Q = \alpha^\mu_{-m}$ acts on a unit operator. The expansion of ∂X^μ gives us

$$\begin{aligned}\alpha^\mu_{-m} &= \left(\frac{2}{\alpha'}\right)^{1/2} \oint \frac{dz}{2\pi} z^{-m} \partial X^\mu(z) \\ &\to \left(\frac{2}{\alpha'}\right)^{1/2} \frac{i}{(m-1)!} \partial^m X^\mu(0) \quad ; m \geq 1.\end{aligned} \tag{3.4.1}$$

Therefore we can map state $\alpha^\mu_{-m}\,|1\rangle$ with an operator,

$$\alpha^\mu_{-m}\,|1\rangle \cong \left(\frac{2}{\alpha'}\right)^{1/2} \frac{i}{(m-1)!} \partial^m X^\mu(0) \quad ; m \geq 1. \tag{3.4.2}$$

Note that a state $\alpha^\mu_m\,|1\rangle$ with positive m gives a zero value to the contour integral, means the operator α^μ_m annihilates the identity state $|1\rangle$, this is the same with the definition of ground state $|0,0\rangle$. Therefore in a bosonic theory ground state is equivalent to identity state, $|0,0\rangle \cong |1\rangle$.

We now consider a Virasoro generator acting on a state $|\mathcal{A}\rangle$ which corresponding to a local operator $\mathcal{A}$,

$$\begin{aligned} L_m \,|\mathcal{A}\rangle &\cong \oint \frac{dz}{2\pi i} z^{m+1} T(z)\mathcal{A}(0,0) \\ &\cong L_m \cdot \mathcal{A}(0,0). \end{aligned} \tag{3.4.3}$$

This gives us a picture of state-operator isomorphism: for every operator acting on Hilbert space must exists an image of those operators forming a contour and giving an operator products. In the case of (3.4.3),

$$T(z)\mathcal{A}(0,0) = \sum_{m=-\infty}^{\infty} z^{-m-2} L_m \cdot \mathcal{A}(0,0), \tag{3.4.4}$$

by comparing this with the previous operator products, we got $\mathcal{A}^{(n)} = L_{n-1} \cdot \mathcal{A}\ ; n \geq 0$. This makes the Ward identity (3.2.12) becomes

$$\delta_\nu \mathcal{A}(z,\bar{z}) = -\epsilon \sum_{n=0}^{\infty} \frac{1}{n!} \left[\partial^n \nu(z) L_{n-1} + \left(\partial^n \nu(z)\right)^* \tilde{L}_{n-1}\right] \cdot \mathcal{A}(z,\bar{z}). \tag{3.4.5}$$

Compare (3.4.5) to (3.2.15) and we got

$$\begin{aligned} L_{-1} \cdot \mathcal{A} = \partial \mathcal{A}, \quad \tilde{L}_{-1} \cdot \mathcal{A} = \bar{\partial} \mathcal{A}, \\ L_0 \cdot \mathcal{A} = h\mathcal{A}, \quad \tilde{L}_0 \cdot \mathcal{A} = \tilde{h}\mathcal{A}. \end{aligned} \tag{3.4.6}$$

In the same way we will consider spectrum of $X^\mu \psi^\mu$ SCFT by considering an action in cylindrical coordinate w,

$$\frac{1}{4\pi} \int d^2 w \left(\psi^\mu \partial_{\bar{w}} \psi_\mu + \tilde{\psi}^\mu \partial_w \tilde{\psi}_\mu\right). \tag{3.4.7}$$

We need this action to be invariant under a transformation $w \sim w + 2\pi$, this implies two kinds of periodicity conditions of ψ^μ:

$$\text{Ramond (R)}: \quad \psi^\mu(w+2\pi) = +\psi^\mu(w) \tag{3.4.8a}$$

$$\text{Neveu-Schwarz (NS)}: \quad \psi^\mu(w+2\pi) = -\psi^\mu(w), \tag{3.4.8b}$$

which can be written as

$$\psi^\mu(w+2\pi) = \exp(2\pi i\nu)\psi^\mu(w) \tag{3.4.9a}$$

$$\tilde{\psi}^\mu(\bar{w}+2\pi) = \exp(-2\pi i\tilde{\nu})\tilde{\psi}^\mu(\bar{w}) \tag{3.4.9b}$$

where ν and $\tilde{\nu}$ are 0 for R and $\frac{1}{2}$ for NS. Their energy-momentum tensors are also periodical,

$$T_F(w+2\pi) = \exp(2\pi i\nu) T_F(w) \tag{3.4.10a}$$

$$\tilde{T}_F(\bar{w}+2\pi) = \exp(-2\pi i\tilde{\nu}) \tilde{T}_F(\bar{w}) \tag{3.4.10b}$$

We now change the fields $\psi^\mu, \tilde{\psi}^\mu$ into z-coordinate and their Laurent expansions are [4]

$$\psi^\mu(z) = \sum_{r\in\mathbb{Z}+\nu} \frac{\psi^\mu_r}{z^{r+1/2}}, \quad \tilde{\psi}^\mu(\bar{z}) = \sum_{r\in\mathbb{Z}+\tilde{\nu}} \frac{\tilde{\psi}^\mu_r}{\bar{z}^{r+1/2}} \tag{3.4.11}$$

with algebra

$$\{\psi^\mu_r, \psi^\nu_s\} = \{\tilde{\psi}^\mu_r, \tilde{\psi}^\nu_s\} = \eta^{\mu\nu}\delta_{r,-s}. \tag{3.4.12}$$

The mode expansion of field X^μ and its algebra are the same with the previous discussion in CFT.

Laurent expansions of T_B and T_F are

$$T_B(z) = \sum_{m=-\infty}^{\infty} \frac{L_m}{z^{m+2}}, \quad \tilde{T}_B(\bar{z}) = \sum_{m=-\infty}^{\infty} \frac{\tilde{L}_m}{z^{m+2}} \tag{3.4.13a}$$

$$T_F(z) = \sum_{r\in\mathbb{Z}+\nu} \frac{G_r}{z^{r+3/2}}, \quad \tilde{T}_F(\bar{z}) = \sum_{r\in\mathbb{Z}+\tilde{\nu}} \frac{\tilde{G}_r}{\bar{z}^{r+3/2}} \tag{3.4.13b}$$

which their mode coefficients L_m, G_r are satisfied mode algebras,

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^2-m)\delta_{m,-n} \tag{3.4.14a}$$

$$\{G_r, G_s\} = 2L_{r+s} + \frac{c}{12}(4r^2-1)\delta_{r,-s} \tag{3.4.14b}$$

$$[L_m, G_r] = \frac{m-2r}{2}G_{m+r}, \tag{3.4.14c}$$

and similar for antiholomorphic generators. The superconformal generators can be written in terms of mode coefficients [4],

$$L_m = \frac{1}{2}\sum_{n\in\mathbb{Z}} \alpha^\mu_{m-n}\alpha_{\mu n} + \frac{1}{4}\sum_{r\in\mathbb{Z}+\nu} (2r-m)\psi^\mu_{m-r}\psi_{\mu r} + a^m\delta_{m,0} \tag{3.4.15a}$$

$$G_r = \sum_{n\in\mathbb{Z}} \alpha^\mu_n \psi_{\mu r-n} \tag{3.4.15b}$$

where a^m is a normal order constant which $a^m = \frac{1}{16}D$ for R sector and $a^m = 0$ for NS sector.

The state-operator mapping of fermionic fields comes from considering a Laurent expansion (3.4.11). The single-valueness implies that unit operator and states building from it must be in the NS sector, because the Laurent expansion in R sector has branch cut while the NS sector has not. One could say that the antiperiodicity of the NS sector has made a conformal transformation which cancels down its branch cut at $z = 0$.

From a contour argument, a state $|1\rangle$, which is corresponding to a unit operator, is satisfied

$$\psi^\mu_r |1\rangle = 0 \quad ; r > 0, r \in \mathbb{Z} + \frac{1}{2}. \tag{3.4.16}$$

Again, we see that the identity state is match a definition of NS ground state, so it is. We can also map

$$\psi^\mu_{-r} \cong \frac{1}{(r-1/2)!}\partial^{r-1/2}\psi^\mu(0). \tag{3.4.17}$$

One could define a Noether relation for superconformal transformation which is analogous within a conformal transformation, (3.4.5)

$$\delta_\eta \mathcal{A}(z,\bar{z}) = -\epsilon \sum_{n=0}^{\infty} \frac{1}{n!}\left[\partial^n \eta(z) G_{n-1/2} + (\partial^n \eta(z))^* \tilde{G}_{n-1/2}\right] \cdot \mathcal{A}(z,\bar{z}). \tag{3.4.18}$$

State-operator mapping for R sector is slightly complicated. It cannot be in a simple form because its Laurent expansion has a branch cut. In two-dimensional theory, however, has a technique called "bosonization": relating properties of the vertex operator with a bosonic winding state vertex operator. Lets consider a holomorphic scalar field $H(z)$ which its operator product is $H(z)H(0) \sim -\ln z$. This is similar to the $X(z)X(0)$ product we have discussed before. Consider operator products of $e^{\pm iH(z)}$,

$$e^{iH(z)}e^{-iH(0)} \sim \frac{1}{z} \tag{3.4.19a}$$

$$e^{iH(z)}e^{iH(0)} = \mathcal{O} \tag{3.4.19b}$$

$$e^{-iH(z)}e^{-iH(0)} = \mathcal{O}. \tag{3.4.19c}$$

Beside consider an operator ψ which is a complex combination of Majorana-Weyl fermion $\psi^{1,2}(z)$,

$$\psi \equiv \frac{1}{2}(\psi^1 + i\psi^2), \quad \bar{\psi} \equiv \frac{1}{2}(\psi^1 - i\psi 2), \tag{3.4.20}$$

which has operator products

$$\psi(z)\bar{\psi}(0) \sim \frac{1}{z} \tag{3.4.21a}$$

$$\psi(z)\psi(0) = O(z) \tag{3.4.21b}$$

$$\bar{\psi}(z)\bar{\psi}(0) = O(z). \tag{3.4.21c}$$

This is identical in form with (3.4.19). Same operator products imply their expectation values on S_2 are identical, so one could say that they are equivalent,

$$\begin{aligned} &\psi(z) \cong e^{iH(z)}, \quad \bar{\psi}(z) \cong e^{-iH(z)}, \\ &\tilde{\psi}(\bar{z}) \cong e^{i\tilde{H}(\bar{z})}, \quad \tilde{\bar{\psi}}(\bar{z}) \cong e^{-i\tilde{H}(\bar{z})}. \end{aligned} \tag{3.4.22}$$

Note that in the notation we are using, $\psi(z)$ and $\bar{\psi}(z)$ are holomorphic fields, $\tilde{\psi}(\bar{z})$ and $\tilde{\bar{\psi}}(\bar{z})$ are antiholomorphic fields. By considering operator products

$$e^{iH(z)}e^{-iH(-z)} = \frac{1}{z} + i\partial H(0) + 2zT_B^H(0) + O(z^2) \tag{3.4.23a}$$

$$\psi(z)\bar{\psi}(-z) = \frac{1}{z} + \psi\bar{\psi}(0) + 2zT_B^{\psi}(0) + O(z^2), \tag{3.4.23b}$$

we can match $\psi\bar{\psi} \cong i\partial H$ and $T_B^{\psi} \cong T_B^H$. The energy-momentum tensor equivalence implies that these two theories are the same CFT.

We now use an equivalence to determine a state-operator mapping of fermionic field. Lets consider a general periodic condition of fermionic complex pair,

$$\psi(w + 2\pi) = \exp(2\pi i\nu)\psi(w) \tag{3.4.24a}$$

$$\bar{\psi}(w + 2\pi) = \exp(-2\pi i\nu)\bar{\psi}(w), \tag{3.4.24b}$$

which ν is now an arbitrary real number. We will later use $\nu = 0, \frac{1}{2}$ for R and NS. For general ν, $+\nu$ and $-\nu$ give different values to the periodicity conditions. Therefore the expansions are different, their Fourier expansions by the definition in [3] are

$$\psi(w) = \frac{1}{\sqrt{i}} \sum_{r \in \mathbb{Z}+\nu} \psi_r \exp(irw) \tag{3.4.25a}$$

$$\bar{\psi}(w) = \frac{1}{\sqrt{i}} \sum_{s \in \mathbb{Z}-\nu} \bar{\psi}_s \exp(irw), \tag{3.4.25b}$$

then their Laurent expansions by changing coordinate $z = e^{-iw}$ are

$$\psi(z) = \sum_{r \in \mathbb{Z}+\nu} \frac{\psi_r}{z^{r+1/2}} \tag{3.4.26a}$$

$$\bar{\psi}(z) = \sum_{s \in \mathbb{Z}-\nu} \frac{\bar{\psi}_s}{z^{s+1/2}}. \tag{3.4.26b}$$

One can find an algebra for complex pair modes as

$$\left\{\psi_r, \bar{\psi}_s\right\} = \delta_{r,-s} \tag{3.4.27}$$

To find a corresponding operator for the ground state, let's define a reference state $|0\rangle_\nu$ by

$$\psi_{n+\nu} |0\rangle_\nu = \bar{\psi}_{n+1-\nu} |0\rangle_\nu = 0 \quad ; n = 0, 1, \ldots. \tag{3.4.28}$$

We see that the first terms in the Laurent expansions of $\psi, \bar{\psi}$ are $r = -1 + \nu$ and $s = -\nu$. Therefore the operator products of local operator $\mathcal{A}_\nu$, which corresponds to reference state $|0\rangle_{nu}$, should be

$$\psi(z)\mathcal{A}_{nu}(0) = O(z^{-\nu+1/2}), \quad \bar{\psi}(z)\mathcal{A}_{nu}(0) = O(z^{\nu-1/2}). \tag{3.4.29}$$

This determines a bosonic equivalent,

$$\exp[i(-\nu + 1/2)H] \cong \mathcal{A}_\nu. \tag{3.4.30}$$

Note that the periodicity condition is the same for ν and $\nu + 1$ but the reference state we have defined is not. The definition of reference state makes a sense that the state $\nu + 1$ is an excited state,

$$|0\rangle_{\nu+1} = \bar{\psi}_{-\nu} |0\rangle_\nu. \tag{3.4.31}$$

This kind of excitation is called "spectral flow". Reference state is a ground state when $0 \leq \nu \leq 1$. For R case, $\nu = 0$, we get two degenerated ground state,

$$|s\rangle \cong e^{isH} \quad ; s = \pm\frac{1}{2} \tag{3.4.32}$$

Superstring theory in 10-dimension has 10 spacetime fermions. Then there are 5 bosons, $H^a, a = 0, \dots, 4$, for bosonization [4],

$$\frac{1}{\sqrt{2}}(\pm\psi^0 + \psi^1) \cong e^{\pm iH^0} \tag{3.4.33a}$$

$$\frac{1}{\sqrt{2}}(\psi^{2a} \pm \psi^{2a+1}) \cong e^{\pm iH^a} \quad ; a = 1, \dots, 4. \tag{3.4.33b}$$

Vertex operator Θ_s for R state $|s\rangle$ is then,

$$\Theta_s \cong \exp\left[i\sum_a s_a H^a\right] \tag{3.4.34}$$

which gives a branch cut in ψ^μ, so called "spin field". For closed string, there is another set of antiholomorphic vertex operator built from $\tilde{H}^a$.

3.5 BRST and Ghosts

Like other gauge theories, path integral in string theory also has redundant degrees of freedom. There are a gauge fixing method to choose a gauge equivalent class in string theory as well as a Fadeev-Popov technique in QFT. This suggests that the full theory must contains ghost fields which are arisen from the Fadeev-Popov method. To study a spectrum theory more generally, we need to find a condition for path integral that is invariant under arbitrary fixed choice of gauge. We found a combination of an original gauge invariant action, gauge fixing action and Fadeev-Popov action is invariant under BRST transformation [4]. The physical states we have defined should not changed with gauge choice, so any transformations should be invariant under BRST transformation in order to make it valid in gauge theory. This is a reason why we have to consider CFTs with an arise of ghosts. For a complete review of path integral, Fadeev-Popov technique and BRST theory, the reader is refer to [3] [17].

One of CFTs family we are considering is bc theory which the fields b and c are anticommuting ghost fields with an action $S = \frac{1}{2\pi}\int d^2z b\bar{\partial}c$. This type of CFT arise from inverse of bosonic path integral when we consider To make the action conformally invariant, the fields b and c should transform as tensors with weight $(h_b, 0)$ and $(h_c, 0)$ where $h_b = \lambda$ and $h_c = 1-\lambda$ for constant λ. From the action, the equations of motion are

$$\bar{\partial}c(z) = \bar{\partial}b(z) = 0 \tag{3.5.1a}$$

$$\bar{\partial}b(z)c(0) = 2\pi\delta^2(z,\bar{z}), \tag{3.5.1b}$$

make their operator products are

$$b(z_1)c(z_2) \sim \frac{1}{z_{12}}, \;\; c(z_1)b(z_2) \sim \frac{1}{2}, \tag{3.5.2}$$

and others are nonsingular,

$$b(z_1)b(z_2) = O(z), \;\; c(z_1)c(z_2) = O(z) \tag{3.5.3}$$

Note that the operator products (3.5.2) are the same because b and c are anticommuting fields, so the sign is canceled with changing $z_1 \leftrightarrow z_2$. One can find an energy-momentum tensor for bc theory from Noether theorem,

$$T(z) = (\partial b)\, c - \lambda\partial\,(bc)\,. \tag{3.5.4}$$

These are holomorphic part in bc CFT. There is also a $\tilde{b}\tilde{c}$ CFT for antiholomorphic theory with an action $S = \frac{1}{2\pi}\int d^2z\tilde{b}\bar{\partial}\tilde{c}$ which is similar to what we discuss before with $z \leftrightarrow \bar{z}$.

Bosonization for bc CFT could be done the same way with ψ by replacing $\psi \to b$ and $\bar{\psi} \to c$ and modify the energy-momentum tensor to be

$$T_B^{(\lambda)} = T_B^{(1/2)} - \left(\lambda - \frac{1}{2}\right)\partial\,(bc) \tag{3.5.5}$$

which is corresponding to an energy-momentum tensor for bosonic operator

$$T_B^{(\lambda)} \cong T_B^H - i(\lambda - \frac{1}{2})\partial^2 H. \tag{3.5.6}$$

This gives us $b \cong e^{iH}, c \cong e^{-iH}$. However, hermitianess of b, c implies that the bosonic field H must be antihermitian, so we redefine the bc bosonization in form of $H \to i\rho : c \cong e^{\rho}, b \cong e^{-\rho}$.

Another family of CFTs is $\beta\gamma$ CFT which is much similar to bc CFT except that they are commuting fields. The $\beta\gamma$ CFT action is $S = \frac{1}{2}\int d^2z\beta\bar{\partial}\gamma$ and β, γ have weights $h_\beta = \lambda - \frac{1}{2}, h_\gamma = \frac{3}{2} - \lambda$. Energy-momentum tensor for the theory is $T(z) = (\partial\beta)\gamma - \frac{1}{2}(2\lambda - 1)\partial(\beta\gamma)$. Operator products for $\beta\gamma$ theory can be found in the same method with bc theory. The statistic of $\beta\gamma$ theory, however, is opposite the bc theory. Thus the operator products are

$$\beta(z_1)\gamma(z_2) \sim -\frac{1}{z_{12}}, \;\; \gamma(z_1)\beta(z_2) \sim \frac{1}{z_{12}} \tag{3.5.7a}$$

$$\beta(z_1)\beta(z_2) = O(z^0) \tag{3.5.7b}$$

$$\gamma(z_1)\gamma(z_2) = O(z^0). \tag{3.5.7c}$$

To construct BRST theory for superstring theory, SCFT should has both anticommuting bc and commuting $\beta\gamma$ theory which arise from bosonic and fermionic theory respectively. In this dissertation, we use $\lambda = 2$ which is a relevant value for superstring [4]. Thus, bc ghosts have weights $h_b = (2, 0), h_c = (-1, 0)$ and $\beta\gamma$ ghosts have weights $h_\beta = (\frac{3}{2}, 0), h_\gamma = (-\frac{1}{2}, 0)$.

Similar to the method we use to find vertex operator in the previous section, we consider expansions of ghosts,

$$b(z) = \sum_{m=-\infty}^{\infty} \frac{b_m}{z^{m+2}}, \;\; c(z) = \sum_{m=-\infty}^{\infty} \frac{c_m}{z^{m-1}} \tag{3.5.8a}$$

$$\beta(z) = \sum_{r\in\mathbb{Z}+\nu} \frac{\beta_r}{z^{r+3/2}}, \;\; \gamma(z) = \sum_{r\in\mathbb{Z}+\nu} \frac{\gamma_r}{z^{r-1/2}}, \tag{3.5.8b}$$

with (anti)commutator

$$\{b_m, c_n\} = \delta_{m,-n}, \;\; [\gamma_r, \beta_s] = \delta_{r,-s}. \tag{3.5.9}$$

We define ground states $|0\rangle_{NS,R}$ by

$$b_m\,|0\rangle_{NS,R} = 0\ ;\ m \geq 0,\ \ c_m\,|0\rangle_{NS,R} = 0\ ;\ m \geq 1, \tag{3.5.10a}$$

$$\beta_r\,|0\rangle_{NS} = 0\ ;\ r \geq \frac{1}{2},\ \ \gamma_r\,|0\rangle_{NS} = 0\ ;\ r \geq \frac{1}{2}, \tag{3.5.10b}$$

$$\beta_r\,|0\rangle_{R} = 0\ ;\ r \geq 0,\ \ \gamma_r\,|0\rangle_{R} = 0\ ;\ r \geq 1. \tag{3.5.10c}$$

By considering (3.5.8a), we got

$$b_m = \oint \frac{dz}{2\pi} z^{m+1} b(z),\ \ c_m = \oint \frac{dz}{2\pi} z^{m-2} c(z), \tag{3.5.11}$$

suggest that the modes b_m, c_m annihilate identity state $|1\rangle$,

$$b_m\,|1\rangle = 0\ ;\ m \geq -1,\ \ c_m\,|1\rangle = 0\ ;\ m \geq 2. \tag{3.5.12}$$

The anticommutator for bc modes implies that the ground state is built from the identity state, $|0\rangle = c_1\,|1\rangle$, so that our Fock space can be generated from $|1\rangle$. For $\beta\gamma$ modes, we got

$$\beta_r = \oint \frac{dz}{2\pi} z^{r+1/2} \beta(z),\ \ \gamma_r = \oint \frac{dz}{2\pi} z^{r-3/2} \gamma(z). \tag{3.5.13}$$

For NS sector $\beta\gamma$, these modes satisfy

$$\beta_r\,|1\rangle = 0\ ;\ r \geq -\frac{1}{2},\ \ \gamma_r\,|1\rangle = 0\ ;\ r \geq \frac{3}{2}. \tag{3.5.14}$$

However, the β_r, γ_r are commuting oscillators, so there is no such a Fock space which is built from $|1\rangle$ using $\gamma_{1/2}$ like what we have done with bc theory. Therefore we have to find a new way to define the $\beta\gamma$ ground state. We will give a try with the idea that the ground state might be constructed from $\gamma(z)$ as same as the bc ghosts NS ground state that mapped to the operator c.

Let's consider properties of the ground state $|0\rangle_{NS}$. Assume that there is an operator $\delta(\gamma)$ corresponding to the state $|0\rangle_{NS}$, the equations (3.5.10b) are mapped to

$$\gamma_r \delta(\gamma(0)) \sim \oint dz z^{r-3/2} \gamma(z) \delta(\gamma(0)) = 0\ \ ;\ r \geq \frac{1}{2} \tag{3.5.15a}$$

$$\beta_r \delta(\gamma(0)) \sim \oint dz z^{r+1/2} \beta(z) \delta(\gamma(0)) = 0\ \ ;\ r \geq \frac{1}{2}, \tag{3.5.15b}$$

so (3.5.10b) can be translated to

$$\gamma(z)\delta(\gamma(0)) = O(z), \;\; \beta(z)\delta(\gamma(0)) = O(z^{-1}). \tag{3.5.16}$$

We use a bosonization technique to find an explicit description of the operator $\delta(\gamma)$, by considering operator product

$$\beta\gamma(z)\beta\gamma(0) \sim -\frac{1}{z^2} \tag{3.5.17}$$

and a holomorphic scalar field ϕ which has an operator product $\phi(z)\phi(0) \sim -ln(z)$. We see that (3.5.17) has the same operator product with $\partial\phi$, so we got an equivalent $\beta\gamma(z) \cong \partial\phi(z)$. From (3.5.7a), one might see that the equivalence is analog with the *bc* bosonization,

$$\beta(z) \stackrel{?}{\cong} e^{-\phi(z)}, \;\; \gamma(z) \stackrel{?}{\cong} e^{\phi(z)}. \tag{3.5.18}$$

However, this gives us wrong operator products,

$$\beta(z)\beta(0) \stackrel{?}{=} O(z^{-1}), \;\; \beta(z)\gamma(0) \stackrel{?}{=} O(z^{1}), \;\; \gamma(z)\gamma(0) \stackrel{?}{=} O(z^{-1}). \tag{3.5.19}$$

We fix this by adding factors,

$$\beta(z) \cong e^{-\phi(z)}\partial\xi(z), \;\; \gamma(z) \cong e^{\phi(z)}\eta(z) \tag{3.5.20}$$

where $\eta(z)$ and $\xi(z)$ must be nonsingular with respect to ϕ in order to conserve the $\beta\gamma$ structure. On the other hand $\eta\xi$ is a new CFT which is decoupled from ϕ CFT. The correct operator products is restored when η, ξ satisfy

$$\eta(z)\xi(0) \sim \frac{1}{2}, \;\; \eta(z)\eta(0) = O(z), \;\; \partial\xi(z)\partial\xi(0) = O(z). \tag{3.5.21}$$

Therefore the operator products of $\eta\xi$ theory is the same with CFT of the *bc* type. The properties (3.5.16) of $\delta(\gamma)$ suggests the bosonization,

$$\delta(\gamma) \cong e^{-\phi} \tag{3.5.22}$$

with weight $h = \frac{1}{2}$ for $\lambda = 2$. We got that the operators $e^{-\phi}, e^{-\phi}e^{\pm iH^a}$ are fermionic parts for tachyon and massless NS vertex operator respectively.

For R ground state $|0\rangle_R$, again we assume the corresponding operator Σ which satisfies

$$\beta(z)\Sigma(0) = O(z^{-1/2}, \ \ \gamma(z)\Sigma(0) = O(z^{1/2}). \tag{3.5.23}$$

One can find that $\Sigma \cong e^{-\phi/2}$ with weight $h = \frac{3}{8}$ and R ground state vertex operator is then $\mathscr{V}_s = e^{-\phi/2}\Theta_s$.

3.6 Vertex Operator

Let us consider superspace/supermanifold formalism, the superconformal symmetry can be described with more geometric interpretation. The world sheet in superspace consists of complex coordinate z and anticommuting complex coordinate θ with property $\theta^2 = \bar{\theta}^2 = \left\{\theta, \bar{\theta}\right\} = 0$. Thus a superfield   has both commuting and anticommuting properties. One could treats the superfield   as a combination of various fields through a Taylor expansion. Under an infinitesimal superconformal transformation $\delta\theta = \epsilon\eta(z)$, a superfield   with weight $(h, \tilde{h})$ transforms as

$$\delta\ (\mathbf{z},\mathbf{z}) = -\epsilon\left[2h\theta\partial\eta(z) + \eta(z)\mathcal{Q}_\theta + 2\tilde{h}\bar{\theta}\bar{\partial}\bar{\eta}(\bar{z}) + \bar{\eta}(\bar{z})\mathcal{Q}_{\bar{\theta}}\right]\ (\mathbf{z},\mathbf{z}) \tag{3.6.1}$$

where $\mathbf{z}$ denotes (z,θ) and $\mathcal{Q}_\theta \equiv \partial_\theta - \theta\partial_z$, $\mathcal{Q}_{\bar{\theta}} \equiv \partial_{\bar{\theta}} - \bar{\theta}\partial_{\bar{z}}$. For simplicity, let's focus on the holomorphic part of a Taylor expansion for the superfield,

$$(\mathbf{z}) = \mathcal{O}(z) + \theta\phi(z). \tag{3.6.2}$$

Note that all the higher order of θ vanished due to its property, so the Taylor expansion for superfield is exact. From (3.6.1), we got

$$\delta\mathcal{O} = -\epsilon\eta\psi, \ \ \delta\psi = -\epsilon[2h\partial\eta\mathcal{O} + \eta\partial\mathcal{O}], \tag{3.6.3}$$

by comparing to (3.4.18), we got

$$G_{-1/2}\cdot\mathcal{O} = \psi, \ \ G_r\cdot\mathcal{O} = 0\ ;\ r \geq \frac{1}{2}, \tag{3.6.4a}$$

$$G_{-1/2}\cdot\psi = \partial\mathcal{O}, \ \ G_{1/2}\cdot\psi = 2h\mathcal{O}, \ \ G_r\cdot\psi = 0\ ;\ r \geq \frac{3}{2} \tag{3.6.4b}$$

We now write vertex operators we will use to find the string amplitude in the new chapter. The string vertex operators we are interested in must describe "physical states" with weight $(1,1)$, for closed strings, to make it offset the transformation of d^2z. Start from the previous state-operator mapping, the NS-NS vertex operator should be in the form of $\delta(\gamma)\delta(\tilde{\gamma})$ times superconformal tensor field of weight $(\frac{1}{2},\frac{1}{2})$ which is the lowest component of the superfield , in other words when $\theta, \bar{theta}$ are fixed to be zero ,

$$\mathscr{V}^{-1,-1} = e^{-\phi-\tilde{\phi}}\mathcal{O}. \tag{3.6.5}$$

Note that the convention we used the vertex operators are label with charge of $\phi, \tilde{phi}$: operator with charge $(q,\tilde{q})$ is said to be in a $(q,\tilde{q})$ picture. In this case, it is a $(-1,-1)$ picture. Another vertex operator is integrated over θ and $\bar{\theta}$ of $(\frac{1}{2},\frac{1}{2})$ superfield. From (3.6.4a), the vertex operator should be in the form of

$$\mathscr{V}^{(0,0)} = G_{-1/2}\tilde{G}_{-1/2}\cdot\mathcal{O} \tag{3.6.6}$$

which is in a $(0,0)$ picture.

Consider a massless state

$$\psi^{\mu}_{-1/2}\tilde{\psi}^{\nu}_{-1/2}\left|0;k\right\rangle_{NS} \tag{3.6.7}$$

which corresponds to a vertex operator

$$\mathscr{V}^{-1,-1} = g_c e^{-\phi-\tilde{\phi}}\psi^{\mu}\tilde{\psi}^{\nu}e^{ik\cdot X}. \tag{3.6.8}$$

For integrated vertex operator, consider a massless state

$$\begin{aligned}
&G_{-1/2}\tilde{G}_{-1/2}\psi^{\mu}_{-1/2}\tilde{\psi}^{\nu}_{-1/2}\left|0;k\right\rangle_{NS}\\
&= -(G_{-\frac{1}{2}}\psi^{\mu}_{-\frac{1}{2}})(\tilde{G}_{-\frac{1}{2}}\tilde{\psi}^{\nu}_{-\frac{1}{2}})\left|0;k\right\rangle_{NS}\\
&= -(\{G_{-\frac{1}{2}},\psi^{\mu}_{-\frac{1}{2}}\} - \psi^{\mu}_{-\frac{1}{2}}G_{-\frac{1}{2}})(\{\tilde{G}_{-\frac{1}{2}},\tilde{\psi}^{\nu}_{-\frac{1}{2}}\} - \tilde{\psi}^{\nu}_{-\frac{1}{2}}\tilde{G}_{-\frac{1}{2}})\left|0;k\right\rangle_{NS}\\
&= -(\alpha^{\mu}_{-1} + \alpha_0\cdot\psi_{-1/2}\psi^{\mu}_{-1/2})(\tilde{\alpha}^{\nu}_{-1} + \tilde{\alpha}_0\cdot\tilde{\psi}_{-1/2}\tilde{\psi}^{\nu}_{-1/2})\left|0;k\right\rangle_{NS}.
\end{aligned} \tag{3.6.9}$$

From the mode expansion of G_r and anticommutator of mode ϕ_r, we got that

$$\begin{aligned}
\left\{G_{-1/2},\psi^{\mu}_{-1/2}\right\} &= \sum_{n\in\mathbb{Z}}\alpha^{\nu}_{n}\left\{\psi_{\nu-1/2-n},\psi^{\mu}_{-1/2}\right\}\\
&= \sum_{n\in\mathbb{Z}}\alpha^{\nu}_{n}\delta^{\mu}_{\nu}\delta_{-1/2-n,+1/2} = \alpha^{\mu}_{-1}
\end{aligned} \tag{3.6.10}$$

and

$$G_{-1/2}\left|0;k\right\rangle_{NS}=\sum_{n\in\mathbb{Z}}\alpha_n\cdot\psi_{-1/2-n}\left|0;k\right\rangle_{NS}=\alpha_0\cdot\psi_{-1/2}\left|0;k\right\rangle_{NS} \tag{3.6.11}$$

and similarly for antiholomorphic part. Thus (3.6.9) is equal to

$$-(\alpha^{\mu}_{-1}+\alpha_0\cdot\psi_{-1/2}\psi^{\mu}_{-1/2})(\tilde{\alpha}^{\nu}_{-1}+\tilde{\alpha}_0\cdot\tilde{\psi}_{-1/2}\tilde{\psi}^{\nu}_{-1/2})\left|0;k\right\rangle_{NS}, \tag{3.6.12}$$

which is corresponding to a vertex operator

$$\mathscr{V}^{0,0}=\frac{2g_c}{\alpha'}(i\partial X^{\mu}+\frac{1}{2}\alpha' k\cdot\psi\psi^{\mu})(i\bar{\partial}X^{\nu}+\frac{1}{2}\alpha' k\cdot\tilde{\psi}\tilde{\psi}^{\nu})e^{ik\cdot X}. \tag{3.6.13}$$

For massless open string vector, vertex operators are then

$$\mathscr{V}^{-1}=g_o e^{-\phi}t^a\psi^{\mu}e^{ik\cdot X}, \tag{3.6.14a}$$

$$\mathscr{V}^{0}=g_o(2\alpha')^{-1/2}t^a(i\dot{X}^{\mu}+2\alpha' k\cdot\psi\psi^{\mu})e^{ik\cdot X} \tag{3.6.14b}$$

where t^a is Chan-Paton factor. Note that for type I/II superstring $g_c=\frac{\kappa}{2\pi};\ \kappa\equiv\kappa_{10}e^{\Phi}$ and $g_o=g_{YM}(2\alpha')^{1/2};\ g_{YM}\equiv g_{10}e^{\Phi/2}$ [3]. These vertex operators will be used to calculate a string amplitude in the next chapter.

CHAPTER IV

String Amplitude and D-brane Decay

We are now ready to calculate string interaction amplitudes. By considering tree-level amplitudes for low energy quanta, which analog with tree diagram of Feynman diagram without quantum correction. We first discuss the Riemann surface where the vertex operators are on, in this case, the disk. Then we calculate amplitudes and set some configurations to make it match the setup of what we have done in quantum field theory in curved spacetime chapter. Final we show the result in a form of an energy emission rate, as same as previous one. The amplitude calculation sections in this chapter are reviewed from [2–4] and the remaining sections are from [1].

4.1 String Amplitudes

In this section we show how we obtain an amplitude using the vertex operators from the previous chapter.

4.1.1 Four-point Disk Amplitudes

The disk D_2 can be constructed by identifying points under a reflection. It is convenient to use the conformal equivalent reflection, $z' = \bar{z}$. This makes the upper-plane become a fundamental domain, the representative region of all points in the plane, and the real axis is the boundary of the disk. One can simply check that a complex coordinate in the upper-half plane v can be mapped to a disk coordinate w, $w = \frac{v-i}{v+i}$, with the real axis being the disk boundary.

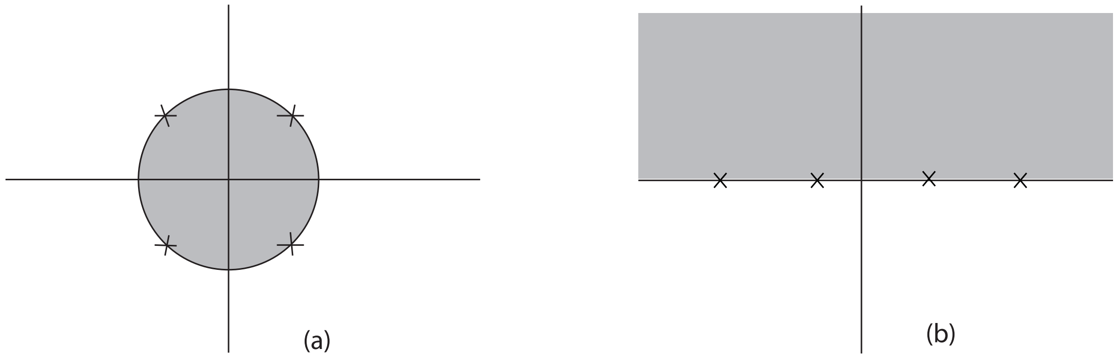


Figure 4.1: Diagram of a disk (a) mapping to an upper-half plane (b) with the boundary and vertex operators lie on it mapped to the real axis.

We calculate a tree amplitude of open strings which correspond to vertex operators on the disk. Since the boundary of disk is mapped to the real axis, the vertex operators are not only a purely holomorphic function but a function of real coordinate as well. The amplitude is assumed to have a form

$$A(\zeta_i, k_i) \sim \int \frac{\prod_i dy_i}{V_{CKG}} \left\langle \prod_i \zeta_i \cdot \mathscr{V}_i(y_i, k_i) \right\rangle \tag{4.1.1}$$

where $\mathscr{V}_i$ is a vertex operator with momentum k_i, ζ_i is a polarization and y_i denoting a coordinate on the real axis. The volume V_{CKG} is a volume of $SL(2, \mathbb{R})$ which is a *conformal Killing group* of a disk.

The calculation could be completed by using the contractions

$$\langle X^\mu(z) X^\nu(w) \rangle = -\eta^{\mu\nu} \ln(z - w) \tag{4.1.2a}$$

$$\langle \psi^\mu(z) \psi^\nu(w) \rangle = -\frac{\eta^{\mu\nu}}{z - w} \tag{4.1.2b}$$

$$\langle \phi(z) \phi(w) \rangle = -ln(z - w). \tag{4.1.2c}$$

where the complex coordinates z and w become real coordinates when the fields are on the disk. Note that we always use $\alpha' = 2$ for the embedding coordinate for both type-I and type-II superstring. The calculation is straightforward but tremendous. To make it simple, let's consider an n-point function of a tachyon correlation function on D_2,

$$A^n_{D_2}(k) \equiv \left\langle \prod_{i=1}^{n} :e^{ik_i\cdot X(y_i)}: \right\rangle_{D_2}, \tag{4.1.3}$$

and a correlation function

$$\begin{aligned}
\left\langle \partial X^\mu(y)\prod_{i=1}^{n} :e^{ik_i\cdot X(y_i)}: \right\rangle_{D_2} &= \left\langle \partial X^\mu(y)\exp(\sum_{i=1}^{n} :ik_i\cdot X(y_i):) \right\rangle_{D_2} \\
&= \left\langle \partial X^\mu(y)\sum_{m=0}^{\infty}\left(\frac{1}{m!}(\sum_{i=1}^{n} :ik_i\cdot X(y_i):)^m\right) \right\rangle_{D_2} \\
&= -iA^n_{D_2}(k)\sum_{i=1}^{n}\frac{k_i^\mu}{y-y_i} + (\text{terms holomorphic in y}).
\end{aligned} \tag{4.1.4}$$

Now the poles of the above equation arise when $y \to y_i$ for some i, for instant let's consider $i = 1$,

$$-iA^n_{D_2}(k)\left(\frac{k_i^\mu}{y-y_1} + \sum_{i=2}^{n}\frac{k_i^\mu}{y_1-y_i} + O(y-y_1)\right) \tag{4.1.5}$$

which should satisfy the operator product,

$$ik_1\cdot\partial X(y):e^{ik_1\cdot X(y)}:= \frac{k_1^2}{y-y_1}:e^{ik_1\cdot X(y)}: + \partial_{y_1}:e^{ik_1\cdot X(y)}: + O(y-y_1). \tag{4.1.6}$$

Thus,

$$\partial_{y_1}A^n_{D_2}(k) = A^n_{D_2}(k)\sum_{i=2}^{n}\frac{k_1\cdot k_n}{y_{1i}}. \tag{4.1.7}$$

Together with momentum conservation, it leads to the result of n-point tachyon correlation function up to normalization,

$$A^n_{D_2}(k) = C_{D_2}(2\pi)^d\delta^d(\sum_i k_i)\prod_{\substack{i,j=1\\ i<j}}^{n}|y_{ij}|^{k_i\cdot k_j} \tag{4.1.8}$$

where $C_{D_2} = 1/g_o^2$ as mention in [3]. One can use make a benefit of this result by express any amplitude with vertex operators contain $exp(ik\cdot X)$ as other contractions time $A^n_{D_2}$.

We are interested in a tree amplitude of 4 massless open strings interaction. Recall from the previous chapter, the massless vertex operators for $-1, 0$-superghost pictures with momentum k are

$$\mathscr{V}_{-1}^{\mu}(y, 2k) = g_o e^{-\phi}\psi^{\mu}e^{i2k\cdot X}(y) \tag{4.1.9a}$$

$$\mathscr{V}_{0}^{\mu}(y, 2k) = g_o\left(\partial X^{\mu} + i2k\cdot\psi\psi^{\mu}\right)e^{i2k\cdot X}(y) \tag{4.1.9b}$$

where $\mu = 0, \ldots, 9$. The four point amplitude of D-brane oscillators has a form [2]

$$A(\zeta_1, k_1; \zeta_2, k_2; \zeta_3, k_3; \zeta_4, k_4) \sim \int \frac{dy_1 dy_2 dy_3 dy_4}{V_{CKG}} \langle \zeta_1\cdot\mathscr{V}_0(2k_1, y_1)\zeta_2\cdot\mathscr{V}_0(2k_2, y_2)\zeta_3\cdot\mathscr{V}_{-1}(2k_3, y_3)\zeta_4\cdot\mathscr{V}_{-1}(2k_4, y_4)\rangle\,. \tag{4.1.10}$$

Other than a tachyon part, all of the contractions are ∂X^{μ} and ψ^{μ} which are only in products of $|y_{ij}|^{-1}$ where $i<j$ and their momenta. Together with the tachyon correlation function, the integration of (4.1.10) is now in a form of

$$\int y_1 y_2 y_3 y_4 \prod_{\substack{i,j=1\\ i<j}}^{n} |y_{ij}|^{k_i\cdot k_j + a_{ij}} \tag{4.1.11}$$

where a_{ij} is a power of z_{ij} from the contraction of ∂X^{μ} and ψ^{μ} terms. The $PSL(2,\mathbf{R})$ CKG has three generators which can be used to fix the three vertex operators to arbitrary positions. However, this group does not change the cyclic ordering of the vertex operators so we have to add all possible orderings, as in figure 4.2. For 4-point functions, it is convenient to take $y_1 = 0$, $y_2 = 1$ and $y_3 \to \infty$. By fixing $y_{1,2,3}$, the integration (4.1.11) is now only integrates over y_4,

$$\int dy_4 |y_4|^{k_1\cdot k_4 + a_{14}}|1-y_4|^{k_2\cdot k_4 + a_{24}} + (k_2 \leftrightarrow k_3). \tag{4.1.12}$$

It is our convenient to write the amplitude in terms of the *Mandelslam variables*,

$$s = -(k_1+k_2)^2, \;\; t = -(k_1+k_3)^2, \;\; u = -(k_1+k_4)^2 \tag{4.1.13}$$

where the momentum conservation gives

$$s + t + u = \sum_i m_i^2 = -\frac{4}{\alpha'}. \tag{4.1.14}$$

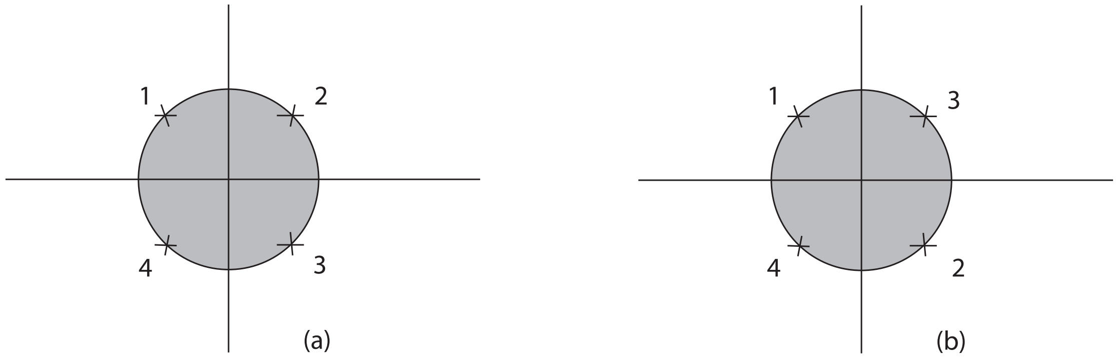


Figure 4.2: Diagram shows the possible orderings of four vertex operators on the disk.

Note that for our interests, each vertex operator is a massless string, so that any $k_i^2 = 0$. The integration (4.1.12) becomes

$$\int_{-\infty}^{\infty} dy_4 |y_4|^{-u+a_{14}} |1-y_4|^{-t+a_{24}} + (t \to s). \tag{4.1.15}$$

The integral splits into three range, $-\infty < y_4 < 0, 0 < y_4 < 1$ and $1 < y_4 < \infty$, which are ordered as in figures 4.3(a), (b) and (c) respectively and the $(t \leftrightarrow s)$ term gives figures 4.3(d), (e) and (f). Möbius invariance, an invariance under $PSL(2, \mathbf{R})$ transformation, can be used to take each of these ranges into any other. These make the integration (4.1.15) becomes

$$\begin{aligned} &2B(-s+a_{34}+1, -t+a_{24}+1) + 2B(-t+a_{24}+1, -u+a_{14}+1) \\ &+ 2B(-u+a_{14}+1, -s+a_{34}+1) \end{aligned} \tag{4.1.16}$$

where $B(a, b)$ is the beta function

$$B(a, b) = \int_0^1 dy y^{a-1} (1-y)^{b-1}, \tag{4.1.17}$$

with properties

$$B(a, b) = B(b, a) \tag{4.1.18a}$$

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \tag{4.1.18b}$$

where $\Gamma(a)$ is the gamma function.

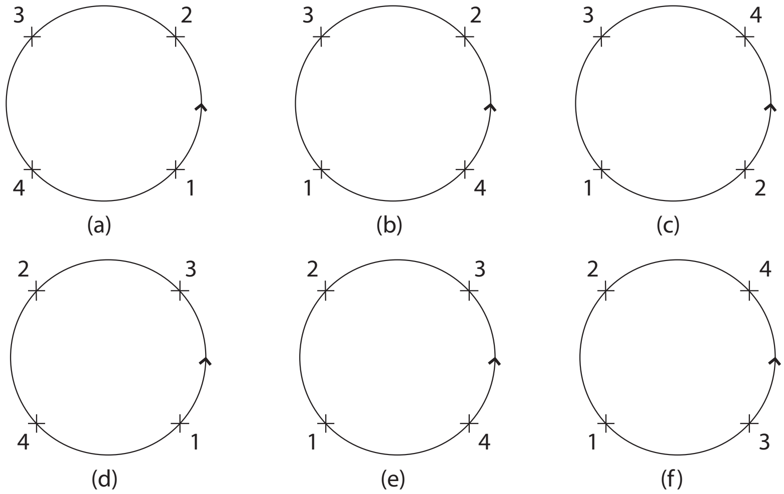


Figure 4.3: Diagrams of cyclical orderings of four vertex operators on the disk when points 1, 2 and 3 are fixed. The coordinate of the real axis increases in the direction of the arrow except that it jumps from $+\infty$ to $-\infty$ at point 3. The range of integration of point 4 separates into (a), (b), (c) and (d), (e), (f) for $(t \to s)$

After all contractions, the result is our 4-point amplitude (4.1.10) [2],

$$A = \frac{\Gamma(4k_1 \cdot k_2)\Gamma(4k_1 \cdot k_4)}{\Gamma(1 + 4k_1 \cdot k_2 + 4k_1 \cdot k_4)} K(\zeta_1, k_1; \zeta_2, k_2; \zeta_3, k_3; \zeta_4, k_4) \tag{4.1.19}$$

where K is a kinematic function

$$\begin{aligned} K =& 4k_2 \cdot k_3 k_2 \cdot k_4 \zeta_1 \cdot \zeta_2 \zeta_3 \cdot \zeta_4 \\ &+ 4k_1 \cdot k_2 (\zeta_1 \cdot k_4 \zeta_3 \cdot k_2 \zeta_2 \cdot \zeta_4 + \zeta_2 \cdot k_3 \zeta_4 \cdot k_1 \zeta_1 \cdot \zeta_3 \\ &+ \zeta_1 \cdot k_3 \zeta_4 \cdot k_2 \zeta_2 \cdot \zeta_3 + \zeta_2 \cdot k_4 \zeta_3 \cdot k_1 \zeta_1 \cdot \zeta_4) \\ &+ \{(1234) \to (1324)\} + \{(1234) \to (1432)\} \end{aligned} \tag{4.1.20}$$

For four world volume photon, we let ζ_l be in the world volume directions. The kinematic function is the same as above. For our calculation, we need scalar scattering, so ζ_l must be in the transverse directions, $\zeta_l \cdot k_m = 0$. This makes the polarization dependent kinematic function become

$$\begin{aligned} K =& 4k_2 \cdot k_3 k_2 \cdot k_4 \zeta_1 \cdot \zeta_2 \zeta_3 \cdot \zeta_4 + 4k_2 \cdot k_3 k_3 \cdot k_4 \zeta_1 \cdot \zeta_3 \zeta_2 \cdot \zeta_4 \\ &+ 4k_4 \cdot k_3 k_2 \cdot k_4 \zeta_1 \cdot \zeta_4 \zeta_2 \cdot \zeta_3. \end{aligned} \tag{4.1.21}$$

4.1.2 Three-point Disk Amplitudes of Two Open Strings and a Closed String

In this section, we use the result of four-point amplitude to calculate the collision of two massless open strings on D-brane to produce a massless closed string state in the bulk. Let's assign momenta p_1 and p_2 to the open string states which are restricted within the D-brane world volume and momentum q for the closed string. Only the momentum along the brane direction is conserved,

$$p_1 + p_2 + q_{\|} = 0 \tag{4.1.22}$$

and the non-conservation of momenta in the transverse direction of the brane gives some freedom for the closed string to rise. The conservation of the longitudinal momenta implies that there is only one kinematic invariant variable, $t = 2p \cdot k = 2q \cdot k = -2p \cdot q$.

The leading order of amplitude for two massless open strings on the disk scatters into a closed string in the bulk is can be modified from the four-point function,

$$A = \int \frac{dz_1 dz_2 d^2 z_3}{V_{CKG}} \left\langle \mathscr{V}_1(z_1) \mathscr{V}_2(z_2) \mathscr{V}_3(z_3, \bar{z}_3) \right\rangle \tag{4.1.23}$$

where

$$\mathscr{V}_1(z_1) = \xi^1_\mu \mathscr{V}^\mu_0(z_1, p) \tag{4.1.24}$$

$$\mathscr{V}_2(z_2) = \xi^2_\nu \mathscr{V}^\nu_0(z_2, q) \tag{4.1.25}$$

$$\mathscr{V}_3(z_3, \bar{z}_3) = \frac{g_c}{g_o^2} \varepsilon_{\sigma\lambda} \mathscr{V}^\sigma_{-1}(z_3, k) \tilde{\mathscr{V}}^\lambda_{-1}(\bar{z}_3, k). \tag{4.1.26}$$

Note that $\tilde{\mathscr{V}}^\lambda_{-1}$ is an antiholomorphic operator of $\mathscr{V}^\lambda_{-1}$ and the closed string vertex operator $\mathscr{V}_3$ uses g_c instead of g_o^2. We can represent antiholomorphic fields in term of holomorphic fields by the rules,

$$\tilde{X}^\mu(\bar{z}) \to D^\mu_\nu X^\nu(\bar{z}), \quad \tilde{\psi}^\mu(\bar{z}) \to D^\mu_\nu \psi^\nu(\bar{z}), \quad \tilde{\phi}(\bar{z}) \to \phi(\bar{z}). \tag{4.1.27}$$

The matrix D^μ_ν is a diagonal matrix which first $p+1$ entries (longitudinal directions) are equal to 1 and the other $9-p$ entries (transverse directions) are equal to -1. Therefore the 3-point function amplitude is

$$A = \int \frac{dz_1 dz_2 dz_3 d\bar{z}_3}{V_{CKG}} \xi^1_\mu \xi^2_\nu \varepsilon_{\sigma\lambda} D^\lambda_\eta \left\langle \mathscr{V}^\mu_0(z_1, 2p) \mathscr{V}^\nu_0(z_2, 2q) \mathscr{V}^\sigma_{-1}(z_3, k) \mathscr{V}^\eta_{-1}(z_4, D \cdot k) \right\rangle . \tag{4.1.28}$$

We can also write this amplitude to be the same with the 4-point correlation function we calculated before by changing some variables,

$$\begin{aligned} 2p \to 2k_1 \quad 2q \to 2k_2 \quad k \to 2k_3 \quad D \cdot q \to 2k_4 \\ \xi_1 \to \zeta_1 \quad \xi_2 \to \zeta_2 \quad \varepsilon \cdot D \to \zeta_3 \otimes \zeta_4, \end{aligned} \tag{4.1.29}$$

with the Mandelslam variables,

$$s = 4k_1 \cdot k_2 = 4k_3 \cdot k_4 = 4p \cdot q, \tag{4.1.30a}$$

$$t = 4k_1 \cdot k_3 = 4k_2 \cdot k_4 = 2p \cdot k, \tag{4.1.30b}$$

$$u = 4k_1 \cdot k_4 = 4k_2 \cdot k_3 = 2q \cdot k. \tag{4.1.30c}$$

Note that $t = 4k_1 \cdot k_3 = 2p_1 \cdot q$ is the same with the previous kinematic quantity.

The three-point function calculation for two open strings and one closed string can be done easily by fixing the operators at $(z_1, z_2, z_3, z_4) = (x, -x, i, -i)$, which corresponds to the open strings vertex operators on the real axis and fixing the closed string vertex operator at $z = i$. Furthermore, there is only one kinematic invariant variable, we can set $s = 4k_1 \cdot k_2 = -2t$ and then $u = t$ by the property of Mandelstam variables. By straightforward calculation or replace the variables, one can show that the three-point amplitude is in the form

$$A \sim g_c \frac{\Gamma[-2t]}{\Gamma[1-t]^2} K(1,2,3) \tag{4.1.31}$$

where $K(1,2,3)$ is the three-point function kinematic factor. We take the polarization of the closed string to be in the D-brane transverse direction, make the kinematic factor more simple and the amplitude is then

$$A \sim g_c \frac{\Gamma[-2t]}{\Gamma[1-t]^2} t^2 \left(\xi^1 \cdot \epsilon \cdot \xi^2 + \xi^2 \cdot \epsilon \cdot \xi^1 \right) \sim g_c \frac{\Gamma[1-2t]}{\Gamma[1-t]^2} t \epsilon_{ij} \xi^i_1 \xi^j_2. \tag{4.1.32}$$

For low t, $t \ll 1$, this amplitude becomes $A \sim g_c t \epsilon_{ij} \xi^i_1 \xi^j_2$ which coincides with the amplitude obtained from the D-brane effective action in the next section when the string coupling $g_c = \kappa/2\pi$ for type I/II superstring [4]. The features of this calculation are the kinematics and a different prescription for fixing the residual $SL(2, \mathbf{R})$ on the world sheet.

4.1.3 Amplitude from D-brane Effective Action

For low energy limit, we can use the Dirac-Born-Infeld action to calculate the amplitude. The DBI action can be written as

$$S_{BI} = T \int d^2\xi e^{-\phi(x)} \sqrt{det[G_{mn}(x) + B_{mn}(x) + F_{mn}(x)} \tag{4.1.33}$$

where F_{mn} is the gauge field strength on the D-string worldsheet, G_{mn} is the background metric, B_{mn} is NS-NS 2 form-fields on the worldsheet,

$$G_{mn} = G^{(s)}_{\mu\nu}(x)\partial_m X^\mu \partial_n X^\nu, \;\; B_{mn} = B_{\mu\nu}(x)\partial_m X^\mu \partial_n X^\nu, \tag{4.1.34}$$

and T is a D-string tension.

By requiring the condition of two open strings interact with a graviton, we can read off the corresponding term from the DBI action,

$$\frac{1}{2}\partial_\alpha X^i \partial^\alpha X^j G_{ij}. \tag{4.1.35}$$

The lowest order interaction between the metric fluctuations and the open string modes can be obtained by expanding the metric around flat space, $G_{\mu\nu} = \eta_{\mu\nu} + 2\kappa h_{\mu\nu}(X)$ and expanding the transverse coordinates $X^i(\xi)$ around the brane position $X^i = 0$, where $h_{\mu\nu}$ and X^i are small. Note that the constant κ is the ten dimensional gravitational coupling. The interaction term with specific polarization, say h_{67}, is then

$$\sqrt{2}\kappa \bar{h}_{67} \partial X^6 \partial X^7. \tag{4.1.36}$$

Note that it is written in a properly normalized form $\bar{h}_{67} = \sqrt{2} h_{67}$. By considering the pair of open strings with worldsheet momenta (p_0, p_1) and (q_0, q_1), the open strings merge into a closed string with momentum $(k_0, k_1, \vec{k})$ where $\vec{k}$ denotes the momentum in transverse direction. One can read off the amplitude from this interaction term,

$$\mathcal{A}_D = \sqrt{2}\kappa A_D = \sqrt{2}\kappa p \cdot q. \tag{4.1.37}$$

4.2 D-brane Thermodynamics

Similar to standard thermodynamics, we are going to assume the statistical mechanics of massless strings on the brane [1]. For large D-brane, we can approximate the ensemble by a canonical ensemble. For a system of n_r particles with energy e_r, let the total momentum P goes along the X^1 direction and p_r is a momentum of a particle r in this direction. We define a partition function $\mathcal{Z}$ by

$$\mathcal{Z} = e^h = \sum_{\text{state}} exp[-\beta \sum_r n_r e_r - \alpha \sum_r n_r p_r]. \tag{4.2.1}$$

The characterizing constants β and α have the same meaning with the inverse temperature and the chemical potential in thermodynamics. They are determined by the total energy E and total momentum P,

$$E = -\frac{\partial h}{\partial \beta}, \quad P = -\frac{\partial h}{\partial \alpha}. \tag{4.2.2}$$

The density of particles n_r in a state (e_r, p_r) is given by

$$\rho(e_r, p_r) = \frac{1}{e^{\beta e_r + \alpha p_r} \pm 1} \tag{4.2.3}$$

where the plus sign goes for fermions and minus for bosons. Then the entropy S is given by the standard thermodynamics relation,

$$S = h + \alpha P + \beta E \tag{4.2.4}$$

For D-string with f bosons and f fermions, E, P and S can be evaluated as [1]

$$E = \frac{fL\pi}{8}[\frac{1}{(\beta+\alpha)^2} + \frac{1}{(\beta-\alpha)^2}] \tag{4.2.5a}$$

$$P = \frac{fL\pi}{8}[\frac{1}{(\beta+\alpha)^2} - \frac{1}{(\beta-\alpha)^2}] \tag{4.2.5b}$$

$$S = \frac{fL\pi}{4}[\frac{1}{\beta+\alpha} + \frac{1}{\beta-\alpha}]. \tag{4.2.5c}$$

The massless particles in one spatial dimension implies that they are either right moving, $e_r = p_r$, or left moving, $e_r = -p_r$, then the distribution functions for right and left moving particles become

$$\rho_R = \frac{1}{e^{(\beta+\alpha)e_r} \pm 1} \tag{4.2.6a}$$

$$\rho_L = \frac{1}{e^{(\beta-\alpha)e_r} \pm 1}. \tag{4.2.6b}$$

We can treat $T_R = 1/(\beta+\alpha)$ and $T_L = 1/(\beta+\alpha)$ as effective temperatures for right and left moving particles. We can also write all of the thermodynamics quantities into left and right moving, $E = E_R + E_L, P = P_R + P_L$ and $S = S_R + S_L$. The equation (4.2.5) implies that

$$T_R = \frac{4S_R}{\pi f L}, \;\; T_L = \frac{4S_L}{\pi f L} \tag{4.2.7a}$$

$$T_R = \sqrt{\frac{8E_R}{\pi f L}}, \;\; T_L = \sqrt{\frac{8E_L}{\pi f L}}. \tag{4.2.7b}$$

This relations will be used in the decay rate calculation in the next section.

4.3 D-brane Decay Rate

We now calculate the decay rate of two open strings on D-brane scattering into a closed string in the bulk. We use the configuration that 5D-brane wrap around T^5 in $(X^5 - X^6 - X^7 - X^8 - X^9)$ directions and D-string wrap around X^5 (which has radius R) Q_1 times with the radius of T^4 in $(X^6 - X^7 - X^8 - X^9)$ much smaller than R. For low energy excitations of the D-brane by massless modes of open strings, their polarizations are in the directions of $(6-7-8-9)$ plane and their ends are on the D-string. Thus there are 4 bosonic and 4 fermionic modes. From the point of view of these dimensions, the emitting gravitons in the bulk are considering as scalar particles.

The S-matrix element is given by

$$S_{fi} = (2\pi)^2 \delta(p_0 + q_0 - k_0)\delta(p_1 + q_1 - k_1) \frac{-i\mathcal{A}_D}{\sqrt{(2p_0 L)(2q_0)(2k_0 V_9)}} \tag{4.3.1}$$

where V_9 is the five dimensional spatial volume and L is the length of the D-string. With the configuration, the length is then $L = 2\pi Q_1 R$ and $V_9 = 2\pi R V_4 \tilde{V}_4$ where V_4 denotes the volume of the spatial noncompact four dimension and $\tilde{V}_4$ is the volume of the compact direction T^4. The decay rate for a pair of open strings producing a graviton with $\mathcal{A}_D = \sqrt{2}\kappa A_D$ is then

$$\Gamma(p,q,k) = \frac{\kappa_5^2(2\pi)^2}{4L}\delta(p_0+q_0-k_0)\delta(p_1+q_1-k_1)\frac{|A_D|^2}{p_0 q_0 k_0 V_4}\frac{V_4[d^4]}{(2\pi)^4} \tag{4.3.2}$$

where $\kappa_5^2 = \frac{\kappa^2}{2\pi R\tilde{V}_4}$.

In order to find a total brane decay rate at a given spectrum, we integrate over momenta of open strings with bosonic string distribution for brane thermodynamics,

$$\Gamma(k) = \int_{-\infty}^{\infty}\frac{Ldp_1}{2\pi}\int_{-\infty}^{\infty}\frac{Ldq_1}{2\pi}\Gamma(p,q,k)\rho(q_0,q_1)\rho(p_0,p_1). \tag{4.3.3}$$

Since we are considering massless modes, so $p_0 = |p_1|, q_0 = |q_1|$ and the integral over dq_1 gives $q_1 = -p_1$. Using $A_D = p\cdot q = p_0 q_0 - p_1 q_1 = 2|p_1|^2$, we get

$$\begin{aligned}
\Gamma(k) &= \frac{\kappa_5^2 L[dk^4]}{k_0(2\pi)^4}\int_{-\infty}^{\infty} dp_1 \delta(2|p_1|-k_0)|p_1|^2\rho(|p_1|,p_1)\rho(|p_1|,-p_1)\\
&= \frac{\kappa_5^2 L[dk^4]}{k_0(2\pi)^4}2\cdot\frac{1}{2}(\frac{k_0}{2})^2\rho(\frac{k_0}{2},\frac{k_0}{2})\rho(\frac{k_0}{2},-\frac{k_0}{2})\\
&= \frac{\kappa_5^2 L[dk^4]}{k_0(2\pi)^4}(\frac{k_0}{2})^2\rho_R(\frac{k_0}{2})\rho_L(\frac{k_0}{2}).
\end{aligned} \tag{4.3.4}$$

where the factor 2 comes from the two values of p_1 in the integration delta function and $\frac{1}{2}$ comes from the Jacobian. The distribution functions are in the forms of left and right distributions defined in (4.2.6).

Now we approximate the state to be slightly nonextremal that $E_L = E_{BPS} + \Delta E$ and $E_R = \Delta E$ for $\Delta E << E_{BPS}$. Together with (4.2.7b) this implies that $\frac{p_0}{T_L}$ is small. For bosonic distribution we can approximate that

$$\rho_L(p_0) \sim \frac{T_L}{p_0} = \frac{4S_L}{p_0\pi f L}. \tag{4.3.5}$$

Moreover since the left distribution is dominant so we can estimate the left entropy near the extremal entropy, $S_L \approx S_{ext} = \frac{A_H}{4G_5}$ where A_H is the area horizon and G_5 is the five dimensional Newton gravitational constant.

Using $\kappa_5^2 = 8\pi G_5$ and $f = 4$, we finally get

$$\begin{aligned}
\Gamma(k_0) &\approx \frac{8\pi G_5 L[dk^4]}{k_0(2\pi)^4}(\frac{k_0}{2})^2 \frac{A_H}{4\pi G_5 L \frac{k_0}{2}} \rho_R(\frac{k_0}{2}) \\
&= \frac{A_H}{(2\pi)^4} 2\pi^2 k_0^3 dk_0 \rho_R(\frac{k_0}{2}) \\
&= \frac{A_H}{8\pi^2} \frac{k_0^3 dk_0}{e^{\beta_H k_0} - 1}
\end{aligned} \tag{4.3.6}$$

where the Hawking temperature $T_H = 1/\beta_H = 2T_R$. The total emitting energy of range k_0 to $k_0 + dk_0$ is then

$$\frac{dE(k)}{dt} = \frac{A_H}{8\pi^2} \frac{k_0^4 dk_0}{e^{\beta_H k_0} - 1} \tag{4.3.7}$$

which is the same with the classical result.

CHAPTER V

Summary and Recent Attempts

5.1 Summary

In this section we summarize and discuss all the results we have made so far. First of all, this dissertation is mainly focused on comparing the results from the D1D5-brane decay and the Hawking radiation of its corresponding black hole. We have discussed the quantum field theory in curved spacetime and given some examples. The important result is that the accelerating observer will observe a thermal radiation of a scalar field with temperature proportional to its acceleration. In the case of black holes, the acceleration is their surface gravity. This gives a stronger statement than the analogy between the laws of black hole mechanics and the laws of thermodynamics. Moreover, there is the graybody effect caused by the spacetime geometry. The semiclassical calculation chapter 2 includes this effect for the low energy quanta emission.

We then discussed the conformal field theory and how to calculate the amplitude using string vertex operators. First, we calculate the 4-point amplitude then gives configurations to obtain the result of two open strings on a brane scatter into a closed string in a bulk, which is the scalar field in the five dimensional point of view. Some configurations are given to make the result the D1D5 system. By considering thermodynamics on the D-brane, we finally calculate the decay rate of the slightly excited D1D5 system. The result is exactly the same with what we got from the semiclassical section.

This dissertation only considers an emission of a scalar particle. However, the original papers [1] mentioned that fermionic open string states are not contributing the decay rate in the lowest order. Particles with spin one and spin two are also not produced in low energy emission.

5.2 Recent Interesting Developments

In this section we briefly introduce some significant examples of the most interesting developments recently appear in black holes physics and string theory.

AdS/CFT correspondence

Recently, the anti-de Sitter/conformal field theory (AdS/CFT) correspondence (also known as gauge/gravity duality) is one of the most outstanding developments in string theory. It is a duality between conformal field theories, which is the gauge theories, and string theory in anti-de Sitter spaces. With this duality, one can study strongly coupled quantum field theories, which cannot be described by perturbative theories, via the gravitational theories which are weakly interacting. This powerful toolkit was introduced by Juan Maldacena in 1997 [18].

The AdS/CFT correspondence is not only interested in the field of high energy physics and string theories, but also used to study other fields, i.e. condense matter physics and nuclear physics. The condensed matter physics also use the formalism of quantum field theory to describe exotic states. However, some phenomena are not easy to describe using the field theories. Some condense matter physicists believe that we can study these phonemena by using the AdS/CFT correspondence [19].

Recently, there has been a lot of study using AdS/CFT correspondence and string theories in the fields of hadron physics [20] and condense matter, including superconductor [21–26] and superfluid [27]. As conventional superconductors are well-described by Cooper pair fluid in BCS theory, the understanding of pairing mechanism and unconventional superconductors are still incomplete since a normal state of some materials is not well-described by the standard Fermi liquid theory. Recently, the AdS/CFT correspondence technique is used to study an unconventional superconductor by introducing a $3 + 1$ black hole with a charged scalar field which is dual with a strongly coupled gauge theory on layered unconventional superconductor. This is well-known as AdS_4/CFT_3 correspondence. With this technique, we can obtain Einstein-Maxwell scalar theory and results of conductivity, phase transition and energy gap.

Complementarity and Firewall

Lately, the black hole complementarity and firewall became popular in the topic of black hole information paradox problem. There are some postulates that describe the complementarity [28, 29]:

- Postulate 1: The process of black hole formation and evaporation could be fully quantum mechanically described by a distant observer.
- Postulate 2: Physics can be well approximated by a set of semi-classical field equations outside the stretched horizon of a massive black hole.
- Postulate 3: A black hole is a quantum system with discrete energy levels to a distant observer.
- Postulate 4: A freely falling observer observes nothing different when crossing the horizon.

To summarize the idea, the complementarity requires that an information can escape the black hole through the Hawking radiation and different observers see the same information in the different places. When an information falls into a black hole, a freely falling observer should see it inside, while an external observer sees it in the Hawking radiation. This was first proposed by Leonard Susskind, Lalus Thorlacius and John Uglum in 1993 [28]. However, the arguments in [29] suggest that the complementarity is not enough, since we know that the information gets out, but we do not know the mechanism. Some physicists come up with a new idea of the falling observers see high-energy quanta firewall at the event horizon. This breaks the entanglement between the outgoing and ingoing pair. Unfortunately, this forces us to give up the Einstein's equivalence principle.

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APPENDIX

APPENDIX A

Supergravity Solution

In this appendix, we will discuss how to derive a SUGRA solution for the corresponding charged black hole we have used in chapter 2.

A.1 The Curvature

Starting from the supergravity action in D-dimension,

$$I = \frac{1}{16\pi G^{(D)}} \int d^D x \sqrt{-g} \left\{ R - \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} \sum_I \frac{1}{n_I!} e^{a_I \phi} F_{n_i}^2 \right\} \tag{A.1.1}$$

where $I = 1, \ldots, \mathcal{N}$, $\mathcal{N}$ is the number of supersymmetries, and n_I is the rank of F_{n_I}. By the least action principle we can make variations with respect to dynamical variables $g_{\mu\nu}, \phi$ and A_{n_I-1}. Together with $F_{n_I} = dA_{n_I-1}$, the variations give equations of motion as

$$R_{\mu\nu} = \frac{1}{2}\partial_\mu \phi \partial_\nu \phi + \frac{1}{2} \sum_I \frac{e^{a_I \phi}}{n_I!} \left(n_I F_{\mu\lambda_2 \ldots \lambda_{n_I}} F_\nu{}^{\lambda_2 \ldots \lambda_{n_I}} - \frac{(n_I - 1)}{D-2} g_{\mu\nu} F_{n_I}^2 \right) \tag{A.1.2a}$$

$$\Box \phi = \frac{1}{2} \sum_I \frac{a_I}{n_I!} e^{a_I \phi} F_{n_I}^2 \tag{A.1.2b}$$

$$\nabla_{\mu_1} (e^{a_I \phi} F^{\mu_1 \ldots \mu_{n_I}}) = 0 \tag{A.1.2c}$$

$$\partial_{[\mu_1} F_{\mu_2 \ldots \mu_{n+1}]} = 0. \tag{A.1.2d}$$

We will solve for p-brane solution by using the ansatz for the dynamical variables $g_{\mu\nu}, \phi$ and A_{n_I-1}. By starting with the general metric in D-dimensional space-time,

$$ds^2 = g_{\mu\nu} dz^\mu dz^\nu, \ \ ; \mu, \nu = 0, 1, \ldots, D-1. \tag{A.1.3}$$

We split the coordinates z^μ into the time-like coordinate, p-brane coordinates and the transverse space coordinates (t, y^i, x^a), $i = 1, \ldots, p$ and $a = p+1, \ldots, D-1$. The metric is required to be invariant under time translation, invariant under translations in the y^i directions, invariant under $SO(D-p-1)$ rotations in the transverse space and invariant under $SO(P)$ rotations in the longitudinal space. By using the spherical coordinates in the transverse space, the symmetries will give us the metric

$$ds^2 = -e^{2B}dt^2 + e^{2C}\delta_{ij}dy^i dy^j + e^{2F}dr^2 + e^{2G}r^2 d\Omega^2_{(d-1)}, \tag{A.1.4}$$

where $d = D - p - 1$ is the dimensions of the transverse space and the functions B, C, F, G depend only on r. Note that the $SO(p)$ symmetry can be relaxed in the solution that has several branes, thus the metric can be generalized to

$$ds^2 = -e^{2B}dt^2 + \sum_{i=1}^{p} e^{2C_i}(dy^i)^2 + e^{2F}dr^2 + e^{2G}r^2 d\Omega^2_{(d-1)}. \tag{A.1.5}$$

Since we are going to do a tetrad calculation for the curvature, it will be more useful if we use a metric in more general form then restore the $SO(p)$ symmetry later. Using the spherical coordinates in transverse space with $r^2 = \delta_{ab}x^a x^b$ and $\delta_{ab}dx^a dx^b = dr^2 + r^2 d\Omega^2_{(d-1)}$ where

$$d\Omega^2_{(d-1)} = d\theta_1^2 + sin^2\theta_1 d\theta_2^2 + \cdots + sin^2\theta_1 sin^2\theta_2 \cdots sin^2\theta_{d-2} d\theta^2_{d-1}, \tag{A.1.6}$$

after reparametrizing the functions $F(r)$ and $G(r)$, the metric then becomes

$$ds^2 = -e^{2B}dt^2 + \sum_{i=1}^{p} e^{2C_i}(dy^i)^2 + e^{2G}\delta_{ab}dx^a dx^b. \tag{A.1.7}$$

To compute the curvature from the metric, we introduce the tetrad components, $e^{\hat{t}} = e^B dt, e^{\hat{i}} = e^{C_i}dy^i, e^{\hat{a}} = e^G dx^a$ with spin connection $de^{\hat{\mu}} = e^{\hat{\nu}} \wedge \omega^{\hat{\mu}}{}_{\hat{\nu}}$. By considering the possible forms of the spin connections, one can show that the only non vanish components are

$$\omega^{\hat{t}}{}_{\hat{a}} = e^{-G}\partial_a B e^{\hat{t}} \tag{A.1.8a}$$

$$\omega^{\hat{i}}{}_{\hat{a}} = e^{-G}\partial_a C_i e^{\hat{i}} \tag{A.1.8b}$$

$$\omega^{\hat{a}}{}_{\hat{b}} = e^{-G}(\partial_b G e^{\hat{a}} - \partial_a G e^{\hat{b}}). \tag{A.1.8c}$$

The Riemann tensor and the Ricci tensor are defined by $R^{\hat{\mu}}{}_{\hat{\nu}} = \frac{1}{2}R^{\hat{\mu}}{}_{\hat{\nu}\hat{\rho}\hat{\sigma}}e^{\hat{\rho}} \wedge e^{\hat{\sigma}}$ where $R^{\hat{\mu}}{}_{\hat{\nu}} = d\omega^{\hat{\mu}}{}_{\hat{\nu}} + \omega^{\hat{\mu}}{}_{\hat{\lambda}} \wedge \omega^{\hat{\lambda}}{}_{\hat{\nu}}$ with properties

$$R_{\hat{\mu}\hat{\nu}} = R^{\hat{\lambda}}{}_{\hat{\mu}\hat{\lambda}\hat{\nu}} \tag{A.1.9a}$$

$$R^{\hat{\mu}}{}_{\hat{\nu}\hat{\lambda}\hat{\sigma}} = -R^{\hat{\mu}}{}_{\hat{\nu}\hat{\sigma}\hat{\lambda}} \tag{A.1.9b}$$

$$R^{\hat{\mu}}{}_{\hat{\nu}\hat{\lambda}\hat{\sigma}} + R^{\hat{\mu}}{}_{\hat{\sigma}\hat{\nu}\hat{\lambda}} + R^{\hat{\mu}}{}_{\hat{\lambda}\hat{\sigma}\hat{\nu}} = 0 \tag{A.1.9c}$$

$$R_{\hat{\mu}\hat{\nu}\hat{\lambda}\hat{\sigma}} = -R_{\hat{\nu}\hat{\mu}\hat{\lambda}\hat{\sigma}} = -R_{\hat{\mu}\hat{\nu}\hat{\sigma}\hat{\lambda}}. \tag{A.1.9d}$$

After the calculation, the non zero components of curvature are

$$R_{\hat{t}\hat{t}} = e^{-2G}\{\partial_a\partial^a B - \partial_a B\partial^a\varphi\} \tag{A.1.10a}$$

$$R_{\hat{i}\hat{j}} = e^{-2G}\delta_{ij}\{-\partial_a\partial^a C_i - \partial_a C_i\partial^a\varphi\} \tag{A.1.10b}$$

$$\begin{aligned} R_{\hat{a}\hat{b}} = e^{-2G}\{&-\partial_a\partial b\varphi + \partial_a G\partial_b\varphi + \partial_b\partial_a\varphi - \partial_a B\partial_b B - \sum_i \partial_a C_i\partial_b C_i \\ &- (d-2)\partial_a G\partial_b G - \partial_c\partial^c G\delta_{ab} - \delta_{ab}\partial_c G\partial^c[\varphi\} \end{aligned} \tag{A.1.10c}$$

where $\varphi = [B + \sum_i C_i + (d-2)G]$. We change the curvature components in transverse directions $R_{\hat{a}\hat{b}}$ into spherical coordinate components $R_{\hat{r}\hat{r}}, R_{\hat{\alpha}\hat{\beta}}$, which is the coordinate transformation $\{x^a\}, a = 1, \ldots, d$ to $\{r, \theta_\alpha\}, \alpha = 1, \ldots, d-1$ where

$$x_a = rsin\theta_1 \cdots sin\theta_{a-1}cos\theta_a \quad ; a = 1, \ldots, d-1 \tag{A.1.11a}$$

$$x_d = rsin\theta_1 \cdots sin\theta_{d-1}. \tag{A.1.11b}$$

By expecting the metric (A.1.7) to be in the form of (A.1.5), we get a condition for reparametrization which made the curvature to be

$$R_{\hat{t}\hat{t}} = e^{-2F}\{B'' + B'[B' + \sum_i C_i' - F' + (d-1)(G' + \frac{1}{r})]\} \tag{A.1.12a}$$

$$R_{\hat{i}\hat{j}} = -e^{-2F}\delta_{ij}\{C_i'' + C_i'[B' - F' + \sum_i C)i' + (d+1)(G' + \frac{1}{r})]\} \tag{A.1.12b}$$

$$\begin{aligned} R_{\hat{r}\hat{r}} =& -e^{-2F}\{B'^2 + \sum_i (C_i'^2 + C_i'' - F'C_i') - B'' - B'F' \\ & + (d-1)[G'' + \frac{2}{r}G' + G'^2 - F'G' - \frac{F'}{r}]\} \end{aligned} \tag{A.1.12c}$$

$$\begin{aligned} R_{\hat{\alpha}\hat{\beta}} =& -e^{-2F}\delta_{\alpha\beta}\{(G' + \frac{1}{r})[B' + \sum_i C_i' - F' \\ & + (d-1)(G' + \frac{1}{r})] + G'' - \frac{d-2}{r^2}e^{2F-2G} - \frac{1}{r^2}\}, \end{aligned} \tag{A.1.12d}$$

A.2 Singly Charged p-brane Solutions

In this section we will solve the solution for the extremal p-brane with singly charged which couples to one gauge field. The equations of motion (A.1.2a) are then

$$R_{\mu\nu} = \frac{1}{2}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}\frac{e^{a\phi}}{n!}\left(nF_{\mu\lambda_2\ldots\lambda_n}F_\nu{}^{\lambda_2\ldots\lambda_n} - \frac{(n-1)}{D-2}g_{\mu\nu}F_n^2\right) \tag{A.2.1a}$$

$$\Box\phi = \frac{1}{2}\frac{a}{n!}e^{a\phi}F_n^2 \tag{A.2.1b}$$

$$\nabla_{\mu_1}(e^{a\phi}F^{\mu_1\ldots\mu_n}) = 0 \tag{A.2.1c}$$

$$\partial_{[\mu_1}F_{\mu_2\ldots\mu_{n+1}]} = 0, \tag{A.2.1d}$$

with $n = p+2$. The ansatz for F_n can be divided into an electric ansatz and a magnetic ansatz which are given by

$$F_{ti_1\cdots i_p r} = \epsilon_{i_1\cdots i_p}\partial_r E(r) \tag{A.2.2a}$$

$$F^{a_1\cdots a_{d-1}} = \frac{1}{\sqrt{-g}}\epsilon^{a_1\cdots a_d}e^{-a\phi}\partial_{a_d}\tilde{E}(r) \tag{A.2.2b}$$

respectively. For the magnetic ansatz in the spherical coordinates, the non-vanishing component is $F_{\theta_1\cdot\theta_{d-1}} \sim f(r)sin^{d-2}\theta_1\cdots sin\theta_{d-2}$. These are equivalent to $F_{tr} = \partial_r E(r)$ and $F_{\theta\phi} = B(r)r^2 sin^2\theta$ in the Reissner-Nordstrom case. By setting all C_i to be equal to restore the $SO(p)$ symmetry and solve the equation (A.2.1c) using (A.2.2a), we obtain

$$F_{ti_1\cdots i_p r} = \epsilon_{i_1\cdots i_p} e^{B+pC+F-a\phi-(d-1)G} \frac{Q}{r^{d-1}} \tag{A.2.3a}$$

$$F^2_{p+2} = -(p+2)! e^{-2a\phi-2(d-1)G} \frac{Q^2}{r^{2(d-1)}} \tag{A.2.3b}$$

$$F_\mu{}^{\lambda_2\cdots\lambda_{p+2}} F_{\nu\lambda_2\cdots\lambda_{p+2}} = -g_{\mu\lambda}(\delta^\lambda_t \delta_t \nu + \delta^\lambda_i \delta_i \nu + \delta^\lambda_r \delta_r \nu)(p+1)! e^{-2a\phi-2(d-1)G} \frac{Q^2}{r^{2(d-1)}}, \tag{A.2.3c}$$

where Q is the integration constant.

We now will solve equations above for the extremal p-brane solution which is the one that its mass is saturated its charged. There is only one parameter in the extremal solution. We cannot introduce any more parameter since we already have defined one parameter Q. From before, we choose the isotropic gauge for the radial coordinate r by letting $F = G$ and restore the Poincare symmetry in the world volume be letting $B = C$. We use $R_{\mu\nu} = e^{\hat{\mu}}_\mu e^{\hat{\nu}}_\nu R_{\hat{\mu}\hat{\nu}}$ for the Ricci tensor component in the coordinate basis. Hence, by considering (A.2.1a), (A.2.3a) and the curvature (A.1.12a), we obtain [35]

$$\phi'' + (d-1)\frac{\phi'}{r} + \phi'[(p+1)B' + (d-2)G'] = -\frac{a}{2} e^{-a\phi-2(d-2)G} \frac{Q^2}{r^{2(d-2)}} \tag{A.2.4a}$$

$$B'' + B'[(p+1)B' + (d-2)G' + \frac{d-1}{r}] = \frac{d-2}{2(D-2)} e^{-a\phi-2(d-2)G} \frac{Q^2}{r^{2(d-1)}} \tag{A.2.4b}$$

$$\begin{aligned} &(p+1)(-B'^2 - B'' + B'G') - (d-1)(G'' + \frac{G'}{r}) - \frac{\phi'^2}{2} \\ &= -\frac{(d-2)}{2(D-2)} e^{-a\phi-2(d-2)G} \frac{Q^2}{r^{2(d-1)}} \end{aligned} \tag{A.2.4c}$$

$$\begin{aligned} &-(G' + \frac{1}{r})[(p+1)B' + (d-2)G'] - G'' - \frac{(d-1)}{r} G' \\ &= \frac{(p+1)}{2(D-2)} e^{-a\phi-2(d-2)G} \frac{Q^2}{r^{2(d-a)}}. \end{aligned} \tag{A.2.4d}$$

By solving these equations with the asymptotically flat solution, $\phi \to 0$ as $r \to \infty$, we finally obtain [35]

$$e^\phi = H^{\frac{a(D-2)}{\Delta}} \tag{A.2.5a}$$

$$e^G = H^{\frac{p+1}{\Delta}} \tag{A.2.5b}$$

$$e^B = H^{-\frac{(d-2)}{\Delta}}, \tag{A.2.5c}$$

where the harmonic function H is defined by

$$H = 1 + \frac{1}{d-2}\sqrt{\frac{\Delta}{2(D-2)}}\frac{|Q|}{r^{d-2}} \equiv 1 + \frac{h^{d-2}}{r^{d-2}}, \tag{A.2.6}$$

and $\Delta \equiv (p+1)(d-2) + \frac{1}{2}a^2(D-2)$. The metric for the extremal p-brane is then

$$ds^2 = H^{-\frac{2(d-2)}{\Delta}}(-dt^2 + dy_1^2 + \cdots + dy_p^2) + H^{\frac{2(p+1)}{\Delta}}(dr^2 + r^2 d\Omega^2_{(d-1)}). \tag{A.2.7}$$

A.2.1 Dp-brane Solution

The general Dp-brane solution can be obtained by choosing $D = 10$ and taking $a = \frac{3-p}{2}$ [35]. The Dp-brane solution in the Einstein frame is then

$$ds^2_{Dp} = H^{-\frac{p-7}{8}}(-dt^2 + dy_1^2 + \cdots + dy_p^2) + H^{\frac{p+1}{8}}(dr^2 + r^2 d\Omega^2_{(8-p)}) \tag{A.2.8a}$$

$$H = 1 + \frac{|Q|}{(7-p)r^{7-p}} \equiv 1 + \frac{Q_{Dp}}{r^{7-p}} \tag{A.2.8b}$$

$$e^{\phi} = H^{\frac{3-p}{4}}, \tag{A.2.8c}$$

for $p < 7$. In the string frame where $ds^2_{Dp} = e^{-\frac{\phi}{2}}ds^2_{D_p string}$, the metric takes more simple form

$$ds^2_{D_p string} = H^{-\frac{1}{2}}(-dt^2 + dy_1^2 + \cdots + dy_p^2) + H^{\frac{1}{2}}(dr^2 + r^2 d\Omega^2_{(8-p)}). \tag{A.2.9}$$

We have to consider an intersection of branes, to be specific, the intersection of D1-brane and D5-brane which is used to derive a Hawking radiation. The metric in the string frame is

$$\begin{aligned} ds^2_{D1\perp D5 string} =& H_1^{-\frac{1}{2}}H_5^{-\frac{1}{2}}(-dt^2 + dy_1^2) + H_1^{\frac{1}{2}}H_5^{-\frac{1}{2}}(dy_2^2 + \cdots + dy_5^2) \\ &+ H_1^{\frac{1}{2}}H_5^{\frac{1}{2}}(dr^2 + r^2 d\Omega^2_{(3)}), \end{aligned} \tag{A.2.10}$$

where H_1 and H_5 are the harmonic functions of the D1-brane and D5-brane respectively and the dilation is given by $e^{\phi} = H_1^{\frac{1}{2}}H_5^{-\frac{1}{2}}$. The metric in the Einstein frame is then

$$\begin{aligned} ds^2_{D1\perp D5} =& H_1^{-\frac{3}{4}}H_5^{-\frac{1}{4}}(-dt^2 + dy_1^2) + H_1^{\frac{1}{4}}H_5^{-\frac{1}{4}}(dy_2^2 + \cdots + dy_5^2) \\ &+ H_1^{\frac{1}{4}}H_5^{\frac{3}{4}}(dr^2 + r^2 d\Omega^2_{(3)}). \end{aligned} \tag{A.2.11}$$

A.3 Dimensional Reduction of the Solution

Since the solutions are obtained from the theory of 10-dimensional spacetime, but the spacetime we can observe has lower dimensions. We have to reduce them by compactify the extra dimension to a circle or a torus. The metric and fields will break into several fields in lower dimensions. Let us consider a case of dimensional reduction from D to $D-1$ dimensions in the transverse direction. By letting z to be the extra dimension coordinate in the transverse direction with identity $z \sim z + 2\pi R$, where R is the size of the extra dimension, and $r = |\vec{x}_{d-1}|$ be the radial coordinates in $D-1$ dimensions. The harmonic function can be written as

$$H = 1 + \sum_{n=-\infty}^{\infty} \frac{h^{d-2}}{(r^2 + (z + 2\pi R)^2)^{\frac{d-2}{2}}}. \tag{A.3.1}$$

For a small extra dimension, $R << r$, we can rewrite the harmonic function and replace the summation by the integral. The harmonic function is then in the form

$$H = 1 + \frac{h^{d-2}}{r^{d-3}} \frac{1}{2\pi R} \frac{\omega_{(d-3)}}{\omega_{(d-4)}} \equiv 1 + \frac{h'^{d-2}}{r^{d-3}}, \tag{A.3.2}$$

where ω_m is the volume of a (m)-sphere, defined by

$$\begin{aligned} \omega_{(m)} &= \int_0^{2\pi} \int_0^{\pi} \cdots \int_0^{\pi} sin^{m-1}\theta_{m-1} sin^{m-2}\theta_{m-2} \cdots sin\theta_1 d\theta_{m-1} \cdots d\theta_1 d\varphi \\ &= \frac{2\pi^{\frac{m+1}{2}}}{\Gamma\left(\frac{m+1}{2}\right)}. \end{aligned} \tag{A.3.3}$$

Note that for 10-dimensional spacetime $n = p + 2 = 7 - d$. We can reduce p dimensions by repeating the procedure and the result is

$$H = 1 + \frac{h^{d-2}}{r^{d-p-2}} \frac{1}{(2\pi R)^p} \frac{\omega_{(d-3)}}{\omega_{(d-p-3)}} \tag{A.3.4}$$

with $d > p + 2$. Note that the dimensional reduction of the longitudinal direction does not affect the harmonic function.

We can obtain how the $D+1$ dimensional dilaton relates to the D dimensional dilaton by considering the action in the string frame,

$$I_{string} = \frac{1}{16\pi G^{(D)}} \int d^{D+1}x \sqrt{-\hat{G}} e^{-2\phi_{D+1}} \left\{ \hat{R} + 4\partial_\mu \phi_{D+1} \partial^\mu \phi_{D+1} - \frac{1}{12} \hat{H}_{\mu\nu\rho} \hat{H}^{\mu\nu\rho} \right\}, \tag{A.3.5}$$

where the "hat" notation indicates the quantities in $D+1$ dimensions. By using $\sqrt{-\hat{G}} = \sqrt{-G}\sqrt{G_{compact}}$ where $G_{compact}$ is the determinant of the metric in the compacted dimension, the relation is then

$$e^{-2\phi_{D+1}} = \sqrt{G_{compact}}e^{-2\phi_D}. \tag{A.3.6}$$

Next, we consider a dimensional reduction of $D_1 \perp D_5$ solution (A.2.10),

$$\begin{aligned} ds^2_{D1\perp D5string} =&(H_1H_5)^{-\frac{1}{2}}(-dt^2+dy_1^2) + H_1^{\frac{1}{2}}H_5^{-\frac{1}{2}}(dy_2^2+\cdots+dy_5^2) \\ &+ (H_1H_5)^{\frac{1}{2}}(dx_1^2+\cdots+dx_4^2), \end{aligned} \tag{A.3.7}$$

with $e^{\phi_{10}} = H_1^{\frac{1}{2}}H_5^{-\frac{1}{2}}$ and $H_1 = 1+\frac{Q_{D1}^{(10)}}{r^6}, H_5 = 1+\frac{Q_{D5}^{(10)}}{r^2}$. We do the reduction in the $D1D5$ directions $(y_1, y_2, y_3, y_4, y_5)$, the metric in the Einstein frame then becomes

$$ds^2_{D1\perp D5} = -(H_1H_5)^{-\frac{2}{3}}dt^2 + (H_1H_5)^{\frac{1}{3}}(dr^2 + r^2 d\Omega^2_{(3)}). \tag{A.3.8}$$

This configuration, however, is for the extremal black hole which is the ground state with zero entropy. We are interested in the near-extremal black hole which is considered as an excitation of the extremal ground state. The solution for this configuration can be obtained by adding the gravitational wave called pp-wave, the plane-fronted wave with parallel ray. The solution for $D1D5$-pp-wave superposition is [30] [31] [32]

$$\begin{aligned} ds^2_{string} =&(H_1H_5)^{-\frac{1}{2}}(-dt^2+dy_1^2+(H_K-1)(dt+dy_1)^2) + H_1^{\frac{1}{2}}H_5^{-\frac{1}{2}}(dy_2^2+\cdots+dy_5^2) \\ &+ (H_1H_5)^{\frac{1}{2}}(dx_1^2+\cdots+dx_4^2), \end{aligned} \tag{A.3.9}$$

with $e^{\phi_{10}} = H_1^{\frac{1}{2}}H_5^{-\frac{1}{2}}$ and $H_1 = 1+\frac{Q_{D1}^{(10)}}{r^6}, H_5 = 1+\frac{Q_{D5}^{(10)}}{r^2}, H_K = 1+\frac{Q_K^{(10)}}{r^6}$. The dimensional reduction procedure for this solution in (y_2, y_3, y_4, y_5) directions is the same as previous. However, the dimensional reduction for the y_1 direction is not since there is the $dtdy_1$ term which give a non-trivial Kaluza-Klein vector A_μ. Let us consider

$$-dt^2+dy_1^2+(H_K-1)(dt+dy_1)^2 = (H_K-2)dt^2 + H_K dy_1^2 + 2(H_K-1)dtdy_1 \tag{A.3.10}$$

and

$$\hat{g}_{\hat{\mu}\hat{\nu}}dx^{\hat{\mu}}dx^{\hat{\nu}} = g_{\mu\nu}dx^{\mu}dx^{\nu} + e^{2\varphi}(A_t dt + dz)^2, \tag{A.3.11}$$

where z is the compact extra dimension coordinate. By comparing the tt component, we obtain

$$e^{2\varphi} = H_K, \;\; A_t = 1 - H_K^{-1}, \;\; g_{tt} = -H_K^{-1}. \tag{A.3.12}$$

The compactified solution in string frame is then

$$ds^2_{string} = -(H_1 H_5)^{-\frac{1}{2}} H_K^{-1} dt^2 + (H_1 H_5)^{\frac{1}{2}}(dx_1^2 + \cdots + dx_4^2), \tag{A.3.13}$$

with

$$H_1 = 1 + \frac{Q_1}{r^2}, \;\; H_5 = 1 + \frac{Q_5}{r^2}, \;\; H_K = 1 + \frac{Q_K}{r^2} \tag{A.3.14a}$$

$$e^{-2\phi_5} = e^{-2\phi_{10}}\sqrt{H_1^2 H_5^{-2}(H_1H_5)^{-\frac{1}{2}} H_K} = H_K^{\frac{1}{2}}(H_1H_5)^{-\frac{1}{4}}. \tag{A.3.14b}$$

Note that $Q_K^{(10)}$ is related to the momentum of the wave, after the compactification of y_1 direction, the momentum must be quantized accordingly, $Q_K \sim \frac{N_K}{R}$ where $N_K \in \mathbb{Z}$. Finally we got the solution in Einstein frame

$$ds^2 = -(H_1 H_5 H_K)^{-\frac{2}{3}} dt^2 + (H_1 H_5 H_K)^{\frac{1}{3}}(dr^2 + r^2 d\Omega^2_{(3)}), \tag{A.3.15}$$

which we have used to compute the Hawking radiation.

Vitae

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