สมการเชิงฟังก์ชันค่าเฉลี่ยศูนย์บนทรงสี่เหลี่ยมด้านขนานหลายมิติ

นางสาวธนิษฐา โกวรรณ์

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ZERO-MEAN FUNCTIONAL EQUATION ON HYPER-PARALLELEPIPED

Miss Thanittha Kowan

A Thesis Submitted in Partial Fulfillment of the Requirements
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ธนิษฐา โกวรรณ์ : สมการเชิงฟังก์ชันค่าเฉลี่ยศูนย์บนทรงสี่เหลี่ยมด้านขนานหลายมิติ (ZERO-MEAN FUNCTIONAL EQUATION ON HYPER-PARALLELEPIPED) อ. ที่ปรึกษาวิทยานิพนธ์หลัก: รศ. ดร. ไพศาล นาคมหาชลาสินธุ์, อ. ที่ปรึกษาวิทยานิพนธ์ร่วม: ผศ. ดร. ณัฐพันธ์ กิติสิน, 25 หน้า.

ให้ $n \in \mathbb{N}$ กำหนดฐาน $\{e_1, \ldots, e_n\}$ สำหรับ $\mathbb{R}^n$ บน $\mathbb{R}$ และฟังก์ชัน $f : \mathbb{R}^n \to \mathbb{R}$ เราศึกษาหาผลเฉลยทั่วไปของสมการเชิงฟังก์ชัน

$$\sum_{\varepsilon_1, \ldots, \varepsilon_n = 0}^{1} f(x + t\varepsilon_1 e_1 + \ldots + t\varepsilon_n e_n) = 0$$

สำหรับทุก $x \in \mathbb{R}^n$ และสำหรับทุก $t > 0$ ทางเรขาคณิตสมการข้างต้นกล่าวว่าค่าเฉลี่ยเลขคณิตของค่าฟังก์ชัน f ที่ส่งจากจุดยอดของรูปทรงสี่เหลี่ยมด้านขนานหลายมิติซึ่งเกิดจากการเลื่อนขนานและการย่อขยายของรูปทรงสี่เหลี่ยมด้านขนานหลายมิติที่กำหนดไว้ค่าหนึ่งซึ่งมีด้านขนานกับ e_i $(i = 1, \ldots, n)$ เท่ากับศูนย์ ดังนั้นเราจะเรียกสมการข้างต้นว่าสมการเชิงฟังก์ชันค่าเฉลี่ยศูนย์บนทรงสี่เหลี่ยมด้านขนานหลายมิติ

ภาควิชา คณิตศาสตร์และวิทยาการคอมพิวเตอร์
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ลายมือชื่อนิสิต ....

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Let $n \in \mathbb{N}$. Given a basis $\{e_1, \ldots, e_n\}$ for $\mathbb{R}^n$ over $\mathbb{R}$ and a function $f : \mathbb{R}^n \to \mathbb{R}$, we determine the general solution of the functional equation

$$\sum_{\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n=0}^{1} f(x + t\varepsilon_1 e_1 + ... + t\varepsilon_n e_n) = 0$$

for all $x \in \mathbb{R}^n$ and for all $t > 0$. Geometrically, the above equation says that the arithmetic mean of the values of f taken at the vertices of any hyper-parallelepiped obtained from translations and dilations of a fixed hyper-parallelepiped, whose sides are parallel to e_i $(i = 1, \ldots, n)$, equals zero. So, we will call the above equation a zero-mean functional equation on hyper-parallelepiped.

Department	: Mathematics and Computer Science	Student's Signature	....
		Advisor's Signature	....
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CONTENTS

CHAPTER I

INTRODUCTION

In all that follows, $\mathbb{N}$ will denote the natural numbers, $\mathbb{R}$ will denote the real numbers, and $\mathbb{C}$ will denote the complex numbers.

1.1 Functional Equations

P.K. Sahoo and Pl. Kannappan [8] have said that

"The functional equations forms a modern branch of mathematics. The origin of functional equations came about the same time as the modern definition of function."

J.Aczél [1] gave the concept of functional equation and system of functional equations in definitions 1.1 to 1.3 as follows:

Definition 1.1. [1]

(a) The independent variables $x_1, \ldots, x_k$ are ***terms***.

(b) Given that $A_1, \ldots, A_m$ are terms and that F is a function of m variables, then $F(A_1, \ldots, A_m)$ is also a ***term***.

(c) There are no other terms.

Definition 1.2. [1] A ***functional equation*** is an equation

$$A_1 = A_2$$

between two terms A_1 and A_2, which contains k independent variables $x_1, \ldots, x_k$ and $n \geq 1$ unknown functions $F_1, \ldots, F_n$ of $j_1, \ldots, j_n$ variables respectively, as well as a finite number of known functions.

k is the rank and n is the number of functions of the functional equation, $j = \min(j_1, \ldots, j_n)$ is the minimal number of the variables in the functions of the functional equation.

The rank must be larger than the minimal number of variables in the functions of the equation.

Definition 1.3. [1] A ***system of functional equations*** consists of $p \geq 2$ functional equations, which contain $n \geq 1$ unknown functions altogether. p is the number of equations, n the number of functions of the system.

P.K. Sahoo and Pl. Kannappan [8] simply explain that

"Functional equations are equations in which the unknowns are functions. To solve a functional equation means to find the unknown function. In order to obtain a solution, the functions must often be restricted to a specific nature (such as analytic, bounded, continuous, convex, differentiable, measurable, and monotonic)."

Now we will show the example of functional equation in one variable.

Example 1.1. *Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the functional equation*

$$f(1-x) + 2f(x) = x + 7 \text{ for all } x \in \mathbb{R}. \tag{1.1}$$

Solution. Assume that there exists a function $f : \mathbb{R} \to \mathbb{R}$ satisfying (1.1).
Replacing x by $1 - x$ in (1.1), we get

$$f(x) + 2f(1-x) = 8 - x. \tag{1.2}$$

Solving (1.1) and (1.2), we have $f(x) = x + 2$ for all $x \in \mathbb{R}$.
Conversely if a function f is given by $f(x) = x + 2$, then

$$f(1-x) + 2f(x) = (1-x) + 2 + 2(x+2) = x + 7. \tag{1.3}$$

Therefore, the function f defined by $f(x) = x + 2$ for all $x \in \mathbb{R}$ is the unique solution of the functional equation (1.1). $\square$

Importantly, we need return to our original equation and verify our supposed solution by replacing it back in the original equation since there can be no solution to the original functional equation.

Example 1.2. *Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the functional equation*

$$4f(10-x) + xf(x) = 3 \text{ for all } x \in \mathbb{R}. \tag{1.4}$$

Solution. Assume that there exists a function $f : \mathbb{R} \to \mathbb{R}$ satisfying (1.4).
Substituting $x = 2$ in (1.4), we have

$$4f(8) + 2f(2) = 3. \tag{1.5}$$

Substituting $x = 8$ in (1.4), we obtain

$$4f(2) + 8f(8) = 3. \tag{1.6}$$

$f(2)$ and $f(8)$ concurrently satisfy (1.5) and (1.6), contradiction.
Thus, there is no function $f : \mathbb{R} \to \mathbb{R}$ satisfying the functional equation (1.4). □

The functional equation in more than one variable is shown in the following example.

Example 1.3. *Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the functional equation*

$$3f(x+2y) = f(3x) + 6y + 6 \tag{1.7}$$

for all $x, y \in \mathbb{R}$.

Solution. Assume that there exists a function $f : \mathbb{R} \to \mathbb{R}$ satisfying (1.7).
Substituting $x = 0$ in (1.7), we have

$$3f(2y) = f(0) + 6y + 6. \tag{1.8}$$

So, there is a constant c such that $f(x) = x + c$ for all $x \in \mathbb{R}$.
Replacing $f(x) = x + c$ in the left side of (1.7), we have

$$3f(x + 2y) = 3x + 6y + 3c. \tag{1.9}$$

Replacing $f(x) = x + c$ in the right side of (1.7), we have

$$f(3x) + 6y + 6 = 3x + c + 6y + 6. \tag{1.10}$$

From (1.9) and (1.10), we obtain $c = 3$.
Therefore, the function f defined by $f(x) = x + 3$ for all $x \in \mathbb{R}$ is the unique solution of the functional equation (1.7). □

Next, we will give an example of functional equations where the function is defined on $\mathbb{R}^3$ and has three variables.

Example 1.4. *Find all functions $f : \mathbb{R}^3 \to \mathbb{R}$ satisfying the functional equation*

$$f(x, y, z) = f(x + y, 0, 0) + f(0, y + z, 0) + f(0, 0, x + z) \tag{1.11}$$

for all $x, y, z \in \mathbb{R}$.

Solution. Assume that there exists a function $f : \mathbb{R}^3 \to \mathbb{R}$ satisfying (1.11).
Substituting $y = 0$ and $z = 0$ in (1.11), we have

$$f(x, 0, 0) = f(x, 0, 0) + f(0, 0, 0) + f(0, 0, x). \tag{1.12}$$

That is $f(0, 0, x) = -f(0, 0, 0)$ for all $x \in \mathbb{R}$.
Substituting $x = 0$ in (1.12), we have $f(0, 0, 0) = 0$. So, we have $f(0, 0, x) = 0$.
Similarly, we get $f(x, 0, 0) = 0$ and $f(0, x, 0) = 0$ for all $x \in \mathbb{R}$.
Hence, $f(x, y, z) = f(x + y, 0, 0) + f(0, y + z, 0) + f(0, 0, x + z) = 0$.
Conversely if a function f is given by $f(x, y, z) = 0$ for all $x, y, z \in \mathbb{R}$, then f satisfy (1.11). Therefore the function f defined by $f(x, y, z) = 0$ for all $x, y, z \in \mathbb{R}$ is the unique solution of the functional equation (1.11). □

1.2 Motivation and Proposed Problem

Geometric functional equations have been studied by several authors. In 1968, J. Aczél, H. Haruki, M.A. McKiernan, and G. N. Sakovič [2] investigated general solution of the functional equation

$$f(x+t, y+t) + f(x+t, y-t) + f(x-t, y+t) + f(x-t, y-t) = 4f(x, y) \quad (1.13)$$

where $f : \mathbb{R}^2 \to \mathbb{R}$ is a function and $x, y, t \in \mathbb{R}$. The general solution of (1.13) is given in terms of arbitrary symmetric multi-additive functions of four variables. In 1969, H. Haruki [4] studied the functional equation

$$\begin{aligned} &f(x+t, y+t, z+t) + f(x+t, y+t, z-t) + f(x+t, y-t, z+t)+ \\ &f(x+t, y-t, z-t) + f(x-t, y+t, z+t) + f(x-t, y+t, z-t)+ \\ &\quad f(x-t, y-t, z+t) + f(x-t, y-t, z-t) = 8f(x, y, z) \end{aligned} \quad (1.14)$$

where $f : \mathbb{R}^3 \to \mathbb{R}$ is a function and $x, y, z, t \in \mathbb{R}$. The general solution of (1.14) under a continuity condition of f is

$$f(x, y, z) = \sum_{0 \le i,j,k \le 5} c_{i\,j\,k} \frac{\partial^{i+j+k}}{\partial x^i \partial y^j \partial z^k} P(x, y, z),$$

where $c_{i\,j\,k}$ are real constants for $0 \le i, j, k \le 5$ and $P(x, y, z) = xyz(y^2 - z^2)(z^2 - x^2)(x^2 - y^2)$.

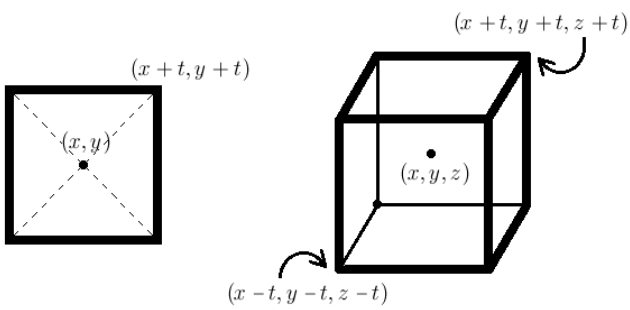

Figure 1.1 : Square and cube

In accordance with Figure 1.1, (1.13) and (1.14) say that for each square (cube) obtained from translations and dilations of a fixed square (cube), the values of the function at its center is the arithmetic mean of its values at all vertices. H. Haruki [4, 5] called (1.13) a "square" functional equation and (1.14) a "cube" functional equation.

In 1974, L. Etigson [3] proved that the "rhombus" functional equation

$$f(x+t,y)+f(x-t,y)+f(x,y-t)+f(x,y-t)=4f(x,y) \tag{1.15}$$

is equivalent to the square functional equation and also proved that the "octahedron" functional equation

$$\begin{aligned} f(x+t,y,z)+f(x-t,y,z)+f(x,y+t,z)+f(x,y-t,z) \\ +f(x,y,z+t)+f(x,y,z-t)=6f(x,y,z) \end{aligned} \tag{1.16}$$

is equivalent to the cube functional equation.

Geometrically, (1.15) and (1.16) say that for each rhombus (octahedron) obtained from translations and dilations of a fixed rhombus (octahedron), the values of the function at its center is the arithmetic mean of its values at all vertices as shown in the following figure.

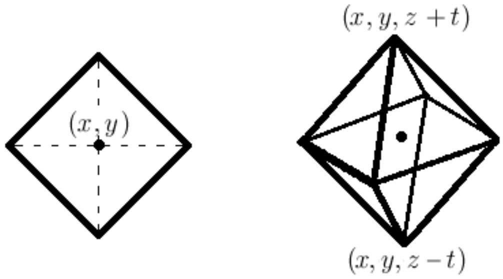

Figure 1.2 : Rhombus and octahedron

In 1991, L. Székelyhidi [9] investigated two geometric functional equations: the n-dimensional octahedron functional equation

$$[\sum_{i=1}^{n}(\tau_i^t+\tau_i^{-t})]f(x)=2nf(x) \tag{1.17}$$

and the n-dimensional cube functional equation

$$[\prod_{i=1}^{n}(\tau_i^t + \tau_i^{-t})]f(x) = 2^n f(x) \tag{1.18}$$

where $t \in \mathbb{R}$, $x \in \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{C}$ is a complex valued function, and τ_i^t is a partial translation operator in the i^{th} variable on $\mathbb{R}^n$ as follows

$$\tau_i^t f(x_1, \ldots, x_n) = \tau_i^t f(x_1, \ldots, x_{i-1}, x_i + t, x_{i+1}, \ldots, x_n).$$

L. Székelyhidi proved that the continuous solutions of the n-dimensional cube equation on $\mathbb{R}^n$ is a linear combination of the partial derivatives of a special given harmonic polynomial Q_n defined by, for each $(x_1, \ldots, x_n) \in \mathbb{R}^n$,

$$Q_n(x_1, \ldots, x_n) = x_1 x_2 \ldots x_n \prod_{i<j}(x_i^2 - x_j^2)$$

as well as proved that the n-dimensional octahedron and cube equation are equivalent.

Later in 2011, R. Kotnara [6] studied a functional equation

$$f(z) + f(z + \lambda a_1) + f(z + \lambda a_2) + f(z + \lambda(a_1 + a_2)) = 0 \tag{1.19}$$

for fixed complex constant a_1, a_2 where $f : \mathbb{C} \to \mathbb{C}$ is a function, $z \in \mathbb{C}$, and $\lambda \in \mathbb{R} \setminus \{0\}$. The functional equation (1.19) says that, for each parallelogram obtained from translations and dilations of an arbitrary fixed parallelogram, the sum of the values of the function at all the vertices is equal to zero as in the following figure.

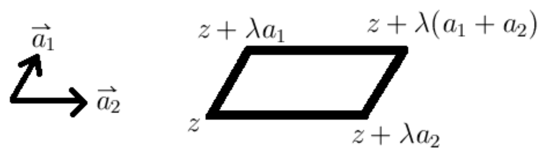

Figure 1.3 : Parallelogram

In this thesis,we extend (1.19) from two dimensions to any n dimensions. Given $n \in \mathbb{N}$, we find the general solution $f : \mathbb{R}^n \to \mathbb{R}$ of the functional equation

$$\sum_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n = 0}^{1} f(x + t\varepsilon_1 e_1 + \ldots + t\varepsilon_n e_n) = 0, \tag{1.20}$$

where $\{e_1, \ldots, e_n\}$ is a basis for $\mathbb{R}^n$ over $\mathbb{R}$, $x \in \mathbb{R}^n$ and $t > 0$. In particular for $n = 2$, the arithmetic mean of the values of f taken at the vertices of any parallelogram obtained from translations and dilations of a fixed parallelogram (whose sides are parallel to e_i) equals zero. Similarly, for $n = 3$ the arithmetic mean of the values of f taken at the vertices of any parallelepiped obtained from translations and dilations of a fixed parallelepiped equals zero as in Figure 1.4. According to the geometric interpretation of (1.20), we will call (1.20) a "zero-mean" functional equation on hyper-parallelepiped.

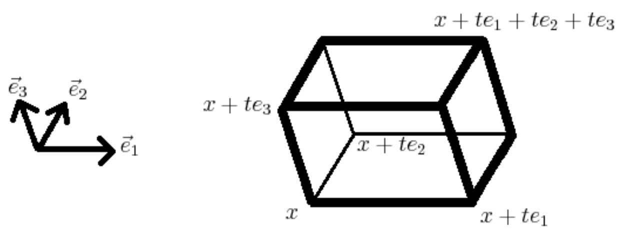

Figure 1.4 : Parallelepiped

CHAPTER II

PRELIMINARIES

2.1 Vector Spaces and Linear Operators

Linear operators are so useful to determine the solution of zero-mean functional equation on hyper-parallelepiped. So, we will state some definitions related to linear operators.

Definition 2.1. A ***vector space*** V over a field F is a set V together with the operations of addition $V \times V \to V$ and scalar multiplication $F \times V \to V$ satisfying the following properties:

(i) ***Commutativity*:** $u + v = v + u$ for all $u, v \in V$;

(ii) ***Associativity*:** $(u+v)+w = u+(v+w)$ and $(ab)v = a(bv)$ for all $u, v, w \in V$ and $a, b \in F$;

(iii) ***Additive identity*:** There exists an element $0 \in V$ such that $0 + v = v$ for all $v \in V$;

(iv) ***Additive inverse*:** For every $v \in V$, there exists an element $w \in V$ such that $v + w = 0$;

(v) ***Multiplicative identity*:** $1v = v$ for all $v \in V$;

(vi) ***Distributivity*:** $a(u + v) = au + av$ and $(a + b)u = au + bu$ for all $u, v \in V$ and $a, b \in F$.

Usually, a vector space over $\mathbb{R}$ is called a ***real vector space*** and a vector space over $\mathbb{C}$ is called a ***complex vector space***. The elements $v \in V$ of a vector space are called ***vectors***.

A ***linear combination*** of vectors $x_1, \ldots, x_m$ of a vector space V is an expression of the form

$$\alpha_1 x_1 + \ldots + \alpha_m x_m$$

where the coefficients $\alpha_1, \ldots, \alpha_m$ are any scalars.

Definition 2.2. [7] Linear independence and dependence of a given set M of vectors $x_1, \ldots, x_r$ $(r \geq 1)$ in a vector space V are defined by means of equation

$$\alpha_1 x_1 + \ldots + \alpha_r x_r = 0, \tag{2.1}$$

where $\alpha_1, \ldots, \alpha_r$ are scalars. Clearly, equation (2.1) holds for $\alpha_1 x_1 = \ldots = \alpha_r x_r = 0$. if this is the only r-tuple of scalars for which (2.1) holds, the set M is said to the ***linearly independent***. M is said to be ***linearly dependent*** if M is not linearly independent.

Any arbitrary subset M of V is said to the linearly independent if every nonempty finite subset of M is linearly independent. M is said to be linearly dependent if M is not linearly independent.

Definition 2.3. [7] A vector space V is said to the ***finite dimensional*** if there is a positive integer n such that V contains a linearly independent set of n vectors whereas any set of $n+1$ or more vectors of V is linearly independent. n is called the ***dimension*** of V, written $n =$ dim V. If V is not finite dimensional, V is said to be ***infinite dimensional***.

Example 2.1. *$\mathbb{R}^n$ and $\mathbb{C}^n$ are n-dimensional.*

If dim $V = n$, a linearly independent n-tuple of vectors of V is called a ***basis*** for V. If $\{e_1, \ldots, e_n\}$ is a basis for V, every $x \in V$ has a unique representation as a linear combination of basis vectors:

$$x = \alpha_1 e_1 + \ldots + \alpha_n e_n.$$

In case of vector spaces, a function is called an ***operator***.

Definition 2.4. [7] A ***linear operator*** T is an operator such that

(i) the domain $\mathfrak{D}(T)$ of T is a vector space and the range $\mathfrak{R}(T)$ lies in a vector space over the same field,

(ii) for all $x, y \in \mathfrak{D}(T)$ and scalars α,

$$T(x + y) = T(x) + T(y)$$
$$T(\alpha x) = \alpha T(x)$$

Now, we will see some examples of linear operators.

Example 2.2. *Let V be a vector space.*

*2.2.1 The **identity operator** $I : V \to V$ is defined by $I(x) = x$ for all $x \in V$.*

*2.2.2 The **zero operator** $0 : V \to V$ is defined by $0(x) = 0$ for all $x \in V$.*

*2.2.3 A **translation operator** $T^t, t \in \mathbb{R}$, takes a function f on $\mathbb{R}$ to its translation f_t, $f_t(x) = f(x + t)$. That is $T^t f(x) = f(x + t)$.*

2.2 Notations and Definitions

Let $n \in \mathbb{N}$. In this thesis, fix a basis $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^n$ over $\mathbb{R}$. For a real-valued function $f : \mathbb{R}^n \to \mathbb{R}$, we define the following operators:

I denotes the identity operator;
For each $i = 1, \ldots, n$, τ_i^t are the translation operators defined by

$$\tau_i^t f(x) = f(x + te_i)$$

for all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$;

For each $i = 1, \ldots, n$, the operators σ_i^t and ρ_i^t are defined by

$$\begin{aligned} \sigma_i^t f(x) &= (I + \tau_i^t) f(x) \\ \rho_i^t f(x) &= (I - \tau_i^t) f(x) \end{aligned}$$

for all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$.

To simplify the notations, we write

$$\begin{aligned} \sigma_{i_1,\ldots,i_m}^t f(x) &= \sigma_{i_1}^t \sigma_{i_2}^t \ldots \sigma_{i_m}^t f(x) \\ \rho_{i_1,\ldots,i_m}^t f(x) &= \rho_{i_1}^t \rho_{i_2}^t \ldots \rho_{i_m}^t f(x) \end{aligned}$$

for all $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, and $i_1, \ldots, i_m$ are elements in $\{1, \ldots, n\}$.

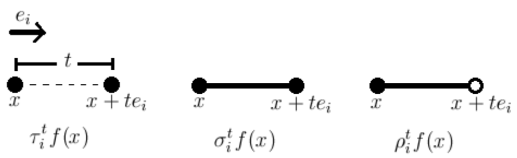

Figure 2.1 : • represents the value of f taken at the point
and ∘ represents the value of $-f$ taken at the point.

Geometrically as in Figure 2.1, $\tau_i^t f(x)$ is the value of f taken at a point $x + te_i$. Moreover, $\sigma_i^t f(x)$ is the sum of the values of f taken at the vertices x and $x + te_i$. Similarly, $\rho_i^t f(x)$ is the sum of the values of f taken at the vertices x and the values of $-f$ taken at the vertices $x + te_i$.

So, we can see that

$$\sigma^t_{1,2,3}f(x) = f(x) + f(x+te_1) + f(x+te_2) + f(x+te_3) +$$
$$f(x+te_1+te_2) + f(x+te_1+te_3) + f(x+te_2+te_3) + f(x+te_1+te_2+te_3)$$

is the sum of the values of f taken at all vertices of any parallelepiped whose sides are parallel to e_i $(i = 1, 2, 3)$ as in the following figure.

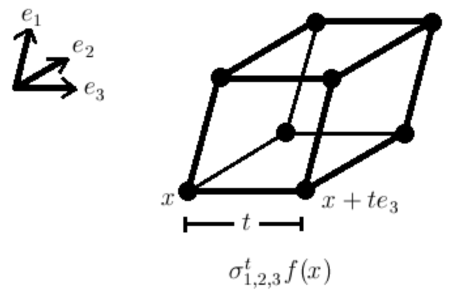

Figure 2.2 : • represents the value of f taken at the point.

From definition of σ^t_i's, we can see that (1.20) is equivalent to $\sigma^t_{1,\ldots,n}f(x) = 0$. In addition, note that, for each $i = 1, \ldots, n$ and $t \in \mathbb{R}$, $I = \tau^0_i$ and I, τ^t_i's, σ^t_i's, ρ^t_i's are commutative and distributive.

Definition 2.5. For each $r \in \mathbb{N}$, a set $\{1, 2, \ldots, r\}$ is called ***r-section*** of $\mathbb{N}$, written $\mathbb{N}_r$.

Example 2.3. $\mathbb{N}_1 = \{1\}$. $\mathbb{N}_2 = \{1, 2\}$. $\mathbb{N}_5 = \{1, 2, 3, 4, 5\}$.

Let A be a nonempty subset of $\mathbb{N}_n$ where $|A| = m$, cardinal number of A. Given $t \in \mathbb{R}$, σ^t_A is defined by

$$\sigma^t_A f(x) = \sigma^t_{i_1,i_2,\ldots,i_m} f(x)$$

for all $x \in \mathbb{R}^n$ where $i_1, i_2, \ldots, i_m$ are distinct integers in A.

Example 2.4. *Suppose* $n > 7$. *Let* $A = \{1, 3, 7\} \subseteq \mathbb{N}_n$ *and* $t \in \mathbb{R}$.

$$\sigma^t_A f(x) = \sigma^t_{1,3,7} f(x)$$

for all $x \in \mathbb{R}^n$.

CHAPTER III

ZERO-MEAN FUNCTIONAL EQUATION ON PARALLELEPIPED

Now, we fix a basis $\{e_1, e_2, e_3\}$ for $\mathbb{R}^3$ over $\mathbb{R}$. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a real-valued function. In this chapter, we determine the general solution $f : \mathbb{R}^3 \to \mathbb{R}$ of the functional equation

$$\sum_{\varepsilon_1,\varepsilon_2,\varepsilon_3=0}^{1} f(x + t\varepsilon_1 e_1 + t\varepsilon_2 e_2 + t\varepsilon_3 e_3) = 0 \tag{3.1}$$

for all $x \in \mathbb{R}^3$ and for all $t > 0$. From (3.1), we observe that the arithmetic mean of the values of f taken at the vertices of any parallelepiped, whose sides are parallel to e_i, is equal to zero. Accordingly, (3.1) will be called a "zero-mean" functional equation on parallelepiped.

First we will prove the following useful proposition.

Proposition 3.1.

$$\rho_i^t \sigma_i^t f(x) = \rho_i^{2t} f(x)$$

for all $x \in \mathbb{R}^3$, *for all* $i \in \mathbb{N}_3$, *and for all* $t \in \mathbb{R}$.

Proof. Let $x \in \mathbb{R}^3$, $t \in \mathbb{R}$, and $i \in \mathbb{N}_3$. Then,

$$\begin{aligned} \rho_i^t \sigma_i^t f(x) &= f(x) + f(x + te_i) - f(x + te_i) - f(x + 2te_i) \\ &= f(x) - f(x + 2te_i) \\ &= \rho_i^{2t} f(x). \end{aligned}$$

□

Lemma 3.2. *If there exists $y \in \mathbb{R}^3$ such that*

$$f(x) + f(x + ty) = 0 \tag{3.2}$$

for all $x \in \mathbb{R}^3$ and for all $t > 0$, then f is identically zero.

Proof. Let $x \in \mathbb{R}^3$ and $t > 0$. Suppose there is $y \in \mathbb{R}^3$ such that f satisfies (3.2). Replacing x by $x + ty$ in (3.2), we have

$$f(x + ty) + f(x + 2ty) = 0. \tag{3.3}$$

From (3.2) and (3.3), we have

$$f(x) - f(x + 2ty) = 0.$$

Since x and t are arbitrary, we get

$$f(x) - f(x + ty) = 0 \tag{3.4}$$

for all $x \in \mathbb{R}^3$ and for all $t > 0$. Combining (3.2) with (3.4), we have $f(x) = 0$. Since x is arbitrary, $f(x) = 0$ for all $x \in \mathbb{R}^3$. ☐

By Lemma 3.2, we obtain the following lemma.

Lemma 3.3. *If a function f satisfies, for each $x \in \mathbb{R}^3$ and for each $t > 0$,*

$$\sigma^t_{i,j} f(x) = 0, \tag{3.5}$$

where i, j are distinct integers of $\mathbb{N}_3$, then f is identically zero.

Proof. Let $x \in \mathbb{R}^3$ and $t > 0$. Suppose f satisfies (3.5). By Proposition 3.1, we get

$$\rho^{2t}_{i,j} f(x) = \rho^t_{i,j} \sigma^t_{i,j} f(x) = 0.$$

Since x and t are arbitrary,

$$\rho_{i,j}^{t} f(x) = 0 \tag{3.6}$$

for all $x \in \mathbb{R}^3$ and for all $t > 0$. From (3.5) and (3.6), we have

$$\begin{aligned}
&2[f(x) + f(x + te_i + te_j)] \\
&\quad = f(x) + f(x + te_i) + f(x + te_j) + f(x + te_i + te_j) \\
&\quad\quad + f(x) - f(x + te_i) - f(x + te_j) + f(x + te_i + te_j) \\
&\quad = \sigma_{i,j}^{t} f(x) + \rho_{i,j}^{t} f(x) = 0.
\end{aligned} \tag{3.7}$$

Therefore, for each $x \in \mathbb{R}^3$ and for each $t > 0$,

$$f(x) + f(x + t(e_i + e_j)) = 0.$$

By Lemma 3.2, $f(x) = 0$ for all $x \in \mathbb{R}^3$. □

The following theorem is our main result in this chapter.

Theorem 3.4. *A function f satisfies (3.1) if and only if f is identically zero.*

Proof. Let $x \in \mathbb{R}^3$ and $t > 0$. Suppose f satisfies (3.1). By Proposition (3.1), we have

$$\rho_{1,2,3}^{2t} f(x) = \rho_{1,2,3}^{t} \sigma_{1,2,3}^{t} f(x) = 0.$$

Since x and t are arbitrary,

$$\rho_{1,2,3}^{t} f(x) = 0 \tag{3.8}$$

for all $x \in \mathbb{R}^3$ and for all $t > 0$. From (3.1) and (3.8), we obtain

$$\begin{aligned}
&2[f(x) + f(x + te_2 + te_3) - f(x + 2te_1) - f(x + 2te_1 + te_2 + te_3)] \\
&= 2[f(x) + f(x + te_1 + te_2) + f(x + te_1 + te_3) + f(x + te_2 + te_3)] \\
&\quad - 2[f(x + 2te_1) + f(x + te_1 + te_2) + f(x + te_1 + te_3) + f(x + 2te_1 + te_2 + te_3)] \\
&= (\sigma_{1,2,3}^{t} + \rho_{1,2,3}^{t}) f(x) - (\sigma_{1,2,3}^{t} - \rho_{1,2,3}^{t}) f(x + te_1) = 0.
\end{aligned} \tag{3.9}$$

And we also obtain

$$\begin{aligned}
&2[f(x+te_2)+f(x+te_3)-f(x+2te_1+te_2)-f(x+2te_1+te_3)]\\
&= \; 2[f(x+te_1)+f(x+te_2)+f(x+te_3)+f(x+te_1+te_2+te_3)]\\
&\quad -2[f(x+te_1)+f(x+2te_1+te_2)+f(x+2te_1+te_3)+f(x+te_1+te_2+te_3)]\\
&= \; (\sigma^t_{1,2,3}-\rho^t_{1,2,3})f(x)-(\sigma^t_{1,2,3}+\rho^t_{1,2,3})f(x+te_1) = \; 0.
\end{aligned} \tag{3.10}$$

From (3.9) and (3.10), we have

$$f(x)+f(x+te_2+te_3)-f(x+2te_1)-f(x+2te_1+te_2+te_3) \;=\; 0, \tag{3.11}$$

$$f(x+te_2)+f(x+te_3)-f(x+2te_1+te_2)-f(x+2te_1+te_3) \;=\; 0, \tag{3.12}$$

for all $x \in \mathbb{R}^3$ and for all $t > 0$.
Replacing x by $x+te_2$ in (3.12), we have

$$f(x+2te_2)+f(x+te_2+te_3)-f(x+2te_1+2te_2)-f(x+2te_1+te_2+te_3) = 0. \tag{3.13}$$

Subtracting (3.13) from (3.11), we get

$$\rho^{2t}_{1,2}f(x) = f(x)-f(x+2te_1)-f(x+2te_2)+f(x+2te_1+2te_2) = 0.$$

Since x and t are arbitrary, $\rho^t_{1,2}f(x) = 0$ for all $x \in \mathbb{R}^3$ and for all $t > 0$. Hence,

$$\begin{aligned}
&2[f(x)+f(x+te_1+te_2)+f(x+te_3)+f(x+te_1+te_2+te_3)]\\
&= \; [f(x)+f(x+te_1)+f(x+te_2)+f(x+te_1+te_2)\\
&\quad +f(x+te_3)+f(x+te_1+te_3)+f(x+te_2+te_3)+f(x+te_1+te_2+te_3)]\\
&\quad +[f(x)-f(x+te_1)-f(x+te_2)+f(x+te_1+te_2)]\\
&\quad +[f(x+te_3)-f(x+te_1+te_3)-f(x+te_2+te_3)+f(x+te_1+te_2+te_3)]\\
&= \; \sigma^t_{1,2,3}f(x)+\rho^t_{1,2}f(x)+\rho^t_{1,2}f(x+te_3) = \; 0.
\end{aligned}$$

Therefore, for each $x \in \mathbb{R}^3$ and for each $t > 0$,

$$f(x) + f(x + te_1 + te_2) + f(x + te_3) + f(x + te_1 + te_2 + te_3) = 0. \qquad (3.14)$$

Let $e'_1 = e_1 + e_2, e'_2 = e_2$, and $e'_3 = e_3$. It is easy to prove that $\{e'_1, e'_2, e'_3\}$ is a basis for $\mathbb{R}^3$ over $\mathbb{R}$. From (3.14) with respect to $\{e'_1, e'_2, e'_3\}$, we have

$$\sigma^t_{1,3} f(x) = 0$$

for all $x \in \mathbb{R}^3$ and for all $t > 0$. By Lemma 3.3, $f(x) = 0$ for all $x \in \mathbb{R}^3$. Conversely, it is obvious that if f is identically zero, then f satisfies (3.1). $\square$

CHAPTER IV

Zero-Mean Functional Equation on Hyper-Parallelepiped

Recall that we fix a basis $\{e_1, \ldots, e_n\}$ for $\mathbb{R}^n$ over $\mathbb{R}$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a real-valued function. In this chapter, we will determine the general solution of zero-mean functional equation on hyper-parallelepiped

$$\sum_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n = 0}^{1} f(x + t\varepsilon_1 e_1 + \ldots + t\varepsilon_n e_n) = 0 \tag{1.20}$$

for all $x \in \mathbb{R}^n$ and for all $t > 0$.

first, we will consider the following proposition which is the generalized Proposition 3.1.

Proposition 4.1.

$$\rho_i^t \sigma_i^t f(x) = \rho_i^{2t} f(x)$$

for all $x \in \mathbb{R}^n$, *for all* $i \in \mathbb{N}_n$, *and for all* $t \in \mathbb{R}$.

Proof. Similar to Proposition 3.1, we can prove this proposition. ☐

The following simple proposition is useful to obtain our main result.

Proposition 4.2.

$$\rho_i^t f(x) + \sigma_i^t f(x) = 2f(x)$$

for all $x \in \mathbb{R}^n$, *for all* $i \in \mathbb{N}_n$, *and for all* $t \in \mathbb{R}$.

Proof. Let $x \in \mathbb{R}^n$, $t > 0$, and $i \in \mathbb{N}_n$. Then,

$$\rho_i^t f(x) + \sigma_i^t f(x) \;=\; [f(x) - f(x + te_i)] + [f(x) + f(x + te_i)] \;=\; 2f(x).$$

☐

Given the conditions in the following lemma, we can consider zero-mean functional equation on hyper-parallelepiped on $\mathbb{R}^m$ whose basis is reduced from $\mathbb{R}^n$ $(m \in \mathbb{N}, m < n)$.

Recall that σ_A^t is defined by

$$\sigma_A^t f(x) = \sigma_{i_1,i_2,\ldots,i_m}^t f(x)$$

for all $x \in \mathbb{R}^r$ and for all $t \in \mathbb{R}$ where A is a nonempty subset of $\mathbb{N}_n$ such that $|A| = m$ and $i_1, i_2, \ldots, i_m$ are distinct integers in A.

Lemma 4.3. *Let A be a nonempty subset of $\mathbb{N}_n$ with $|A| = m < n$. Assume that a real-valued function f satisfies* (1.20) *and*

$$\rho_i^t f(x) = 0$$

for all $i \in A$, for all $x \in \mathbb{R}^n$, and for all $t > 0$. Then

$$\sigma_{\mathbb{N}_n \setminus A}^t f(x) = 0$$

for all $x \in \mathbb{R}^n$ and for all $t > 0$.

Proof. Let $t > 0$. By assumption, let $i_1, \ldots, i_n$ be distinct integers in $\mathbb{N}_n$ such that $i_j \in A$ for all $0 < j \leq m$. Then, we have

$$\sigma_{1,\ldots,n}^t f(x) = 0 \text{ and } \rho_{i_1}^t f(x) = 0$$

for all $x \in \mathbb{R}^n$. By Proposition 4.2, for each $x \in \mathbb{R}^n$ we get

$$\begin{aligned} 2\sigma_{i_2,\ldots,i_n}^t f(x) &= \sigma_{1,\ldots,n}^t f(x) + \rho_{i_1}^t \sigma_{i_2,\ldots,i_n}^t f(x) \\ &= \sigma_{1,\ldots,n}^t f(x) + \rho_{i_1}^t [f(x) + \ldots + f(x + te_{i_2} + \ldots + te_{i_n})] \\ &= 0. \end{aligned}$$

That is $\sigma_{i_2,\ldots,i_n}^t f(x) = 0$ for all $x \in \mathbb{R}^n$.

Continuing this process inductively, for k^{th} step where $k \leq m$ we have

$$\sigma^t_{i_k,\ldots,i_n} f(x) = 0 \text{ and } \rho^t_{i_k} f(x) = 0$$

for all $x \in \mathbb{R}^n$. By Proposition 4.2, for each $x \in \mathbb{R}^n$ we get

$$\begin{aligned} 2\sigma^t_{i_{k+1},\ldots,i_n} f(x) &= \sigma^t_{i_k,\ldots,i_n} f(x) + \rho^t_{i_k} \sigma^t_{i_{k+1},\ldots,i_n} f(x) \\ &= \sigma^t_{i_k,\ldots,i_n} f(x) + \rho^t_{i_k} [f(x) + \ldots + f(x + te_{i_{k+1}} + \ldots + te_{i_n})] \\ &= 0. \end{aligned}$$

That is $\sigma^t_{i_{k+1},\ldots,i_n} f(x) = 0$ for all $x \in \mathbb{R}^n$.
Finally, for $k = m$ we get $\sigma^t_{i_{m+1},\ldots,i_n} f(x) = 0$ for all $x \in \mathbb{R}^n$.
Therefore, $\sigma^t_{\mathbb{N}_n \smallsetminus A} f(x) = 0$ for all $x \in \mathbb{R}^n$ and for all $t > 0$. □

Repeatedly applying Lemma 4.3, we obtain the following Lemma.

Lemma 4.4. *If a real-valued function f satisfies* (1.20)*, then*

$$\rho^t_i f(x) = 0$$

for all $x \in \mathbb{R}^n$, for all $t > 0$, and for all integers $i \in \mathbb{N}_n$.

Proof. Let $x \in \mathbb{R}^n$. By Proposition 4.1, for each $t > 0$ we have

$$\begin{aligned} \rho^{2t}_{1,\ldots,n} f(x) &= \rho^t_{1,\ldots,n} \sigma^t_{1,\ldots,n} f(x) \\ &= \sigma^t_{1,\ldots,n} [f(x) - f(x + te_1) + \ldots + (-1)^n f(x + te_1 + \ldots + te_n)] \\ &= 0. \end{aligned}$$

Since t is arbitrary, we obtain $\rho^t_{1,\ldots,n} f(x) = 0$ for all $t > 0$. Now consider

$$\begin{aligned} \sigma^t_{1,\ldots,n} \rho^t_{i_1,\ldots,i_{n-1}} f(x) &= \sigma^t_{1,\ldots,n} [f(x) + \ldots + (-1)^{n-1} f(x + te_{i_1} + \ldots + te_{i_{n-1}})] \\ &= 0 \end{aligned}$$

for all distinct integers $i_1, \ldots, i_{n-1} \in \mathbb{N}_n$ and for all $t > 0$. So, we have

$$\rho^t_{1,\ldots,n} f(x) = 0 \text{ and } \sigma^t_{1,\ldots,n} \rho^t_{i_1,\ldots,i_{n-1}} f(x) = 0$$

for all distinct integers $i_1, \ldots, i_{n-1} \in \mathbb{N}_n$ and for all $t > 0$.

Thus, $\rho^t_{i_1,\ldots,i_{n-1}} f$ satisfies (1.20) and $\rho^t_{i_n} \rho^t_{i_1,\ldots,i_{n-1}} f(x) = 0$. Lemma 4.3 implies

$$\sigma^t_{i_1,\ldots,i_{n-1}} \rho^t_{i_1,\ldots,i_{n-1}} f(x) = 0$$

for all distinct integers $i_1, \ldots, i_{n-1} \in \mathbb{N}_n$ and for all $t > 0$.

By Proposition 4.1, we obtain

$$\rho^t_{i_1,\ldots,i_{n-1}} f(x) = 0$$

for all distinct integers $i_1, \ldots, i_{n-1} \in \mathbb{N}_n$ and for all $t > 0$.

Continuing this process inductively, for $(k)^{th}$ step where $k < n$ we have

$$\rho^t_{i_j} \rho^t_{i_1,\ldots,i_{n-k}} f(x) = 0 \text{ for all } j > n-k, \tag{4.1}$$

$$\text{and } \sigma^t_{1,\ldots,n} \rho^t_{i_1,\ldots,i_{n-k}} f(x) = 0$$

for all distinct integers $i_1, \ldots, i_{n-k} \in \mathbb{N}_n$ and for all $t > 0$.

Thus, $\rho^t_{i_1,\ldots,i_{n-k}} f$ satisfies (1.20) and (4.1). Lemma 4.3 implies

$$\sigma^t_{i_1,\ldots,i_{n-k}} \rho^t_{i_1,\ldots,i_{n-k}} f(x) = 0$$

for all distinct integers $i_1, \ldots, i_{n-k} \in \mathbb{N}_n$ and for all $t > 0$.

By Proposition 4.1, we obtain

$$\rho^t_{i_1,\ldots,i_{n-k}} f(x) = 0$$

for all distinct integers $i_1, \ldots, i_{n-k} \in \mathbb{N}_n$ and for all $t > 0$.

Therefore, $\rho^t_i f(x) = 0$ for all $x \in \mathbb{R}^n$, for all $t > 0$, and for all $i \in \mathbb{N}_n$. $\square$

In the following lemma, we will solve an essential functional equation to obtain the main theorem in this chapter.

Lemma 4.5. *If a real-valued function f satisfies*

$$\rho_i^t f(x) = 0 \tag{4.2}$$

for all $x \in \mathbb{R}^n$, for all $t > 0$, and for all $i \in \mathbb{N}_n$, then f is a constant function.

Proof. Let $x \in \mathbb{R}^n$. Then $x = \alpha_1 e_1 + \alpha_2 e_2 + \ldots + \alpha_n e_n$ for some $\alpha_1, \ldots, \alpha_n$ in $\mathbb{R}$.
From (4.2), we have

$$f(x) = f(x + te_i) \tag{4.3}$$

for all $x \in \mathbb{R}^n$, for all $t \geq 0$, and for all integers $i \in \mathbb{N}_n$.
Repeatedly using (4.3) by replacing t by $|\alpha_i|$ where $i = 1, \ldots, n$, we have

$$\begin{aligned} f(x) &= f(x + |\alpha_1|e_1 + \ldots. + |\alpha_n|e_n) \\ &= f((\alpha_1 + |\alpha_1|)e_1 + \ldots. + (\alpha_n + |\alpha_n|)e_n). \end{aligned} \tag{4.4}$$

For each $i = 1, \ldots, n$, if $\alpha_i \leq 0$, then $\alpha_i + |\alpha_i| = 0$; otherwise, $\alpha_i + |\alpha_i| > 0$.
Repeatedly using (4.3) in (4.4), we have $f(x) = f(0)$.
Since x is arbitrary, we obtain $f(x) = f(0)$ for all $x \in \mathbb{R}^n$.
Therefore, f is a constant function. □

Finally, we are ready to establish our main theorem.

Theorem 4.6. *A real-valued function f satisfies* (1.20) *if and only if f is identically zero.*

Proof. Let $x \in \mathbb{R}^n$ and $t > 0$. Assume that f satisfies (1.20).
By Lemma 4.4, we have $\rho_i^t f(x) = 0$ for all $i \in \mathbb{N}_n$.
By Lemma 4.5, we obtain f is a constant function.
So, there exists $c \in \mathbb{R}$ such that $f(x) = c$ for all $x \in \mathbb{R}^n$. Since f satisfies (1.20), we have $2^n c = 0$. That is $c = 0$. Hence, $f(x) = 0$ for all $x \in \mathbb{R}^n$.
Conversely, it is obvious that if f is identically zero, then (1.20) holds. □

REFERENCES

[1] Aczél, J.: *Lectures on functional equations and their applications*, Academic Press, New York, 1966.

[2] Aczél, J., Haruki, H., Mckiernan, M.A., Sakovič, G.N.: General and regular solutions of functional equations characterizing harmonic polynomials, *Aequationes Math.* **1** (1968), 37-53.

[3] Etigson, L.: Equivalence of 'Cube' and 'Octahedron' functional equations, *Aequationes Math.* **10** (1974), 50-56.

[4] Haruki, H.: On a 'Cube Functional Equation', *Aequationes Math.* **3** (1969), 156-159.

[5] Haruki, H.: On a relation between the "square" functional equation and the "square" mean-value property, *Canad. Math. Bull.* **14**(2) (1971), 161-165.

[6] Kotnara, R.: *Functional Equation on Planar Quadrilaterals*, Master's Thesis, Department of Mathematics and Computer Science, Faculty of Science, Chulalongkorn University, 2011.

[7] Kreyszig, E.: *Introductory functional analysis with applications*, John Wiley & Sons, Canada, 1978.

[8] Sahoo, P.K., Kannappan, Pl.: *Introduction to Functional Equations*, CRC Press - Taylor and Francis Group, Boca Raton, 2011.

[9] Székelyhidi, L.: Mean-value type functional equations, *Aequationes Math.* **42** (1991), 23-36.

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