ทฤษฎีบทบางบทในกึ่งมอดูลเสมือนบนกึ่งริง

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SOME THEOREMS IN SKEW-SEMIMODULES OVER SEMIRINGS

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กึ่งมอดูลเสมือน M บนกึ่งริง S คือโมนอยด์ M ภายใต้การดำเนินการการบวก และมีการ กระทำทางซ้าย $S \times M \to M$ ซึ่งกำหนดโดย $(s,m) \mapsto sm$ ที่มีสมบัติว่า สำหรับแต่ละ $r,s \in S$ และ $m,n \in M$ (i) $(r+s)m = rm+sm$ (ii) $s(m+n) = sm+sn$ (iii) $(rs)m = r(sm)$ และ (iv) $s0 = 0$

เราเรียกสับเซตไม่ว่าง A ของกึ่งมอดูลเสมือน M บนกึ่งริง S ว่า **ไอดีลของ** M ก็ต่อเมื่อ $A+M, M+A$ และ S^*A เป็นสับเซตของ A โดยที่ $S^* = S \setminus \{0\}$ และเมื่อกำหนดให้ A เป็นไอดีลของ M **ความสัมพันธ์สมมูลรีส์บน** M **ที่ก่อกำเนิดโดย** A คือความสัมพันธ์ที่กำหนดโดย $R_A = \big\{(m,n) \in M \times M \mid m = n$ หรือ $m,n \in A\big\}$

กำหนดให้ M และ N เป็นกึ่งมอดูลเสมือนบนกึ่งริง S การส่ง $\varphi : M \to N$ เป็น **โฮโมมอร์ฟิซึม** ก็ต่อเมื่อ สำหรับทุกๆ $m,n \in M$ และ $s \in S$ (i) $\varphi(m+n) = \varphi(m)+\varphi(n)$ (ii) $\varphi(sm) = s\varphi(m)$ และ (iii) $\varphi(0) = 0$ นอกจากนี้เราเรียกเซตของ $m \in M$ โดยที่ $\varphi(m) = 0$ ว่า **เซตศูนย์ของ** φ ซึ่งจะเขียนแทนด้วย $Zs\varphi$ และ **เคอร์เนลของ** φ คือความสัมพันธ์ $Ker\varphi = \big\{(m,n) \in M \times M \,\big|\, \varphi(m) = \varphi(n)\big\}$

กำหนดให้ M N และ P เป็นกรุปและเป็นกึ่งมอดูลเสมือนบนกึ่งริง S เรากล่าวว่าลำดับ $M \xrightarrow{f} N \xrightarrow{g} P$ ของกึ่งมอดูลเสมือนและโฮโมมอร์ฟิซึมเป็น **เอกแซคท์ที่** N ก็ต่อเมื่อ $Imf = Zsg$

เรากล่าวว่าลำดับ $A_1 \subseteq A_2 \subseteq \cdots$ หรือ $A_1 \supseteq A_2 \supseteq \cdots$ ของสับเซตของกึ่งมอดูลเสมือน M บนกึ่งริง S เป็น **อนุกรมไอดีล** ก็ต่อเมื่อ A_i เป็นไอดีลของ M สำหรับทุกๆ จำนวนนับ i

ผลสำคัญของงานวิจัยนี้คือการทำให้ทฤษฎีบทไอโซมอร์ฟิซึม สมบัติการส่งทั่วไปของผลคูณตรง ผลบวกตรง และมอดูลเสรี ทฤษฎีบทบางบทของลำดับเอกแซคท์ มอดูลอาร์ทิเนียน และนอทิเรียน เป็นกรณีทั่วไปในกึ่งมอดูลเสมือน

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A *skew-semimodule* M *over a semiring* S is an additive monoid M with a left action $S \times M \to M$, defined by $(s,m) \mapsto sm$, such that for all $r,s \in S$ and $m,n \in M$ (i) $(r+s)m = rm + sm$, (ii) $s(m+n) = sm + sn$, (iii) $(rs)m = r(sm)$ and (iv) $s0 = 0$ where 0 is the identity of M.

A non-empty subset A of a skew-semimodule M over a semiring S is said to be an *ideal of* M if $A+M, M+A$ and S^*A are subsets of A where $S^* = S \setminus \{0\}$. Moreover, given an ideal A of M, the *Rees congruence on* M *generated by* A is the congruence relation $R_A = \{(m,n) \in M \times M \mid m = n \text{ or } m,n \in A\}$.

Let M and N be skew-semimodules over a semiring S. A mapping $\varphi : M \to N$ is called a *homomorphism* if (i) $\varphi(m+n) = \varphi(m) + \varphi(n)$, (ii) $\varphi(sm) = s\varphi(m)$ and (iii) $\varphi(0) = 0$ for all $m,n \in M$ and $s \in S$. The set of $m \in M$ such that $\varphi(m) = 0$ is called the *zero set of* φ, denoted by $Zs\varphi$. In addition, the *kernel of* φ is the relation $Ker\varphi = \{(m,n) \in M \times M \mid \varphi(m) = \varphi(n)\}$.

Let M, N and P be groups and skew-semimodules over a semiring S. A sequence $M \xrightarrow{f} N \xrightarrow{g} P$ of skew-semimodules and homomorphisms is said to be *exact at* N if $Imf = Zsg$.

A chain $A_1 \subseteq A_2 \subseteq \cdots$ or $A_1 \supseteq A_2 \supseteq \cdots$ of subsets of a skew-semimodule M over a semiring S is said to be an *ideal series* of M if A_i is an ideal of M for all positive integers i.

The main purpose of this research is to generalize of Isomorphism Theorems, the universal mapping properties of direct products, direct sums and free modules, some theorems of exact sequences and Artinian and Noetherian modules to those of skew-semimodules.

Department **Mathematics**	Student's signature..............................
Field of study **Mathematics**	Advisor's signature..............................
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CONTENTS

INTRODUCTION

A very important algebraic structure is that of a module over a ring. In addition, there are generalizations of some theorems in module theory to those in skewmodules over skewrings and in semimodules over semirings which were introduced in [2] and [5], respectively.

Recall that a module M over a ring S is an abelian group M with a left action $S \times M \to M$ defined by $(s,m) \mapsto sm$ such that for all $m,n \in M$ and $r,s \in S$,

(i) $(r+s)m = rm + sm$,

(ii) $s(m+n) = sm + sn$, and

(iii) $(rs)m = r(sm)$.

In [2], a structure $(S,+,\cdot)$ is a skewring if $(S,+)$ is a group, $(S,\cdot)$ is a semigroup and the operation $\cdot$ is distributive over $+$; moreover, a skewmodule M over a skewring S is a group with a left action $S \times M \to M$ defined similarly to a module over a ring. Also, in [5], a semimodule M over a semiring S is defined analogously where $(M,+)$ is a commutative monoid and $(S,+,\cdot)$ is a semiring, i.e., $(S,+)$ is a commutative monoid, $(S,\cdot)$ is a monoid and the operation $\cdot$ is distributive over $+$. For our research, we are interested in a more general structure.

We define a skew-semimodule M over a semiring S analogously to those structures, i.e., $(M,+)$ is a monoid and $(S,+,\cdot)$ is a semiring which $(S,+)$ and $(S,\cdot)$ are semigroups and the operation $\cdot$ is distributive over $+$. Notice that a semiring S in this research is given differently from the mentioned above. Moreover, we study which theorems in module theory can be generalized to ones in skew-semimodules. However, our concern goes to skew-subsemimodules, homomorphisms, quotient skew-semimodules, direct products, direct sums, free skew-semimodules and exact sequences. Furthermore, we investigate concept of ideals of skew-semimodules and introduce idea of Artinian and Noetherian skew-semimodules.

There are four chapters in this thesis. In Chapter 1, we give definitions and prove some theorems regarding skew-semimodules, skew-subsemimodules, ideals, homomorphisms, congruence relations and quotient skew-semimodules.

In Chapter 2, we study direct products and direct sums of families of skew-semimodules. Moreover, a free skew-semimodule is defined in this chapter and some theorems involving this are proved such as the Universal Mapping Property of Free Skew-semimodules.

In Chapter 3, we consider particular skew-semimodules which are also groups. Doing this lead us to define an exact sequence and prove theorems parallel to those of exact sequences in module theory.

Finally, in Chapter 4, we define an ideal series of skew-semimodules. Moreover, we prove some elementary theorems of Artinian and Noetherian skew-semimodules.

CHAPTER I

SKEW-SEMIMODULES OVER SEMIRINGS

This chapter covers basic results about skew-semimodules and is divided into two sections. In section 1.1, we introduce skew-semimodules, skew-subsemimodules, ideals and homomorphisms. In section 1.2, we discuss congruence relations on skew-semimodules and quotient skew-semimodules. Many results in this chapter play important roles in order to study other topics in this thesis.

1.1. Skew-semimodules and Homomorphisms

This section gives basic definitions and theorems concerning skew-semimodules including skew-subsemimodules, ideals and homomorphisms. The concepts of ideals of skew-semimodules are not found in the theory of semimodules over semirings.

Definition 1.1.1. A triple $(S,+,\cdot)$ is a *semiring* if

(i) $(S,+)$ is a semigroup,

(ii) $(S,\cdot)$ is a semigroup, and

(iii) $r\cdot(s+t)=r\cdot s+r\cdot t$ and $(s+t)\cdot r=s\cdot r+t\cdot r$ for all $r,s,t\in S$.

A semiring S has the *zero* if there exists an element $0\in S$ such that $0+s=s=s+0$ and $0\cdot s=0=s\cdot 0$ for all $s\in S$. Also, S has the *identity* if there exists an element $1\in S$ such that $1\cdot s=s=s\cdot 1$ for all $s\in S$. If S is a semiring with zero (and identity), then we write that S is a semiring with 0 (and 1).

Example 1.1.2. The set $\mathbb{N}$ of all natural numbers with the usual addition and multiplication is a semiring with 1. For each $n\in\mathbb{N}$, given $[n]=\{m\in\mathbb{N}\,|\,m>n\}$.

Then for all $n \in \mathbb{N}$, we also obtain that $[n]$ with the usual addition and multiplication on $\mathbb{N}$ is a semiring without 0 and 1.

Definition 1.1.3. Let S be a semiring. A *(left) skew-semimodule M over S* or *S-skew-semimodule* is an additive monoid M with a left action $S \times M \to M$, called a *scalar multiplication*, given by $(s, m) \mapsto sm$ which satisfies the following condition: for all $r, s \in S$ and all $m, n \in M$,

(i) $(r + s)m = rm + sm$,

(ii) $s(m + n) = sm + sn$,

(iii) $(rs)m = r(sm)$, and

(iv) $s0 = 0$ where 0 is the identity of M.

A *(right) skew-semimodule M over S* is defined in a similar way by replacing the left action by the right action with corresponding properties.

If S is a semiring with 0 and 1 such that $0 \neq 1$ and M is a skew-semimodule over S satisfying

$$(\star) \qquad 0m = 0 \qquad \text{and} \qquad 1m = m \qquad \text{for all } m \in M,$$

then we say that M is a *skew-semimodule over S satisfying* $(\star)$ or an *$S^\star$-skew-semimodule*.

In this research, we study on left skew-semimodules over semirings only.

Example 1.1.4. If S is a semiring with 0, then S is a skew-semimodule over itself.

Example 1.1.5. Let S be a semiring and M a monoid. Define $sm = 0$ for all $s \in S$ and $m \in M$. Hence it is easy to see that M is a skew-semimodule over S.

Example 1.1.6. Let X be the right zero semigroup, i.e., $x + y = y$ for all $x, y \in X$. Given $X^0 = X \cup \{0\}$ where $x + 0 = x = 0 + x$ and $0 + 0 = 0$ for all $x \in X$. Then

X^0 is a monoid. Let $n \in \mathbb{N} \cup \{0\}$ which is a semiring with 0 and 1. Define $nx = x$ for all $x \in X^0$. Therefore, X^0 is a skew-semimodule over $\mathbb{N} \cup \{0\}$.

We can see that the skew-semimodule X^0 in Example 1.1.6 does not satisfy $(\star)$ but the following example gives an $S^\star$-skew-semimodule.

Example 1.1.7. Let X be a monoid and $n \in \mathbb{N} \cup \{0\}$. We define a scalar multiplication by

$$nx = \begin{cases} x, & \text{if } n \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

for all $x \in X$. Hence it is easy to see that X is an skew-semimodule over $\mathbb{N} \cup \{0\}$ satisfying $(\star)$.

Definition 1.1.8. Let M be an S-skew-semimodule and N a non-empty subset of M. Then N is a *skew-subsemimodule of M* if

(i) N is a monoid under the same operation with the same identity as M, and

(ii) $sn \in N$ for all $s \in S$ and $n \in N$.

Remark 1.1.9. Let M be an S-skew-semimodule.

(i) If N is a skew-subsemimodule of M, then N is a submonoid of M with the same identity and is an S-skew-semimodule.

(ii) Trivial skew-subsemimodules of M are M and $\{0\}$.

(iii) If P and Q are skew-subsemimodules of M such that $P+Q = Q+P$, then $P+Q$ is a skew-subsemimodule of M.

Proposition 1.1.10. *Let $(N_i)_{i\in I}$ be a family of skew-subsemimodules of an S-skew-semimodule M. Then $\bigcap_{i\in I} N_i$ is a skew-subsemimodule of M.*

Proof. Since N_i is a submonoid of M with the same identity for all $i \in I$, it follows that $0 \in N_i$ for all $i \in I$, where 0 is the identity of M, so that $0 \in \bigcap_{i\in I} N_i$. Let

$n, n' \in \bigcap_{i\in I} N_i$ and $s \in S$. Then $n, n' \in N_i$ so $n + n', sn \in N_i$ since N_i is a skew-subsemimodule of M for all $i \in I$. Hence $n + n', sn \in \bigcap_{i\in I} N_i$. That is $\bigcap_{i\in I} N_i$ is a skew-subsemimodule of M. □

The previous proposition suggests the following definition.

Definition 1.1.11. Let N be a subset of an S-skew-semimodule M. Then the *skew-subsemimodule of* M *generated by* N, denoted by $[N]$, is the intersection of all skew-subsemimodules of M containing N. If $[N] = M$, then we say that N *generates* M.

Moreover, M is *finitely generated* if there exists a finite subset N of M such that N generates M.

Remark 1.1.12. Let M be an S-skew-semimodule and $N \subseteq M$. Then $[N]$ is the smallest skew-subsemimodule of M containing N. Moreover, $[\emptyset] = \{0\}$ and $[M] = M$.

If S is a semiring with 0 and 1 such that $0 \neq 1$ and N is a non-empty subset of an $S^\star$-skew-semimodule M, then we can describe elements of $[N]$ explicitly.

Proposition 1.1.13. *Let S be a semiring with 0 and 1 such that $0 \neq 1$ and N a non-empty subset of an $S^\star$-skew-semimodule M. Then*

$$[N] = \left\{ \sum_{i=1}^{m} s_i n_i \;\middle|\; m \in \mathbb{N} \text{ and } s_i \in S, n_i \in N \text{ for all } i = 1, 2, \ldots, m \right\}.$$

Proof. Given

$$P = \left\{ \sum_{i=1}^{m} s_i n_i \;\middle|\; m \in \mathbb{N} \text{ and } s_i \in S, n_i \in N \text{ for all } i = 1, 2, \ldots, m \right\}.$$

Since $N \subseteq [N]$ which is a skew-subsemimodule of M, it follows that $P \subseteq [N]$.

It is clear that P is a skew-subsemimodule of M since $0 \in P$. Moreover, $N \subseteq P$ because $1 \in S$. Hence $[N] \subseteq P$ by Remark 1.1.12. Therefore, $[N] = P$. $\square$

Proposition 1.1.13 is very useful in proving theorems about free skew-semimodules. The purpose of the following definition is replacing a normal subskewmodule in [2].

Definition 1.1.14. Let A be a non-empty subset of an S-skew-semimodule M. Then A is an *ideal of* M if $A + M \subseteq A, M + A \subseteq A$ and $S^*A \subseteq A$ where $S^* = S \setminus \{0\}$ if S has the zero and $S^* = S$ otherwise.

Remark 1.1.15. Let M be an S-skew-semimodule.

(i) M is an ideal of M.

(ii) If A is a proper ideal of M, then $0 \notin A$ so that A is not a skew-subsemimodule of M.

(iii) If A is an ideal of M, then $A^0 := A \cup \{0\}$ is a skew-subsemimodule of M.

The following two propositions are very important basic results which will be referred later.

Proposition 1.1.16. *Let $(A_i)_{i \in I}$ be a family of ideals of an S-skew-semimodule M.*

(i) *If $\bigcap_{i \in I} A_i \neq \emptyset$, then $\bigcap_{i \in I} A_i$ is an ideal of M.*

(ii) *$\bigcup_{i \in I} A_i$ is an ideal of M.*

Proof. Let M be an S-skew-semimodule and $(A_i)_{i \in I}$ be a family of ideals of M.

(i) Assume that $\bigcap_{i \in I} A_i \neq \emptyset$. Let $m \in M, s \in S^*$ and $a \in \bigcap_{i \in I} A_i$. Then $a + m, m + a, sa \in A_i$ since A_i is an ideal of M for all $i \in I$. Thus $a + m, m + a, sa \in \bigcap_{i \in I} A_i$. Hence $\bigcap_{i \in I} A_i$ is an ideal of M.

(ii) Let $m \in M, s \in S^*$ and $a \in \bigcup_{i \in I} A_i$. Then $a \in A_i$ for some $i \in I$, say A_{i_0}, so that $a + m, m + a, sa \in A_{i_0}$ since A_{i_0} is an ideal of M. Thus $a + m, m + a, sa \in \bigcup_{i \in I} A_i$. Hence $\bigcup_{i \in I} A_i$ is an ideal of M. $\square$

Proposition 1.1.17. *Let M be an S-skew-semimodule, N a skew-subsemimodule of M, A and B ideals of M.*

(i) *If $A \subseteq N$, then A is an ideal of N.*

(ii) *If $A \cap N \neq \emptyset$, then $A \cap N$ is an ideal of N.*

(iii) *$A + B$ is an ideal of M.*

(iv) *$N + A^0$ is a skew-subsemimodule of M.*

Proof. Let N be a skew-subsemimodule of an S-skew-semimodule M, A and B ideals of M.

(i) This is obvious.

(ii) Assume that $A \cap N \neq \emptyset$. Let $n \in N, s \in S^*$ and $a \in A \cap N$. Then $a+n, n+a, sa \in N$ since $a, n \in N$ which is a skew-subsemimodule of M. Moreover, $a \in A$ which is an ideal of M so that $a+n, n+a, sa \in A$. Thus $a+n, n+a, sa \in A \cap N$.

Therefore, $A \cap N$ is an ideal of N.

(iii) Note that A and B are ideals of M. Thus

$$(A+B)+M = A+(B+M) \subseteq A+B,$$

$$M+(A+B) = (M+A)+B \subseteq A+B, \text{ and}$$

$$S^*(A+B) = S^*A + S^*B \subseteq A+B.$$

Therefore, $A+B$ is an ideal of M.

(iv) Clearly that $0 \in N + A^0$. Let $m+a, n+b \in N+A^0$ where $m, n \in N$ and $a, b \in A^0$. If $a = 0$, then $(m+a)+(n+b) = (m+n)+b \in N+A^0$. If $a \neq 0$, then since A is an ideal of M, we obtain that

$$\begin{aligned}(m+a)+(n+b) &= a'+n+b && \text{for some } a' \in A\\ &= b'+b && \text{for some } b' \in A\\ &= 0+c && \text{for some } c \in A\\ &\in N+A^0.\end{aligned}$$

Hence $N + A^0$ is a monoid having the same identity as M.

Next, let $m \in M, a \in A^0$ and $s \in S$. Then $s(m + a) = sm + sa \in N + A^0$. Therefore, $N + A^0$ is a skew-subsemimodule of M. $\square$

The rest of this section will be about homomorphisms.

Definition 1.1.18. Let M and N be S-skew-semimodules. Then $\varphi : M \to N$ is an *S-homomorphism* if for each $s \in S$ and $m, n \in M$,

(i) $\varphi(m + n) = \varphi(m) + \varphi(n)$,

(ii) $\varphi(sm) = s\varphi(m)$, and

(iii) $\varphi(0) = 0$.

The set of all elements m in M such that $\varphi(m) = 0$ is call the *zero set of* φ, denoted by $Zs\varphi$, i.e., $Zs\varphi = \{m \in M \mid \varphi(m) = 0\}$.

Proposition 1.1.19. *Let M and N be S-skew-semimodules and $\varphi : M \to N$ an S-homomorphism.*

(i) *If L is a skew-subsemimodule of M, then $\varphi[L]$ is a skew-subsemimodule of N. In particular, $Im\varphi$ is a skew-subsemimodule of N.*

(ii) *If P is a skew-subsemimodule of N, then $\varphi^{-1}[P]$ is a skew-subsemimodule of M. In particular, $Zs\varphi$ is a skew-subsemimodule of M.*

(iii) *If B is an ideal of N, then $\varphi^{-1}[B]$ is an ideal of M.*

(iv) *If A is an ideal of M and φ is surjective, then $\varphi[A]$ is an ideal of N.*

Proof. Let M and N be S-skew-semimodules and $\varphi : M \to N$ an S-homomorphism.

(i) Assume that L is a skew-subsemimodule of M. Then $0 \in L$ and $0 = \varphi(0) \in \varphi[L]$. Let $k, l \in L$ and $s \in S$. Then $k+l, sl \in L$ so that $\varphi(k)+\varphi(l) = \varphi(k+l) \in \varphi[L]$, and $s\varphi(l) = \varphi(sl) \in \varphi[L]$. Note also that $\varphi[L] \subseteq N$. Therefore, $\varphi[L]$ is a skew-subsemimodule of N.

In particular, $Im\varphi = \varphi[M]$ is a skew-subsemimodule of N.

(ii) Assume that P is a skew-subsemimodule of N. Then $0 \in P$, so $\varphi(0) = 0 \in P$ implies that $0 \in \varphi^{-1}[P]$. Let $m, n \in \varphi^{-1}[P]$ and $s \in S$. Then $\varphi(m), \varphi(n) \in P$. Thus $\varphi(m+n) = \varphi(m) + \varphi(n) \in P$ and $\varphi(sm) = s\varphi(m) \in P$. Hence $m+n, sm \in \varphi^{-1}[P]$. Recall that $\varphi^{-1}[P] \subseteq M$. Therefore, $\varphi^{-1}[P]$ is a skew-subsemimodule of M.

In particular, $\varphi^{-1}[\{0\}] = Zs\varphi$ is a skew-subsemimodule of M.

(iii) Assume that B is an ideal of N. Then $\emptyset \neq \varphi^{-1}[B] \subseteq M$. Thus

$$\varphi\big[M + \varphi^{-1}[B]\big] = \varphi[M] + \varphi\big[\varphi^{-1}[B]\big] \subseteq \varphi[M] + B \subseteq N + B \subseteq B,$$

$$\varphi\big[\varphi^{-1}[B] + M\big] = \varphi\big[\varphi^{-1}[B]\big] + \varphi[M] \subseteq B + \varphi[M] \subseteq B + N \subseteq B, \text{ and}$$

$$\varphi\big[S^*\varphi^{-1}[B]\big] = S^*\varphi\big[\varphi^{-1}[B]\big] \subseteq S^*B \subseteq B.$$

Hence $M + \varphi^{-1}[B], \varphi^{-1}[B] + M$ and $S^*\varphi^{-1}[B]$ are subsets of $\varphi^{-1}[B]$.

Therefore, $\varphi^{-1}[B]$ is an ideal of M.

(iv) Assume that A is an ideal of M and φ is surjective. Then $\emptyset \neq \varphi[A] \subseteq N$ and

$$\varphi[A] + N = \varphi[A] + \varphi[M] = \varphi[A + M] \subseteq \varphi[A],$$

$$N + \varphi[A] = \varphi[M] + \varphi[A] = \varphi[M + A] \subseteq \varphi[A], \text{ and}$$

$$S^*\varphi[A] = \varphi[S^*A] \subseteq \varphi[A].$$

Therefore, $\varphi[A]$ is an ideal of N. $\square$

Definition 1.1.20. An S-homomorphism $\varphi : M \to N$ is called an *epimorphism* if φ is surjective; φ is called a *monomorphism* if φ is injective; and φ is called an *isomorphism* if φ is bijective, in this case, we say that M is *isomorphic* to N, denoted by $M \cong N$.

Remark 1.1.21. If $\varphi : M \to N$ is a monomorphism, then $\varphi^{-1} : Im\varphi \to M$ is also a monomorphism. In particular, φ^{-1} is an isomorphism if and only if φ is an isomorphism.

Lemma 1.1.22. *Let N be a skew-subsemimodule of an S-skew-semimodule M. Then the inclusion map $1_{N,M} : N \to M$ is a monomorphism.*

1.2. Quotient Skew-semimodules over Semirings

In this section, we define a congruence relation on a skew-semimodule, a quotient skew-semimodule and the kernel of a homomorphism similarly to those objects of semimodules over semirings defined in [5]. Although almost of results are analogous concepts appeared in [5], results about ideals are excluded. Moreover, we give the definition of the Rees congruence in the same way found in [3] which will play a major part in the proof of the Isomorphism Theorems and most of results in Chapter 4.

Definition 1.2.1. Let M be a skew-semimodule over a semiring S. An equivalence relation ρ on M is called a *congruence relation on M* if for any $m, n \in M$ such that $(m,n) \in \rho$ implies $(m+l, n+l) \in \rho, (l+m, l+n) \in \rho$ and $(sm, sn) \in \rho$ for all $l \in M$ and $s \in S$.

Remark 1.2.2. Let M be a skew-semimodule over a semiring S. Then $M \times M$ and $\{(m,m) \mid m \in M\}$ are the largest and the smallest congruence relations on M, respectively.

Definition 1.2.3. Let $\varphi : M \to N$ be an S-homomorphism. Then the *kernel of φ*, denoted by $Ker\varphi$, is the relation $\{(m,n) \in M \times M \mid \varphi(m) = \varphi(n)\}$.

Lemma 1.2.4. *Let $\varphi : M \to N$ be an S-homomorphism. Then*

(i) *$Ker\varphi$ is a congruence relation on M, and*

(ii) *$Ker\varphi = \{(m,m) \mid m \in M\}$ if and only if φ is injective, i.e., $Ker\varphi$ is the smallest congruence relation on M if and only if φ is injective.*

Proof. Let $\varphi : M \to N$ be an S-homomorphism.

(i) Clearly, $Ker\varphi$ is an equivalence relation on M. Let $m,n \in M$ be such that $\varphi(m) = \varphi(n)$. Let $l \in M$ and $s \in S$. Then

$$\varphi(m+l) = \varphi(m) + \varphi(l) = \varphi(n) + \varphi(l) = \varphi(n+l),$$

$$\varphi(l+m) = \varphi(l) + \varphi(m) = \varphi(l) + \varphi(n) = \varphi(l+n), \text{ and}$$

$$\varphi(sm) = s\varphi(m) = s\varphi(n) = \varphi(sn).$$

Therefore, $Ker\varphi$ is a congruence relation on M.

(ii) First, assume that $Ker\varphi = \{(m,m) \mid m \in M\}$. Let $p,q \in M$ be such that $\varphi(p) = \varphi(q)$. Thus $(p,q) \in Ker\varphi$, hence $p = q$. Therefore, φ is injective.

Conversely, assume that φ is injective. Note that $\{(m,m) \mid m \in M\} \subseteq Ker\varphi$. It remains to show only that $Ker\varphi \subseteq \{(m,m) \mid m \in M\}$. Let $(p,q) \in Ker\varphi$. Then $\varphi(p) = \varphi(q)$ which implies that $p = q$ since φ is injective. Hence $(p,q) \in \{(m,m) \mid m \in M\}$.

Therefore, $\{(m,m) \mid m \in M\} = Ker\varphi$. □

Lemma 1.2.5. *Let A be an ideal of an S-skew-semimodule M. Define a relation R_A on M by $(m,n) \in R_A$ if and only if $m = n$ or $m,n \in A$ for all $m,n \in M$. Then R_A is a congruence relation on M.*

Proof. Clearly, R_A is an equivalence relation on M. Let $(m,n) \in R_A$, then $m = n$ or $m,n \in A$. Let $l \in M$ and $s \in S$. If $m = n$, then $m+l = n+l$, $l+m = l+n$ and $sm = sn$. Now, assume that $m,n \in A$. Then $m+l$, $n+l$, $l+m$, $l+n$, sm and sn belong to A since A is an ideal of M. Hence $(m+l, n+l), (l+m, l+n)$ and (sm, sn) are in R_A. Therefore, R_A is a congruence relation on M. □

The previous lemma suggests the following important definition, the Rees congruence relation.

Definition 1.2.6. Let A be an ideal of an S-skew-semimodule M. Then the relation R_A on M defined in Lemma 1.2.5 is called the *Rees congruence relation generated by A*.

Proposition 1.2.7. *Let ρ be a congruence relation on an S-skew-semimodule M. For each $m \in M$, recall that $[m]_\rho = \{n \in M \,|\, (m,n) \in \rho\}$ is the equivalence class of m, and $M/\rho = \{[m]_\rho \,|\, m \in M\}$. Define an addition and a scalar multiplication on M/ρ by $[m]_\rho + [n]_\rho = [m+n]_\rho$ and $s[m]_\rho = [sm]_\rho$ for all $m,n \in M$ and $s \in S$, respectively. Then M/ρ is an S-skew-semimodule.*

Proof. Let $m, m', n, n' \in M$ and $s \in S$. Assume that $[m]_\rho = [m']_\rho$ and $[n]_\rho = [n']_\rho$. Then $(m,m'), (n,n') \in \rho$ so that $(m+n, m'+n), (m'+n, m'+n'), (sm, sm') \in \rho$ since ρ is a congruence relation. Thus $(m+n, m'+n') \in \rho$ because of the transitivity of ρ. Hence $[m+n]_\rho = [m'+n']_\rho$ and $[sm]_\rho = [sm']_\rho$. Hence the addition and scalar multiplication are well-defined.

We claim that $[0]_\rho$ is the identity of M/ρ. Let $m \in M$. Then $[0]_\rho + [m]_\rho = [0+m]_\rho = [m]_\rho = [m+0]_\rho = [m]_\rho + [0]_\rho$. Moreover, it is easy to verify that M/ρ satisfies the associative rule. Hence M/ρ is a monoid.

Next, we will show that the scalar multiplication satisfies the S-skew-semimodule conditions. Let $r, s \in S$ and $[m]_\rho,\ [n]_\rho \in M/\rho$. Then

(i) $(r+s)[m]_\rho = \big[(r+s)m\big]_\rho = [rm+sm]_\rho = [rm]_\rho + [sm]_\rho = r[m]_\rho + s[m]_\rho$,

(ii) $s\big([m]_\rho + [n]_\rho\big) = s\big([m+n]_\rho\big) = [s(m+n)]_\rho = [sm+sn]_\rho = s[m]_\rho + s[n]_\rho$,

(iii) $(rs)[m]_\rho = \big[(rs)m\big]_\rho = \big[r(sm)\big]_\rho = r[sm]_\rho = r\big(s[m]_\rho\big)$, and

(iv) $s[0]_\rho = [s0]_\rho = [0]_\rho$.

Therefore, M/ρ is an S-skew-semimodule. $\square$

Definition 1.2.8. An S-skew-semimodule N is called a *quotient skew-semimodule* of an S-skew-semimodule M if there exists a congruence relation ρ on M such that $N = M/\rho$ with the addition and scalar multiplication defined in Proposition 1.2.7.

Note that if A is an ideal of an S-skew-semimodule M, then M/R_A is a quotient skew-semimodule of M.

Proposition 1.2.9. *Let M be an S-skew-semimodule and ρ a congruence relation on M. Then the mapping $\pi : M \to M/\rho$, defined by $\pi(m) = [m]_\rho$ for all $m \in M$ is an epimorphism with $Ker\pi = \rho$.*

Proof. Clearly, π is surjective. To show that π is an S-homomorphism, let $m, n \in M$ and $s \in S$. Then

$$\pi(m+n) = [m+n]_\rho = [m]_\rho + [n]_\rho = \pi(m) + \pi(n),$$

$$\pi(sm) = [sm]_\rho = s[m]_\rho = s\pi(m), \text{ and}$$

$$\pi(0) = [0]_\rho.$$

Hence π is an epimorphism.

Finally, let $m, n \in M$. Then

$$(m,n) \in Ker\pi \Leftrightarrow \pi(m) = \pi(n) \Leftrightarrow [m]_\rho = [n]_\rho \Leftrightarrow (m,n) \in \rho.$$

This shows that $Ker\pi = \rho$. □

Definition 1.2.10. Let ρ be a congruence relation on an S-skew-semimodule M. The epimorphism π defined in Proposition 1.2.9 is called the *natural* or *canonical surjection* of M onto M/ρ.

Proposition 1.2.11. *Let M be an S-skew-semimodule and A a non-singleton ideal of M. Then, for each $m \in M$,*

(i) $[m]_{R_A} = A$ *if and only if* $m \in A$, *and*

(ii) $[m]_{R_A} = \{m\}$ *if and only if* $m \notin A$.

Proof. Let $m \in M$.

(i) Obviously, if $[m]_{R_A} = A$, then $m \in A$. Thus, assume that $m \in A$. Let $n \in [m]_{R_A}$. Then $(m,n) \in R_A$ that is $m = n$ or $m, n \in A$. Thus $n \in A$ so that

$[m]_{R_A} \subseteq A$. Next, let $n \in A$. Then $(m,n) \in R_A$ so that $n \in [m]_{R_A}$. Therefore, $[m]_{R_A} = A$.

(ii) Assume that $[m]_{R_A} = \{m\}$. Suppose that $m \in A$. By (i), we obtain that $A = [m]_{R_A} = \{m\}$ which is singleton. This leads to a contradiction. Therefore, $m \notin A$.

Conversely, assume that $m \notin A$. Clearly, $\{m\} \subseteq [m]_{R_A}$. Let $n \in [m]_{R_A}$. Then $(m,n) \in R_A$ so that $m = n$ since $m \notin A$. Hence $[m]_{R_A} = \{m\}$. $\square$

Remark 1.2.12. If a singleton set A is an ideal of an S-skew-semimodule M, then $[m]_{R_A} = \{m\}$ for all $m \in M$.

We can classify the quotient skew-semimodule M/R_A of an S-skew-semimodule M where A is an ideal of M.

Corollary 1.2.13. *Let A be an ideal of an S-skew-semimodule M. Then*

$$M/R_A = \{A\} \cup \left(\bigcup_{m \in M \setminus A} \{\{m\}\} \right).$$

Now, we are ready to study Isomorphism Theorems.

Theorem 1.2.14. ***First Isomorphism Theorem***

Let $\varphi : M \to N$ be an S-homomorphism. Then $M/Ker\varphi \cong Im\varphi$.

Proof. Recall from Lemma 1.2.4 that $Ker\varphi$ is a congrunce relation on M so that $M/Ker\varphi$ exists. Define $\psi : M/Ker\varphi \to Im\varphi$ by

$$\psi\big([m]_{Ker\varphi}\big) = \varphi(m) \quad \text{for all } m \in M.$$

Note that, for each $m, n \in M$, we obtain that

$$[m]_{Ker\varphi} = [n]_{Ker\varphi} \Leftrightarrow (m,n) \in Ker\varphi \Leftrightarrow \varphi(m) = \varphi(n).$$

Hence ψ is well-defined and injective. Clearly, ψ is onto. It remains to show only that ψ is an S-homomorphism. Let $m, n \in M$ and $s \in S$. Then

$$\begin{aligned}\psi\big([m]_{Ker\varphi} + [n]_{Ker\varphi}\big) &= \psi\big([m+n]_{Ker\varphi}\big) = \varphi(m+n) \\ &= \varphi(m) + \varphi(n) = \psi\big([m]_{Ker\varphi}\big) + \psi\big([n]_{Ker\varphi}\big),\end{aligned}$$

$$\psi\big(s[m]_{Ker\varphi}\big) = \psi\big([sm]_{Ker\varphi}\big) = \varphi(sm) = s\varphi(m) = s\psi\big([m]_{Ker\varphi}\big), \text{ and}$$

$$\psi\big([0]_{Ker\varphi}\big) = \varphi(0) = 0.$$

Therefore, ψ is an isomorphism, i.e., $M/Ker\varphi \cong Im\varphi$. □

The analogous First Isomorphism Theorem is found in [1], [2], [4] and [5]. Moreover, the following statements in this section are parallel to the statements in [1], [2] and [5] by using concepts of ideals and the Rees congruences.

Corollary 1.2.15. *Let M and N be S-skew-semimodules and A an ideal of N. If $\varphi : M \to N$ is an epimorphism and $Ker\varphi \subseteq R_{\varphi^{-1}[A]}$, then $M/R_{\varphi^{-1}[A]} \cong N/R_A$.*

Proof. Assume that $\varphi : M \to N$ is an epimorphism and $Ker\varphi \subseteq R_{\varphi^{-1}[A]}$. Define $\psi : M \to N/R_A$ by

$$\psi(m) = \big[\varphi(m)\big]_{R_A} \quad \text{for all } m \in M.$$

Then ψ is well-defined.

Next, we show that ψ is an epimorphism. Let $m, n \in M$ and $s \in S$. Then

$$\begin{aligned}\psi(m+n) &= \big[\varphi(m+n)\big]_{R_A} = \big[\varphi(m) + \varphi(n)\big]_{R_A} \\ &= \big[\varphi(m)\big]_{R_A} + \big[\varphi(n)\big]_{R_A} = \psi(m) + \psi(n),\end{aligned}$$

$$\psi(sm) = \big[\varphi(sm)\big]_{R_A} = s\big[\varphi(m)\big]_{R_A} = s\psi(m), \text{ and}$$

$$\psi(0) = \big[\varphi(0)\big]_{R_A} = [0]_{R_A}.$$

Hence ψ is an S-homomorphism.

Let $[x]_{R_A} \in N/R_A$ where $x \in N$. Then there exists $m \in M$ such that $\varphi(m) = x$ since φ is surjective. Thus $\psi(m) = \big[\varphi(m)\big]_{R_A} = [x]_{R_A}$. This shows that ψ is surjective. Hence ψ is an epimorphism.

To show that $Ker\psi = R_{\varphi^{-1}[A]}$, first, let $(m,n) \in Ker\psi$. Then

$$\big[\varphi(m)\big]_{R_A} = \psi(m) = \psi(n) = \big[\varphi(n)\big]_{R_A}.$$

Thus $\big(\varphi(m), \varphi(n)\big) \in R_A$, i.e., $\varphi(m) = \varphi(n)$ or $\varphi(m), \varphi(n) \in A$. If $\varphi(m), \varphi(n) \in A$, then $m, n \in \varphi^{-1}[A]$ so that $(m,n) \in R_{\varphi^{-1}[A]}$. If $\varphi(m) = \varphi(n)$, then $(m,n) \in Ker\varphi \subseteq R_{\varphi^{-1}[A]}$. Hence $Ker\psi \subseteq R_{\varphi^{-1}[A]}$.

Next, let $(m,n) \in R_{\varphi^{-1}[A]}$. Then $m = n$ or $m, n \in \varphi^{-1}[A]$. If $m = n$, then $\varphi(m) = \varphi(n)$ so that $(m,n) \in Ker\varphi$. If $m, n \in \varphi^{-1}[A]$, then $\varphi(m), \varphi(n) \in A$, so $\big[\varphi(m)\big]_{R_A} = \big[\varphi(n)\big]_{R_A}$ that is $\psi(m) = \psi(n)$. Thus $(m,n) \in Ker\psi$. Hence $R_{\varphi^{-1}[A]} \subseteq Ker\psi$. As a result, $Ker\psi = R_{\varphi^{-1}[A]}$.

Therefore, by the First Isomorphism Theorem, $M/R_{\varphi^{-1}[A]} \cong N/R_A$. □

Theorem 1.2.16. ***Second Isomorphism Theorem***

Let N be a skew-subsemimodule of an S-skew-semimodule M and A an ideal of M. If $N \cap A \neq \emptyset$, then $N/R_{N\cap A} \cong (N + A^0)/R_A$.

Proof. Let $a \in A$. Then $a = 0 + a$ so that $A \subseteq N + A^0$ which is a skew-subsemimodule of M by Proposition 1.1.17 (iv). Moreover, A is an ideal of $N + A^0$ from Proposition 1.1.17 (i). Define $\varphi : N \to (N + A^0)/R_A$ by

$$\varphi(n) = [n]_{R_A} \quad \text{for all } n \in N.$$

Then φ is well-defined. To show that φ is an S-homomorphism, let $m, n \in N$ and $s \in S$. Then

$$\varphi(m+n) = [m+n]_{R_A} = [m]_{R_A} + [n]_{R_A} = \varphi(m) + \varphi(n),$$

$$\varphi(sm) = [sm]_{R_A} = s[m]_{R_A} = s\varphi(m), \text{ and}$$

$$\varphi(0) = [0]_{R_A}.$$

Hence φ is an S-homomorphism.

Let $[m]_{R_A} \in (N+A^0)/R_A$ where $m \in N+A^0$. Then $m=n+a$ for some $n \in N$ and $a \in A^0$. If $a=0$, then $m=n$ that is $\varphi(n)=[n]_{R_A}=[m]_{R_A}$. Next, assume that $a \in A$ which is an ideal of M. Then $m=n+a \in A$ so that $[m]_{R_A}=A$ by Corollary 1.2.13. Since $N \cap A \neq \emptyset$, there exists $p \in N \cap A$. Then $p \in N$ and $p \in A$ so that $\varphi(p)=[p]_{R_A}=A=[m]_{R_A}$. This shows that φ is surjective.

Finally, we show that $Ker\varphi = R_{N\cap A}$. Note from Proposition 1.1.17 (ii) that $N \cap A$ is an ideal of N so that $R_{N\cap A} \subseteq N \times N$. Let $(m,n) \in Ker\varphi$. Then $m,n \in N$ and $\varphi(m)=\varphi(n)$ so that $[m]_{R_A}=[n]_{R_A}$ which implies that $(m,n) \in R_A$. Thus $m=n$ or $m,n \in A$. If $m=n$, then $(m,n) \in R_{N\cap A}$. If $m,n \in A$, then $m,n \in N \cap A$ so $(m,n) \in R_{N\cap A}$. Hence $Ker\varphi \subseteq R_{N\cap A}$. On the other hand, let $(m,n) \in R_{N\cap A}$. Then $m=n$ or $m,n \in N \cap A$. If $m=n$, then $\varphi(m)=\varphi(n)$ so that $(m,n) \in Ker\varphi$. If $m,n \in A$, then $\varphi(m)=[m]_{R_A}=A=[n]_{R_A}=\varphi(n)$ that is $(m,n) \in Ker\varphi$. Hence $R_{N\cap A} \subseteq Ker\varphi$. This shows that $Ker\varphi = R_{N\cap A}$.

Therefore, $N/R_{N\cap A} \cong (N+A^0)/R_A$ by the First Isomorphism Theorem. $\square$

Lemma 1.2.17. *Let A and B be ideals of an S-skew-semimodule M such that $A \subseteq B$. Define a relation ρ_A on B by $\rho_A = \{(a,b) \in B \times B \mid a=b \text{ or } a,b \in A\}$. Then*

(i) *ρ_A is a congruence relation on B so that B/ρ_A exists. Moreover, $\rho_A \subseteq R_A$ and $[b]_{\rho_A}=[b]_{R_A}$ for all $b \in B$,*

(ii) *B/ρ_A is an ideal of M/R_A, and*

(iii) *if $[m]_{R_A} \in B/\rho_A$, then $m \in B$.*

Proof. (i) Standard.

(ii) It is clear from (i) that $B/\rho_A \subseteq M/R_A$. Let $[b]_{\rho_A} \in B/\rho_A$, $[m]_{R_A} \in M/R_A$ and $s \in S$ where $b \in B$ and $m \in M$. Thus $b+m, m+b, sb \in B$ since B is an ideal of M. Moreover, $[b]_{\rho_A}=[b]_{R_A} \in M/R_A$. Then

$$[b]_{\rho_A} + [m]_{R_A} = [b]_{R_A} + [m]_{R_A} = [b+m]_{R_A} = [b+m]_{\rho_A} \in B/\rho_A,$$

$$[m]_{R_A} + [b]_{\rho_A} = [m]_{R_A} + [b]_{R_A} = [m+b]_{R_A} = [m+b]_{\rho_A} \in B/\rho_A, \text{ and}$$

$$s[b]_{\rho_A} = s[b]_{R_A} = [sb]_{R_A} = [sb]_{\rho_A} \in B/\rho_A.$$

Therefore, B/ρ_A is an ideal of M/R_A.

(iii) Assume that $[m]_{R_A} \in B/\rho_A$. Then there exists $b \in B$ such that $[m]_{R_A} = [b]_{\rho_A}$. Thus $[m]_{R_A} = [b]_{R_A}$ from (i) so that $(m,b) \in R_A$. As a result, $m = b \in B$ or $m, b \in A \subseteq B$. □

Theorem 1.2.18. ***Third Isomorphism Theorem***

Let A and B be ideals of an S-skew-semimodule M such that $A \subseteq B$. Then $M/R_B \cong \dfrac{M/R_A}{R_{(B/\rho_A)}}$ where ρ_A is defined in Lemma 1.2.17.

Proof. Define $\varphi : M/R_A \to M/R_B$ by

$$\varphi\big([m]_{R_A}\big) = [m]_{R_B} \quad \text{for all } m \in M.$$

Let $m, n \in M$ be such that $[m]_{R_A} = [n]_{R_A}$. Then $(m,n) \in R_A$. Since $A \subseteq B$, it follows that $R_A \subseteq R_B$. Thus $(m,n) \in R_B$ so that $[m]_{R_B} = [n]_{R_B}$. Hence φ is well-defined. Moreover, it is obvious that φ is surjective.

To show that φ is an S-homomorphism, let $m, n \in M$ and $s \in S$. Then

$$\begin{aligned}\varphi\big([m]_{R_A} + [n]_{R_A}\big) &= \varphi\big([m+n]_{R_A}\big) = [m+n]_{R_B} \\ &= [m]_{R_B} + [n]_{R_B} = \varphi\big([m]_{R_A}\big) + \varphi\big([n]_{R_A}\big), \text{ and}\end{aligned}$$

$$\varphi\big(s[m]_{R_A}\big) = \varphi\big([sm]_{R_A}\big) = [sm]_{R_B} = s[m]_{R_B} = s\varphi\big([m]_{R_A}\big).$$

Clearly, $\varphi\big([0]_{R_A}\big) = [0]_{R_B}$. Hence φ is an S-homomorphism.

By Lemma 1.2.17, it follows that B/ρ_A is an ideal of M/R_A. Recall that

$$R_{B/\rho_A} = \Big\{ \big([m]_{R_A}, [n]_{R_A}\big) \in M/R_A \times M/R_A \;\Big|\; [m]_{R_A} = [n]_{R_A} \text{ or } [m]_{R_A}, [n]_{R_A} \in B/\rho_A \Big\}.$$

Next, we will show that $Ker\varphi = R_{(B/\rho_A)}$. Let $\varphi\big([m]_{R_A}\big) = \varphi\big([n]_{R_A}\big)$ where $m, n \in M$. Then $[m]_{R_B} = [n]_{R_B}$ which implies that $(m, n) \in R_B$. Thus $m = n$ or $m, n \in B$. If $m = n$, then $[m]_{R_A} = [n]_{R_A}$ so that $\big([m]_{R_A}, [n]_{R_A}\big) \in R_{(B/\rho_A)}$. Assume that $m, n \in B$, then $[m]_{R_A} = [m]_{\rho_A}, [n]_{R_A} = [n]_{\rho_A} \in B/\rho_A$ so that $\big([m]_{R_A}, [n]_{R_A}\big) \in R_{(B/\rho_A)}$. Hence $Ker\varphi \subseteq R_{(B/\rho_A)}$.

Suppose that $\big([m]_{R_A}, [n]_{R_A}\big) \in R_{(B/\rho_A)}$. Then $[m]_{R_A} = [n]_{R_A}$ or $[m]_{R_A},\ [n]_{R_A} \in B/\rho_A$. If $[m]_{R_A} = [n]_{R_A}$, then we are done. Assume that $[m]_{R_A},\ [n]_{R_A} \in B/\rho_A$. From Lemma 1.2.17 (iii), we obtain that $m, n \in B$ which is an ideal of M, so $(m, n) \in R_B$. Thus $[m]_{R_B} = [n]_{R_B}$ that is $\varphi\big([m]_{R_A}\big) = \varphi\big([n]_{R_A}\big)$. Hence $R_{(B/\rho_A)} \subseteq Ker\varphi$. Thus $Ker\varphi = R_{(B/\rho_A)}$.

Therefore, by the First Isomorphism Theorem, $M/R_B \cong \dfrac{M/R_A}{R_{(B/\rho_A)}}$. $\square$

The next theorem leads to the Universal Mapping Property of Quotients.

Theorem 1.2.19. *Let $\varphi : M \to N$ and $\psi : M \to P$ be S-homomorphisms. If ψ is surjective and $Ker\psi \subseteq Ker\varphi$, then*

(i) *there exists a unique S-homomorphism $\mu : P \to N$ such that $\varphi = \mu \circ \psi$, i.e., the following diagram commutes:*

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[d, "\psi"'] & N \\ P \arrow[ur, dotted, "\mu"'] & \end{tikzcd}$$

(ii) *$Zs\mu = \psi[Zs\varphi]$ and $Im\varphi = Im\mu$, and*

(iii) *μ is injective if and only if $Ker\varphi = Ker\psi$.*

Proof. Assume that ψ is surjective and $Ker\psi \subseteq Ker\varphi$.

(i) Since ψ is surjective, for each $p \in P$ there exists $m_p \in M$ such that $\psi(m_p) = p$. Define $\mu : P \to N$ by

$$\mu(p) = \varphi(m_p) \quad \text{for all } p \in P.$$

Let $p \in P$ and $m_1, m_2 \in M$ be such that $\psi(m_1) = \psi(m_2) = p$. Then $(m_1, m_2) \in Ker\psi \subseteq Ker\varphi$. Thus $\varphi(m_1) = \varphi(m_2)$. Hence μ is well-defined. Moreover, $\mu \circ \psi(m) = \mu\big(\psi(m)\big) = \varphi(m)$ for all $m \in M$. Thus $\mu \circ \psi = \varphi$.

Next, we will show that μ is an S-homomorphism. Let $p_1, p_2 \in P$ and $s \in S$. Then $\psi(m_1) = p_1$ and $\psi(m_2) = p_2$ for some $m_1, m_2 \in M$. Then

$$\begin{aligned}\mu(p_1 + p_2) &= \mu\big(\psi(m_1) + \psi(m_2)\big) = \mu\big(\psi(m_1 + m_2)\big) = \varphi(m_1 + m_2)\\ &= \varphi(m_1) + \varphi(m_2) = \mu\big(\psi(m_1)\big) + \mu\big(\psi(m_2)\big) = \mu(p_1) + \mu(p_2),\\ \mu(sp_1) &= \mu\big(s\psi(m_1)\big) = \mu\big(\psi(sm_1)\big) = \varphi(sm_1) = s\varphi(m_1) = s\mu\big(\psi(m_1)\big), \text{ and}\\ \mu(0) &= \varphi(0) = 0.\end{aligned}$$

Hence μ is an S-homomorphism. Since ψ is surjective, μ is unique.

Therefore, there exists a unique S-homomorphism $\mu : P \to N$ such that $\varphi = \mu\circ\psi$.

(ii) First, let $p \in Zs\mu$. Then $\mu(p) = 0$ and there exists $m_p \in M$ such that $\psi(m_p) = p$. Thus $\varphi(m_p) = \mu\big(\psi(m_p)\big) = \mu(p) = 0$. This shows that $m_p \in Zs\varphi$. Hence $p \in \psi[Zs\varphi]$. Next, let $q \in \psi[Zs\varphi]$. Then there exists $m_q \in M$ such that $\psi(m_q) = q$ and $\varphi(m_q) = 0$. Thus $\mu(q) = \mu\big(\psi(m_q)\big) = \varphi(m_q) = 0$. Hence $q \in Zs\mu$. Therefore, $Zs\mu = \psi[Zs\varphi]$.

In addition, $\mu[P] = \mu\big[\psi[M]\big] = \varphi[M]$ since ψ is surjective. Hence $Im\mu = Im\varphi$.

(iii) First, assume that μ is an injection. It is enough to show that $Ker\varphi \subseteq Ker\psi$. Let $(m_1, m_2) \in Ker\varphi$. Then $\varphi(m_1) = \varphi(m_2)$. Thus $\mu\big(\psi(m_1)\big) = \mu\big(\psi(m_2)\big)$, so $\big(\psi(m_1), \psi(m_2)\big) \in Ker\mu$. By Lemma 1.2.4 (ii) and μ is injective, $\psi(m_1) = \psi(m_2)$ so $(m_1, m_2) \in Ker\psi$. Therefore, $Ker\varphi = Ker\psi$.

Conversely, assume that $Ker\varphi = Ker\psi$. We will show that μ is injective. Let $m_1, m_2 \in M$ be such that $\mu\big(\psi(m_1)\big) = \mu\big(\psi(m_2)\big)$. Then $\varphi(m_1) = \varphi(m_2)$, so $(m_1, m_2) \in Ker\varphi = Ker\psi$. Thus $\psi(m_1) = \psi(m_2)$. Hence μ is injective. $\square$

Corollary 1.2.20. ***The Universal Mapping Property of Quotients***

Let $\varphi : M \to N$ be an S-homomorphism and ρ a congruence relation on M such that $\rho \subseteq Ker\varphi$. Then there exists a unique S-homomorphism $\psi : M/\rho \to N$ such that $\varphi = \psi \circ \pi$, where π is the canonical surjection of M onto M/ρ. Moreover, $Im\varphi = Im\psi$. Equivalently, the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[d, "\pi"'] & N \\ M/\rho \arrow[ur, dotted, "\psi"'] & \end{tikzcd}$$

Proof. Recall from Proposition 1.2.9 that π is an epimorphism and $Ker\pi = \rho$. Hence the result follows immediately from Theorem 1.2.19. $\square$

Theorem 1.2.21. *Let $\varphi : M \to N$ and $\psi : P \to N$ be S-homomorphisms. If ψ is injective and $Im\varphi \subseteq Im\psi$, then*

(i) *there exists a unique S-homomorphism $\mu : M \to P$ such that $\varphi = \psi \circ \mu$, i.e., the following diagram commutes:*

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[d, dotted, "\mu"'] & N \\ P \arrow[ur, "\psi"'] & \end{tikzcd}$$

(ii) *$Zs\mu = Zs\varphi$ and $\psi^{-1}[Im\varphi] = Im\mu$, and*

(iii) *μ is injective if and only if φ is injective.*

Proof. Assume that ψ is injective and $Im\varphi \subseteq Im\psi$. Then $\psi^{-1} : Im\psi \to P$ is a monomorphism.

(i) First, we claim that for each $m \in M$ there exists a unique $p_m \in P$ such that $\psi(p_m) = \varphi(m)$. Let $m \in M$. Then $\varphi(m) \in Im\varphi \subseteq Im\psi$. Thus there is $p_m \in P$ such that $\psi(p_m) = \varphi(m)$. Suppose that $\psi(q) = \varphi(m)$ for some $q \in P$. Then

$\psi(p_m) = \psi(q)$ so that $p_m = q$ since ψ is injective. Hence the claim is proved. Define $\mu : M \to P$ by

$$\mu(m) = \psi^{-1}\big(\varphi(m)\big) \quad \text{for all } m \in M.$$

Then μ is well-defined from the above claim.

To show that μ is an S-homomorphism, let $m_1, m_2 \in M$ and $s \in S$. Then

$$\begin{aligned}\mu(m_1 + m_2) &= \psi^{-1}\big(\varphi(m_1 + m_2)\big) = \psi^{-1}\big(\varphi(m_1) + \varphi(m_2)\big)\\ &= \psi^{-1}\big(\varphi(m_1)\big) + \psi^{-1}\big(\varphi(m_2)\big) = \mu(m_1) + \mu(m_2),\end{aligned}$$

$$\mu(sm_1) = \psi^{-1}\big(\varphi(sm_1)\big) = \psi^{-1}\big(s\varphi(m_1)\big) = s\psi^{-1}\big(\varphi(m_1)\big) = s\mu(m_1), \text{ and}$$

$$\mu(0) = 0.$$

Hence μ is an S-homomorphism. Since ψ is an injection, $\psi \circ \mu(m) = \psi \circ \psi^{-1}\big(\varphi(m)\big) = \varphi(m)$ for all $m \in M$. This shows that $\varphi = \psi \circ \mu$. Next, suppose that $\gamma : M \to P$ is an S-homomorphism such that $\psi \circ \gamma = \varphi$. Then $\psi \circ \gamma = \varphi = \psi \circ \mu$. Since ψ is injective, $\gamma = \mu$.

Therefore, there exists a unique S-homomorphism $\mu : M \to P$ such that $\varphi = \psi \circ \mu$.

(ii) If $m \in Zs\mu$, then $\mu(m) = 0$, so $\varphi(m) = \psi\big(\mu(m)\big) = \psi(0) = 0$ which implies that $m \in Zs\varphi$. This shows that $Zs\mu \subseteq Zs\varphi$. Conversely, let $m \in Zs\varphi$. Then $\psi\big(\mu(m)\big) = \varphi(m) = 0$. Since ψ is injective, $\mu(m) = 0$ so that $m \in Zs\mu$. Hence $Zs\varphi \subseteq Zs\mu$. As a result, $Zs\mu = Zs\varphi$.

Since $\psi\big[\mu[M]\big] = \varphi[M]$ and ψ is injective, $\mu[M] = \psi^{-1}\big[\varphi[M]\big]$. Hence $Im\mu = \psi^{-1}[Im\varphi]$.

(iii) First, assume that μ is an injection. Note that ψ is injective and $\varphi = \psi \circ \mu$ from (i). Hence φ is injective.

Next, assume that φ is an injection. Let $m_1, m_2 \in M$ be such that $\mu(m_1) = \mu(m_2)$. Then $\psi^{-1}\big(\varphi(m_1)\big) = \psi^{-1}\big(\varphi(m_2)\big)$. Since ψ^{-1} and φ are injections, $m_1 = m_2$. Hence μ is injective. $\square$

CHAPTER II

DIRECT PRODUCTS, DIRECT SUMS AND FREE SKEW-SEMIMODULES

This chapter, we discuss direct products and direct sums of skew-semimodules over semirings. Moreover, if a semiring has 0 and 1, then we can define a free skew-semimodule over such a semiring similarly to a free module over a ring. In addition, most of parallel basic properties still hold.

2.1. Direct Products and Direct Sums

In this section, we define a direct product, a direct sum of skew-semimodules over semirings and the canonical mappings. Besides, the universal mapping properties of direct products and direct sums are satisfied for the case of skew-semimodules. Furthermore, the results in this section are important for the next section and Chapter 3.

Proposition 2.1.1. *Let $(M_i)_{i\in I}$ be a family of S-skew-semimodules. Then the set $M = \prod_{i\in I} M_i$ is an S-skew-semimodule under the additive and scalar multiplication defined by*

$$(m_i)_{i\in I} + (n_i)_{i\in I} = (m_i + n_i)_{i\in I},$$

$$s(m_i)_{i\in I} = (sm_i)_{i\in I},$$

for all $(m_i)_{i\in I}, (n_i)_{i\in I} \in M$ and $s \in S$.

Proof. It is easy to see that M is a monoid with $(0)_{i\in I} := (0_i)_{i\in I}$ as the identity where 0_i is the identity of M_i for all $i \in I$.

Let $(m_i)_{i\in I}, (n_i)_{i\in I} \in M$ and $r, s \in S$. Then

$$\begin{aligned}
(r+s)(m_i)_{i\in I} &= \big((r+s)m_i\big)_{i\in I} = (rm_i + sm_i)_{i\in I} \\
&= (rm_i)_{i\in I} + (sm_i)_{i\in I} = r(m_i)_{i\in I} + s(m_i)_{i\in I}, \\
s\big((m_i)_{i\in I} + (n_i)_{i\in I}\big) &= s(m_i + n_i)_{i\in I} = \big(s(m_i+n_i)\big)_{i\in I} = (sm_i + sn_i)_{i\in I} \\
&= (sm_i)_{i\in I} + (sn_i)_{i\in I} = s(m_i)_{i\in I} + s(n_i)_{i\in I}, \\
(rs)(m_i)_{i\in I} &= \big((rs)(m_i)\big)_{i\in I} = \big(r(sm_i)\big)_{i\in I} = r(sm_i)_{i\in I}, \text{ and} \\
s(0)_{i\in I} &= (s0)_{i\in I} = (0)_{i\in I}.
\end{aligned}$$

Therefore, M is an S-skew-semimodule. □

Definition 2.1.2. Let $(M_i)_{i\in I}$ be a family of S-skew-semimodules. The S-skew-semimodule $M = \prod_{i\in I} M_i$ defined in Proposition 2.1.1 is called the *direct product* of $(M_i)_{i\in I}$.

Definition 2.1.3. Let $M = \prod_{i\in I} M_i$ be the direct product of a family $(M_i)_{i\in I}$ of S-skew-semimodules. For each $i_0 \in I$, the mapping $\pi_{i_0} : M \to M_{i_0}$ defined by $\pi_{i_0}\big((m_i)_{i\in I}\big) = m_{i_0}$ for all $(m_i)_{i\in I} \in M$ is called the *natural* or *canonical projection* of M onto M_{i_0}.

For each $j_0 \in I$, the mapping $\lambda_{j_0} : M_{j_0} \to M$ defined by $\lambda_{j_0}(m) = (m_i)_{i\in I}$ for all $m \in M_{j_0}$, where $m_i = 0$ for all $i \neq j_0$ and $m_{j_0} = m$ is called the *natural* or *canonical injection* of M_{j_0} into M.

Remark 2.1.4. Let $M = \prod_{i\in I} M_i$ be the direct product of a family $(M_i)_{i\in I}$ of S-skew-semimodules. Then the canonical projection is an epimorphism and the canonical injection is a monomorphism. Moreover, for all $i, j \in I$, with $i \neq j$, $\pi_i \circ \lambda_i = 1_{M_i}$ and $\pi_i \circ \lambda_j = 0$.

Proposition 2.1.5. ***The Universal Mapping Property of Direct Products***

Let $M = \prod\limits_{i \in I} M_i$ *be the direct product of a family* $(M_i)_{i \in I}$ *of* S*-skew-semimodules and* $\pi_i : M \to M_i$ *be the canonical projection for all* $i \in I$*. If* N *is an* S*-skew-semimodule and* $\psi_i : N \to M_i$ *is an* S*-homomorphism for all* $i \in I$*, then there exists a unique* S*-homomorphism* $\varphi : N \to M$ *such that* $\psi_i = \pi_i \circ \varphi$ *for all* $i \in I$*, i.e., the following diagram commutes for all* $i \in I$*.*

$$\begin{tikzcd} N \arrow[r, dotted, "\varphi"] \arrow[dr, "\psi_i"'] & M = \prod\limits_{i \in I} M_i \arrow[d, "\pi_i"] \\ & M_i \end{tikzcd}$$

Proof. Define $\varphi : N \to M$ by

$$\varphi(n) = \big(\psi_i(n)\big)_{i \in I} \quad \text{for all } n \in N.$$

It is clear that φ is well-defined since ψ_i is well-defined for all $i \in I$. Let $m, n \in N$ and $s \in S$. Then

$$\begin{aligned} \varphi(m+n) &= \big(\psi_i(m+n)\big)_{i \in I} \\ &= \big(\psi_i(m)\big)_{i \in I} + \big(\psi_i(n)\big)_{i \in I} \\ &= \varphi(m) + \varphi(n), \\ \varphi(sm) &= \big(\psi_i(sm)\big)_{i \in I} = \big(s\psi_i(m)\big)_{i \in I} \\ &= s\big(\psi_i(m)\big)_{i \in I} = s\varphi(m), \text{ and} \\ \varphi(0) &= \big(\psi_i(0)\big)_{i \in I} = (0)_{i \in I}. \end{aligned}$$

Hence φ is an S-homomorphism.

Let $n \in N$. For each $i \in I$, we can see that

$$\pi_i \circ \varphi(n) = \pi_i\big(\varphi(n)\big) = \pi_i\Big(\big(\psi_j(n)\big)_{j\in I}\Big) = \psi_i(n).$$

Thus $\psi_i = \pi \circ \varphi_i$ for all $i \in I$.

To verify the uniqueness of φ, suppose that $\mu : N \to M$ such that $\pi_i \circ \mu = \psi_i$ for all $i \in I$. Let $n \in N$. Then $\mu(n) = (n_j)_{j\in I}$ for some $(n_j)_{j\in I} \in M$. Thus $\psi_i(n) = \pi_i \circ \mu(n) = \pi_i\big((n_j)_{j\in I}\big) = n_i$ for all $i \in I$. Hence $\mu(n) = (n_i)_{i\in I} = \big(\psi_i(n)\big)_{i\in I} = \varphi(n)$ so that $\varphi = \mu$. □

Definition 2.1.6. If B is a set, then we say that a particular property holds for *almost all* elements in B if there is a finite subset F of B such that the property holds for every element in $B \setminus F$.

Definition 2.1.7. Let $(M_i)_{i\in I}$ be a family of S-skew-semimodules. Then the subset

$$\sum_{i\in I} M_i = \left\{ (m_i)_{i\in I} \in \prod_{i\in I} M_i \,\middle|\, m_i = 0 \text{ for almost all indices } i \in I \right\}$$

of $\prod_{i\in I} M_i$ is called the *direct sum* of $(M_i)_{i\in I}$.

Proposition 2.1.8. *Let $(M_i)_{i\in I}$ be a family of S-skew-semimodules. Then the direct sum $\sum_{i\in I} M_i$ is a skew-subsemimodule of the direct product $\prod_{i\in I} M_i$.*

Proof. Note that for each $(m_i)_{i\in I} \in \sum_{i\in I} M_i$, there exist only finite $i \in I$ such that $m_i \neq 0$. The result follows. □

Proposition 2.1.9. ***The Universal Mapping Property of Direct Sums***

Let $M = \sum_{i\in I} M_i$ be the direct sum of a family $(M_i)_{i\in I}$ of S-skew-semimodules and $\lambda_i : M_i \to M$ be the canonical injection for all $i \in I$. If N is an S-skew-semimodule and $\varphi_i : M_i \to N$ is an S-homomorphism for all $i \in I$, then there exists a unique

S-homomorphism $\psi : M \to N$ such that $\varphi_i = \psi \circ \lambda_i$ for all $i \in I$, i.e., the following diagram commutes for all $i \in I$.

$$\begin{tikzcd} M_i \arrow[r, "\varphi_i"] \arrow[d, "\lambda_i"'] & N \\ M = \sum_{i\in I} M_i \arrow[ur, dotted, "\psi"'] & \end{tikzcd}$$

Proof. Remark here that for each $(m_i)_{i\in I} \in M$, there are exactly finite $i \in I$ such that $m_i \neq 0$. Define $\psi : M \to N$ by

$$\psi((m_i)_{i\in I}) = \sum_{k\in I} \varphi_k\Big(\pi_k\big((m_i)_{i\in I}\big)\Big) \quad \text{for all } (m_i)_{i\in I} \in M.$$

Then ψ is well-defined since φ_k is well-defined for all $k \in I$ and the sum is finite.

Let $(m_i)_{i\in I}, (n_i)_{i\in I} \in M$ and $s \in S$. Then

$$\begin{aligned}
\psi\big((m_i)_{i\in I} + (n_i)_{i\in I}\big) &= \psi\big((m_i + n_i)_{i\in I}\big) = \sum_{k\in I} \varphi_k\big(\pi_k(m_i + n_i)_{i\in I}\big) \\
&= \sum_{k\in I} \varphi_k(m_k + n_k) = \sum_{k\in I} \varphi_k(m_k) + \sum_{k\in I} \varphi_k(n_k) \\
&= \sum_{k\in I} \varphi_k\big(\pi_k(m_i)_{i\in I}\big) + \sum_{k\in I} \varphi_k\big(\pi_k(n_i)_{i\in I}\big) \\
&= \psi\big((m_i)_{i\in I}\big) + \psi\big((n_i)_{i\in I}\big), \\
\psi\big(s(m_i)_{i\in I}\big) &= \psi\big((sm_i)_{i\in I}\big) = \sum_{k\in I} \varphi_k\big(\pi_k(sm_i)_{i\in I}\big) \\
&= \sum_{k\in I} \varphi_k(sm_k) = s\sum_{k\in I} \varphi_k(m_k) \\
&= s\sum_{k\in I} \varphi_k\big(\pi_k(m_i)_{i\in I}\big) = s\psi\big((m_i)_{i\in I}\big), \text{ and} \\
\psi\big((0)_{i\in I}\big) &= 0.
\end{aligned}$$

Hence ψ is an S-homomorphism.

For each $i, j \in I$, with $i \neq j$, note that $\pi_i \circ \lambda_i = 1_{M_i}$ and $\pi_i \circ \lambda_j = 0$. Fix $i \in I$ and let $m_i \in M_i$. Then $\lambda_i(m_i) \in M$ so that $\psi\big(\lambda_i(m_i)\big) = \sum_{k \in I} \varphi_k\Big(\pi_k\big(\lambda_i(m_i)\big)\Big) = \varphi_i(m_i)$. Hence $\varphi_i = \psi \circ \lambda_i$ for all $i \in I$.

Finally, suppose that there exists $\mu : M \to N$ such that $\varphi_i = \mu \circ \lambda_i$ for all $i \in I$. Let $(m_i)_{i \in I} \in M$. If $(m_i)_{i \in I} = (0)_{i \in I}$, then it is clear that $\psi((m_i)_{i \in I}) = \mu((m_i)_{i \in I})$. Assume that $(m_i)_{i \in I} \neq (0)_{i \in I}$. Then there exist $n \in \mathbb{N}$ and $m_{i_k} \in M_{i_k}$ such that $m_{i_k} \neq 0$ for all $k = 1, 2, \ldots, n$. Thus $\lambda_{i_k}\Big(\pi_{i_k}\big((m_i)_{i \in I}\big)\Big) = \lambda_{i_k}(m_{i_k}) = (p_i)_{i \in I}$ where $p_i = 0$ for all $i \neq i_k$ and $p_{i_k} = m_{i_k}$ for all $k = 1, 2, \ldots, n$. Then $(m_i)_{i \in I} = \sum_{k=1}^{n} \lambda_{i_k}\Big(\pi_{i_k}\big((m_i)_{i \in I}\big)\Big)$. Thus

$$
\begin{aligned}
\mu\big((m_i)_{i \in I}\big) &= \mu\left(\sum_{k=1}^{n} \lambda_{i_k}\Big(\pi_{i_k}\big((m_i)_{i \in I}\big)\Big)\right) \\
&= \sum_{k=1}^{n} \mu\Big(\lambda_{i_k}\Big(\pi_{i_k}\big((m_i)_{i \in I}\big)\Big)\Big), \\
&= \sum_{k=1}^{n} \varphi_{i_k}\Big(\pi_{i_k}\big((m_i)_{i \in I}\big)\Big) \\
&= \sum_{i \in I} \varphi_i\Big(\pi_i\big((m_i)_{i \in I}\big)\Big) \\
&= \psi\big((m_i)_{i \in I}\big).
\end{aligned}
$$

Hence $\psi = \mu$. $\square$

Remark 2.1.10. Let $(M_i)_{i \in I}$ be a family of skew-semimodules over a semiring S. Then $\sum_{i \in I} M_i = \prod_{i \in I} M_i$ if and only if I is finite.

2.2. Free Skew-semimodules over Semirings

In this section, we assume that each semiring has 0 and 1 such that $0 \neq 1$ and each skew-semimodule over a semiring satisfies $(\star)$ given in Definition 1.1.3. The notion of free skew-semimodules over semirings is slightly different from the one of free modules over rings and free semimodules over semirings.

Proposition 2.2.1. *Let M be an $S^\star$-skew-semimodule and B a non-empty subset of M such that for each $m \in M$, there exists a unique family $(s_b)_{b\in B}$ of elements of S such that $s_b = 0$ almost all $b \in B$ and $m = \sum_{b\in B} s_b b$. Define an addition $\oplus$ and a scalar multiplication by*

$$m \oplus n = \sum_{b\in B}(s_b + t_b)b, \text{ and}$$

$$sm = \sum_{b\in B}(ss_b)b,$$

for all $m, n \in M$ and $s \in S$ where $(s_b)_{b\in B}$ and $(t_b)_{b\in B}$ are unique families of elements of S such that $s_b = 0$ and $t_{b'} = 0$ almost all $b, b' \in B$ and $m = \sum_{b\in B} s_b b$ and $n = \sum_{b\in B} t_b b$. Then the monoid $(M, \oplus)$ is an $S^\star$-skew-semimodule.

Proof. It is clear that $(M, \oplus)$ is a monoid. Next, we will show that M is an $S^\star$-skew-semimodule. Let $r, s \in S$ and $m, n \in M$ be such that $m = \sum_{b\in B} s_b b$ and $n = \sum_{b\in B} t_b b$ where $(s_b)_{b\in B}$ and $(t_b)_{b\in B}$ are unique families of elements of S such that $s_b = 0$ and $t_{b'} = 0$ almost all $b, b' \in B$. Then

(i) $(r+s)m = \sum_{b\in B}\big((r+s)s_b\big)b = \sum_{b\in B}(rs_b + ss_b)b = \sum_{b\in B}\big((rs_b)b + (ss_b)b\big)$
$= \sum_{b\in B}(rs_b)b \oplus \sum_{b\in B}(ss_b)b = rm \oplus sm$,

(ii) $s(m \oplus n) = s\left(\sum_{b\in B}(s_b + t_b)b\right) = \sum_{b\in B}\big(s(s_b + t_b)\big)b = \sum_{b\in B}(ss_b + st_b)b$
$= \sum_{b\in B}\big((ss_b)b + (st_b)b\big) = \sum_{b\in B}(ss_b)b \oplus \sum_{b\in B}(st_b)b = sm \oplus sn$,

(iii) $(rs)m = \sum_{b\in B}\big((rs)s_b\big)b = \sum_{b\in B}\big(r(ss_b)\big)b = r\left(\sum_{b\in B}(ss_b)b\right) = r(sm)$,

(iv) $s0 = 0$, and

($\star$) $0m = \sum_{b\in B}(0s_b)b = 0$, and $1m = \sum_{b\in B}(1s_b)b = \sum_{b\in B} s_b b = m$.

Hence the monoid $(M, \oplus)$ is an $S^\star$-skew-semimodule. $\square$

Definition 2.2.2. Let M be an $S^\star$-skew-semimodule and B a non-empty subset of M such that for each $m \in M$, there exists a unique family $(s_b)_{b\in B}$ of elements of S such that $s_b = 0$ almost all $b \in B$ and $m = \sum\limits_{b\in B} s_b b$. Then the monoid M under $\oplus$ and the scalar multiplication defined in Proposition 2.2.1 is called a *free $S^\star$-skew-semimodule with a basis B.*

Proposition 2.2.3. ***The Universal Mapping Property of Free Skew-semimodules***

If M is a free $S^\star$-skew-semimodule with a basis B and $f : B \to N$ is a mapping into an $S^\star$-skew-semimodule N, then there exists a unique S-homomorphism $\varphi : M \to N$ which extends f, i.e., the following diagram commutes:

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d, "i_{B,M}"'] & N \\ M \arrow[ur, dotted, "\varphi"'] & \end{tikzcd}$$

where $i_{B,M}$ is the inclusion map of B into M.

Proof. Let M be a free $S^\star$-skew-semimodule with a basis B and $f : B \to N$ a mapping into an $S^\star$-skew-semimodule N. Recall that for each $m \in M$, there exists a unique family $(s_b)_{b\in B}$ of elements of S such that $s_b = 0$ almost all $b \in B$ and $m = \sum\limits_{b\in B} s_b b$. Define $\varphi : M \to N$ by

$$\varphi(m) = \sum_{b\in B} s_b f(b) \quad \text{for all } m \in M.$$

Since $(s_b)_{b\in B}$ is the unique family of elements of S such that $s_b = 0$ almost all $b \in B$ and f is a function, φ is well-defined.

To show that φ is an S-homomorphism, let $m, n \in M$ and $s \in S$. Then there exist unique families $(s_b)_{b\in B}$ and $(t_b)_{b\in B}$ of elements of S such that $s_b = 0$ and $t_{b'} = 0$ almost all $b, b' \in B$, $m = \sum\limits_{b\in B} s_b b$ and $n = \sum\limits_{b\in B} t_b b$. Then

$$\varphi(m+n) = \varphi\left(\sum_{b\in B}(s_b+t_b)b\right) = \sum_{b\in B}(s_b+t_b)f(b) = \sum_{b\in B}\big(s_bf(b)+t_bf(b)\big)$$

$$= \sum_{b\in B} s_bf(b) + \sum_{b\in B} t_bf(b) = \varphi(m)+\varphi(n),$$

$$\varphi(sm) = \varphi\left(\sum_{b\in B}(ss_b)b\right) = \sum_{b\in B}(ss_b)f(b) = s\sum_{b\in B}(s_b)f(b) = s\varphi(m), \text{ and}$$

$$\varphi(0) = \varphi(0m) = \sum_{b\in B}(0s_b)f(b) = 0.$$

Hence φ is an S-homomorphism.

Next, let $a \in B$. Then $a = \sum_{b\in B} s_b b$ where $s_b = 0$ for all $b \neq a$ and $s_a = 1$. Then $\varphi \circ i_{B,M}(a) = \varphi(a) = \sum_{b\in B} s_bf(b) = f(a)$. This shows that φ is an extension of f.

Suppose that there is an S-homomorphism $\mu : M \to N$ such that μ extends f. Let $m \in M$. Then there exists a unique family $(s_b)_{b\in B}$ of elements of S such that $s_b = 0$ almost all $b \in B$ and $m = \sum_{b\in B} s_b b$. Then

$$\mu(m) = \mu\left(\sum_{b\in B} s_b b\right) = \sum_{b\in B} s_b\mu(b) = \sum_{b\in B} s_bf(b) = \varphi\left(\sum_{b\in B} s_b b\right) = \varphi(m).$$

Hence $\mu = \varphi$. □

Proposition 2.2.4. *Let S be a semiring with 0 and 1 such that $0 \neq 1$, B be a non-empty set and $M_b = S$ for all $b \in B$. Then $\sum_{b\in B} M_b$ is a free $S^\star$-skew-semimodule. Moreover, for each $b \in B$, let $f_b \in \sum_{b\in B} M_b$ be defined by*

$$f_b(b') = \begin{cases} 1, & \text{if } b' = b, \\ 0, & \text{if } b' \neq b. \end{cases}$$

Then $\{f_b \mid b \in B\}$ *is a basis of* $\sum_{b\in B} M_b$ *and the map* $b \mapsto f_b$ *is a bijection from* B *onto* $\{f_b \mid b \in B\}$.

Proof. Since S has 0 and 1, we obtain that $\sum_{b\in B} M_b$ is an $S^\star$-skew-semimodule. Let $(m_b)_{b\in B} \in \sum_{b\in B} M_b$. Then there are $n \in \mathbb{N}$ and $b_1, b_2, \ldots, b_n \in B$ such that $m_{b_i} \neq 0$ and $m_b = 0$ for all $b \neq b_i$ and $i = 1, 2, \ldots, n$. Then

$$(m_b)_{b\in B} = \sum_{i=1}^{n} m_{b_i} f_{b_i} = \sum_{b\in B} m_b f_b.$$

Suppose that there exists a family $(s_b)_{b\in B}$ of elements of S such that $s_b = 0$ almost all $b \in B$ and $(m_b)_{b\in B} = \sum_{b\in B} s_b f_b$ which is a finite sum. Then for each $b \in B$,

$$m_b = \pi_b\left(\sum_{b'\in B} m_{b'} f_{b'}\right) = \pi_b\left(\sum_{b'\in B} s_{b'} f_{b'}\right) = s_b.$$

Hence $(m_b)_{b\in B} = (s_b)_{b\in B}$. By Proposition 2.2.1, $\sum_{b\in B} M_b$ is a free $S^\star$-skew-semimodule with a basis $\{f_b \mid b \in B\}$.

Finally, it is easy to verify that the map $b \mapsto f_b$ is a bijection from B onto $\{f_b \mid b \in B\}$. $\square$

Proposition 2.2.5. *Let* F *be a free* $S^\star$*-skew-semimodule with a basis* B. *For each* $b \in B$, *let* $M_b = S$. *Then* $\sum_{b\in B} M_b \cong F$.

Proof. We obtain from Proposition 2.2.4 that $\sum_{b\in B} M_b$ is a free $S^\star$-skew-semimodule with a basis $B^* = \{f_b \mid b \in B\}$. Moreover, there exists a bijection between B and B^*.

By The Universal Mapping Property of Free Skew-semimodules, there exist S-homomorphisms $\varphi : F \to \sum_{b\in B} M_b$ and $\psi : \sum_{b\in B} M_b \to F$ such that $\varphi(b) = f_b$ and $\psi(f_b) = b$ for all $b \in B$.

Now we have $\psi \circ \varphi : F \to F$ such that $\psi \circ \varphi(b) = b$ for all $b \in B$, then by the uniqueness part of The Universal Mapping Property of Free Skew-semimodules, $\psi \circ \varphi = 1_F$. Similarly, $\varphi \circ \psi = 1_{\sum_{b \in B} M_b}$. Therefore, φ is an isomorphism, i.e., $\sum\limits_{b \in B} M_b \cong F$. $\square$

Proposition 2.2.6. *Let M be an $S^\star$-skew-semimodule. Then there exist a free $S^\star$-skew-semimodule F over S and an epimorphism $\varphi : F \to M$.*

Moreover, if M is finitely generated, it is possible to choose F with a finite basis.

Proof. Let X generate M. Note that $X \neq \emptyset$. For each $x \in X$, let $M_x = S$. Then $F = \sum\limits_{x \in X} M_x$ is a free $S^\star$-skew-semimodule with a basis $\{f_x \,|\, x \in X\}$ defined in Proposition 2.2.4, and the map $f_x \mapsto x$ is a bijection. By The Universal Mapping Property of Free Skew-semimodules, there exists a unique S-homomorphism $\varphi : F \to M$ such that $\varphi(f_x) = x$ for all $x \in X$. Let $m \in M$. By Proposition 1.1.13, we can write $m = \sum\limits_{i=1}^{k} r_i x_i$ where $k \in \mathbb{N}$, $r_i \in S$ and $x_i \in X$ for all $i \in \{1, 2, \ldots, k\}$. Then $m = \sum\limits_{i=1}^{k} r_i x_i = \sum\limits_{x \in X} s_x x$ where $s_x = 0$ if $x \neq x_i$ for all $i \in \{1, 2, \ldots, k\}$ and $s_x = r_i$ otherwise. Thus

$$m = \sum_{x \in X} s_x \varphi(f_x) = \sum_{x \in X} \varphi(s_x f_x) = \varphi\left(\sum_{x \in X} s_x f_x\right).$$

Hence φ is surjective. Moreover, it is clear that $\{f_x | x \in X\}$ is finite if X is finite. $\square$

CHAPTER III

EXACT SEQUENCES

In this chapter, we consider a particular skew-semimodule which is also a group. Doing this leads us to define an exact sequence of skew-semimodules over semirings and homomorphisms.

3.1. Definitions and The Four Lemma

From now on, only this chapter, we assume that each S-skew-semimodule is not only a monoid but also a group. Recall that the zero set of a homomorphism $\varphi : M \to N$ of skew-semimodules is $\{m \in M \mid \varphi(m) = 0\}$. Moreover, we can see that the zero set of a homomorphism of skew-semimodules over semirings is defined in the same way as the kernel of a homomorphism of modules over rings. An interesting property is that for a given homomorphism φ of modules, φ is injective if and only if the kernel of φ is $\{0\}$. However, the analogous property does not hold for the case of a homomorphism of skew-semimodules. Nevertheless, assuming that skew-semimodules M and N are groups gives the same nice result.

Proposition 3.1.1. *Let $\varphi : M \to N$ be an S-homomorphism. Then φ is a monomorphism if and only if $Zs\varphi = \{0\}$.*

Proof. It remains to verify the necessary part. Assume that $Zs\varphi = \{0\}$. Let $m, n \in M$ be such that $\varphi(m) = \varphi(n)$. Then $\varphi(m - n) = \varphi(m) - \varphi(n) = 0$, so $m - n \in Zs\varphi = \{0\}$ that is $m - n = 0$ which implies that $m = n$. Hence φ is a monomorphism. $\square$

Definition 3.1.2. A sequence $M \xrightarrow{f} N \xrightarrow{g} P$ of S-skew-semimodules and S-homomorphisms is said to be *exact at* N provided $Imf = Zsg$. A finite sequence $M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \xrightarrow{f_3} \cdots \xrightarrow{f_n} M_n$ of S-skew-semimodules and S-homomorphisms is *exact* provided $Imf_i = Zsf_{i+1}$ for $i = 1, 2, \ldots, n-1$.

Remark 3.1.3. If a sequence $M \xrightarrow{f} N \xrightarrow{g} P$ of S-skew-semimodules and S-homomorphisms is exact, then $g \circ f = 0$.

Notation 3.1.4. From now on, the zero skew-semimodule will be denoted by 0. Moreover, $0 \longrightarrow M$ and $M \longrightarrow 0$ stand for the inclusion map and the zero map, respectively.

Proposition 3.1.5. *Let $f : M \to N$ be an S-homomorphism. Then the following statements hold:*

(i) *f is a monomorphism if and only if $0 \longrightarrow M \xrightarrow{f} N$ is exact,*

(ii) *f is an epimorphism if and only if $M \xrightarrow{f} N \longrightarrow 0$ is exact,*

(iii) *f is an isomorphism if and only if $0 \longrightarrow M \xrightarrow{f} N \longrightarrow 0$ is exact.*

Proof. Obvious. $\square$

Definition 3.1.6. Let M, N and P be S-skew-semimodules. The exact sequence of the form $0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} P \longrightarrow 0$ is called a *short exact sequence.*

Theorem 3.1.7. *Given the diagram of S-skew-semimodules and S-homomorphisms*

$$\begin{tikzcd} & & Q \arrow[dl, dotted, "\varphi"'] \arrow[d, "h"] & \\ 0 \arrow[r] & M \arrow[r, "f"] & N \arrow[r, "g"] & P \end{tikzcd}$$

in which the row is exact and $g \circ h = 0$. Then there exists a unique S-homomorphism $\varphi : Q \to M$ such that the complete diagram is commutative, i.e., $f \circ \varphi = h$.

Proof. Since $g \circ h = 0$ and the row is exact at N, it follows that $Imh \subseteq Zsg = Imf$. Then for each $q \in Q$ there exists $m_q \in M$ such that $f(m_q) = h(q)$. Thus we define $\varphi : Q \to M$ by

$$\varphi(q) = m_q \quad \text{for all } q \in Q.$$

Let $q_1, q_2 \in Q$ be such that $q_1 = q_2$. Then $h(q_1) = h(q_2)$ and there exist $m_{q_1}, m_{q_2} \in M$ such that $f(m_{q_1}) = h(q_1)$ and $f(m_{q_2}) = h(q_2)$ so $f(m_{q_1}) = f(m_{q_2})$. By the exactness at M, we obtain that f is injective which implies that $m_{q_1} = m_{q_2}$. Thus φ is well-defined.

To show that $f \circ \varphi = h$, let $q \in Q$. Then $f(m_q) = h(q)$ for some $m_q \in M$. Hence $f \circ \varphi(q) = f(m_q) = h(q)$.

Next, we will show that φ is an S-homomorphism. Let $q_1, q_2 \in Q$ and $s \in S$. Then

$$f\big(\varphi(q_1+q_2)\big) = h(q_1+q_2) = h(q_1)+h(q_2) = f\big(\varphi(q_1)\big)+f\big(\varphi(q_2)\big) = f\big(\varphi(q_1)+\varphi(q_2)\big)$$

and

$$f\big(\varphi(sq_1)\big) = h(sq_1) = sh(q_1) = sf\big(\varphi(q_1)\big) = f\big(s\varphi(q_1)\big).$$

Thus

$$\varphi(q_1+q_2) = \varphi(q_1)+\varphi(q_2), \varphi(sq_1) = s\varphi(q_1) \text{ and } \varphi(0) = 0$$

since f is injective. Therefore, φ is an S-homomorphism.

Finally, the uniqueness of φ is immediate from the injectivity of f. $\square$

We can state and prove The Four Lemma and its corollaries in the same way as those in module theory.

Theorem 3.1.8. *The Four Lemma*

Suppose that the following diagram of S-skew-semimodules and S-homomorphisms

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\alpha"] & N \arrow[r, "g"] \arrow[d, "\beta"] & P \arrow[r, "h"] \arrow[d, "\gamma"] & Q \arrow[d, "\delta"] \\ M' \arrow[r, "f'"] & N' \arrow[r, "g'"] & P' \arrow[r, "h'"] & Q' \end{tikzcd}$$

is commutative and has exact rows. Then

(i) *if α, γ are epimorphisms and δ is a monomorphism, then β is an epimorphism,*

(ii) *if α is an epimorphism and β, δ are monomorphisms, then γ is a monomorphism.*

Proof. (i) Assume that α and γ are epimorphisms and δ is a monomorphism. Let $n' \in N'$. Then $g'(n') \in P'$. Since γ is an epimorphism, there exists $p \in P$ such that $\gamma(p) = g'(n')$. By the commutativity of the right-square, we have

$$\delta\big(h(p)\big) = h'\big(\gamma(p)\big) = h'\big(g'(n')\big) = 0,$$

since $h' \circ g' = 0$. Then $h(p) \in Zs\delta$ so that $h(p) = 0$ since δ is a monomorphism. Thus $p \in Zsh = Img$ because of the exactness at P. Hence there is $n \in N$ such that $g(n) = p$. By the commutativity of the middle square,

$$g'(n') = \gamma(p) = \gamma\big(g(n)\big) = g'\big(\beta(n)\big).$$

Then $g'\big(n' - \beta(n)\big) = g'(n') - g'\big(\beta(n)\big) = 0$. Thus $n' - \beta(n) \in Zsg' = Imf'$. Then there exists $m' \in M'$ such that $f'(m') = n' - \beta(n)$. Since α is an epimorphism, $\alpha(m) = m'$ for some $m \in M$. By the commutativity of the left-square,

$$n' - \beta(n) = f'(m') = f'\big(\alpha(m)\big) = \beta\big(f(m)\big).$$

Hence $n' = \beta\big(f(m)\big) + \beta(n) = \beta\big(f(m) + n\big)$ where $f(m) + n \in N$.

Therefore, β is an epimorphism.

(ii) Assume that α is an epimorphism and β, δ are monomorphisms. Let $p \in Zs\gamma$. By the commutativity of the right-square,

$$\delta\big(h(p)\big) = h'\big(\gamma(p)\big) = h'(0) = 0,$$

so that $h(p) = 0$ since δ is a monomorphism. Then $p \in Zsh = Img$ because of the exactness at P. Hence $g(n) = p$ for some $n \in N$. By the commutativity of the middle square,

$$g'\big(\beta(n)\big) = \gamma\big(g(n)\big) = \gamma(p) = 0,$$

which implies that $\beta(n) \in Zsg' = Imf'$ which is obtained from the exactness at N'. Then there exists $m' \in M'$ such that $f'(m') = \beta(n)$. Since α is an epimorphism, $\alpha(m) = m'$ for some $m \in M$. By the commutativity of the left-square,

$$\beta\big(f(m)\big) = f'\big(\alpha(m)\big) = f'(m') = \beta(n).$$

This shows that $f(m) = n$ since β is a monomorphism. Hence $p = g(n) = g\big(f(m)\big) = 0$ because of the exactness at N.

Therefore, γ is a monomorphism. $\square$

Corollary 3.1.9. ***The Five Lemma***

Suppose that the following diagram of S-skew-semimodules and S-homomorphisms

$$\begin{tikzcd}
K \arrow[r, "f_1"] \arrow[d, "\alpha_1"] & M \arrow[r, "f_2"] \arrow[d, "\alpha_2"] & N \arrow[r, "f_3"] \arrow[d, "\alpha_3"] & P \arrow[r, "f_4"] \arrow[d, "\alpha_4"] & Q \arrow[d, "\alpha_5"] \\
K' \arrow[r, "f'_1"'] & M' \arrow[r, "f'_2"'] & N' \arrow[r, "f'_3"'] & P' \arrow[r, "f'_4"'] & Q'
\end{tikzcd}$$

is commutative and has exact rows. If $\alpha_1, \alpha_2, \alpha_4, \alpha_5$ are isomorphisms, then so is α_3.

Proof. Applying The Four Lemma (i) to the right-hand three squares, we obtain that α_3 is an epimorphism. Again applying The Four Lemma (ii) to the left-hand three squares, we see that α_3 is a monomorphism. Therefore, α_3 is an isomorphism. □

Corollary 3.1.10. ***The Short Five Lemma***

Suppose that the following diagram of S-skew-semimodules and S-homomorphisms

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "f"] \arrow[d, "\alpha"] & N \arrow[r, "g"] \arrow[d, "\beta"] & P \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & M' \arrow[r, "f'"'] & N' \arrow[r, "g'"'] & P' \arrow[r] & 0
\end{tikzcd}$$

is commutative and has exact rows. If α and γ are isomorphisms, then so is β.

Proof. Obvious. □

3.2. Isomorphic Short Exact Sequences

We investigate when given two short exact sequences are isomorphic.

Definition 3.2.1. Given two short exact sequences $0 \to M \xrightarrow{f} N \xrightarrow{g} P \to 0$ and $0 \to M' \xrightarrow{f'} N' \xrightarrow{g'} P' \to 0$. Then they are said to be *isomorphic* if there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "f"] \arrow[d, "\alpha"] & N \arrow[r, "g"] \arrow[d, "\beta"] & P \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & M' \arrow[r, "f'"'] & N' \arrow[r, "g'"'] & P' \arrow[r] & 0
\end{tikzcd}$$

such that α, β and γ are isomorphisms.

Remark 3.2.2. From the definition of isomorphic short exact sequences, it is easy to see that the diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "f"] & N \arrow[r, "g"] & P \arrow[r] & 0 \\
0 \arrow[r] & M' \arrow[r, "f'"'] \arrow[u, "\alpha^{-1}"'] & N' \arrow[r, "g'"'] \arrow[u, "\beta^{-1}"'] & P' \arrow[r] \arrow[u, "\gamma^{-1}"'] & 0
\end{tikzcd}$$

is also commutative.

The statements and proofs of the following results are similar to those in module theory.

Let M and N be S-skew-semimodules. Then we denote the direct sum of M and N by $M \oplus N$.

Theorem 3.2.3. *Let M and N be S-skew-semimodules. Then the sequence*

$$0 \rightarrow M \xrightarrow{\lambda} M \oplus N \xrightarrow{\pi} N \rightarrow 0$$

is exact, where λ and π are canonical injection and projection, respectively.

Moreover, the given sequence is called the direct sum short exact sequence*.*

Proof. It remains to show that the sequence is exact at $M \oplus N$. Note that $\lambda(m) = (m, 0)$ and $\pi(m, n) = n$ for all $m \in M$ and $n \in N$. Then

$$Im\lambda = \{(m, 0) \mid m \in M\} = Zs\pi.$$

Hence the sequence is exact. $\square$

Theorem 3.2.4. *Let $0 \rightarrow M_1 \xrightarrow{f} N \xrightarrow{g} M_2 \rightarrow 0$ be a short exact sequence of S-skew-semimodules and S-homomorphisms. Then the following conditions are equivalent.*

(i) *There is an S-homomorphism $h : M_2 \rightarrow N$ with $g \circ h = 1_{M_2}$.*

(ii) *There is an* S*-homomorphism* $k : N \to M_1$ *with* $k \circ f = 1_{M_1}$.

(iii) *The given sequence is isomorphic to the direct sum short exact sequence*

$$0 \longrightarrow M_1 \xrightarrow{\lambda_1} M_1 \oplus M_2 \xrightarrow{\pi_2} M_2 \longrightarrow 0$$

Proof. First of all, note that

$$0 \longrightarrow M_1 \xrightarrow{\lambda_1} M_1 \oplus M_2 \xrightarrow{\pi_2} M_2 \longrightarrow 0$$

is exact from Theorem 3.2.3.

(i)$\Rightarrow$(iii) Assume that (i) holds. Now, we obtain that $f : M_1 \to N$ and $h : M_2 \to N$ are S-homomorphisms. Thus by The Universal Mapping Property of Direct Sums, there exists an S-homomorphism $\varphi : M_1 \oplus M_2 \to N$, given by

$$\varphi(m_1, m_2) = f(m_1) + h(m_2) \quad \text{for all } (m_1, m_2) \in M_1 \oplus M_2.$$

We will show that the following diagram

$$\begin{tikzcd}
0 \arrow[r] & M_1 \arrow[r, "\lambda_1"] \arrow[d, "1_{M_1}"] & M_1 \oplus M_2 \arrow[r, "\pi_2"] \arrow[d, "\varphi"] & M_2 \arrow[r] \arrow[d, "1_{M_2}"] & 0 \\
0 \arrow[r] & M_1 \arrow[r, "f"'] & N \arrow[r, "g"'] & M_2 \arrow[r] & 0
\end{tikzcd}$$

is commutative. We can see that

$$\varphi\big(\lambda_1(m_1)\big) = \varphi\big((m_1, 0)\big) = f(m_1) + h(0) = f(m_1) = f\big(1_{m_1}(m_1)\big)$$

for all $m_1 \in M_1$, and since $g \circ h = 1_{M_2}$, it follows that

$$\begin{aligned}
g\Big(\varphi\big((m_1, m_2)\big)\Big) &= g\big(f(m_1) + h(m_2)\big) = g\big(f(m_1)\big) + g\big(h(m_2)\big) = 0 + m_2 \\
&= m_2 = \pi_2\big((m_1, m_2)\big) = 1_{M_2}\Big(\pi_2\big((m_1, m_2)\big)\Big)
\end{aligned}$$

for all $(m_1, m_2) \in M_1 \oplus M_2$. This shows that the above diagram is commutative.

By The Short Five Lemma, φ is an isomorphism. Therefore, the two sequences are isomorphic.

(ii)$\Rightarrow$(iii) Assume that (ii) holds. Now, we obtain that $k : N \to M_1$ and $g : N \to M_2$ are S-homomorphisms. Thus by The Universal Mapping Property of Direct Products, there exists an S-homomorphism $\psi : N \to M_1 \times M_2 = M_1 \oplus M_2$, given by

$$\psi(n) = \big(k(n), g(n)\big) \qquad \text{for all } n \in N.$$

We will show that the following diagram

$$\begin{tikzcd}
0 \arrow[r] & M_1 \arrow[r, "f"] \arrow[d, "1_{M_1}"] & N \arrow[r, "g"] \arrow[d, "\psi"] & M_2 \arrow[r] \arrow[d, "1_{M_2}"] & 0 \\
0 \arrow[r] & M_1 \arrow[r, "\lambda_1"'] & M_1 \oplus M_2 \arrow[r, "\pi_2"'] & M_2 \arrow[r] & 0
\end{tikzcd}$$

is commutative. Since $k \circ f = 1_{M_1}$, we can see that

$$\psi\big(f(m_1)\big) = \Big(k\big(f(m_1)\big), g\big(f(m_1)\big)\Big) = (m_1, 0) = \lambda_1(m_1) = \lambda_1\big(1_{M_1}(m_1)\big)$$

for all $m_1 \in M_1$, and

$$\pi_2\big(\psi(n)\big) = \pi_2\big(k(n), g(n)\big) = g(n) = 1_{M_2}\big(g(n)\big)$$

for all $n \in N$. This shows that the above diagram is commutative. By The Short Five Lemma, ψ is an isomorphism.

Therefore, the two sequences are isomorphic.

(iii)$\Rightarrow$(i)&(ii) Given a commutative diagram with exact rows and an isomorphism α

$$\begin{tikzcd}
0 \arrow[r] & M_1 \arrow[r, shift left, "\lambda_1"] \arrow[d, "1_{M_1}"] & M_1 \oplus M_2 \arrow[l, shift left, "\pi_1"] \arrow[r, shift left, "\pi_2"] \arrow[d, "\alpha"] & M_2 \arrow[l, shift left, "\lambda_2"] \arrow[r] \arrow[d, "1_{M_2}"] & 0 \\
0 \arrow[r] & M_1 \arrow[r, "f"'] & N \arrow[r, "g"'] & M_2 \arrow[r] & 0
\end{tikzcd}$$

Define $h : M_2 \to N$ and $k : N \to M_1$ by

$$h(m_2) = \alpha\big(\lambda_2(m_2)\big) \quad \text{for all } m_2 \in M_2, \text{ and}$$

$$k(n) = \pi_1\big(\alpha^{-1}(n)\big) \quad \text{for all } n \in N.$$

Since $\alpha, \alpha^{-1}, \lambda_2$ and π_1 are S-homomorphisms, h and k are also S-homomorphisms. Let $m_2 \in M_2$ and $m_1 \in M_1$. Then

$$g\big(h(m_2)\big) = g\Big(\alpha\big(\lambda_2(m_2)\big)\Big) = g\Big(\alpha\big((0, m_2)\big)\Big) = 1_{M_2}\Big(\pi_2\big((0, m_2)\big)\Big) = m_2, \quad \text{and}$$

$$k\big(f(m_1)\big) = \pi_1\Big(\alpha^{-1}\big(f(m_1)\big)\Big) = \pi_1\Big(\lambda_1\big(1_{M_1}(m_1)\big)\Big) = m_1.$$

Therefore, $g \circ h = 1_{M_2}$ and $k \circ f = 1_{M_1}$. $\square$

CHAPTER IV

INTRODUCTION TO ARTINIAN AND NOETHERIAN SKEW-SEMIMODULES

The concepts of Artinian and Noetherian modules over rings have been found in module theory. Moreover, Artinian and Noetherian skewmodules over skewrings have been introduced in [2]. These were studied regarding chains of submodules and of normal subskewmodules, respectively. In this chapter, we define Artinian and Noetherian skew-semimodules over semirings involving chains of ideals of skew-semimodules. We can also prove some basic theorems.

4.1. Artinian and Noetherian Skew-semimodules

Definition 4.1.1. Let M be an S-skew-semimodule. A chain $A_1 \subseteq A_2 \subseteq \cdots$ or $A_1 \supseteq A_2 \supseteq \cdots$ of subsets of M is said to be an *ideal series* of M if A_i is an ideal of M for all $i \in \mathbb{N}$.

Definition 4.1.2. An S-skew-semimodule M is said to be *Artinian* if every decreasing ideal series $A_1 \supseteq A_2 \supseteq \cdots$ of M, there exists $n \in \mathbb{N}$ such that $A_i = A_n$ for all integers $i \geq n$.

An S-skew-semimodule M is said to be *Noetherian* if every increasing ideal series $A_1 \subseteq A_2 \subseteq \cdots$ of M, there exists $n \in \mathbb{N}$ such that $A_i = A_n$ for all integers $i \geq n$.

Theorem 4.1.3. *Let M be an S-skew-semimodule. Then M is Artinian if and only if for every non-empty collection of ideals of M has a minimal element.*

Proof. Assume that M is Artinian. Let Y be a non-empty collection of ideals of M. Then we choose $A_1 \in Y$. If A_1 is not minimal, then there exists $A_2 \in Y$ such that

$A_2 \subsetneq A_1$. If we choose $A_i \in Y$ which is not minimal, then there exists $A_{i+1} \in Y$ such that $A_{i+1} \subsetneq A_i$. After a finite step, we obtain a minimal element of Y. If not, then we would have an infinite chain of ideals of M such that $A_1 \supsetneq A_2 \supsetneq \cdots$ which contradicts the assumption that M is Artinian.

Conversely, assume that every non-empty collection of ideals of M has a minimal element. Let $A_1 \supseteq A_2 \supseteq \cdots$ be a decreasing ideal series of M. Then the set $\{A_1, A_2, \ldots\}$ has a minimal element, say A_n. Hence $A_n = A_{n+i}$ for all $i \in \mathbb{N}$.

Therefore, M is Artinian. $\square$

Theorem 4.1.4. *Let M be an S-skew-semimodule. Then M is Noetherian if and only if for every non-empty collection of ideals of M has a maximal element.*

Proof. This can be verified similarly to the proof of Theorem 4.1.3. $\square$

Theorem 4.1.5. *Let M be an S-skew-semimodule. If every skew-subsemimodule of M is finitely generated, then M is Noetherian.*

Proof. Let $\mathcal{C} : A_1 \subseteq A_2 \subseteq \cdots$ be an increasing ideals series of M. By Proposition 1.1.16 (ii), we obtain that $\bigcup_{i\in\mathbb{N}} A_i$ is an ideal of M. Let $A = \bigcup_{i\in\mathbb{N}} A_i$. Then A^0 is a skew-subsemimodule of M. By assumption, $A^0 = [B]$, where B is a finite subset of M, so A^0 is the smallest skew-subsemimodule of M containing B. Given $B = \{b_1, b_2, \ldots, b_k\}$. Then for each $j \in \{1, 2, \ldots, k\}$, there is $A_{i_j} \in \mathcal{C}$ such that $b_j \in A_{i_j}$. Hence there exists $n \in \mathbb{N}$ such that $b_j \in A_n$ for all $j \in \{1, 2, \ldots, k\}$ so that $B \subseteq A_n \subseteq A_n^0$. Then $A^0 \subseteq A_n^0$ that is $A \subseteq A_n \subseteq A$. Hence $A_n = A = \bigcup_{i\in\mathbb{N}} A_i$. It follows that $A_n = A_i$ for all integers $i \geq n$. Therefore, M is Noetherian. $\square$

Theorem 4.1.6. ***Correspondence Theorem***

Let A be an ideal of an S-skew-semimodule M. Then there is an inclusion-preserving bijection between the collection of ideals of M/R_A and the collection of ideals of M containing A.

Proof. Let X and Y be the collection of all ideals of M containing A and the collection of all ideals of M/R_A, respectively. By Lemma 1.2.17, for each $B \in X$, we obtain that B/ρ_A is an ideal of M/R_A where $\rho_A = \big\{(a,b) \in B \times B \,\big|\, a = b \text{ or } a,b \in A\big\}$. Define $\varphi : X \to Y$ by

$$\varphi(B) = B/\rho_A \qquad \text{for all } B \in X.$$

Then φ is well-defined and it is easy to verify that φ is inclusion-preserving.

Next, we will show that φ is injective. Let $B_1, B_2 \in X$ be such that $\varphi(B_1) = \varphi(B_2)$. Then $B_1/\rho_A = B_2/\rho_A$. Let $b \in B_1$. Then $[b]_{\rho_A} \in B_1/\rho_A = B_2/\rho_A$ so that $b \in B_2$. This shows that $B_1 \subseteq B_2$. Similarly, we can show that $B_2 \subseteq B_1$. Hence $B_1 = B_2$ which implies that φ is injective.

It remains to show that φ is surjective. Let $D \in Y$. Then it is clear that $A \in D$. Moreover, we obtain from Proposition 1.1.19 (iii) that $\pi^{-1}[D]$ is an ideal of M where $\pi : M \to M/R_A$ is the canonical surjection. Let $a \in A$. Then by Proposition 1.2.11 $\pi(a) = [a]_{R_A} = A \in D$. Thus $a \in \pi^{-1}[D]$. This shows that $A \subseteq \pi^{-1}[D]$. Hence $\pi^{-1}[D] \in X$. We claim that $\pi^{-1}[D]/\rho_A = \varphi\big(\pi^{-1}[D]\big) = D$. First, let $[d]_{\rho_A} \in \pi^{-1}[D]/\rho_A$ where $d \in \pi^{-1}[D]$, so $[d]_{\rho_A} = [d]_{R_A} = \pi(d) \in D$. Next, let $[c]_{R_A} \in D$. Then $\pi(c) = [c]_{R_A} \in D$ so $c \in \pi^{-1}[D]$ which implies that $[c]_{R_A} = [c]_{\rho_A} \in \pi^{-1}[D]/\rho_A$. This shows that φ is surjective. $\square$

Theorem 4.1.7. *Let A be an ideal of an Artinian S-skew-semimodule M. Then,*

(i) *for every chain $A_1 \supseteq A_2 \supseteq \cdots$ of ideals of A^0 such that A_i is an ideal of M for all $i \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that $A_n = A_i$ for all integers $i \geq n$,*

(ii) *M/R_A is Artinian.*

Proof. Let A be an ideal of an Artinian S-skew-semimodule M.

(i) Note that such a decreasing chain of ideals of A^0 is a decreasing ideals series of M. The result follows immediately.

(ii) Let $A_1 \supseteq A_2 \supseteq \cdots$ be any decreasing ideal series of M/R_A. By the Correspondence Theorem, for each $i \in \mathbb{N}$ there exists an ideal B_i of M containing A such that $B_i/\rho_A = A_i$. Moreover, we obtain that $B_1 \supseteq B_2 \supseteq \cdots$ is a decreasing ideal series of M. Since M is Artinian, there exists $n \in \mathbb{N}$ such that $B_n = B_i$ for all integers $i \geq n$. It follows that $A_n = B_n/\rho_A = B_i/\rho_A = A_i$ for all integers $i \geq n$. Therefore, M/R_A is Artinian. □

Theorem 4.1.8. *Let A be an ideal of a Noetherian S-skew-semimodule M. Then,*

(i) *for every chain $A_1 \subseteq A_2 \subseteq \cdots$ of ideals of A^0 such that A_i is an ideal of M for all $i \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that $A_n = A_i$ for all integers $i \geq n$,*

(ii) *M/R_A is Noetherian.*

Proof. This can be verified similarly to the proof of Theorem 4.1.7. □

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