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TEMPLATES AND PROGRAM EXTRACTION FROM PROOFS IN HIGHER ORDER SYSTEMS

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เทอมเคอร์รี-โฮวาร์ดเป็นเทอมแลมบ์ดาชนิดไทป์ซึ่งสามารถใช้เป็นตัวแทนบทพิสูจน์รูปนัยในระบบนิรนัยธรรมชาติ และการสร้างเทอมเคอร์รี-โฮวาร์ดนี้เป็นวิธีการมาตรฐานในการสร้างโปรแกรมจากบทพิสูจน์

ในการพิสูจน์ทางคณิตศาสตร์ แบบรูปที่เหมือนกันมักเกิดขึ้นบ่อยครั้ง ดังนั้นในการสร้างโปรแกรมจากบทพิสูจน์จะเป็นประโยชน์ถ้ามีการนิยามแบบรูปหรือเทมเพลตอย่างเป็นแบบแผนแล้วเติมเข้าไปในระบบ เพื่อที่เราจะได้ไม่ต้องทำสิ่งที่เหมือนๆกันซ้ำแล้วซ้ำอีก ยิ่งกว่านั้นการทำเช่นนี้ยังสะท้อนสิ่งที่ปฏิบัติกันตามปรกติในคณิตศาสตร์

ในงานวิจัยนี้เราสร้างระบบใหม่สำหรับการสร้างโปรแกรมจากบทพิสูจน์ ด้วยการขยายระบบของครอสสลีย์และเชเพอร์ดสันซึ่งสร้างในภาษาของแคลคูลัสภาคแสดงอันดับที่หนึ่ง(ใน[3]) ไปสู่ตรรกศาสตร์อันดับที่สองและเติมเทมเพลตในฐานะกฎใหม่เข้าไปในระบบที่ขยายนี้

เราสร้างเทมเพลตสองชนิดคือเทมเพลตแบบอุปนัยและเทมเพลตแบบการย่อ เทมเพลตแบบแรกทำให้เราสามารถใช้อุปนัยแบบต่างๆในระบบรูปนัยโดยไม่ต้องผ่านจำนวนธรรมชาติ และเทมเพลตแบบหลังทำให้เราสามารถย่อประโยคทางคณิตศาสตร์ในบทพิสูจน์รูปนัยด้วยเพรดิเคตใหม่

เทอมเคอร์รี-โฮวาร์ดที่สร้างขึ้นในระบบใหม่นี้ยังคงมีสมบัติพื้นฐานรวมทั้งยังสอดคล้องกับทฤษฎีนอร์มาไลเซชันแบบเข้ม เราจึงสามารถขยายการสร้างโปรแกรมไปสู่ระบบตรรกศาสตร์ที่กว้างกว่ามาก

ภาควิชา **คณิตศาสตร์**	ลายมือชื่อนิสิต..
สาขาวิชา **คณิตศาสตร์**	ลายมือชื่ออาจารย์ที่ปรึกษา..
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Curry-Howard terms are typed-lambda terms, which are a way of representing formal proofs in a natural deduction system. The standard approach to extracting programs from proofs is by constructing Curry-Howard terms.

In carrying out mathematical proofs, the same patterns frequently recur. Therefore in extracting programs from proofs it would be helpful to formally define what a pattern, or template, is and then add it into the system so that we do not have to repeat what is essentially the same argument. Moreover, this mirrors ordinary mathematical practice.

In this research, we create a new system for extracting programs from proofs by extending Crossley and Shepherdson's system (in [3]) in the language of first-order predicate calculus to second-order logic and adding templates into the extended system as new rules.

We introduce two kinds of templates: induction templates and abbreviation templates. The former templates allow us to use various kinds of induction in the formal system without going through the natural numbers. The latter templates enable us to abbreviate formulae by new predicates in formal proofs.

The Curry-Howard terms produced in the new system still satisfy all the basic properties including the strong normalization theorem so we can extend the extraction of programs to the greatly expanded logical system.

Department **Mathematics** Student's signature...

Field of study **Mathematics** Advisor's signature..

Academic year **2003** Co-advisors' signatures ....

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CONTENTS

CHAPTER I

INTRODUCTION

There is a close correspondence between intuitionistic logic and typed-lambda calculi. Curry (see [4]) and subsequently Howard (see [8]) noticed the correspondence between term formation rules in lambda calculus and the rules of inference in intuitionistic propositional calculus. Such a correspondence leads to the idea of *extracting programs from proofs* to which a brief introduction is as follows.

The standard approach to *extracting programs from proofs* is by constructing *Curry-Howard terms*.

Curry-Howard terms are typed-lambda terms which are defined to correspond to formal proofs in the natural deduction system. Such a correspondence is called *Curry-Howard isomorphism* (see [7] for more details). As in [3], the natural deduction system used here is the version of Gentzen's intuitionistic system given by Prawitz (see [12]) and types of Curry-Howard terms are first-order formulae. By defining a Curry-Howard term formation rule corresponding to each rule in the natural deduction system we will get the correspondence between proofs and Curry-Howard terms. In order to give some ideas about the correspondence, we will give some examples by using rules for the connective $\wedge$. The full version which deals with every connective and quantifier in first-order logic is in [3].

($\wedge$ Introduction)

Natural deduction rule	C-H term formation rule
$\dfrac{\begin{matrix}\vdots & \vdots\\ \alpha & \beta\end{matrix}}{\alpha \wedge \beta}(\wedge \text{ Intro})$	$\dfrac{F^{\alpha} \quad G^{\beta}}{(F^{\alpha}, G^{\beta})^{\alpha\wedge\beta}}$

($\wedge$ Elimination)

Natural deduction rules	C-H term formation rules
$\dfrac{\begin{matrix}\vdots\\ \alpha \wedge \beta\end{matrix}}{\alpha}(\wedge_1 \text{ Elim})$	$\dfrac{F^{\alpha\wedge\beta}}{(\pi_1 F^{\alpha\wedge\beta})^{\alpha}}$
$\dfrac{\begin{matrix}\vdots\\ \alpha \wedge \beta\end{matrix}}{\beta}(\wedge_2 \text{ Elim})$	$\dfrac{F^{\alpha\wedge\beta}}{(\pi_2 F^{\alpha\wedge\beta})^{\beta}}$

A proof of α from premises α, β	Construction of a corresponding C-H term
$\dfrac{\dfrac{\alpha \quad \beta}{\alpha \wedge \beta}(\wedge \text{ Intro})}{\alpha}(\wedge_1 \text{ Elim})$	$\dfrac{\dfrac{X^{\alpha} \quad Y^{\beta}}{(X^{\alpha}, Y^{\beta})^{\alpha\wedge\beta}}}{(\pi_1(X^{\alpha}, Y^{\beta})^{\alpha\wedge\beta})^{\alpha}}$

In the above proof, we can see that there are unnecessary steps, since if α is a premise, we can deduce α from the premise α, so the above proof is redundant and can be reduced to the uppermost α of which the corresponding Curry-Howard term is X^{α}.

We use the notation $\succ$ for *reduces* and write the above reduction for the Curry-Howard term as follows.

$$(\pi_1(X^{\alpha}, Y^{\beta})^{\alpha\wedge\beta})^{\alpha} \succ X^{\alpha}$$

A Curry-Howard term is said to be *normal* if it cannot be reduced.

In order to see the connexion between programs and proofs, let us consider the following example.

Suppose we have a proof of $\forall x \exists y \alpha$. As explained above, we can construct a corresponding Curry-Howard term $F^{\forall x \exists y \alpha}$. When a closed individual term t is given, by applying the ($\forall$ Elim) rule (see [3]) to the last line of the proof, we obtain a proof of $\exists y \alpha(x/t)$, where $\alpha(x/t)$ denotes the result of substituting t for all free occurrences of x in α subject to avoiding clashes of variables, and the corresponding Curry-Howard term is $(F^{\forall x \exists y \alpha}(t))^{\exists y \alpha(x/t)}$. As shown in the proof of Theorem 6.6 in [3], $(F^{\forall x \exists y \alpha}(t))^{\exists y \alpha(x/t)}$ can be reduced to a normal term which is of the form $(I(u, G^{\alpha(x/t)(y/u)}))^{\exists y \alpha(x/t)}$ for some closed individual term u.

From the above process we can see that $F^{\forall x \exists y \alpha}$ is a program extracted from the proof of $\forall x \exists y \alpha$ and when a value t of x is given, we can extract the corresponding value u of y by reducing $(F^{\forall x \exists y \alpha}(t))^{\exists y \alpha(x/t)}$ to normal form. We can think of the process of computing the value of y from a given value of x as reducing $(F^{\forall x \exists y \alpha}(t))^{\exists y \alpha(x/t)}$ to normal form and then extracting the y from the final term.

For computational purposes every term must eventually reduce in a finite number of steps to a normal form. A Curry-Howard term F is *strongly normalizable* if all reduction sequences beginning with F are finite. A calculus satisfies the *strong normalization theorem* if every term is strongly normalizable.

Takeuti (see [17]) first formulated a conjecture in 1954 that it would be possible to prove strong normalization (otherwise known as cut-elimination) for simple type theory and in 1966 Tait (see [14]) proved cut-elimination for second-order logic. The theorem for higher order logic/simple type theory was published by Prawitz [13] and Takahashi [15] and the full theorem was subsequently proved by Girard in his thesis [6] and published in [5].

In [3] Crossley and Shepherdson gave a proof of strong normalization for first-order logic that provides a more user-friendly calculus from which to derive computer programs. They use Girard's *candidates for reducibility* (see [7]).

In carrying out mathematical proofs the same patterns frequently recur. Therefore in extracting programs from proofs it would be helpful to characterize what a pattern or template is. We will integrate templates into the system so that we do not have to repeat what are essentially the same arguments. Moreover, this mirrors ordinary mathematical practice.

In this thesis, we introduce two kinds of templates namely *induction templates* and *abbreviation templates*.

The idea of induction templates comes from the induction used in ordinary mathematical proofs. Adding induction on natural numbers or lists can be found in [1], [3], [9], and [11]. The new induction templates are more general than those inductions because they can be used on natural numbers, lists, and other inductively defined predicates.

In ordinary mathematical proofs, we often abbreviate formulae by predicates. We will introduce abbreviation templates for this purpose. Analogous templates can be found as meta-rules in [9] and [11]. Now we will add them as formal rules. It is essential that the systems to which the templates are added must be higher order. Therefore, in this thesis, we do program extraction from proofs for higher order systems of logic, specifically systems of full second-order predicate logic.

The new Curry-Howard terms produced in the systems to which templates are added will still satisfy all the basic properties including the *strong normalization theorem*.

The thesis is arranged as follows.

In Chapter II we introduce our second-order language and define substitution for second-order formulae. We prove some lemmas concerning substitutions that establish basic properties and will be used in the following chapters.

Chapter III is separated into three sections. In the first section, we start with the second-order natural deduction system, NJ_2, extended from the first-order natural deduction system, NJ. In the second section, we define associated Curry-Howard terms. We then give definitions of substitutions, legitimate changes of bound variables, and equivalence of the Curry-Howard terms together with some lemmas concerning them. We give reduction rules for the new Curry-Howard terms as well as some basic lemmas in the last section.

Chapter IV discusses strong normalization. It has two sections. We give some basic definitions in the first section and a proof of the strong normalization theorem in the second section.

Chapter V is about templates. We introduce two kinds of templates namely induction templates and abbreviation templates as mentioned above.

Chapter VI summarizes the results of our work, and suggests possibilities for further research.

CHAPTER II

SECOND-ORDER LANGUAGE AND SUBSTITUTIONS

In this chapter we set up the language and establish basic lemmas for substitutions. We follow the approach of [3].

We define L_2 to be a second-order language extended from a first order language L as follows.

We take the basic symbols of L as $\wedge, \vee, \supset, \perp, \forall, \exists, =, (,)$, and an infinite sequence of variables $x, y, z, x_1, \ldots$, called *individual variables*. We also call terms of L *individual terms*.

We define a new class of *predicate variables* of all arities. For each arity n, we use $P^n, Q^n, R^n, P_1^n, \ldots$ to denote n-ary predicate variables. The superscript n may be omitted if we do not want to state the arity or it is clear in the context of which arity it is. We also add two new quantifier symbols $\forall_2$ and $\exists_2$.

Definition 2.1. *An* ***atomic formula*** *of L_2 is either an atomic formula of L or an expression of the form $P(t_1, \ldots, t_n)$ where P is an n-ary predicate variable and $t_1, \ldots, t_n$ are individual terms.*

Definition 2.2. *The* ***formulae*** *of L_2 form the smallest set of expressions containing the atomic formulae and $\perp$, closed under the following formation rules.*

i. If α and β are formulae so are the expressions $(\alpha \wedge \beta), (\alpha \vee \beta)$, and $(\alpha \supset \beta)$.

ii. If α is a formula and x is an individual variable, then $(\forall x \alpha)$ and $(\exists x \alpha)$ are formulae.

iii. If α is a formula and P is a predicate variable, then $(\forall_2 P \alpha)$ and $(\exists_2 P \alpha)$

are formulae.

Notes.

a. Parentheses will be omitted when there is no ambiguity.

b. An occurrence of an individual variable x (respectively a predicate variable P) in a formula α is *bound* if it occurs in a subformula of α of the form $\forall x\beta$ or $\exists x\beta$ (respectively $\forall_2 P\beta$ or $\exists_2 P\beta$), otherwise it is *free*. We call β the *scope* of the quantifier.

c. From now on when we say "α is a formula"or "α is a second-order formula"we mean "α is a formula of L_2 for some extension L_2"unless otherwise stated.

d. When we say "x is the first individual variable ... "we mean "x is the first individual variable in some fixed ordering of individual variables ... ". Similarly for predicate variables of each arity.

e. When we give a definition or a proof that proceeds by induction on the construction of a formula α, we will omit the case α is $\perp$ whenever it is similar to the case where α is an atomic formula.

f. When we say "induction on a formula α"we mean "induction on the construction of a formula α".

g. Sometimes we use $x', y', z', x'', \ldots$ or $x^*, y^*, z^*, x^{**}, \ldots$ to denote individual variables. Similarly for predicate variables and any variables in this thesis.

Notation. We use

a. $fv(\alpha)$ (respectively $FV(\alpha)$) to denote the set of free individual variables (respectively the set of free predicate variables) of a formula α; similarly for $fv(t)$ where t is an individual term;

b. $\{\underline{x}\}$ to denote the set $\{x_1, \ldots, x_n\}$, where $\underline{x} = x_1, \ldots, x_n$ are individual variables; similarly for $\{\underline{P}\}$, where $\underline{P}$ is a list of predicate variables;

c. $fv(\underline{t})$ to denote $\bigcup_{i=1}^{n} fv(t_i)$, where $\underline{t} = t_1, \ldots, t_n$ are individual terms;

d. $\alpha[\underline{x}/_r\underline{t}]$ (respectively $u[\underline{x}/\underline{t}]$) to denote the simple simultaneous replacements of all free occurrences of (distinct) individual variables $\underline{x} = x_1, \ldots, x_n$ in a formula α (respectively an individual term u) by individual terms $\underline{t} = t_1, \ldots, t_n$, respectively;

e. $\alpha[\underline{P}/_r\underline{R}]$ to denote the simple simultaneous replacements of all free occurrences of (distinct) predicate variables $\underline{P} = P_1^{m_1}, \ldots, P_n^{m_n}$ in a formula α by predicate variables $\underline{R} = R_1^{m_1}, \ldots, R_n^{m_n}$, respectively.

Note. When we write "$\underline{a}^*$ is the sublist of $\underline{a}$ ... ", $\underline{a}^*$ could be empty and every definition and notation used for $\underline{a}$ can also be used with $\underline{a}^*$ in a natural way e.g. $fv(\underline{t}^*) = \varnothing$ if $\underline{t}^*$ is the empty sublist of a list of individual terms $\underline{t}$.

Definition 2.3. *Let α be a formula, $\underline{x} = x_1, \ldots, x_n$ be distinct individual variables, and $\underline{t} = t_1, \ldots, t_n$ be individual terms. The result of simultaneously substituting $t_1, \ldots, t_n$ for all free occurrences of $x_1, \ldots, x_n$, respectively, in α, denoted by $\alpha[x_1/t_1, \ldots, x_n/t_n]$ or $\alpha[\underline{x}/\underline{t}]$, is defined inductively as follows.*

i. If α is an atomic formula, then $\alpha[\underline{x}/\underline{t}] = \alpha[\underline{x}/_r\underline{t}]$.

ii. $(\beta \wedge \gamma)[\underline{x}/\underline{t}] = \beta[\underline{x}/\underline{t}] \wedge \gamma[\underline{x}/\underline{t}]$.

Similarly for $(\beta \vee \gamma)[\underline{x}/\underline{t}]$ and $(\beta \supset \gamma)[\underline{x}/\underline{t}]$.

iii. $(\forall y\beta)[\underline{x}/\underline{t}] = \forall y'(\beta[y/y'][\underline{x}^*/\underline{t}^*])$,

where $\underline{x}^*$ is the sublist of $\underline{x}$ consisting of those x_i's which are in $fv(\forall y\beta)$, $\underline{t}^*$ is the corresponding sublist of $\underline{t}$, and y' is y if $y \notin fv(\underline{t}^*)$, otherwise y' is the first individual variable which is not in $fv(\beta) \cup fv(\underline{t}^*)$.

Similarly for $(\exists y\beta)[\underline{x}/\underline{t}]$.

iv. $(\forall_2 P\beta)[\underline{x}/\underline{t}] = \forall_2 P(\beta[\underline{x}/\underline{t}])$.

Similarly for $(\exists_2 P\beta)[\underline{x}/\underline{t}]$.

Note. From the above definition, it can be easily proved by induction on α that

a. $\alpha[\underline{x}/\underline{t}]$ is a formula and is of the same form as α;

b. $\alpha[\underline{x}/\underline{t}] = \alpha[\underline{x}^*/\underline{t}^*]$, where $\underline{x}^*$ is the sublist of $\underline{x}$ consisting of those x_i's which are in $fv(\alpha)$ and $\underline{t}^*$ is the corresponding sublist of $\underline{t}$;

c. $\alpha[\underline{x}/\underline{x}] = \alpha$;

d. if $\underline{x}^*$ is the sublist of $\underline{x}$ consisting of those variables which are in $fv(\alpha)$ and $\underline{t}^*$ is the corresponding sublist of $\underline{t}$, then $fv(\alpha[\underline{x}/\underline{t}]) = (fv(\alpha) - \{\underline{x}^*\}) \cup fv(\underline{t}^*)$;

e. $FV(\alpha[\underline{x}/\underline{t}]) = FV(\alpha)$.

Notation. $\alpha[\underline{x}/\underline{t}, \underline{y}/\underline{u}]$ will abbreviate $\alpha[x_1/t_1, \ldots, x_n/t_n, y_1/u_1, \ldots, y_m/u_m]$, where $\underline{x} = x_1, \ldots, x_n$, $\underline{t} = t_1, \ldots, t_n$, $\underline{y} = y_1, \ldots, y_m$, and $\underline{u} = u_1, \ldots, u_m$.

Following Takeuti [17], we extend our language and define *abstraction terms.*

Definition 2.4. *If α is a formula with $fv(\alpha) = \{x_1, \ldots, x_n\}$, then $\lambda x_1, \ldots, x_n\alpha$ is called an* ***abstraction term****.*

Note. All occurrences of $x_i, 1 \leq i \leq n$, in $\lambda x_1, \ldots, x_n\alpha$ are bound, so every abstraction term contains no free individual variable.

Notation.

a. The set of free predicate variables of an abstraction term $T = \lambda x_1, \ldots, x_n\alpha$, denoted by $FV(T)$, is the set $FV(\alpha)$.

b. We use $FV(\underline{T})$ to denote $\bigcup_{i=1}^{m} FV(T_i)$, where $\underline{T} = T_1, \ldots, T_m$ are abstraction terms.

c. We may write an abstraction term of the form $\lambda x_1, \ldots, x_n R^n(x_1, \ldots, x_n)$ as R^n when there is no ambiguity.

Definition 2.5. *Let α be a formula, $\underline{P} = P_1^{n_1}, \ldots, P_m^{n_m}$ be distinct predicate variables, and $\underline{T} = T_1, \ldots, T_m$, where $T_i = \lambda x_1^i, \ldots, x_{n_i}^i\delta_i$, $1 \leq i \leq m$, be abstraction terms. We define $\alpha[P_1/T_1, \ldots, P_m/T_m]$, which can be written as $\alpha[\underline{P}/\underline{T}]$, inductively as follows.*

i. If α is an atomic formula, then

$$\alpha[\underline{P}/\underline{T}] = \begin{cases} \delta_q[x_1^q/t_1, \ldots, x_{n_q}^q/t_{n_q}] & \text{if } \alpha = P_q(t_1, \ldots, t_{n_q}) \text{ for some } 1 \leq q \leq m \\ & \text{and some individual terms } t_1, \ldots, t_{n_q}, \\ \alpha & \text{otherwise.} \end{cases}$$

ii. $(\beta \wedge \gamma)[\underline{P}/\underline{T}] = \beta[\underline{P}/\underline{T}] \wedge \gamma[\underline{P}/\underline{T}]$.

Similarly for $(\beta \vee \gamma)[\underline{P}/\underline{T}]$ *and* $(\beta \supset \gamma)[\underline{P}/\underline{T}]$.

iii. $(\forall x\beta)[\underline{P}/\underline{T}] = \forall x(\beta[\underline{P}/\underline{T}])$.

Similarly for $(\exists x\beta)[\underline{P}/\underline{T}]$.

iv. $(\forall_2 Q\beta)[\underline{P}/\underline{T}] = \forall_2 Q'(\beta[Q/Q'][\underline{P}^*/\underline{T}^*])$,

where $\underline{P}^*$ *is the sublist of* $\underline{P}$ *consisting of those* P_i*'s which are in* $FV(\forall_2 Q\beta)$, $\underline{T}^*$ *is the corresponding sublist of* $\underline{T}$, *and* Q' *is* Q *if* $Q \notin FV(\underline{T}^*)$, *otherwise* Q' *is the first predicate variable with the same arity as* Q *which is not in* $FV(\beta) \cup FV(\underline{T}^*)$.

Similarly for $(\exists_2 Q\beta)[\underline{P}/\underline{T}]$.

Notation.

a. If $U = \lambda y_1, \ldots, y_k \gamma$ is an abstraction term, we use $U[\underline{P}/\underline{T}]$ to denote $\lambda y_1, \ldots, y_k(\gamma[\underline{P}/\underline{T}])$.

b. $\forall y\beta[\underline{x}/\underline{t}]$ will abbreviate $\forall y(\beta[\underline{x}/\underline{t}])$. Similarly for $\forall y\beta[\underline{P}/\underline{T}]$.

$\forall y$ in the above statement can also be replaced by $\exists y$, $\forall_2 Q$, or $\exists_2 Q$.

Note. From the above definition, it can be easily proved by induction on α that

a. $\alpha[\underline{P}/\underline{T}]$ is a formula;

b. $\alpha[\underline{P}/\underline{T}] = \alpha[\underline{P}^*/\underline{T}^*]$, where $\underline{P}^*$ is the sublist of $\underline{P}$ consisting of those P_i's which are in $FV(\alpha)$ and $\underline{T}^*$ is the corresponding sublist of $\underline{T}$;

c. $\alpha[\underline{P}/\underline{P}] = \alpha$;

d. if $\underline{P}^*$ is the sublist of $\underline{P}$ consisting of those variables which are in $FV(\alpha)$ and $\underline{T}^*$ is the corresponding sublist of $\underline{T}$, then $FV(\alpha[\underline{P}/\underline{T}]) = (FV(\alpha) - \{\underline{P}^*\}) \cup$

$FV(\underline{T}^*)$;

e. $fv(\alpha[\underline{P}/\underline{T}]) = fv(\alpha)$.

Definition 2.6. *Let α be a formula.*

*If $\underline{x} = x_1, \ldots, x_n$ are distinct individual variables, and $\underline{t} = t_1, \ldots, t_n$ are individual terms, we say $\underline{t}$ is **free** for $\underline{x}$ in α if no free occurrence of any x_i, $1 \leq i \leq n$, in α is within the scope of a quantifier $\forall y$ or $\exists y$ where y occurs in t_i.*

*If $\underline{P} = P_1^{n_1}, \ldots, P_m^{n_m}$ are distinct predicate variables, and $\underline{T} = T_1, \ldots, T_m$, where $T_i = \lambda x_1^i, \ldots, x_{n_i}^i \delta_i$, $1 \leq i \leq m$, are abstraction terms, we say $\underline{T}$ is **free** for $\underline{P}$ in α if no free occurrence of P_i, $1 \leq i \leq n$, in α is within the scope of a quantifier $\forall_2 Q$ or $\exists_2 Q$ where Q occurs free in T_i.*

Definition 2.7. *Suppose $\forall x\beta$ or $\exists x\beta$ (respectively $\forall_2 P\beta$ or $\exists_2 P\beta$) is a subformula of a formula α. A change of an occurrence of $\forall x\beta$ to $\forall x'\beta[x/_r x']$ or $\exists x\beta$ to $\exists x'\beta[x/_r x']$ (respectively $\forall_2 P\beta$ to $\forall_2 P'\beta[P/_r P']$ or $\exists_2 P\beta$ to $\exists_2 P'\beta[P/_r P']$, where P and P' are of the same arity) in α is called **legitimate** if x' (respectively P') does not occur free in β and x' is free for x (respectively P' is free for P) in β.*

Definition 2.8. *If a formula α' can be obtained from a formula α by a finite sequence of legitimate changes of bound individual variables or bound predicate variables, we say α is **equivalent** to α', and write $\alpha \equiv \alpha'$.*

Note. It can be proved by induction on α that

a. if $\alpha \equiv \alpha'$, then $fv(\alpha) = fv(\alpha')$ and $FV(\alpha) = FV(\alpha')$;

b. if $\underline{t}$ is free for $\underline{x}$ (respectively $\underline{R}$ is free for $\underline{P}$) in α, then $\alpha[\underline{x}/\underline{t}] = \alpha[\underline{x}/_r\underline{t}]$ (respectively $\alpha[\underline{P}/\underline{R}] = \alpha[\underline{P}/_r\underline{R}]$).

Lemma 2.9. $\equiv$ *is an equivalence relation.*

Proof. It is clear that $\equiv$ is reflexive and transitive. To prove that $\equiv$ is symmetric, it is enough to show this for a single change of bound variable. Suppose an occurrence of $\forall x\beta$ in a formula α is replaced by $\forall x'\beta[x/_r x']$, where x' is free for x and does not occur free in β, and the result is α'. Since $x \notin fv(\beta[x/_r x'])$ and x is free for x' in $\beta[x/_r x']$, the change from $\forall x'\beta[x/_r x']$ to $\forall x\beta[x/_r x'][x'/_r x]$ which is $\forall x\beta$ is also legitimate. Thus α can be obtained from α' by a legitimate change of bound variable. Similarly, if the replaced subformula is of the form $\exists x\beta$, $\forall_2 P\beta$, or $\exists_2 P\beta$. □

Notation. We use $[\alpha]$ to denote the *equivalence class* of a formula α.

Note. When we prove by induction on α and the proofs for the cases $\alpha = \forall y\beta$ and $\alpha = \exists y\beta$ are similar, we will prove only the case $\alpha = \forall y\beta$ and omit the case $\alpha = \exists y\beta$. Similarly for the cases $\alpha = \forall_2 Q\beta$ and $\alpha = \exists_2 Q\beta$.

Lemma 2.10. *For any formula α,*

a. if $\underline{x} = x_1, \ldots, x_n$ are distinct individual variables and $\underline{t} = t_1, \ldots, t_n$ are individual terms, then $\alpha[\underline{x}/\underline{t}] = \alpha'[\underline{x}/\underline{t}]$ for some formula α' such that $\alpha' \equiv \alpha$ and $\underline{t}$ is free for $\underline{x}$ in α';

b. if $\underline{P} = P_1^{n_1}, \ldots, P_m^{n_m}$ are distinct predicate variables and $\underline{T} = T_1, \ldots, T_m$, where $T_i = \lambda x_1^i, \ldots, x_{n_i}^i \delta_i$, $1 \le i \le m$, are abstraction terms, then $\alpha[\underline{P}/\underline{T}] = \alpha'[\underline{P}/\underline{T}]$ for some formula α' such that $\alpha' \equiv \alpha$ and $\underline{T}$ is free for $\underline{P}$ in α'.

Proof. Let α be a formula. We will prove this by induction on α.

a: Let $\underline{x} = x_1, \ldots, x_n$ be distinct individual variables and $\underline{t} = t_1, \ldots, t_n$ be individual terms.

If α is an atomic formula, then $\underline{t}$ is free for $\underline{x}$ in α. The cases where α is $\beta \wedge \gamma$, $\beta \vee \gamma$, $\beta \supset \gamma$, $\forall_2 Q\beta$, or $\exists_2 Q\beta$ follow straightforwardly by the induction hypothesis.

The remaining cases are $\alpha = \forall y\beta$ and $\alpha = \exists y\beta$ for which the proofs are similar.

$\alpha = \forall y\beta$:

Then $\alpha[\underline{x}/\underline{t}] = \forall y'\beta[y/y'][\underline{x}^*/\underline{t}^*]$, where $\underline{x}^*$ is the sublist of $\underline{x}$ consisting of those x_i's which are in $fv(\alpha)$, $\underline{t}^*$ is the corresponding sublist of $\underline{t}$, and y' is y if $y \notin fv(\underline{t}^*)$, otherwise y' is the first individual variable which is not in $fv(\beta) \cup fv(\underline{t}^*)$.

By the induction hypothesis, $\beta[y/y'] = \beta'[y/y']$ and $\beta'[y/y'][\underline{x}^*/\underline{t}^*] = \beta^*[\underline{x}^*/\underline{t}^*]$ for some formulae β' and β^* such that $\beta' \equiv \beta$, $\beta^* \equiv \beta'[y/y']$, y' is free for y in β', and $\underline{t}^*$ is free for $\underline{x}^*$ in β^*. Hence $\alpha[\underline{x}/\underline{t}] = \forall y'\beta^*[\underline{x}^*/\underline{t}^*] = (\forall y'\beta^*)[\underline{x}/\underline{t}]$ and $\forall y'\beta^* \equiv \forall y'\beta'[y/y'] \equiv \forall y\beta' \equiv \forall y\beta = \alpha$. Since $\underline{t}^*$ is free for $\underline{x}^*$ in β^* and $y' \notin fv(\underline{t}^*)$, $\underline{t}$ is free for $\underline{x}$ in $\forall y'\beta^*$.

b: The proof is similar to (a). □

Corollary 2.11. *For any formula β,*

a. if y and y' are individual variables such that $y' \notin fv(\beta)$, then $\forall y\beta \equiv \forall y'\beta[y/y']$ and $\exists y\beta \equiv \exists y'\beta[y/y']$;

b. if P and P' are predicate variables with the same arity and $P' \notin FV(\beta)$, then $\forall_2 P\beta \equiv \forall_2 P'\beta[P/P']$ and $\exists_2 P\beta \equiv \exists_2 P'\beta[P/P']$.

Proof. Let β be a formula.

a: Let y and y' be individual variables such that $y' \notin fv(\beta)$. By the above lemma, $\beta[y/y'] = \beta'[y/y']$ for some formula β' such that $\beta' \equiv \beta$ (so $y' \notin fv(\beta')$) and y' is free for y in β'. Hence $\forall y\beta \equiv \forall y\beta' \equiv \forall y'\beta'[y/_r y'] = \forall y'\beta'[y/y'] = \forall y'\beta[y/y']$. Similarly, $\exists y\beta \equiv \exists y'\beta[y/y']$.

b: The proof is similar to (a). □

Lemma 2.12. *Let α be a formula, $\underline{x} = x_1, \ldots, x_n$ be distinct individual variables, $\underline{t} = t_1, \ldots, t_n$ be individual terms, $\underline{P} = P_1^{r_1}, \ldots, P_m^{r_m}$ be distinct predicate variables, and $\underline{R} = R_1^{r_1}, \ldots, R_m^{r_m}$ be predicate variables.*

Then $\alpha[\underline{P}/\underline{R}][\underline{x}/\underline{t}] = \alpha[\underline{x}/\underline{t}][\underline{P}/\underline{R}]$.

Proof. We will prove this by induction on α. The cases where α is $\beta \wedge \gamma$, $\beta \vee \gamma$, or $\beta \supset \gamma$ follow straightforwardly by the induction hypothesis.

(i) α is an atomic formula.

If $\alpha = P_q(u_1, \ldots, u_{r_q})$ for some $1 \leq q \leq m$ and some individual terms $u_1, \ldots, u_{r_q}$, then

$$\begin{aligned}
\alpha[\underline{P}/\underline{R}][\underline{x}/\underline{t}] &= P_q(u_1, \ldots, u_{r_q})[\underline{P}/\underline{R}][\underline{x}/\underline{t}] \\
&= R_q(u_1, \ldots, u_{r_q})[\underline{x}/\underline{t}] \\
&= R_q(u_1[\underline{x}/\underline{t}], \ldots, u_{r_q}[\underline{x}/\underline{t}]) \\
&= P_q(u_1[\underline{x}/\underline{t}], \ldots, u_{r_q}[\underline{x}/\underline{t}])[\underline{P}/\underline{R}] \\
&= P_q(u_1, \ldots, u_{r_q})[\underline{x}/\underline{t}][\underline{P}/\underline{R}] = \alpha[\underline{x}/\underline{t}][\underline{P}/\underline{R}],
\end{aligned}$$

otherwise $\alpha[\underline{P}/\underline{R}][\underline{x}/\underline{t}] = \alpha[\underline{x}/\underline{t}] = \alpha[\underline{x}/\underline{t}][\underline{P}/\underline{R}]$.

(ii) $\alpha = \forall y\beta$.

By the induction hypothesis, we have

$$\begin{aligned}
\alpha[\underline{P}/\underline{R}][\underline{x}/\underline{t}] &= (\forall y\beta[\underline{P}/\underline{R}])[\underline{x}/\underline{t}] \\
&= \forall y'\beta[\underline{P}/\underline{R}][y/y'][\underline{x}^*/\underline{t}^*] \\
&= \forall y'\beta[y/y'][\underline{P}/\underline{R}][\underline{x}^*/\underline{t}^*] \\
&= \forall y'\beta[y/y'][\underline{x}^*/\underline{t}^*][\underline{P}/\underline{R}] \\
&= (\forall y'\beta[y/y'][\underline{x}^*/\underline{t}^*])[\underline{P}/\underline{R}] \\
&= (\forall y\beta)[\underline{x}/\underline{t}][\underline{P}/\underline{R}] = \alpha[\underline{x}/\underline{t}][\underline{P}/\underline{R}],
\end{aligned}$$

where $\underline{x}^*$ is the sublist of $\underline{x}$ consisting of those x_i's which are in $fv(\alpha)$ ($= fv(\alpha[\underline{P}/\underline{R}])$), $\underline{t}^*$ is the corresponding sublist of $\underline{t}$, and y' is y if $y \notin fv(\underline{t}^*)$, otherwise y' is the first individual variable which is not in $fv(\beta) \cup fv(\underline{t}^*)$ ($= fv(\beta[\underline{P}/\underline{R}]) \cup fv(\underline{t}^*)$).

(iii) $\alpha = \forall_2 Q\beta$.

By the induction hypothesis, we have

$$\begin{aligned}\alpha[\underline{P}/\underline{R}][\underline{x}/\underline{t}] &= (\forall_2 Q'\beta[Q/Q'][\underline{P}^*/\underline{R}^*])[\underline{x}/\underline{t}] \\ &= \forall_2 Q'\beta[Q/Q'][\underline{P}^*/\underline{R}^*][\underline{x}/\underline{t}] \\ &= \forall_2 Q'\beta[Q/Q'][\underline{x}/\underline{t}][\underline{P}^*/\underline{R}^*] \\ &= \forall_2 Q'\beta[\underline{x}/\underline{t}][Q/Q'][\underline{P}^*/\underline{R}^*] \\ &= (\forall_2 Q\beta[\underline{x}/\underline{t}])[\underline{P}/\underline{R}] \\ &= (\forall_2 Q\beta)[\underline{x}/\underline{t}][\underline{P}/\underline{R}] = \alpha[\underline{x}/\underline{t}][\underline{P}/\underline{R}],\end{aligned}$$

where $\underline{P}^*$ is the sublist of $\underline{P}$ consisting of those P_i's which are in $FV(\alpha)$ ($= FV(\alpha[\underline{x}/\underline{t}])$), $\underline{R}^*$ is the corresponding sublist of $\underline{R}$, and Q' is Q if $Q \notin \{\underline{R}^*\}$, otherwise Q' is the first predicate variable with the same arity as Q which is not in $FV(\beta) \cup \{\underline{R}^*\}$ ($= FV(\beta[\underline{x}/\underline{t}]) \cup \{\underline{R}^*\}$). □

Lemma 2.13. *Let α be a formula, $\underline{x} = x_1, \ldots, x_m$ and $\underline{y} = y_1, \ldots, y_n$ be sequences of distinct individual variables, and $\underline{t} = t_1, \ldots, t_m$ and $\underline{u} = u_1, \ldots, u_n$ be individual terms.*

Then $\alpha[\underline{x}/\underline{t}][\underline{y}/\underline{u}] \equiv \alpha[x_1/t_1[\underline{y}/\underline{u}], \ldots, x_m/t_m[\underline{y}/\underline{u}], y_{i_1}/u_{i_1}, \ldots, y_{i_k}/u_{i_k}]$, where $y_{i_1}, \ldots, y_{i_k}$ is the sublist of $\underline{y}$ consisting of those y_j's which are in $fv(\alpha) - \{\underline{x}\}$.

Lemma 2.14. *Let α and α' be formulae, $\underline{x} = x_1, \ldots, x_n$ be distinct individual variables, and $\underline{t} = t_1, \ldots, t_n$ be individual terms.*

If $\alpha \equiv \alpha'$, then $\alpha[\underline{x}/\underline{t}] \equiv \alpha'[\underline{x}/\underline{t}]$.

Proof. We will prove these two lemmas simultaneously by induction on α.

Proof of Lemma 2.13. The cases where α is $\beta \wedge \gamma$, $\beta \vee \gamma$, $\beta \supset \gamma$, $\forall_2 Q\beta$, or $\exists_2 Q\beta$ follow straightforwardly by the induction hypothesis. The remaining cases are as follows.

(i) α is an atomic formula.

This is clear from the definition since all substitutions are simple. For this case we obtain the lemma with $\equiv$ replaced by $=$.

(ii) $\alpha = \forall z\beta$.

Suppose $\underline{x}^* = x_{j_1}, \ldots, x_{j_l}$ and $\underline{y}^*$ are the sublists of $\underline{x}$ and $\underline{y}$, respectively, consisting of those variables which are in $fv(\alpha)$ and $fv(\alpha[\underline{x}/\underline{t}])$, respectively, and $\underline{t}^*$ and $\underline{u}^*$ are the corresponding sublists of $\underline{t}$ and $\underline{u}$, respectively.

We have $(\forall z\beta)[\underline{x}/\underline{t}][\underline{y}/\underline{u}] = \forall z''\beta[z/z'][\underline{x}^*/\underline{t}^*][z'/z''][\underline{y}^*/\underline{u}^*]$, where z' is z (respectively z'' is z') if $z \notin fv(\underline{t}^*)$ (respectively $z' \notin fv(\underline{u}^*)$), otherwise z' (respectively z'') is the first individual variable which is not in $fv(\beta) \cup fv(\underline{t}^*)$ (respectively $fv(\beta[z/z'][\underline{x}^*/\underline{t}^*]) \cup fv(\underline{u}^*)$).

We have $(\forall z\beta)[x_1/t_1[\underline{y}/\underline{u}], \ldots, x_n/t_n[\underline{y}/\underline{u}], y_{i_1}/u_{i_1}, \ldots, y_{i_k}/u_{i_k}] =$ $\forall z'''\beta[z/z'''][x_{j_1}/t_{j_1}[\underline{y}/\underline{u}], \ldots, x_{j_l}/t_{j_l}[\underline{y}/\underline{u}], y_{i_1}/u_{i_1}, \ldots, y_{i_k}/u_{i_k}]$, where z''' is z if $z \notin \bigcup_{r=1}^{l} fv(t_{j_r}[\underline{y}/\underline{u}]) \cup \bigcup_{s=1}^{k} fv(u_{i_s})$, otherwise z''' is the first individual variable which is not in $fv(\beta) \cup \bigcup_{r=1}^{l} fv(t_{j_r}[\underline{y}/\underline{u}]) \cup \bigcup_{s=1}^{k} fv(u_{i_s})$.

Let z^* be an individual variable which does not occur in $\underline{x}^*$, $\underline{y}^*$,
$\beta[z/z'''][x_{j_1}/t_{j_1}[\underline{y}/\underline{u}], \ldots, x_{j_l}/t_{j_l}[\underline{y}/\underline{u}], y_{i_1}/u_{i_1}, \ldots, y_{i_k}/u_{i_k}]$, or
$\beta[z/z'][\underline{x}^*/\underline{t}^*][z'/z''][\underline{y}^*/\underline{u}^*]$. Then, by the induction hypothesis, we have

$$\begin{aligned}
\alpha[\underline{x}/\underline{t}][\underline{y}/\underline{u}] &= \forall z''\beta[z/z'][\underline{x}^*/\underline{t}^*][z'/z''][\underline{y}^*/\underline{u}^*] \\
&\equiv \forall z^*\beta[z/z'][\underline{x}^*/\underline{t}^*][z'/z''][\underline{y}^*/\underline{u}^*][z''/z^*] \\
&\equiv \forall z^*\beta[z/z'][\underline{x}^*/\underline{t}^*][z'/z''][z''/z^*][\underline{y}^*/\underline{u}^*] \\
&\equiv \forall z^*\beta[z/z'][\underline{x}^*/\underline{t}^*][z'/z^*][\underline{y}^*/\underline{u}^*] \\
&\equiv \forall z^*\beta[z/z'][z'/z^*][\underline{x}^*/\underline{t}^*][\underline{y}^*/\underline{u}^*] \\
&\equiv \forall z^*\beta[z/z^*][\underline{x}^*/\underline{t}^*][\underline{y}^*/\underline{u}^*], \text{ and}
\end{aligned}$$

$\alpha[x_1/t_1[\underline{y}/\underline{u}], \ldots, x_m/t_m[\underline{y}/\underline{u}], y_{i_1}/u_{i_1}, \ldots, y_{i_k}/u_{i_k}]$

$$\begin{aligned}
&= \forall z''' \beta[z/z'''][x_{j_1}/t_{j_1}[\underline{y}/\underline{u}], \ldots, x_{j_l}/t_{j_l}[\underline{y}/\underline{u}], y_{i_1}/u_{i_1}, \ldots, y_{i_k}/u_{i_k}] \\
&\equiv \forall z^* \beta[z/z'''][x_{j_1}/t_{j_1}[\underline{y}/\underline{u}], \ldots, x_{j_l}/t_{j_l}[\underline{y}/\underline{u}], y_{i_1}/u_{i_1}, \ldots, y_{i_k}/u_{i_k}][z'''/z^*] \\
&\equiv \forall z^* \beta[z/z'''][z'''/z^*][x_{j_1}/t_{j_1}[\underline{y}/\underline{u}], \ldots, x_{j_l}/t_{j_l}[\underline{y}/\underline{u}], y_{i_1}/u_{i_1}, \ldots, y_{i_k}/u_{i_k}] \\
&\equiv \forall z^* \beta[z/z^*][x_{j_1}/t_{j_1}[\underline{y}/\underline{u}], \ldots, x_{j_l}/t_{j_l}[\underline{y}/\underline{u}], y_{i_1}/u_{i_1}, \ldots, y_{i_k}/u_{i_k}].
\end{aligned}$$

Since $z^* \notin \{\underline{y}^*\}$, $(fv(\beta[z/z^*]) - \{\underline{x}^*\}) \cap \{\underline{y}^*\} = (fv(\forall z\beta) - \{\underline{x}\}) \cap \{\underline{y}\} = \{y_{i_1}, \ldots, y_{i_k}\}$. Hence, by the induction hypothesis,

$\beta[z/z^*][\underline{x}^*/\underline{t}^*][\underline{y}^*/\underline{u}^*]$

$$\begin{aligned}
&\equiv \beta[z/z^*][x_{j_1}/t_{j_1}[\underline{y}^*/\underline{u}^*], \ldots, x_{j_l}/t_{j_l}[\underline{y}^*/\underline{u}^*], y_{i_1}/u_{i_1}, \ldots, y_{i_k}/u_{i_k}] \\
&= \beta[z/z^*][x_{j_1}/t_{j_1}[\underline{y}/\underline{u}], \ldots, x_{j_l}/t_{j_l}[\underline{y}/\underline{u}], y_{i_1}/u_{i_1}, \ldots, y_{i_k}/u_{i_k}].
\end{aligned}$$

Proof of Lemma 2.14. Suppose $\alpha \equiv \alpha'$. The cases where α is $\beta \wedge \gamma$, $\beta \vee \gamma$, or $\beta \supset \gamma$ follow straightforwardly by the induction hypothesis.

By Lemma 2.10, we may assume that $\underline{t}$ is free for $\underline{x}$ in both α and α'.

The lemma is trivial if there is no change of bound variable. We can assume there exists a sequence of formulae $\alpha = \alpha_0, \alpha_1, \ldots, \alpha_m = \alpha'$, $m \geq 1$, such that α_i is obtained from α_{i-1} by a single legitimate change of bound variable.

For the remaining cases, we proceed by induction on m. We will prove only the case $m = 1$ since the case $m > 1$ follows straightforwardly by the subsidiary induction hypothesis and the case $m = 1$.

(i) $\alpha = \forall y\beta$.

Then $\alpha[\underline{x}/\underline{t}] = \forall y\beta[\underline{x}^*/\underline{t}^*]$, where $\underline{x}^*$ is the sublist of $\underline{x}$ consisting of those x_i's which are in $fv(\alpha)$ and $\underline{t}^*$ is the corresponding sublist of $\underline{t}$.

Case 1. $\alpha' = \forall y\beta'$ where $\beta' \equiv \beta$.

By the main induction hypothesis, $\beta[\underline{x}^*/\underline{t}^*] \equiv \beta'[\underline{x}^*/\underline{t}^*]$. Hence $\alpha[\underline{x}/\underline{t}] = \forall y\beta[\underline{x}^*/\underline{t}^*] \equiv \forall y\beta'[\underline{x}^*/\underline{t}^*] = \alpha'[\underline{x}/\underline{t}]$.

Case 2. $\alpha' = \forall z\beta[y/z]$ where z is free for y and does not occur free in β.

Then $\alpha'[\underline{x}/\underline{t}] = \forall z\beta[y/z][\underline{x}^*/\underline{t}^*]$.

Let z^* be an individual variable which does not occur in $\underline{x}^*$, $\beta[\underline{x}^*/\underline{t}^*]$, or $\beta[y/z][\underline{x}^*/\underline{t}^*]$.

By the main induction hypothesis, we have

$$\begin{aligned}
\alpha'[\underline{x}/\underline{t}] &= \forall z\beta[y/z][\underline{x}^*/\underline{t}^*] \\
&\equiv \forall z^*\beta[y/z][\underline{x}^*/\underline{t}^*][z/z^*] \\
&\equiv \forall z^*\beta[y/z][z/z^*][\underline{x}^*/\underline{t}^*] \\
&\equiv \forall z^*\beta[y/z^*][\underline{x}^*/\underline{t}^*] \\
&\equiv \forall z^*\beta[\underline{x}^*/\underline{t}^*][y/z^*] \\
&\equiv \forall y\beta[\underline{x}^*/\underline{t}^*] = \alpha[\underline{x}/\underline{t}].
\end{aligned}$$

(ii) $\alpha = \forall_2 Q\beta$.

Case 1. $\alpha' = \forall_2 Q\beta'$ where $\beta' \equiv \beta$.

This case follows straightforwardly by the main induction hypothesis.

Case 2. $\alpha' = \forall_2 Q'\beta[Q/Q']$ where Q' is a predicate variable with the same arity as Q which is free for Q and does not occur free in β (similarly for $\beta[\underline{x}/\underline{t}]$).

$$\begin{aligned}
\text{Then } \alpha[\underline{x}/\underline{t}] &= \forall_2 Q\beta[\underline{x}/\underline{t}] \\
&\equiv \forall_2 Q'\beta[\underline{x}/\underline{t}][Q/Q'] \\
&= \forall_2 Q'\beta[Q/Q'][\underline{x}/\underline{t}] && \text{(by Lemma 2.12)} \\
&= (\forall_2 Q'\beta[Q/Q'])[\underline{x}/\underline{t}] = \alpha'[\underline{x}/\underline{t}].
\end{aligned}$$

□

Lemma 2.15. *Let α be a formula, $\underline{x} = x_1, \ldots, x_n$ be distinct individual variables, $\underline{t} = t_1, \ldots, t_n$ be individual terms, $\underline{P} = P_1^{r_1}, \ldots, P_m^{r_m}$ be distinct predicate variables, and $\underline{T} = T_1, \ldots, T_m$, where $T_i = \lambda y_1^i, \ldots, y_{r_i}^i \delta_i$, $1 \leq i \leq m$, be abstraction terms.*

Then $\alpha[\underline{P}/\underline{T}][\underline{x}/\underline{t}] \equiv \alpha[\underline{x}/\underline{t}][\underline{P}/\underline{T}]$.

Proof. We proceed by induction on α. The cases where α is $\beta\wedge\gamma$, $\beta\vee\gamma$, or $\beta\supset\gamma$ follow straightforwardly by the induction hypothesis.

(i) α is an atomic formula.

If $\alpha = P_q(u_1,\ldots,u_{r_q})$ for some $1 \le q \le m$ and some individual terms $u_1,\ldots,u_{r_q}$, then

$$\begin{aligned}
\alpha[\underline{P}/\underline{T}][\underline{x}/\underline{t}] &= P_q(u_1,\ldots,u_{r_q})[\underline{P}/\underline{T}][\underline{x}/\underline{t}] \\
&= \delta_q[y_1^q/u_1,\ldots,y_{r_q}^q/u_{r_q}][\underline{x}/\underline{t}] \\
&\equiv \delta_q[y_1^q/u_1[\underline{x}/\underline{t}],\ldots,y_{r_q}^q/u_{r_q}[\underline{x}/\underline{t}]] \qquad \text{(by Lemma 2.13)} \\
&= P_q(u_1[\underline{x}/\underline{t}],\ldots,u_{r_q}[\underline{x}/\underline{t}])[\underline{P}/\underline{T}] \\
&= P_q(u_1,\ldots,u_{r_q})[\underline{x}/\underline{t}][\underline{P}/\underline{T}] = \alpha[\underline{x}/\underline{t}][\underline{P}/\underline{T}],
\end{aligned}$$

otherwise $\alpha[\underline{P}/\underline{T}][\underline{x}/\underline{t}] = \alpha[\underline{x}/\underline{t}] = \alpha[\underline{x}/\underline{t}][\underline{P}/\underline{T}]$.

(ii) $\alpha = \forall y\beta$.

By the induction hypothesis, we have

$$\begin{aligned}
\alpha[\underline{P}/\underline{T}][\underline{x}/\underline{t}] &= (\forall y\beta[\underline{P}/\underline{T}])[\underline{x}/\underline{t}] \\
&= \forall y'\beta[\underline{P}/\underline{T}][y/y'][\underline{x}^*/\underline{t}^*] \\
&\equiv \forall y'\beta[y/y'][\underline{P}/\underline{T}][\underline{x}^*/\underline{t}^*] \qquad \text{(by Lemma 2.14)} \\
&\equiv \forall y'\beta[y/y'][\underline{x}^*/\underline{t}^*][\underline{P}/\underline{T}] \\
&= (\forall y'\beta[y/y'][\underline{x}^*/\underline{t}^*])[\underline{P}/\underline{T}] \\
&= (\forall y\beta)[\underline{x}/\underline{t}][\underline{P}/\underline{T}] = \alpha[\underline{x}/\underline{t}][\underline{P}/\underline{T}],
\end{aligned}$$

where $\underline{x}^*$ is the sublist of $\underline{x}$ consisting of those x_i's which are in $fv(\alpha)$ $(= fv(\alpha[\underline{P}/\underline{T}]))$, $\underline{t}^*$ is the corresponding sublist of $\underline{t}$, and y' is y if $y \notin fv(\underline{t}^*)$, otherwise y' is the first individual variable which is not in $fv(\beta)\cup fv(\underline{t}^*)$ $(= fv(\beta[\underline{P}/\underline{T}])\cup fv(\underline{t}^*))$.

(iii) $\alpha = \forall_2 Q\beta$.

By the induction hypothesis,

$$\begin{aligned}\alpha[\underline{P}/\underline{T}][\underline{x}/\underline{t}] &= (\forall_2 Q'\beta[Q/Q'][\underline{P}^*/\underline{T}^*])[\underline{x}/\underline{t}] \\ &= \forall_2 Q'\beta[Q/Q'][\underline{P}^*/\underline{T}^*][\underline{x}/\underline{t}] \\ &\equiv \forall_2 Q'\beta[Q/Q'][\underline{x}/\underline{t}][\underline{P}^*/\underline{T}^*] \\ &= \forall_2 Q'\beta[\underline{x}/\underline{t}][Q/Q'][\underline{P}^*/\underline{T}^*] \qquad \text{(by Lemma 2.12)} \\ &= (\forall_2 Q\beta[\underline{x}/\underline{t}])[\underline{P}/\underline{T}] \\ &= (\forall_2 Q\beta)[\underline{x}/\underline{t}][\underline{P}/\underline{T}] = \alpha[\underline{x}/\underline{t}][\underline{P}/\underline{T}],\end{aligned}$$

where $\underline{P}^*$ is the sublist of $\underline{P}$ consisting of those P_i's which are in $FV(\alpha)$ (= $FV(\alpha[\underline{x}/\underline{t}])$), $\underline{T}^*$ is the corresponding sublist of $\underline{T}$, and Q' is Q if $Q \notin FV(\underline{T}^*)$, otherwise Q' is the first predicate variable with the same arity as Q which is not in $FV(\beta) \cup FV(\underline{T}^*)$ (= $FV(\beta[\underline{x}/\underline{t}]) \cup FV(\underline{T}^*)$). □

Lemma 2.16. *Let α be a formula, $\underline{P} = P_1^{r_1}, \ldots, P_m^{r_m}$ and $\underline{R} = R_1^{l_1}, \ldots, R_n^{l_n}$ be sequences of distinct predicate variables, $\underline{T} = T_1, \ldots, T_m$, where $T_i = \lambda x_1^i, \ldots, x_{r_i}^i \delta_i$, $1 \le i \le m$, and $\underline{U} = U_1, \ldots, U_n$, where $U_j = \lambda y_1^j, \ldots, y_{l_j}^j \sigma_j$, $1 \le j \le n$, be abstraction terms.*

Then $\alpha[\underline{P}/\underline{T}][\underline{R}/\underline{U}] \equiv \alpha[P_1/T_1[\underline{R}/\underline{U}], \ldots, P_m/T_m[\underline{R}/\underline{U}], R_{i_1}/U_{i_1}, \ldots, R_{i_k}/U_{i_k}]$, where $R_{i_1}, \ldots, R_{i_k}$ is the sublist of $\underline{R}$ consisting of those R_j's which are in $FV(\alpha) - \{\underline{P}\}$.

Lemma 2.17. *Let α and α' be formulae, $\underline{P} = P_1^{r_1}, \ldots, P_m^{r_m}$ be distinct predicate variables, and $\underline{T} = T_1, \ldots, T_m$, where $T_j = \lambda x_1^j, \ldots, x_{r_j}^j \delta_j$, $1 \le j \le m$, be abstraction terms.*

If $\alpha \equiv \alpha'$, then $\alpha[\underline{P}/\underline{T}] \equiv \alpha'[\underline{P}/\underline{T}]$.

Proof. We will prove these two lemmas simultaneously by induction on α.

Proof of Lemma 2.16. The proof is similar to the proof of Lemma 2.13 except for the following case.

α is an atomic formula:

If $\alpha = P_q(t_1, \ldots, t_{r_q})$ for some $1 \leq q \leq m$ and some individual terms $t_1, \ldots, t_{r_q}$, then

$$\begin{aligned}
\alpha[\underline{P}/\underline{T}][\underline{R}/\underline{U}] &= P_q(t_1, \ldots, t_{r_q})[\underline{P}/\underline{T}][\underline{R}/\underline{U}] \\
&= \delta_q[x_1^q/t_1, \ldots, x_{r_q}^q/t_{r_q}][\underline{R}/\underline{U}] \\
&\equiv \delta_q[\underline{R}/\underline{U}][x_1^q/t_1, \ldots, x_{r_q}^q/t_{r_q}] \qquad \text{(by Lemma 2.15)} \\
&= P_q(t_1, \ldots, t_{r_q})[P_1/T_1[\underline{R}/\underline{U}], \ldots, P_m/T_m[\underline{R}/\underline{U}]] \\
&= \alpha[P_1/T_1[\underline{R}/\underline{U}], \ldots, P_m/T_m[\underline{R}/\underline{U}]], \text{ otherwise}
\end{aligned}$$

$$\begin{aligned}
&\alpha[\underline{P}/\underline{T}][\underline{R}/\underline{U}] \\
&= \alpha[\underline{R}/\underline{U}] \\
&= \begin{cases} \alpha[R_s/U_s] & \text{if } FV(\alpha) = \{R_s\} \text{ for some } 1 \leq s \leq n, \\ \alpha & \text{otherwise,} \end{cases} \\
&= \begin{cases} \alpha[P_1/T_1[\underline{R}/\underline{U}], \ldots, P_m/T_m[\underline{R}/\underline{U}], R_s/U_s] & \text{if } FV(\alpha) = \{R_s\} \text{ for some } \\ & 1 \leq s \leq n, \\ \alpha[P_1/T_1[\underline{R}/\underline{U}], \ldots, P_m/T_m[\underline{R}/\underline{U}]] & \text{otherwise.} \end{cases}
\end{aligned}$$

Lemma 2.17 can be proved in the same way as Lemma 2.14. □

Note. By using Lemmas 2.13, 2.14, 2.16, and 2.17, it can be proved by induction on α that

a. if $\beta \equiv \forall x\alpha$, then $\beta = \forall y\alpha'$ for some formula α' and some individual variable y such that $\alpha' \equiv \alpha[x/y]$ and $y \notin fv(\forall x\alpha)$; similarly if $\beta \equiv \exists x\alpha$;

b. if $\beta \equiv \forall_2 P\alpha$, then $\beta = \forall_2 Q\alpha'$ for some formula α' and some predicate variable Q, which is of the same arity as P, such that $\alpha' \equiv \alpha[P/Q]$ and

$Q \notin FV(\forall_2 P\alpha)$; similarly if $\beta \equiv \exists_2 P\alpha$.

Next, we will give some definitions of substitutions which allow a wider range of changed bound variables. In order to define these substitutions we proceed in two stages. The first is a simultaneous definition and is only for replacing predicate variables by predicate variables. In the second we extend this to substitutions by abstraction terms.

Definition 2.18. *Let α be a formula, $\underline{x} = x_1, \ldots, x_n$ be distinct individual variables, $\underline{t} = t_1, \ldots, t_n$ be individual terms, $\underline{P} = P_1^{r_1}, \ldots, P_m^{r_m}$ be distinct predicate variables, and $\underline{R} = R_1^{r_1}, \ldots, R_m^{r_m}$ be predicate variables.*

Part A. *We define $\alpha(x_1/t_1, \ldots, x_n/t_n)$, which can be written as $\alpha(\underline{x}/\underline{t})$, inductively as follows.*

i. If α is an atomic formula, then $\alpha(\underline{x}/\underline{t}) = \alpha[\underline{x}/_r\underline{t}]$.

ii. $(\beta \wedge \gamma)(\underline{x}/\underline{t}) = (\beta(\underline{x}/\underline{t}) \wedge \gamma(\underline{x}/\underline{t}))$.

Similarly for $(\beta \vee \gamma)(\underline{x}/\underline{t})$ and $(\beta \supset \gamma)(\underline{x}/\underline{t})$.

iii. $(\forall y\beta)(\underline{x}/\underline{t}) = \forall y'(\beta(y/y', \underline{x}^*/\underline{t}^*))$,

where $\underline{x}^*$ is the sublist of $\underline{x}$ consisting of those x_j's which are in $fv(\forall y\beta)$, $\underline{t}^*$ is the corresponding sublist of $\underline{t}$, and y' is any individual variable which is not in $(fv(\forall y\beta) - \{\underline{x}^*\}) \cup fv(\underline{t}^*)$.

Similarly for $(\exists y\beta)(\underline{x}/\underline{t})$.

iv. $(\forall_2 Q\beta)(\underline{x}/\underline{t}) = \forall_2 Q'(\beta(Q/Q')(\underline{x}/\underline{t}))$,

where Q' is any predicate variable with the same arity as Q which is not in $FV(\forall_2 Q\beta)$.

Similarly for $(\exists_2 Q\beta)(\underline{x}/\underline{t})$.

Part B. *We define $\alpha(P_1/R_1, \ldots, P_m/R_m)$, which can be written as $\alpha(\underline{P}/\underline{R})$, inductively as follows.*

i. If α is an atomic formula, then

$$\alpha(\underline{P}/\underline{R}) = \begin{cases} R_q(t_1, \ldots, t_{r_q}) & \text{if } \alpha = P_q(t_1, \ldots, t_{r_q}) \text{ for some } 1 \leq q \leq m \text{ and} \\ & \text{some individual terms } t_1, \ldots, t_{r_q}, \\ \alpha & \text{otherwise.} \end{cases}$$

ii. $(\beta \wedge \gamma)(\underline{P}/\underline{R}) = (\beta(\underline{P}/\underline{R}) \wedge \gamma(\underline{P}/\underline{R}))$.

Similarly for $(\beta \vee \gamma)(\underline{P}/\underline{R})$ *and* $(\beta \supset \gamma)(\underline{P}/\underline{R})$.

iii. $(\forall y\beta)(\underline{P}/\underline{R}) = \forall y'(\beta(y/y')(\underline{P}/\underline{R}))$,

where y' is any individual variable which is not in $fv(\forall y\beta)$.

Similarly for $(\exists y\beta)(\underline{P}/\underline{R})$.

iv. $(\forall_2 Q\beta)(\underline{P}/\underline{R}) = \forall_2 Q'(\beta(Q/Q', \underline{P}^*/\underline{R}^*))$,

where $\underline{P}^*$ is the sublist of $\underline{P}$ consisting of those P_j's which are in $FV(\forall_2 Q\beta)$, $\underline{R}^*$ is the corresponding sublist of $\underline{R}$, and Q' is any predicate variable with the same arity as Q which is not in $(FV(\forall_2 Q\beta) - \{\underline{P}^*\}) \cup \{\underline{R}^*\}$.

Similarly for $(\exists_2 Q\beta)(\underline{P}/\underline{R})$.

Definition 2.19. *Let α be a formula, $\underline{P} = P_1^{n_1}, \ldots, P_m^{n_m}$ be distinct predicate variables, and $\underline{T} = T_1, \ldots, T_m$, where $T_i = \lambda x_1^i, \ldots, x_{n_i}^i \delta_i$, $1 \leq i \leq m$, be abstraction terms. We define $\alpha(P_1/T_1, \ldots, P_m/T_m)$, which can be written as $\alpha(\underline{P}/\underline{T})$, inductively as follows.*

i. If α is an atomic formula, then

$$\alpha(\underline{P}/\underline{T}) = \begin{cases} \delta_q(x_1^q/t_1, \ldots, x_{n_q}^q/t_{n_q}) & \text{if } \alpha = P_q(t_1, \ldots, t_{n_q}) \text{ for some } 1 \leq q \leq m \\ & \text{and some individual terms } t_1, \ldots, t_{n_q}, \\ \alpha & \text{otherwise.} \end{cases}$$

ii. $(\beta \wedge \gamma)(\underline{P}/\underline{T}) = (\beta(\underline{P}/\underline{T}) \wedge \gamma(\underline{P}/\underline{T}))$.

Similarly for $(\beta \vee \gamma)(\underline{P}/\underline{T})$ *and* $(\beta \supset \gamma)(\underline{P}/\underline{T})$.

iii. $(\forall y\beta)(\underline{P}/\underline{T}) = \forall y'(\beta(y/y')(\underline{P}/\underline{T}))$,

where y' *is any individual variable which is not in* $fv(\forall y\beta)$.

Similarly for $(\exists y\beta)(\underline{P}/\underline{T})$.

iv. $(\forall_2 Q\beta)(\underline{P}/\underline{T}) = \forall_2 Q'(\beta(Q/Q', \underline{P}^*/\underline{T}^*))$,

where $\underline{P}^*$ *is the sublist of* $\underline{P}$ *consisting of those* P_j*'s which are in* $FV(\forall_2 Q\beta)$, $\underline{T}^*$ *is the corresponding sublist of* $\underline{T}$, *and* Q' *is any predicate variable with the same arity as* Q *which is not in* $(FV(\forall_2 Q\beta) - \{\underline{P}^*\}) \cup FV(\underline{T}^*)$.

Similarly for $(\exists_2 Q\beta)(\underline{P}/\underline{T})$.

Note. From the above definitions, it is easy to see that

a. $\alpha(\underline{x}/\underline{t})$ is not unique if α contains bound variables;

b. $\alpha(\underline{P}/\underline{T})$ is not unique if α contains bound variables or δ_i contains bound variables for some $1 \leq i \leq m$ where $P_i \in FV(\alpha)$.

Notation.

a. The notations and abbreviations used for the substitutions defined prviously will also be used for the substitutions in the above definitions. Also, we may write $u(\underline{x}/\underline{t})$ instead of $u[\underline{x}/\underline{t}]$ where u is an individual term.

b. When we write "$\alpha(\underline{x}/\underline{t})$" we mean "some formula which can be denoted by $\alpha(\underline{x}/\underline{t})$". Similarly for $\alpha(\underline{P}/\underline{T})$.

c. We use $\{\alpha(\underline{x}/\underline{t})\}$ to denote the set of all formulae which can be denoted by $\alpha(\underline{x}/\underline{t})$. Similarly for $\{\alpha(\underline{P}/\underline{T})\}$.

Note. From the above definitions, it can be proved by induction on α that

a. $\alpha(\underline{x}/\underline{t})$ and $\alpha(\underline{P}/\underline{T})$ are formulae;

b. if $\underline{t}$ is free for $\underline{x}$ in α, then $\alpha[\underline{x}/_r\underline{t}] \in \{\alpha(\underline{x}/\underline{t})\}$;

similarly if $\underline{R}$ is free for $\underline{P}$ in α, then $\alpha[\underline{P}/_r\underline{R}] \in \{\alpha(\underline{P}/\underline{R})\}$;

c. $FV(\alpha(\underline{x}/\underline{t})) = FV(\alpha)$ and $fv(\alpha(\underline{P}/\underline{T})) = fv(\alpha)$;

d. if $\underline{x}^*$ is the sublist of $\underline{x}$ consisting of those x_j's which are in $fv(\alpha)$ and $\underline{t}^*$ is the corresponding sublist of $\underline{t}$, then $fv(\alpha(\underline{x}/\underline{t})) = (fv(\alpha) - \{\underline{x}^*\}) \cup fv(\underline{t}^*)$;

similarly for $FV(\alpha(\underline{P}/\underline{T}))$;

e. $\alpha[\underline{x}/\underline{t}]$ is unique and it is one of the formulae in $\{\alpha(\underline{x}/\underline{t})\}$; similarly for $\alpha[\underline{P}/\underline{T}]$.

The aim of the rest of this chapter is to show that $[\alpha[\underline{x}/\underline{t}]] = [\alpha(\underline{x}/\underline{t})] = \{\alpha(\underline{x}/\underline{t})\}$ and, similarly, $[\alpha[\underline{P}/\underline{T}]] = [\alpha(\underline{P}/\underline{T})] = \{\alpha(\underline{P}/\underline{T})\}$. First we need the following lemmas.

Lemma 2.20. *Let α be a formula, $\underline{x} = x_1, \ldots, x_m$ and $\underline{y} = y_1, \ldots, y_n$ be sequences of distinct individual variables, and $\underline{t} = t_1, \ldots, t_m$ and $\underline{u} = u_1, \ldots, u_n$ be individual terms.*

Then $\{\alpha(\underline{x}/\underline{t})(\underline{y}/\underline{u})\} = \{\alpha(x_1/t_1(\underline{y}/\underline{u}), \ldots, x_m/t_m(\underline{y}/\underline{u}), y_{i_1}/u_{i_1}, \ldots, y_{i_k}/u_{i_k})\}$, where $y_{i_1}, \ldots, y_{i_k}$ is the sublist of $\underline{y}$ consisting of those y_j's which are in $fv(\alpha) - \{\underline{x}\}$.

Lemma 2.21. *Let α be a formula, $\underline{x} = x_1, \ldots, x_m$ be distinct individual variables, $\underline{t} = t_1, \ldots, t_m$ be individual terms, $\underline{P} = P_1, \ldots, P_n$ be distinct predicate variables, and $\underline{R} = R_1, \ldots, R_n$ be predicate variables such that P_i and R_i are of the same arity for all $1 \leq i \leq n$.*

Then $\{\alpha(\underline{P}/\underline{R})(\underline{x}/\underline{t})\} = \{\alpha(\underline{x}/\underline{t})(\underline{P}/\underline{R})\}$.

Lemma 2.22. *Let α be a formula, $\underline{P} = P_1, \ldots, P_m$ and $\underline{R} = R_1^{r_1}, \ldots, R_n^{r_n}$ be sequences of distinct predicate variables, $\underline{P}' = P_1', \ldots, P_m'$ be predicate variables such that P_j and P_j' are of the same arity for all $1 \leq j \leq m$, and $\underline{T} = T_1, \ldots, T_n$, where $T_j = \lambda x_1^j, \ldots, x_{r_j}^j \delta_j$, $1 \leq j \leq n$, be abstraction terms.*

Then $\{\alpha(\underline{P}/\underline{P}')(\underline{R}/\underline{T})\} = \{\alpha(P_1/U_1, \ldots, P_m/U_m, R_{i_1}/T_{i_1}, \ldots, R_{i_k}/T_{i_k})\}$, where $R_{i_1}, \ldots, R_{i_k}$ is the sublist of $\underline{R}$ consisting of those R_j's which are in $FV(\alpha)-$

$\{\underline{P}\}$ *and for all* $1 \leq j \leq m$, $U_j = \begin{cases} T_q & \textit{if } P'_j = R_q \textit{ for some } 1 \leq q \leq n, \\ P'_j & \textit{otherwise.} \end{cases}$

Proof. We will prove these three lemmas simultaneously by induction on α. The cases where α is $\beta \wedge \gamma$, $\beta \vee \gamma$, or $\beta \supset \gamma$ follow straightforwardly by the induction hypothesis.

Proof of Lemma 2.20. This is clear from the definition if α is an atomic formula, since all substitutions are simple. The remaining cases are as follows.

(i) $\alpha = \forall z\beta$.

Suppose $\underline{x}^* = x_{j_1}, \ldots, x_{j_l}$ and $\underline{y}^*$ are the sublists of $\underline{x}$ and $\underline{y}$, respectively, consisting of all those variables which are in $fv(\alpha)$ and $fv(\alpha(\underline{x}/\underline{t}))$, respectively, and $\underline{t}^*$ and $\underline{u}^*$ are the corresponding sublists of $\underline{t}$ and $\underline{u}$, respectively.

Note that $\{\underline{y}^*\} \supseteq (fv(\alpha) - \{\underline{x}\}) \cap \{\underline{y}\} = \{y_{i_1}, \ldots, y_{i_k}\}$.

Let $\alpha^* \in \{\alpha(\underline{x}/\underline{t})(\underline{y}/\underline{u})\}$. Then $\alpha^* = \forall z''\beta(z/z', \underline{x}^*/\underline{t}^*)(z'/z'', \underline{y}^*/\underline{u}^*)$, where z' and z'' are individual variables such that $z' \notin (fv(\forall z\beta) - \{\underline{x}^*\}) \cup fv(\underline{t}^*)$ and $z'' \notin (fv(\forall z'\beta(z/z', \underline{x}^*/\underline{t}^*)) - \{\underline{y}^*\}) \cup fv(\underline{u}^*)$, so $z'' \notin \bigcup_{r=1}^{l} fv(t_{j_r}(\underline{y}^*/\underline{u}^*)) \cup \bigcup_{s=1}^{k} fv(u_{i_s})$ and $z'' \notin fv(\forall z\beta) - (\{\underline{x}^*\} \cup \{y_{i_1}, \ldots, y_{i_k}\})$.

By the induction hypothesis, we have

$$\begin{aligned} \alpha^* &= \forall z''\beta(z/z', \underline{x}^*/\underline{t}^*)(z'/z'', \underline{y}^*/\underline{u}^*) \\ &= \forall z''\beta(z/z'', x_{j_1}/t_{j_1}(\underline{y}^*/\underline{u}^*), \ldots, x_{j_l}/t_{j_l}(\underline{y}^*/\underline{u}^*), y_{i_1}/u_{i_1}, \ldots, y_{i_k}/u_{i_k}) \\ &= (\forall z\beta)(x_1/t_1(\underline{y}^*/\underline{u}^*), \ldots, x_m/t_m(\underline{y}^*/\underline{u}^*), y_{i_1}/u_{i_1}, \ldots, y_{i_k}/u_{i_k}) \\ &= \alpha(x_1/t_1(\underline{y}/\underline{u}), \ldots, x_m/t_m(\underline{y}/\underline{u}), y_{i_1}/u_{i_1}, \ldots, y_{i_k}/u_{i_k}). \end{aligned}$$

Now, let $\alpha^* \in \{\alpha(x_1/t_1(\underline{y}/\underline{u}), \ldots, x_m/t_m(\underline{y}/\underline{u}), y_{i_1}/u_{i_1}, \ldots, y_{i_k}/u_{i_k})\}$. Then $\alpha^* = \forall z'\beta(z/z', x_{j_1}/t_{j_1}(\underline{y}/\underline{u}), \ldots, x_{j_l}/t_{j_l}(\underline{y}/\underline{u}), y_{i_1}/u_{i_1}, \ldots, y_{i_k}/u_{i_k})$, where z' is an individual variable such that $z' \notin (fv(\forall z\beta) - (\{\underline{x}^*\} \cup \{y_{i_1}, \ldots, y_{i_k}\})) \cup \bigcup_{r=1}^{l} fv(t_{j_r}(\underline{y}/\underline{u})) \cup \bigcup_{s=1}^{k} fv(u_{i_s})$.

Let z^* be an individual variable such that $z^* \notin (fv(\forall z\beta) - \{\underline{x}^*\}) \cup fv(\underline{t}^*)$. Since $z' \notin (fv(\forall z\beta) - \{\underline{x}^*\}) \cup \bigcup_{r=1}^{l} fv(t_{j_r}(\underline{y}/\underline{u}))$,
$z' \notin (fv(\forall z^*\beta(z/z^*, \underline{x}^*/\underline{t}^*)) - \{\underline{y}^*\}) \cup fv(\underline{u}^*)$. By the induction hypothesis,

$$\begin{aligned}
\alpha^* &= \forall z'\beta(z/z', x_{j_1}/t_{j_1}(\underline{y}/\underline{u}), \ldots, x_{j_l}/t_{j_l}(\underline{y}/\underline{u}), y_{i_1}/u_{i_1}, \ldots, y_{i_k}/u_{i_k}) \\
&= \forall z'\beta(z/z', x_{j_1}/t_{j_1}(\underline{y}^*/\underline{u}^*), \ldots, x_{j_l}/t_{j_l}(\underline{y}^*/\underline{u}^*), y_{i_1}/u_{i_1}, \ldots, y_{i_k}/u_{i_k}) \\
&= \forall z'\beta(z/z^*, \underline{x}^*/\underline{t}^*)(z^*/z', \underline{y}^*/\underline{u}^*) \\
&= (\forall z^*\beta(z/z^*, \underline{x}^*/\underline{t}^*))(\underline{y}/\underline{u}) \\
&= (\forall z\beta)(\underline{x}/\underline{t})(\underline{y}/\underline{u}) = \alpha(\underline{x}/\underline{t})(\underline{y}/\underline{u}).
\end{aligned}$$

(ii) $\alpha = \forall_2 Q\beta$.

We will prove that $\{\alpha(\underline{x}/\underline{t})(\underline{y}/\underline{u})\} \subseteq \{\alpha(x_1/t_1(\underline{y}/\underline{u}), \ldots, x_m/t_m(\underline{y}/\underline{u}))\}$ and omit the proof of the converse which easily follows by the induction hypothesis.

Let $\alpha^* \in \{\alpha(\underline{x}/\underline{t})(\underline{y}/\underline{u})\}$. Then $\alpha^* = \forall_2 Q''\beta(Q/Q')(\underline{x}/\underline{t})(Q'/Q'')(\underline{y}/\underline{u})$, where Q' and Q'' are predicate variables with the same arity as Q such that $Q' \notin FV(\forall_2 Q\beta)$ and $Q'' \notin FV(\forall_2 Q'\beta(Q/Q')(\underline{x}/\underline{t}))$, so $Q'' \notin FV(\forall_2 Q\beta)$.

By the induction hypothesis,

$$\begin{aligned}
\alpha^* &= \forall_2 Q''\beta(Q/Q')(\underline{x}/\underline{t})(Q'/Q'')(\underline{y}/\underline{u}) \\
&= \forall_2 Q''\beta(Q/Q')(Q'/Q'')(\underline{x}/\underline{t})(\underline{y}/\underline{u}) \\
&= \forall_2 Q''\beta(Q/Q'')(\underline{x}/\underline{t})(\underline{y}/\underline{u}) \\
&= \forall_2 Q''\beta(Q/Q'')(x_1/t_1(\underline{y}/\underline{u}), \ldots, x_m/t_m(\underline{y}/\underline{u}), y_{i_1}/u_{i_1}, \ldots, y_{i_k}/u_{i_k}) \\
&= (\forall_2 Q\beta)(x_1/t_1(\underline{y}/\underline{u}), \ldots, x_m/t_m(\underline{y}/\underline{u}), y_{i_1}/u_{i_1}, \ldots, y_{i_k}/u_{i_k}) \\
&= \alpha(x_1/t_1(\underline{y}/\underline{u}), \ldots, x_m/t_m(\underline{y}/\underline{u}), y_{i_1}/u_{i_1}, \ldots, y_{i_k}/u_{i_k}).
\end{aligned}$$

Proof of Lemma 2.21.

(i) α is an atomic formula.

If $\alpha = P_h(u_1, \ldots, u_g)$ for some $1 \leq h \leq n$ and some individual terms $u_1, \ldots, u_g$, where g is the arity of P_h, then

$$\begin{aligned}
\alpha(\underline{P}/\underline{R})(\underline{x}/\underline{t}) &= R_h(u_1, \ldots, u_g)(\underline{x}/\underline{t}) \\
&= R_h(u_1(\underline{x}/\underline{t}), \ldots, u_g(\underline{x}/\underline{t})) \\
&= P_h(u_1(\underline{x}/\underline{t}), \ldots, u_g(\underline{x}/\underline{t}))(\underline{P}/\underline{R}) \\
&= P_h(u_1, \ldots, u_g)(\underline{x}/\underline{t})(\underline{P}/\underline{R}) = \alpha(\underline{x}/\underline{t})(\underline{P}/\underline{R}),
\end{aligned}$$

otherwise $\alpha(\underline{P}/\underline{R})(\underline{x}/\underline{t}) = \alpha(\underline{x}/\underline{t}) = \alpha(\underline{x}/\underline{t})(\underline{P}/\underline{R})$.

For the remaining cases, we will show that $\{\alpha(\underline{P}/\underline{R})(\underline{x}/\underline{t})\} \subseteq \{\alpha(\underline{x}/\underline{t})(\underline{P}/\underline{R})\}$ and omit the proof of the converse which can be proved similarly.

Let $\alpha^* \in \{\alpha(\underline{P}/\underline{R})(\underline{x}/\underline{t})\}$.

(ii) $\alpha = \forall y\beta$.

Then $\alpha^* = \forall y''\beta(y/y')(\underline{P}/\underline{R})(y'/y'', \underline{x}^*/\underline{t}^*)$, where $\underline{x}^*$ is the sublist of $\underline{x}$ consisting of those x_i's which are in $fv(\alpha(\underline{P}/\underline{R}))$, $\underline{t}^*$ is the corresponding sublist of $\underline{t}$, and y' and y'' are individual variables such that $y' \notin fv(\forall y\beta)$ and $y'' \notin (fv(\forall y'\beta(y/y')(\underline{P}/\underline{R})) - \{\underline{x}^*\}) \cup fv(\underline{t}^*)$, so $y'' \notin (fv(\forall y\beta) - \{\underline{x}^*\}) \cup fv(\underline{t}^*)$.

By the induction hypothesis, we have

$$\begin{aligned}
\alpha^* &= \forall y''\beta(y/y')(\underline{P}/\underline{R})(y'/y'', \underline{x}^*/\underline{t}^*) \\
&= \forall y''\beta(y/y')(y'/y'', \underline{x}^*/\underline{t}^*)(\underline{P}/\underline{R}) \\
&= \forall y''\beta(y/y'', \underline{x}^*/\underline{t}^*)(\underline{P}/\underline{R}) \\
&= (\forall y''\beta(y/y'', \underline{x}^*/\underline{t}^*))(\underline{P}/\underline{R}) \\
&= (\forall y\beta)(\underline{x}/\underline{t})(\underline{P}/\underline{R}) = \alpha(\underline{x}/\underline{t})(\underline{P}/\underline{R}).
\end{aligned}$$

(iii) $\alpha = \forall_2 Q\beta$.

We have $\alpha^* = \forall_2 Q''\beta(Q/Q', \underline{P}^*/\underline{R}^*)(Q'/Q'')(\underline{x}/\underline{t})$, where $\underline{P}^*$ is the sublist of $\underline{P}$ consisting of those P_i's which are in $fv(\alpha)$, $\underline{R}^*$ is the corresponding sublist of

$\underline{R}$, and Q' and Q'' are predicate variables with the same arity as Q such that $Q' \notin (FV(\forall_2 Q\beta) - \{\underline{P}^*\}) \cup \{\underline{R}^*\}$, and $Q'' \notin FV(\forall_2 Q'\beta(Q/Q', \underline{P}^*/\underline{R}^*))$, so $Q'' \notin (FV(\forall_2 Q\beta(\underline{x}/\underline{t})) - \{\underline{P}^*\}) \cup \{\underline{R}^*\}$.

By the induction hypothesis, we have

$$\begin{aligned}
\alpha^* &= \forall_2 Q''\beta(Q/Q', \underline{P}^*/\underline{R}^*)(Q'/Q'')(\underline{x}/\underline{t}) \\
&= \forall_2 Q''\beta(Q/Q'', \underline{P}^*/\underline{R}^*)(\underline{x}/\underline{t}) \\
&= \forall_2 Q''\beta(\underline{x}/\underline{t})(Q/Q'', \underline{P}^*/\underline{R}^*) \\
&= (\forall_2 Q\beta(\underline{x}/\underline{t}))(\underline{P}/\underline{R}) \\
&= (\forall_2 Q\beta)(\underline{x}/\underline{t})(\underline{P}/\underline{R}) = \alpha(\underline{x}/\underline{t})(\underline{P}/\underline{R}).
\end{aligned}$$

Proof of Lemma 2.22.

For all $1 \leq j \leq m$, let $U_j = \begin{cases} T_q & \text{if } P'_j = R_q \text{ for some } 1 \leq q \leq n, \\ P'_j & \text{otherwise.} \end{cases}$

(i) α is an atomic formula.

Let $\alpha^* \in \{\alpha(\underline{P}/\underline{P'})(\underline{R}/\underline{T})\}$.

If $\alpha = P_h(t_1, \ldots, t_g)$ for some $1 \leq h \leq m$ and some individual terms $t_1, \ldots, t_g$, where g is the arity of P_h, then

$$\begin{aligned}
\alpha^* &= P'_h(t_1, \ldots, t_g)(\underline{R}/\underline{T}) \\
&= \begin{cases} \delta_q(x_1^q/t_1, \ldots, x_g^q/t_g) & \text{if } P'_h = R_q \text{ for some } 1 \leq q \leq n, \\ P'_h(t_1, \ldots, t_g) & \text{otherwise,} \end{cases} \\
&\in \{\alpha(P_1/U_1, \ldots, P_m/U_m)\},
\end{aligned}$$

otherwise $\alpha^* = \alpha(\underline{R}/\underline{T})$

$$= \begin{cases} \alpha(R_g/T_g) & \text{if } FV(\alpha) = \{R_g\} \text{ for some } 1 \leq g \leq n, \\ \alpha & \text{otherwise}, \end{cases}$$

$$\in \begin{cases} \{\alpha(P_1/U_1, \ldots, P_m/U_m, R_g/T_g)\} & \text{if } FV(\alpha) = \{R_g\} \text{ for some } 1 \leq g \leq n, \\ \{\alpha(P_1/U_1, \ldots, P_m/U_m)\} & \text{otherwise.} \end{cases}$$

The converse can be proved similarly.

For the remaining cases, we will show that $\{\alpha(\underline{P}/\underline{P}')(\underline{R}/\underline{T})\} \subseteq \{\alpha(P_1/U_1, \ldots, P_m/U_m, R_{i_1}/T_{i_1}, \ldots, R_{i_k}/T_{i_k})\}$ and omit the proof of the converse which can be proved similarly.

Let $\alpha^* \in \{\alpha(\underline{P}/\underline{P}')(\underline{R}/\underline{T})\}$.

(ii) $\alpha = \forall z\beta$.

Then $\alpha^* = \forall z''\beta(z/z')(\underline{P}/\underline{P}')(z'/z'')(\underline{R}/\underline{T})$, where z' and z'' are individual variables such that $z' \notin fv(\forall z\beta)$ and $z'' \notin fv(\forall z'\beta(z/z')(\underline{P}/\underline{P}'))$.

By the induction hypothesis,

$$\begin{aligned} \alpha^* &= \forall z''\beta(z/z')(\underline{P}/\underline{P}')(z'/z'')(\underline{R}/\underline{T}) \\ &= \forall z''\beta(z/z')(z'/z'')(\underline{P}/\underline{P}')(\underline{R}/\underline{T}) \\ &= \forall z''\beta(z/z'')(\underline{P}/\underline{P}')(\underline{R}/\underline{T}) \\ &= \forall z''\beta(z/z'')(P_1/U_1, \ldots, P_m/U_m, R_{i_1}/T_{i_1}, \ldots, R_{i_k}/T_{i_k}) \\ &= (\forall z\beta)(P_1/U_1, \ldots, P_m/U_m, R_{i_1}/T_{i_1}, \ldots, R_{i_k}/T_{i_k}) \\ &= \alpha(P_1/U_1, \ldots, P_m/U_m, R_{i_1}/T_{i_1}, \ldots, R_{i_k}/T_{i_k}). \end{aligned}$$

(iii) $\alpha = \forall_2 Q\beta$.

Then $\alpha^* = \forall_2 Q''\beta(Q/Q', \underline{P}^*/\underline{P}'^*)(Q'/Q'', \underline{R}^*/\underline{T}^*)$, where $\underline{P}^* = P_{j_1}, \ldots, P_{j_l}$ and $\underline{R}^*$ are the sublists of $\underline{P}$ and $\underline{R}$, respectively, consisting of those variables which are in $FV(\alpha)$ and $FV(\alpha(\underline{P}/\underline{P}'))$, respectively, $\underline{P}'^*$ and $\underline{T}^*$ are the corresponding sublists of $\underline{P}'$ and $\underline{T}$, respectively, and Q' and Q'' are predicate variables with the same arity as Q such that $Q' \notin (FV(\forall_2 Q\beta) - \{\underline{P}^*\}) \cup \{\underline{P}'^*\}$ and $Q'' \notin (FV(\forall_2 Q'\beta(Q/Q', \underline{P}^*/\underline{P}'^*)) - \{\underline{R}^*\}) \cup FV(\underline{T}^*)$.

By the induction hypothesis,

$$\begin{aligned}
\alpha^* &= \forall_2 Q''\beta(Q/Q', \underline{P}^*/\underline{P}'^*)(Q'/Q'', \underline{R}^*/\underline{T}^*) \\
&= \forall_2 Q''\beta(Q/Q'', P_{j_1}/U_{j_1}, \ldots, P_{j_l}/U_{j_l}, R_{i_1}/T_{i_1}, \ldots, R_{i_k}/T_{i_k}) \\
&= (\forall_2 Q\beta)(P_1/U_1, \ldots, P_m/U_m, R_{i_1}/T_{i_1}, \ldots, R_{i_k}/T_{i_k}) \\
&= \alpha(P_1/U_1, \ldots, P_m/U_m, R_{i_1}/T_{i_1}, \ldots, R_{i_k}/T_{i_k}).
\end{aligned}$$

□

Lemma 2.23. *Let α be a formula, $\underline{x} = x_1, \ldots, x_m$ be distinct individual variables, $\underline{t} = t_1, \ldots, t_m$ be individual terms, $\underline{P} = P_1^{r_1}, \ldots, P_n^{r_n}$ be distinct predicate variables, and $\underline{T} = T_1, \ldots, T_n$, where $T_i = \lambda y_1^i, \ldots, y_{r_i}^i \delta_i$, $1 \leq i \leq n$, be abstraction terms.*

Then $\{\alpha(\underline{P}/\underline{T})(\underline{x}/\underline{t})\} = \{\alpha(\underline{x}/\underline{t})(\underline{P}/\underline{T})\}$.

Proof. The proof is similar to the proof of Lemma 2.21 except for the following case.

$\alpha = P_q(u_1, \ldots, u_{r_q})$ for some $1 \leq q \leq n$ and some individual terms $u_1, \ldots, u_{r_q}$.

We will show that $\{\alpha(\underline{P}/\underline{T})(\underline{x}/\underline{t})\} \subseteq \{\alpha(\underline{x}/\underline{t})(\underline{P}/\underline{T})\}$ and omit the proof of the converse which can be proved similarly. Let $\alpha^* \in \{\alpha(\underline{P}/\underline{T})(\underline{x}/\underline{t})\}$.

$$\begin{aligned}
\text{Then } \alpha^* &= P_q(u_1, \ldots, u_{r_q})(\underline{P}/\underline{T})(\underline{x}/\underline{t}) \\
&= \delta_q(y_1^q/u_1, \ldots, y_{r_q}^q/u_{r_q})(\underline{x}/\underline{t}) \\
&= \delta_q(y_1^q/u_1(\underline{x}/\underline{t}), \ldots, y_{r_q}^q/u_{r_q}(\underline{x}/\underline{t})) \qquad \text{(by Lemma 2.20)} \\
&= P_q(u_1(\underline{x}/\underline{t}), \ldots, u_{r_q}(\underline{x}/\underline{t}))(\underline{P}/\underline{T}) \\
&= P_q(u_1, \ldots, u_{r_q})(\underline{x}/\underline{t})(\underline{P}/\underline{T}) = \alpha(\underline{x}/\underline{t})(\underline{P}/\underline{T}).
\end{aligned}$$

□

Lemma 2.24. *Let α be a formula, $\underline{P} = P_1^{r_1}, \ldots, P_m^{r_m}$ and $\underline{R} = R_1^{l_1}, \ldots, R_n^{l_n}$ be sequences of distinct predicate variables, and $\underline{T} = T_1, \ldots, T_m$, where $T_i = \lambda x_1^i, \ldots, x_{r_i}^i \delta_i$, $1 \leq i \leq m$, and $\underline{U} = U_1, \ldots, U_n$, where $U_j = \lambda y_1^j, \ldots, y_{l_j}^j \sigma_j$, $1 \leq j \leq m$, be abstraction terms. Then*

$\{\alpha(\underline{P}/\underline{T})(\underline{R}/\underline{U})\} = \{\alpha(P_1/T_1(\underline{R}/\underline{U}), \ldots, P_m/T_m(\underline{R}/\underline{U}), R_{i_1}/U_{i_1}, \ldots, R_{i_k}/U_{i_k})\}$, where $R_{i_1}, \ldots, R_{i_k}$ is the sublist of $\underline{R}$ consisting of those R_j's which are in $FV(\alpha) - \{\underline{P}\}$.

Proof. The proof is similar to the proof of Lemma 2.22 except for the following case.

$\alpha = P_q(t_1, \ldots, t_{r_q})$ for some $1 \leq q \leq m$ and some individual terms $t_1, \ldots, t_{r_q}$:

As usual we will show that

$\{\alpha(\underline{P}/\underline{T})(\underline{R}/\underline{U})\} \subseteq \{\alpha(P_1/T_1(\underline{R}/\underline{U}), \ldots, P_m/T_m(\underline{R}/\underline{U}))\}$ and omit the proof of the converse.

Let $\alpha^* \in \{\alpha(\underline{P}/\underline{T})(\underline{R}/\underline{U})\}$.

$$\begin{aligned}
\text{Then } \alpha^* &= P_q(t_1, \ldots, t_{r_q})(\underline{P}/\underline{T})(\underline{R}/\underline{U}) \\
&= \delta_q(x_1^q/t_1, \ldots, x_{r_q}^q/t_{r_q})(\underline{R}/\underline{U}) \\
&= \delta_q(\underline{R}/\underline{U})(x_1^q/t_1, \ldots, x_{r_q}^q/t_{r_q}) \qquad \text{(by Lemma 2.23)} \\
&= P_q(t_1, \ldots, t_{r_q})(P_q/T_q(\underline{R}/\underline{U})) \\
&= \alpha(P_1/T_1(\underline{R}/\underline{U}), \ldots, P_m/T_m(\underline{R}/\underline{U})).
\end{aligned}$$

□

Lemma 2.25. *Let α and α' be formulae, $\underline{x} = x_1, \ldots, x_m$ be distinct individual variables, $\underline{t} = t_1, \ldots, t_m$ be individual terms, $\underline{P} = P_1^{r_1}, \ldots, P_n^{r_n}$ be distinct predicate variables, and $\underline{T} = T_1, \ldots, T_n$, where $T_i = \lambda z_1^i, \ldots, z_{r_i}^i \delta_i$, $1 \leq i \leq n$, be abstraction terms.*

If $\alpha \equiv \alpha'$, then $\{\alpha(\underline{x}/\underline{t})\} = \{\alpha'(\underline{x}/\underline{t})\}$ and $\{\alpha(\underline{P}/\underline{T})\} = \{\alpha'(\underline{P}/\underline{T})\}$.

Lemma 2.26. *Let α, β, and γ be formulae.*

a. If β and γ are in $\{\alpha(\underline{x}/\underline{t})\}$, where $\underline{x} = x_1, \ldots, x_n$ are distinct individual variables and $\underline{t} = t_1, \ldots, t_n$ are individual terms, then $\beta \equiv \gamma$.

b. If β and γ are in $\{\alpha(\underline{P}/\underline{R})\}$, where $\underline{P} = P_1^{n_1}, \ldots, P_m^{n_m}$ are distinct predicate variables and $\underline{R} = R_1^{n_1}, \ldots, R_m^{n_m}$ are predicate variables, then $\beta \equiv \gamma$.

Proof. We will prove both lemmas simultaneously by induction on α. The cases where α is $\alpha_1 \wedge \alpha_2$, $\alpha_1 \vee \alpha_2$, or $\alpha_1 \supset \alpha_2$ follow straightforwardly by the induction hypothesis.

Proof of Lemma 2.25. Suppose $\alpha \equiv \alpha'$.

This is trivial if $\alpha = \alpha'$. Suppose there exists a sequence of formulae $\alpha = \alpha_0, \alpha_1, \ldots, \alpha_k = \alpha'$, $k \geq 1$, such that α_i is obtained from α_{i-1} by a single legitimate change of bound variable for all $1 \leq i \leq k$.

For the remaining cases, we proceed by induction on k. We will prove only the case $k = 1$ since the case $k > 1$ follows easily by the subsidiary induction hypothesis and the case $k = 1$.

(i) $\alpha = \forall y\beta$.

Suppose $\underline{x}^*$ is the sublist of $\underline{x}$ consisting of those x_i's which are in $fv(\alpha)$ $(= fv(\alpha'))$.

Case 1. $\alpha' = \forall y\beta'$ where $\beta' \equiv \beta$.

$\{\alpha(\underline{x}/\underline{t})\} \subseteq \{\alpha'(\underline{x}/\underline{t})\}$:

Let $\alpha^* \in \{\alpha(\underline{x}/\underline{t})\}$. Then $\alpha^* = \forall y'\beta(y/y', \underline{x}^*/\underline{t}^*)$ for some individual variable y' such that $y' \notin (fv(\forall y\beta) - \{\underline{x}^*\}) \cup fv(\underline{t}^*)$.

By the main induction hypothesis, we have

$\alpha^* = \forall y'\beta(y/y', \underline{x}^*/\underline{t}^*) = \forall y'\beta'(y/y', \underline{x}^*/\underline{t}^*) = (\forall y\beta')(\underline{x}/\underline{t}) = \alpha'(\underline{x}/\underline{t})$.

Similarly, we can prove that $\{\alpha'(\underline{x}/\underline{t})\} \subseteq \{\alpha(\underline{x}/\underline{t})\}$.

$\{\alpha(\underline{P}/\underline{T})\} \subseteq \{\alpha'(\underline{P}/\underline{T})\}$:

Let $\alpha^* \in \{\alpha(\underline{P}/\underline{T})\}$. Then $\alpha^* = \forall y'\beta(y/y')(\underline{P}/\underline{T})$ for some individual variable y' such that $y' \notin fv(\forall y\beta)$.

By the main induction hypothesis, we have

$$\begin{aligned}\alpha^* &= \forall y'\beta(y/y')(\underline{P}/\underline{T}) \\ &= \forall y'\beta'(y/y')(\underline{P}/\underline{T}) \\ &= (\forall y\beta')(\underline{P}/\underline{T}) = \alpha'(\underline{P}/\underline{T}).\end{aligned}$$

Similarly, we can prove that $\{\alpha'(\underline{P}/\underline{T})\} \subseteq \{\alpha(\underline{P}/\underline{T})\}$.

Case 2. $\alpha' = \forall z\beta[y/_r z]$, where z is an individual variable which is free for y and does not occur free in β.

$\{\alpha(\underline{x}/\underline{t})\} \subseteq \{\alpha'(\underline{x}/\underline{t})\}$:

Let $\alpha^* \in \{\alpha(\underline{x}/\underline{t})\}$. Then $\alpha^* = \forall y'\beta(y/y', \underline{x}^*/\underline{t}^*)$ for some individual variable y' such that $y' \notin (fv(\forall y\beta) - \{\underline{x}^*\}) \cup fv(\underline{t}^*)$.

Since $\beta[y/_r z] \in \{\beta(y/z)\}$, by the main induction hypothesis, for any formula β' in $\{\beta(y/z)\}$, $\beta' \equiv \beta[y/_r z]$ and so $\{\beta'(z/y', \underline{x}/\underline{t})\} = \{\beta[y/_r z](z/y', \underline{x}/\underline{t})\}$. Hence

$$\begin{aligned}\alpha^* &= \forall y'\beta(y/y', \underline{x}^*/\underline{t}^*) \\ &= \forall y'\beta(y/z)(z/y', \underline{x}^*/\underline{t}^*) \qquad \text{(by Lemma 2.20)} \\ &= \forall y'\beta[y/_r z](z/y', \underline{x}^*/\underline{t}^*) \\ &= (\forall z\beta[y/_r z])(\underline{x}/\underline{t}) = \alpha'(\underline{x}/\underline{t}).\end{aligned}$$

$\{\alpha'(\underline{x}/\underline{t})\} \subseteq \{\alpha(\underline{x}/\underline{t})\}$:

Let $\alpha^* \in \{\alpha'(\underline{x}/\underline{t})\}$. Then $\alpha^* = \forall y'\beta[y/_r z](z/y', \underline{x}^*/\underline{t}^*)$ for some individual variable y' such that $y' \notin (fv(\forall z\beta[y/_r z]) - \{\underline{x}^*\}) \cup fv(\underline{t}^*)$. Then

$$\begin{aligned} \alpha^* &= \forall y'\beta[y/_r z](z/y', \underline{x}^*/\underline{t}^*) \\ &= \forall y'\beta(y/z)(z/y', \underline{x}^*/\underline{t}^*) \\ &= \forall y'\beta(y/y', \underline{x}^*/\underline{t}^*) \qquad\qquad \text{(by Lemma 2.20)} \\ &= (\forall y\beta)(\underline{x}/\underline{t}) = \alpha(\underline{x}/\underline{t}). \end{aligned}$$

$\{\alpha(\underline{P}/\underline{T})\} \subseteq \{\alpha'(\underline{P}/\underline{T})\}$:

Let $\alpha^* \in \{\alpha(\underline{P}/\underline{T})\}$. Then $\alpha^* = \forall y'\beta(y/y')(\underline{P}/\underline{T})$ for some individual variable y' such that $y' \notin fv(\forall y\beta)$.

Similar to the above proof, by the main induction hypothesis, $\{\beta[y/_r z](z/y')(\underline{P}/\underline{T})\} = \{\beta'(z/y')(\underline{P}/\underline{T})\}$ for any formula β' in $\{\beta(y/z)\}$. Hence

$$\begin{aligned} \alpha^* &= \forall y'\beta(y/y')(\underline{P}/\underline{T}) \\ &= \forall y'\beta(y/z)(z/y')(\underline{P}/\underline{T}) \qquad\qquad \text{(by Lemma 2.20)} \\ &= \forall y'\beta[y/_r z](z/y')(\underline{P}/\underline{T}) \\ &= (\forall z\beta[y/_r z])(\underline{P}/\underline{T}) = \alpha'(\underline{P}/\underline{T}). \end{aligned}$$

$\{\alpha'(\underline{P}/\underline{T})\} \subseteq \{\alpha(\underline{P}/\underline{T})\}$:

Let $\alpha^* \in \{\alpha'(\underline{P}/\underline{T})\}$. Then $\alpha^* = \forall y'\beta[y/_r z](z/y')(\underline{P}/\underline{T})$ for some individual variable y' such that $y' \notin fv(\forall z\beta[y/_r z])$. Then

$$\begin{aligned} \alpha^* &= \forall y'\beta[y/_r z](z/y')(\underline{P}/\underline{T}) \\ &= \forall y'\beta(y/z)(z/y')(\underline{P}/\underline{T}) \\ &= \forall y'\beta(y/y')(\underline{P}/\underline{T}) \qquad\qquad \text{(by Lemma 2.20)} \\ &= (\forall y\beta)(\underline{P}/\underline{T}) = \alpha(\underline{P}/\underline{T}). \end{aligned}$$

(ii) $\alpha = \forall_2 Q\beta$.

This case can be proved in the same way as the above case.

Proof of Lemma 2.26. We will prove (a) and (b) simultaneously.

a: Suppose β and γ are in $\{\alpha(\underline{x}/\underline{t})\}$, where $\underline{x} = x_1, \ldots, x_n$ are distinct individual variables and $\underline{t} = t_1, \ldots, t_n$ are individual terms.

(i) α is an atomic formula.

Then $\beta = \alpha[\underline{x}/_r\underline{t}] = \gamma$.

(ii) $\alpha = \forall y \sigma$.

Suppose $\underline{x}^*$ is the sublist of $\underline{x}$ consisting of those x_i's which are in $fv(\alpha)$ and $\underline{t}^*$ is the corresponding sublist of $\underline{t}$.

Then $\beta = \forall y' \sigma'$ and $\gamma = \forall y'' \sigma''$ for some individual variables y' and y'' which are not in $(fv(\forall y \sigma) - \{\underline{x}^*\}) \cup fv(\underline{t}^*)$ and some formulae σ' and σ'' in $\{\sigma(y/y', \underline{x}^*/\underline{t}^*)\}$ and $\{\sigma(y/y'', \underline{x}^*/\underline{t}^*)\}$, respectively.

Let y^* be an individual variable which does not occur in σ' or σ'', so y^* is free for y' and y'' in σ' and σ'', respectively.

Then $\beta = \forall y' \sigma' \equiv \forall y^* \sigma'[y'/_r y^*]$ and $\gamma = \forall y'' \sigma'' \equiv \forall y^* \sigma''[y''/_r y^*]$. By Note on page 24 and Lemma 2.20,

$$\begin{aligned} \sigma'[y'/_r y^*] &\in \{\sigma'(y'/y^*)\} \\ &\subseteq \{\sigma(y/y', \underline{x}^*/\underline{t}^*)(y'/y^*)\} \\ &= \{\sigma(y/y^*, \underline{x}^*/\underline{t}^*)\}. \end{aligned}$$

Similarly, $\sigma''[y''/_r y^*] \in \{\sigma(y/y^*, \underline{x}^*/\underline{t}^*)\}$. Hence, by the induction hypothesis, $\sigma'[y'/_r y^*] \equiv \sigma''[y''/_r y^*]$. Thus $\beta \equiv \gamma$.

(iii) $\alpha = \forall_2 Q \sigma$.

Then $\beta = \forall_2 Q' \sigma'$ and $\gamma = \forall_2 Q'' \sigma''$ for some predicate variables Q' and Q'' which are of the same arity as Q and are not in $FV(\forall_2 Q \sigma)$ and some formulae σ' and σ'' in $\{\sigma(Q/Q')(\underline{x}/\underline{t})\}$ and $\{\sigma(Q/Q'')(\underline{x}/\underline{t})\}$, respectively.

Let Q^* be a predicate variable with the same arity as Q which does not occur in σ' or σ''. Then

$$\begin{aligned}\sigma'[Q'/_rQ^*] &\in \{\sigma'(Q'/Q^*)\}\\ &\subseteq \{\sigma(Q/Q')(\underline{x}/\underline{t})(Q'/Q^*)\}\\ &= \{\sigma(Q/Q')(Q'/Q^*)(\underline{x}/\underline{t})\} &&\text{(by Lemma 2.21)}\\ &= \{\sigma(Q/Q^*)(\underline{x}/\underline{t})\}. &&\text{(by Lemma 2.22)}\end{aligned}$$

Similarly, $\sigma''[Q''/_rQ^*] \in \{\sigma(Q/Q^*)(\underline{x}/\underline{t})\}$.

Then $\sigma'[Q'/_rQ^*] \in \{\sigma_1(\underline{x}/\underline{t})\}$ and $\sigma''[Q''/_rQ^*] \in \{\sigma_2(\underline{x}/\underline{t})\}$ for some formulae σ_1 and σ_2 in $\{\sigma(Q/Q^*)\}$. By the induction hypothesis, we have $\sigma_1 \equiv \sigma_2$, so $\{\sigma_1(\underline{x}/\underline{t})\} = \{\sigma_2(\underline{x}/\underline{t})\}$, and hence $\sigma'[Q'/_rQ^*] \equiv \sigma''[Q''/_rQ^*]$. Thus $\beta = \forall_2 Q'\sigma' \equiv \forall_2 Q^*\sigma'[Q'/_rQ^*] \equiv \forall_2 Q^*\sigma''[Q''/_rQ^*] \equiv \forall_2 Q''\sigma'' = \gamma$.

b: The proof is similar to (a) except for the following case.

α is an atomic formula:

If $\alpha = P_q(t_1, \ldots, t_{n_q})$ for some $1 \leq q \leq m$ and some individual terms $t_1, \ldots, t_{n_q}$, then $\beta = R_q(t_1, \ldots, t_{n_q}) = \gamma$, otherwise $\beta = \alpha = \gamma$. □

Lemma 2.27. *Let β and γ be formulae, $\underline{P} = P_1^{n_1}, \ldots, P_m^{n_m}$ be distinct predicate variables and $\underline{T} = T_1, \ldots, T_m$, where $T_i = \lambda x_1^i, \ldots, x_{n_i}^i \delta_i$, $1 \leq i \leq m$, be abstraction terms*

If β and γ are in $\{\alpha(\underline{P}/\underline{T})\}$, then $\beta \equiv \gamma$.

Proof. Suppose β and γ are in $\{\alpha(\underline{P}/\underline{T})\}$.

The proof is similar to the proof of Lemma 2.26 except the following case.

α is an atomic formula:

If $\alpha = P_q(t_1, \ldots, t_{n_q})$ for some $1 \leq q \leq m$ and some individual terms $t_1, \ldots, t_{n_q}$, then β and γ are in $\{\delta_q(x_1^q/t_1, \ldots, x_{n_q}^q/t_{n_q})\}$ and hence $\beta \equiv \gamma$ by Lemma 2.26, otherwise $\beta = \alpha = \gamma$. □

Lemma 2.28. *For any formulae α and β,*

a. if $\beta \equiv \alpha^*$ for some α^* in $\{\alpha(\underline{x}/\underline{t})\}$, where $\underline{x} = x_1, \ldots, x_n$ are distinct individual variables and $\underline{t} = t_1, \ldots, t_n$ are individual terms, then $\beta \in \{\alpha(\underline{x}/\underline{t})\}$;

b. if $\beta \equiv \alpha^*$ for some α^* in $\{\alpha(\underline{P}/\underline{T})\}$, where $\underline{P} = P_1^{n_1}, \ldots, P_m^{n_m}$ are distinct predicate variables and $\underline{T} = T_1, \ldots, T_m$, where $T_i = \lambda x_1^i, \ldots, x_{n_i}^i \delta_i$, $1 \leq i \leq m$, are abstraction terms, then $\beta \in \{\alpha(\underline{P}/\underline{T})\}$.

Proof. Let α and β be formulae. We proceed by induction on α.

a: Suppose $\beta \equiv \alpha^*$ for some α^* in $\{\alpha(\underline{x}/\underline{t})\}$, where $\underline{x} = x_1, \ldots, x_n$ are distinct individual variables and $\underline{t} = t_1, \ldots, t_n$ are individual terms.

It is trivial if there is no change of bound variable. Suppose there exists a sequence of formulae $\alpha^* = \alpha_0, \alpha_1, \ldots, \alpha_k = \beta$, $k \geq 1$, such that for all $1 \leq i \leq k$ α_i is obtained from α_{i-1} by a single legitimate change of bound variable.

By our assumption, α is not atomic. The cases where α is $\alpha_1 \wedge \alpha_2, \alpha_1 \vee \alpha_2$, or $\alpha_1 \supset \alpha_2$ follow straightforwardly by the induction hypothesis.

For the remaining cases, we proceed by induction on k. As usual, we will prove only the case $k = 1$ since the other case follows easily by the subsidiary induction hypothesis.

(i) $\alpha = \forall y \gamma$.

Suppose $\underline{x}^*$ is the sublist of $\underline{x}$ consisting of those x_i's which are in $fv(\alpha)$ and $\underline{t}^*$ is the corresponding sublist of $\underline{t}$.

We have $\alpha^* = \forall y' \gamma^*$ for some individual variable y' and some formula γ^* such that $y' \notin (fv(\forall y \gamma) - \{\underline{x}^*\}) \cup fv(\underline{t}^*)$ and $\gamma^* \in \{\gamma(y/y', \underline{x}^*/\underline{t}^*)\}$.

Case 1. $\beta = \forall y' \gamma'$ where $\gamma' \equiv \gamma^*$.

By the main induction hypothesis, $\gamma' \in \{\gamma(y/y', \underline{x}^*/\underline{t}^*)\}$. Hence $\beta = \forall y' \gamma' = \forall y' \gamma(y/y', \underline{x}^*/\underline{t}^*) = (\forall y \gamma)(\underline{x}/\underline{t}) = \alpha(\underline{x}/\underline{t})$.

Case 2. $\beta = \forall z \gamma^*[y'/_r z]$ where z is an individual variable which is free for y'

and does not occur free in γ^*.

Since $\gamma^* \in \{\gamma(y/y', \underline{x}^*/\underline{t}^*)\}$ and $z \notin fv(\gamma^*)$, $z \notin (fv(\forall y\gamma) - \{\underline{x}^*\}) \cup fv(\underline{t}^*)$.

$$\begin{aligned} \text{Hence } \beta = \forall z\gamma^*[y'/_r z] &= \forall z\gamma^*(y'/z) \\ &= \forall z\gamma(y/y', \underline{x}/\underline{t})(y'/z) \\ &= \forall z\gamma(y/z, \underline{x}/\underline{t}) \qquad \text{(by Lemma 2.20)} \\ &= (\forall y\gamma)(\underline{x}/\underline{t}) = \alpha(\underline{x}/\underline{t}). \end{aligned}$$

(ii) $\alpha = \forall_2 Q\gamma$.

We have $\alpha^* = \forall_2 Q'\gamma^*$ for some predicate variable Q' which is of the same arity as Q and is not in $FV(\forall_2 Q\gamma)$ and some formula γ^* in $\{\gamma(Q/Q')(\underline{x}/\underline{t})\}$.

Case 1. $\beta = \forall_2 Q'\gamma'$ where $\gamma' \equiv \gamma^*$.

By the main induction hypothesis, $\gamma' \in \{\gamma(Q/Q')(\underline{x}/\underline{t})\}$. Hence $\beta = \forall_2 Q'\gamma' = \forall_2 Q'\gamma(Q/Q')(\underline{x}/\underline{t}) = (\forall_2 Q\gamma)(\underline{x}/\underline{t}) = \alpha(\underline{x}/\underline{t})$.

Case 2. $\beta = \forall_2 R\gamma^*[Q'/_r R]$ where R is a predicate variable with the same arity as Q' which is free for Q' and does not occur free in γ^*.

$$\begin{aligned} \text{Then } \beta = \forall_2 R\gamma^*[Q'/_r R] &= \forall_2 R\gamma^*(Q'/R) \\ &= \forall_2 R\gamma(Q/Q')(\underline{x}/\underline{t})(Q'/R) \\ &= \forall_2 R\gamma(Q/Q')(Q'/R)(\underline{x}/\underline{t}) \qquad \text{(by Lemma 2.21)} \\ &= \forall_2 R\gamma(Q/R)(\underline{x}/\underline{t}) \qquad \text{(by Lemma 2.22)} \\ &= (\forall_2 Q\gamma)(\underline{x}/\underline{t}) = \alpha(\underline{x}/\underline{t}). \end{aligned}$$

b: Suppose $\beta \equiv \alpha^*$ for some α^* in $\{\alpha(\underline{P}/\underline{T})\}$, where $\underline{P} = P_1^{n_1}, \ldots, P_m^{n_m}$ are distinct predicate variables and $\underline{T} = T_1, \ldots, T_m$, where $T_i = \lambda x_1^i, \ldots, x_{n_i}^i \delta_i$, $1 \leq i \leq m$, are abstraction terms.

By using Lemmas 2.20 and 2.23, (b) can be proved in the same way as (a) except for the following case.

$\alpha = P_q(t_1, \ldots, t_{n_q})$ for some $1 \leq q \leq m$ and some individual terms $t_1, \ldots, t_{n_q}$:

Then $\beta \equiv \alpha^*$ where $\alpha^* \in \{\delta_q(x_1^q/t_1, \ldots, x_{n_q}^q/t_{n_q})\}$. By (a), $\beta \in \{\delta_q(x_1^q/t_1, \ldots, x_{n_q}^q/t_{n_q})\} = \{\alpha(\underline{P}/\underline{T})\}$. $\square$

Corollary 2.29. *For any formula α,*

a. if $\underline{x} = x_1, \ldots, x_n$ are distinct individual variables and $\underline{t} = t_1, \ldots, t_n$ are individual terms, then $[\alpha[\underline{x}/\underline{t}]] = [\alpha(\underline{x}/\underline{t})] = \{\alpha(\underline{x}/\underline{t})\}$;

b. if $\underline{P} = P_1^{n_1}, \ldots, P_m^{n_m}$ are distinct predicate variables and $\underline{T} = T_1, \ldots, T_m$, where $T_i = \lambda x_1^i, \ldots, x_{n_i}^i \delta_i$, $1 \leq i \leq m$, are abstraction terms, then $[\alpha[\underline{P}/\underline{T}]] = [\alpha(\underline{P}/\underline{T})] = \{\alpha(\underline{P}/\underline{T})\}$.

Proof. It can be easily proved by induction on α that $\alpha[\underline{x}/\underline{t}] \in \{\alpha(\underline{x}/\underline{t})\}$ and $\alpha[\underline{P}/\underline{T}] \in \{\alpha(\underline{P}/\underline{T})\}$. Then (a) follows by Lemmas 2.26 and 2.28 and (b) follows by Lemmas 2.27 and 2.28. $\square$

By extending the work in [3], we have introduced our second-order language. We have defined substitutions for second-order formulae and have proved some lemmas that establish basic properties and will be used in the following chapters.

CHAPTER III

CURRY-HOWARD TERMS

We have dealt with the technicalities of substitutions in the previous chapter, now we can define Curry-Howard terms.

3.1 The formal calculus

We take NJ to be Gentzen's intuitionistic natural deduction system given by Prawitz (see [12]). We will extend NJ to a second-order system, denoted by NJ_2.

The rules of NJ_2 include the rules of NJ, extended to second-order formulae, which are as follows.

- **Atomic deductions**

For every formula α, α is a deduction (with uncancelled premise α).

- **Introduction and elimination rules for the various connectives**

($\wedge$ Intro)

$$\begin{array}{c} \vdots \quad \vdots \\ \alpha \quad \beta \\ \hline \alpha \wedge \beta \end{array}$$

($\wedge$ Elim)

$$\begin{array}{c} \vdots \\ \alpha \wedge \beta \\ \hline \alpha \end{array} (\wedge_1 \text{ Elim}) \qquad\qquad \begin{array}{c} \vdots \\ \alpha \wedge \beta \\ \hline \beta \end{array} (\wedge_2 \text{ Elim})$$

(⊃ Intro)

$$\dfrac{\begin{array}{c}[\alpha]\\ \vdots\\ \beta\end{array}}{\alpha \supset \beta}$$

(⊃ Elim)

$$\dfrac{\begin{array}{cc}\vdots & \vdots\\ \alpha \supset \beta & \alpha\end{array}}{\beta}$$

(∀ Intro)

$$\dfrac{\begin{array}{c}\vdots\\ \alpha\end{array}}{\forall x \alpha}$$

provided x is not free in any uncancelled premise.

(∀ Elim)

$$\dfrac{\begin{array}{c}\vdots\\ \forall x \alpha\end{array}}{\alpha(x/t)}$$

where t is an individual term.

(∨ Intro)

$$\dfrac{\begin{array}{c}\vdots\\ \alpha\end{array}}{\alpha \vee \beta}(\vee_1 \text{ Elim}) \qquad \dfrac{\begin{array}{c}\vdots\\ \beta\end{array}}{\alpha \vee \beta}(\vee_2 \text{ Elim})$$

(∨ Elim)

$$\dfrac{\begin{array}{ccc} & [\alpha] & [\beta]\\ \vdots & \vdots & \vdots\\ \alpha \vee \beta & \gamma & \gamma\end{array}}{\gamma}$$

(∃ Intro)

$$\frac{\begin{array}{c}\vdots\\ \alpha(x/t)\end{array}}{\exists x\alpha}$$

(∃ Elim)

$$\frac{\begin{array}{cc} & [\alpha]\\ \vdots & \vdots\\ \exists x\alpha & \gamma\end{array}}{\gamma}$$

provided x is not free in γ nor in any uncancelled premise of the given deduction of γ.

(⊥ Elim)

$$\frac{\begin{array}{c}\vdots\\ \bot\end{array}}{\alpha}$$

Notation. $[\alpha]$ in a proof means that none or some or all occurrences of premise α may be cancelled. We also use $[\alpha]$ for the equivalence class of α but the context makes clear the intention.

We now add rules for second-order quantifiers $\forall_2$ and $\exists_2$ as follows.

($\forall_2$ Intro)

$$\frac{\begin{array}{c}\vdots\\ \alpha\end{array}}{\forall_2 P\alpha}$$

provided P is not free in any uncancelled premise.

($\forall_2$ Elim)

$$\frac{\begin{array}{c}\vdots\\ \forall_2 P^n\alpha\end{array}}{\alpha(P^n/T)}$$

where $T = \lambda x_1, \ldots, x_n\beta$ is an abstraction term.

($\exists_2$ Intro)

$$\frac{\begin{array}{c}\vdots\\ \alpha(P/T)\end{array}}{\exists_2 P\alpha}$$

($\exists_2$ Elim)

$$\frac{\begin{array}{cc} & [\alpha] \\ \vdots & \vdots \\ \exists_2 P\alpha & \gamma\end{array}}{\gamma}$$

provided P is not free in γ nor in any uncancelled premise of the given deduction of γ.

3.2 Basic properties of Curry-Howard terms

First we will give rules (taken from [3]) for the formation of the original Curry-Howard terms which correspond to the rules of the natural deduction system in first-order logic.

Note. Sometimes we write "C-H term"or just "term"instead of "Curry-Howard term".

Rules for the formation of the original Curry-Howard terms:

(Atomic) For each formula α, the *term variables* $X^\alpha, Y^\alpha, Z^\alpha, \ldots$ are terms of type $[\alpha]$.

($\wedge$ Intro) If F^α and G^β are terms of types $[\alpha]$ and $[\beta]$, respectively, then (F^α, G^β) is a term of type $[\alpha \wedge \beta]$.

($\wedge$ Elim) If $F^{\alpha\wedge\beta}$ is a term of type $[\alpha \wedge \beta]$, then $\pi_1 F^{\alpha\wedge\beta}$ and $\pi_2 F^{\alpha\wedge\beta}$ are terms of types $[\alpha]$ and $[\beta]$, respectively.

(⊃ **Intro)** If X^α is a term variable of type $[\alpha]$ and F^β is a term of type $[\beta]$, then $\lambda X^\alpha.F^\beta$ is a term of type $[\alpha \supset \beta]$.

Note. All occurrences of $X^{\alpha'}$, where $[\alpha'] = [\alpha]$, in this term are bound.

(⊃ **Elim)** If $F^{\alpha\supset\beta}$ and $G^{\alpha'}$ are terms of types $[\alpha \supset \beta]$ and $[\alpha']$, respectively, where $[\alpha'] = [\alpha]$, then $F^{\alpha\supset\beta}(G^{\alpha'})$ is a term of type $[\beta]$.

(∀ **Intro)** If F^α is a term of type $[\alpha]$ and x is an individual variable which does not occur free in the type superscript of any free term variable of F^α, then $\lambda x.F^\alpha$ is a term of type $[\forall x\alpha]$.

Note. All occurrences of x in this term are bound.

(∀ **Elim)** If $F^{\forall x\alpha}$ is a term of type $[\forall x\alpha]$ and t is an individual term, then $F^{\forall x\alpha}(t)$ is a term of type $[\alpha(x/t)]$.

(∨ **Intro)** If F^α is a term of type $[\alpha]$ and β is a formula, then $(\mu_1 F^\alpha)^{\alpha\vee\beta}$ is a term of type $[\alpha\vee\beta]$; if F^β is a term of type $[\beta]$ and α is a formula, then $(\mu_2 F^\beta)^{\alpha\vee\beta}$ is a term of type $[\alpha \vee \beta]$.

(∨ **Elim)** If F^γ, $G^{\gamma'}$, and $H^{\alpha\vee\beta}$ are terms of types $[\gamma]$, $[\gamma']$, and $[\alpha \vee \beta]$, respectively, where $[\gamma] = [\gamma']$, and $X^{\alpha'}$ and $Y^{\beta'}$ are term variables of types $[\alpha']$ and $[\beta']$, respectively, where $[\alpha'] = [\alpha]$ and $[\beta'] = [\beta]$, then $\oplus(X^{\alpha'}.F^\gamma, Y^{\beta'}.G^{\gamma'}, H^{\alpha\vee\beta})$ is a term of type $[\gamma]$.

Note. All occurrences of $X^{\alpha''}$, where $[\alpha''] = [\alpha']$, in $X^{\alpha'}.F^\gamma$ and all occurrences of $Y^{\beta''}$, where $[\beta''] = [\beta']$, in $Y^{\beta'}.G^{\gamma'}$ are bound.

(∃ **Intro)** If $F^{\alpha(x/t)}$ is a term of type $[\alpha(x/t)]$, then $I(t, F^{\alpha(x/t)})^{\exists x\alpha}$ is a term of type $[\exists x\alpha]$.

(∃ **Elim)** If F^γ is a term of type $[\gamma]$, X^α is a term variable of type $[\alpha]$, x is an individual variable which does not occur free in γ or in the type superscript of any free term variable of F^γ except X^α, and G^{α^*} is a term of type $[\alpha^*]$, where $[\alpha^*] = [\exists x\alpha]$, then $ST(x.X^\alpha.F^\gamma, G^{\alpha^*})$ is a term of type $[\gamma]$.

Note. All occurrences of x in $x.X^{\alpha}.F^{\gamma}$ and all occurrences of $X^{\alpha'}$, where $[\alpha'] = [\alpha]$, in $X^{\alpha}.F^{\gamma}$ are bound.

($\perp$ Elim) If $F^{\perp}$ is a term of type $[\perp]$ and α is a formula, then $F^{\perp}(\alpha)$ is a term of type $[\alpha]$.

The new Curry-Howard terms include the terms of the above forms with the extension of first-order formulae to second-order formulae. We also add term formation rules for second-order quantifiers as follows.

($\forall_2$ Intro) If F^{α} is a term of type $[\alpha]$ and P is a predicate variable which does not occur free in the type superscript of any free term variable of F^{α}, then $\lambda P.F^{\alpha}$ is a term of type $[\forall_2 P\alpha]$.

Note. All occurrences of P in this term are bound.

($\forall_2$ Elim) If $F^{\forall_2 P\alpha}$ is a term of type $[\forall_2 P\alpha]$ and $T = \lambda x_1, \ldots, x_n\beta$ is an abstraction term, where n is the arity of P, then $F^{\forall_2 P\alpha}(T)$ is a term of type $[\alpha(P/T)]$.

($\exists_2$ Intro) If $F^{\alpha(P/T)}$ is a term of type $[\alpha(P/T)]$, then $I(T, F^{\alpha(P/T)})^{\exists_2 P\alpha}$ is a term of type $[\exists_2 P\alpha]$.

($\exists_2$ Elim) If F^{γ} is a term of type $[\gamma]$, X^{α} is a term variable of type $[\alpha]$, P is a predicate variable which does not occur free in γ or in the type superscript of any free term variable of F^{γ} except X^{α}, and G^{α^*} is a term of type $[\alpha^*]$, where $[\alpha^*] = [\exists_2 P\alpha]$, then $ST(P.X^{\alpha}.F^{\gamma}, G^{\alpha^*})$ is a term of type $[\gamma]$.

Note. All occurrences of P in $P.X^{\alpha}.F^{\gamma}$ and all occurrences of $X^{\alpha'}$, where $[\alpha'] = [\alpha]$, in $X^{\alpha}.F^{\gamma}$ are bound.

Notes.

a. We may omit the type superscript of a Curry-Howard term when we do not want to state the type or it it is clear in the context of which type it is. When we write "F^α"we mean "$[\alpha]$ is the type of F^α".

b. When we say "X^α is the first term variable of type $[\alpha]\ldots$ "we mean "X^α is the first term variable of type $[\alpha]$ in some fixed ordering ... ".

c. We may write each type superscript as any formula in the same equivalence class since the type of every Curry-Howard term depends on the types (not on the formulae which are the type superscripts) of terms from which it is constructed. We will prove the following cases and omit the others which can be proved easily.

Proof.

($\forall$ Elim) Suppose $F_1^{\beta_1}$ and $F_2^{\beta_2}$ are terms where $[\beta_1] = [\forall x\alpha] = [\beta_2]$. By Note on page 21, $\beta_i = \forall y_i\alpha_i$ for some formula α_i and some individual variable y_i such that $\alpha_i \equiv \alpha[x/y_i]$, and $y_i \notin fv(\forall x\alpha)$ for all $i = 1, 2$. Let t be an individual term. Thus, for all $i = 1, 2$, $F_i^{\beta_i}(t)$ is a term of type $[\alpha_i(y_i/t)] = [\alpha_i[y_i/t]] = [\alpha[x/y_i][y_i/t]] = [\alpha[x/t]] = [\alpha(x/t)]$ by Lemmas 2.13 and 2.14.

By using Lemmas 2.16 and 2.17, the case ($\forall_2$ Elim) can be proved similarly.

($\exists$ Intro) Suppose $F_1^{\beta_1}$ and $F_2^{\beta_2}$ are terms where $[\beta_1] = [\alpha(x/t)] = [\beta_2]$. By Lemma 2.28, $\beta_i \in \{\alpha(x/t)\}$ for all $i = 1, 2$. Hence $I(t, F_1^{\beta_1})^{\exists x\alpha}$ and $I(t, F_2^{\beta_2})^{\exists x\alpha}$ are terms of type $[\exists x\alpha]$.

Similarly for ($\exists_2$ Intro). □

Definition 3.2.1. *A **context** is an expression which is of one of the following forms: $x.F^\alpha$, $P.F^\alpha$, $X^\beta.F^\alpha$, $x.X^\beta.F^\alpha$, and $P.X^\beta.F^\alpha$.*

Note. An occurrence of a term variable $X^{\alpha'}$ in a Curry-Howard term is *bound* if it occurs in a context of the form $X^\alpha.F^\beta$, $x.X^\alpha.F^\beta$, or $P.X^\alpha.F^\beta$, where $\alpha \equiv \alpha'$, otherwise it is free, and we say the binding of X^α has *scope* F^β.

Definition 3.2.2. *The set of* ***free individual variables*** *of a Curry-Howard term* F^α*, denoted by* $fv(F^\alpha)$*, is defined inductively as follows.*

i. $fv(X^\alpha) = fv(\alpha)$.

ii. $fv(G^\beta, H^\gamma) = fv(G^\beta) \cup fv(H^\gamma)$.

Similarly for $fv(\pi_i G^{\beta\wedge\gamma})$*,* $i = 1, 2$*, and* $fv(G^{\beta\supset\alpha}(H^\beta))$.

iii. $fv((\mu_1 G^\beta)^{\beta\vee\gamma}) = fv(G^\beta) \cup fv(\gamma)$.

Similarly for $fv((\mu_2 G^\beta)^{\gamma\vee\beta})$.

iv. $fv(G^{\forall x\beta}(t)) = fv(G^{\forall x\beta}) \cup fv(t)$.

Similarly for $fv(I(t, G^{\beta(x/t)}))$.

v. $fv(G^{\forall_2 P\beta}(T)) = fv(G^{\forall_2 P\beta})$.

Similarly for $fv(I(T, G^{\beta(P/T)}))$.

vi. $fv(G^\perp(\alpha)) = fv(G^\perp) \cup fv(\alpha)$.

vii. $fv(\lambda X^\beta.G^\gamma) = fv(X^\beta.G^\gamma)$*, where* $fv(X^\beta.G^\gamma) = fv(\beta) \cup fv(G^\gamma)$.

viii. $fv(\oplus(X^\beta.G^\alpha, Y^\gamma.H^\alpha, K^{\beta\vee\gamma})) = fv(X^\beta.G^\alpha) \cup fv(Y^\gamma.H^\alpha) \cup fv(K^{\beta\vee\gamma})$.

ix. $fv(\lambda x.G^\beta) = fv(x.G^\beta)$*, where* $fv(x.G^\beta) = fv(G^\beta) - \{x\}$.

x. $fv(\lambda P.G^\beta) = fv(P.G^\beta)$*, where* $fv(P.G^\beta) = fv(G^\beta)$.

xi. $fv(ST(x.X^\beta.G^\alpha, H^{\exists x\beta})) = fv(x.X^\beta.G^\alpha) \cup fv(H^{\exists x\beta})$*, where* $fv(x.X^\beta.G^\alpha) = fv(X^\beta.G^\alpha) - \{x\}$.

xii. $fv(ST(P.X^\beta.G^\alpha, H^{\exists_2 P\beta})) = fv(P.X^\beta.G^\alpha) \cup fv(H^{\exists_2 P\beta})$*, where* $fv(P.X^\beta.G^\alpha) = fv(X^\beta.G^\alpha)$.

Definition 3.2.3. *The set of* ***free predicate variables*** *of a Curry-Howard term* F^α*, denoted by* $FV(F^\alpha)$*, is defined inductively as follows.*

i. $FV(X^\alpha) = FV(\alpha)$.

ii. $FV(G^\beta, H^\gamma) = FV(G^\beta) \cup FV(H^\gamma)$.

Similarly for $FV(\pi_i G^{\beta\wedge\gamma})$*,* $i = 1, 2$*, and* $FV(G^{\beta\supset\alpha}(H^\beta))$.

iii. $FV((\mu_1 G^\beta)^{\beta\vee\gamma}) = FV(G^\beta) \cup FV(\gamma)$.

Similarly for $FV((\mu_2 G^\beta)^{\gamma\vee\beta})$.

iv. $FV(G^{\forall x\beta}(t)) = FV(G^{\forall x\beta})$.

Similarly for $FV(I(t, G^{\beta(x/t)}))$.

v. $FV(G^{\forall_2 P\beta}(T)) = FV(G^{\forall_2 P\beta}) \cup FV(T)$.

Similarly for $FV(I(T, G^{\beta(P/T)}))$.

vi. $FV(G^\perp(\alpha)) = FV(G^\perp) \cup FV(\alpha)$.

vii. $FV(\lambda X^\beta.G^\gamma) = FV(X^\beta.G^\gamma)$, *where* $FV(X^\beta.G^\gamma) = FV(\beta) \cup FV(G^\gamma)$.

viii. $FV(\oplus(X^\beta.G^\alpha, Y^\gamma.H^\alpha, K^{\beta\vee\gamma}))$

$$= FV(X^\beta.G^\alpha) \cup FV(Y^\gamma.H^\alpha) \cup FV(K^{\beta\vee\gamma}).$$

ix. $FV(\lambda x.G^\beta) = FV(x.G^\beta)$, *where* $FV(x.G^\beta) = FV(G^\beta)$.

x. $FV(\lambda P.G^\beta) = FV(P.G^\beta)$, *where* $FV(P.G^\beta) = FV(G^\beta) - \{P\}$.

xi. $FV(ST(x.X^\beta.G^\alpha, H^{\exists x\beta})) = FV(x.X^\beta.G^\alpha) \cup FV(H^{\exists x\beta})$, *where* $FV(x.X^\beta.G^\alpha) = FV(X^\beta.G^\alpha)$.

xii. $FV(ST(P.X^\beta.G^\alpha, H^{\exists_2 P\beta})) = FV(P.X^\beta.G^\alpha) \cup FV(H^{\exists_2 P\beta})$, *where* $FV(P.X^\beta.G^\alpha) = FV(X^\beta.G^\alpha) - \{P\}$.

Notation. We use $fv(\underline{K})$ to denote $\bigcup_{i=1}^{n} fv(K_i)$, where $\underline{K} = K_1, \ldots, K_n$ is a sequence of C-H terms; similarly for $FV(\underline{K})$.

Notes.

a. All occurrences of x (respectively P) in $x.F^\beta$ and $x.X^\alpha.F^\beta$ (respectively $P.F^\beta$ and $P.X^\alpha.F^\beta$) are *bound* and we say the binding of x (respectively P) has *scope* F^β and $X^\alpha.F^\beta$, respectively.

b. It can be proved by induction on F^α that

(b1) $fv(\alpha) \subseteq fv(F^\alpha)$ and $FV(\alpha) \subseteq FV(F^\alpha)$;

(b2) if X^β is a free term variable of F^α and $x \in fv(\beta)$ (respectively $P \in FV(\beta)$), then each free occurrence of x (respectively P) in the free occurrences of X^β is also free in F^α.

Definition 3.2.4. *Term variables X^α and $X^{\alpha'}$ are* ***equivalent****, denoted by $X^\alpha \equiv X^{\alpha'}$, if $\alpha \equiv \alpha'$.*

Notation. Let

a. $F^\alpha[\underline{x}/_r\underline{t}]$ (respectively $F^\alpha[\underline{P}/_r\underline{R}]$) denote the simple simultaneous replacements of all free occurrences of (distinct) individual variables $\underline{x} = x_1, \ldots, x_n$ (respectively (distinct) predicate variables $\underline{P} = P_1^{m_1}, \ldots, P_n^{m_n}$) in a C-H term F^α by individual terms $\underline{t} = t_1, \ldots, t_n$ (respectively predicate variables $\underline{R} = R_1^{m_1}, \ldots, R_n^{m_n}$), respectively;

b. $F^\alpha[\underline{X}/_{rt}\underline{K}]$ denote the simple simultaneous replacements of all free occurrences of term variables which are equivalent to (inequivalent) term variables $\underline{X} = X_1^{\delta_1}, \ldots, X_n^{\delta_n}$ in a C-H term F^α by C-H terms $\underline{K} = K_1^{\delta_1'}, \ldots, K_n^{\delta_n'}$, respectively, where $\delta_i \equiv \delta_i'$ for all $1 \leq i \leq n$.

Definition 3.2.5. *If $\underline{x} = x_1, \ldots, x_n$ are distinct individual variables (respectively $\underline{P} = P_1^{r_1}, \ldots, P_m^{r_m}$ are distinct predicate variables) and $\underline{t} = t_1, \ldots, t_n$ are individual terms (respectively $\underline{T} = T_1, \ldots, T_m$, where $T_j = \lambda y_1^j, \ldots, y_{r_j}^j \delta_j$, $1 \leq j \leq m$, are abstraction terms), we say $\underline{t}$ is* ***free*** *for $\underline{x}$ (respectively $\underline{T}$ is* ***free*** *for $\underline{P}$) in an expression E of the form F^α or $X^\beta.F^\alpha$, if no free occurrence of any x_i, $1 \leq i \leq n$, (respectively P_j, $1 \leq j \leq m$) in E is within the scope of a bound variable y (respectively Q) of E where y occurs in t_i (respectively Q occurs free in T_j).*

Let F^α be a C-H term, $\underline{X} = X_1^{\delta_1}, \ldots, X_n^{\delta_n}$ be inequivalent term variables, $\underline{K} = K_1^{\delta_1'}, \ldots, K_n^{\delta_n'}$ be C-H terms, where $\delta_i \equiv \delta_i'$ for all $1 \leq i \leq n$. We say $\underline{K}$ is ***free*** *for $\underline{X}$ in F^α if no free occurrence of any term variable equivalent to $X_i^{\delta_i}$, $1 \leq i \leq n$, in F^α is within the scope of a bound term variable Y^β of F^α, where Y^β is equivalent to some free term variable of $K_i^{\delta_i'}$.*

For substitution purposes, we want to treat equivalent term variables as the same. Moreover, substitutions for Curry-Howard terms must satisfy all the basic properties of substitutions including the property that every free variable should not become bound after each substitution.

Since it could happen that α and β are inequivalent formulae but $\alpha[x/t] \equiv \beta[x/t]$ for some individual variable x and individual term t (e.g. $\alpha = P(x)$ and $\beta = P(t)$ where $x \notin fv(t)$), so $X^\alpha \not\equiv X^\beta$ but $X^{\alpha[x/t]} \equiv X^{\beta[x/t]}$. Consider the term $\lambda X^\alpha.X^\beta$. In this term X^β is free. If we simply substitute t for free occurrences of x in $\lambda X^\alpha.X^\beta$, this term becomes $\lambda X^{\alpha[x/t]}.X^{\beta[x/t]}$. Since $X^{\alpha[x/t]} \equiv X^{\beta[x/t]}$, $X^{\beta[x/t]}$ in the new term is bound. We do not want this to happen. In order to overcome this problem, we need to change the bound term variable X^α to some other term variable which will not cause the problem before substituting t for x.

We first define *replaceability* which is needed in defining *legitimate changes* of *bound variables* later.

Definition 3.2.6. ***Replaceability*** *of distinct individual variables* $\underline{x} = x_1, \ldots, x_n$ *by individual terms* $\underline{t} = t_1, \ldots, t_n$ *in a C-H term* F^α *is defined inductively as follows.*

i. If F^α *is a term variable, then* $\underline{x}$ *is replaceable by* $\underline{t}$ *in* F^α.

ii. $\underline{x}$ *is replaceable by* $\underline{t}$ *in* (G^β, H^γ) *if* $\underline{x}$ *is replaceable by* $\underline{t}$ *in* G^β *and* H^γ. *Similarly for* $\pi_i G^{\beta\wedge\gamma}$, $\mu_i G^\beta$, $i = 1, 2$, *and* $G^{\beta\supset\alpha}(H^\beta)$.

iii. $\underline{x}$ *is replaceable by* $\underline{t}$ *in* $G^{\forall y\beta}(u)$ *if* $\underline{x}$ *is replaceable by* $\underline{t}$ *in* $G^{\forall y\beta}$. *Similarly for* $I(u, G^{\beta(y/u)})$, $G^{\forall_2 Q\beta}(U)$, $I(U, G^{\beta(Q/U)})$, *and* $G^\perp(\alpha)$.

iv. $\underline{x}$ *is replaceable by* $\underline{t}$ *in* $\lambda X^\beta.G^\gamma$ *if* $\underline{x}$ *is replaceable by* $\underline{t}$ *in* $X^\beta.G^\gamma$ *i.e.* $\underline{x}$ *is replaceable by* $\underline{t}$ *in* G^γ *and* $X^\beta.G^\gamma$ *has no free term variable* X^σ *such that* $\sigma[\underline{x}/\underline{t}] \equiv \beta[\underline{x}/\underline{t}]$.

v. $\underline{x}$ *is replaceable by* $\underline{t}$ *in* $\oplus(X^\beta.G^\alpha, Y^\gamma.H^\alpha, K^{\beta\vee\gamma})$ *if* $\underline{x}$ *is replaceable by* $\underline{t}$ *in* $X^\beta.G^\alpha$, $Y^\gamma.H^\alpha$, *and* $K^{\beta\vee\gamma}$.

vi. $\underline{x}$ *is replaceable by* $\underline{t}$ *in* $\lambda Q.G^\beta$ *if* $\underline{x}$ *is replaceable by* $\underline{t}$ *in* G^β.

vii. $\underline{x}$ *is replaceable by* $\underline{t}$ *in* $ST(Q.X^{\beta}.G^{\alpha}, H^{\exists_2 Q\beta})$ *if* $\underline{x}$ *is replaceable by* $\underline{t}$ *in* $X^{\beta}.G^{\alpha}$ *and* $H^{\exists_2 Q\beta}$.

viii. $\underline{x}$ *is replaceable by* $\underline{t}$ *in* $\lambda y.G^{\beta}$ *if* $\underline{x}^*$ *is replaceable by* $\underline{t}^*$ *in* G^{β}, *where* $\underline{x}^*$ *is the sublist of* $\underline{x}$ *consisting of those* x_i*'s which are in* $fv(\lambda y.G^{\beta})$ *and* $\underline{t}^*$ *is the corresponding sublist of* $\underline{t}$.

ix. $\underline{x}$ *is replaceable by* $\underline{t}$ *in* $ST(y.X^{\beta}.G^{\alpha}, H^{\exists y\beta})$ *if* $\underline{x}$ *is replaceable by* $\underline{t}$ *in* $H^{\exists y\beta}$ *and* $\underline{x}^*$ *is replaceable by* $\underline{t}^*$ *in* $X^{\beta}.G^{\alpha}$, *where* $\underline{x}^*$ *is the sublist of* $\underline{x}$ *consisting of those* x_i*'s which are in* $fv(y.X^{\beta}.G^{\alpha})$ *and* $\underline{t}^*$ *is the corresponding sublist of* $\underline{t}$.

Replaceability *of distinct predicate variables* $\underline{P} = P_1^{r_1}, \ldots, P_m^{r_m}$ *by abstraction terms* $\underline{T} = T_1, \ldots, T_m$, *where* $T_j = \lambda y_1^j, \ldots, y_{r_j}^j \delta_j$, $1 \leq j \leq m$, *in a C-H term* F^{α} *is defined similarly.*

Definition 3.2.7. *A replacement of an occurrence of a context* $x.F^{\alpha}$ *(respectively* $x.X^{\beta}.F^{\alpha}$*) in a C-H term by* $x'.F^{\alpha}[x/_r x']$ *(respectively* $x'.X^{\beta[x/_r x']}.F^{\alpha}[x/_r x']$*) is called a* ***legitimate change*** *of* ***bound individual variable*** *if* x' *is an individual variable such that* x *is replaceable by* x', x' *is free for* x, *and* x' *does not occur free in* F^{α} *(respectively* $X^{\beta}.F^{\alpha}$*).*

A replacement of an occurrence of a context $P.F^{\alpha}$ *(respectively* $P.X^{\beta}.F^{\alpha}$*) in a C-H term by* $P'.F^{\alpha}[P/_r P']$ *(respectively* $P'.X^{\beta[P/_r P']}.F^{\alpha}[P/_r P']$*), where* P' *is a predicate variable with the same arity as* P, *is called a* ***legitimate change*** *of* ***bound predicate variable*** *if* P *is replaceable by* P', P' *is free for* P, *and* P' *does not occur free in* F^{α} *(respectively* $X^{\beta}.F^{\alpha}$*).*

A replacement of a context $X^{\beta}.F^{\alpha}$ *(respectively* $x.X^{\beta}.F^{\alpha}$ *and* $P.X^{\beta}.F^{\alpha}$*) in a C-H term by* $Y^{\beta'}.F^{\alpha}[X^{\beta}/_{rt} Y^{\beta'}]$ *(respectively* $x.Y^{\beta'}.F^{\alpha}[X^{\beta}/_{rt} Y^{\beta'}]$ *and* $P.Y^{\beta'}.F^{\alpha}[X^{\beta}/_{rt} Y^{\beta'}]$*), where* $Y^{\beta'}$ *is a term variable such that* $\beta \equiv \beta'$, *is called a* ***legitimate change*** *of* ***bound term variable*** *if* $Y^{\beta'}$ *is free for* X^{β} *in* F^{α} *and is not equivalent to any free term variable of* F^{α}.

Definition 3.2.8. *If there exists a sequence of C-H terms* $F = F_0, F_1, \ldots, F_n = F'$, $n \geq 1$, *such that for each* $1 \leq i \leq n$, F_i *is obtained from* F_{i-1} *either by replacing some occurrences of term variables by equivalent term variables or by a legitimate change of bound individual variable, bound predicate variable, or bound term variable, we say* F *is* ***equivalent*** *to* F', *denoted by* $F \equiv F'$.

This relation is defined similarly for contexts.

Note. It can be proved by induction on F that if $F \equiv F'$, then

a. $fv(F) = fv(F')$, $FV(F) = FV(F')$, and every free term variable of F is equivalent to some free term variable of F' and vice versa;

b. F and F' are of the same type.

Lemma 3.2.9. $\equiv$ *is an equivalence relation.*

Proof. It is easy to see that $\equiv$ is reflexive and transitive. It remains to show that $\equiv$ is symmetric.

First we need the following claim.

Claim. For any individual variables x and x', if x is replaceable by x' in a term F^α, then x' is replaceable by x in $F^\alpha[x/_r x']$.

Proof of the claim. We will prove by induction on F^α.

(i) F^α is a term variable.

Then $F^\alpha[x/_r x']$ is also a term variable and hence x' is replaceable by x in $F^\alpha[x/_r x']$.

(ii) $F^\alpha = (G^\beta, H^\gamma)$.

This case follows by the induction hypothesis.

Similarly for $\pi_i G^{\beta\wedge\gamma}$, $\mu_i G^\beta$, $i = 1, 2$, $G^{\beta\supset\alpha}(H^\beta)$, $G^{\forall y\beta}(u)$, $I(u, G^{\beta(y/u)})$, $G^{\forall_2 Q\beta}(U)$, $I(U, G^{\beta(Q/U)})$, $\lambda Q.G^\beta$, and $G^\perp(\alpha)$.

(iii) $F^\alpha = \lambda X^\beta.G^\gamma$.

Then $F^\alpha[x/_r x'] = \lambda X^{\beta[x/_r x']}.G^\gamma[x/_r x']$. We have to show that x' is replaceable by x in $X^{\beta[x/_r x']}.G^\gamma[x/_r x']$.

By the induction hypothesis, x' is replaceable by x in $G^\gamma[x/_r x']$. Suppose for a contradiction that $X^{\beta[x/_r x']}.G^\gamma[x/_r x']$ has a free occurrence of a term variable X^σ such that $\sigma[x'/x] \equiv \beta[x/_r x'][x'/x]$. Since X^σ occurs free in $G^\gamma[x/_r x']$ and $x \notin fv(G^\gamma[x/_r x'])$, by Note (b2) on page 49, $x \notin fv(\sigma)$. Hence, by Lemmas 2.13 and 2.14, $\sigma \equiv \sigma[x'/x][x/x'] \equiv \beta[x/_r x'][x'/x][x/x'] \equiv \beta[x/_r x']$. This is a contradiction. Thus x' is replaceable by x in $F^\alpha[x/_r x']$.

(iv) $F^\alpha = \oplus(X^\beta.G^\alpha, Y^\gamma.H^\alpha, K^{\beta\vee\gamma})$.

Then $F^\alpha[x/_r x'] = \oplus(X^{\beta[x/_r x']}.G^\alpha[x/_r x'], Y^{\gamma[x/_r x']}.H^\alpha[x/_r x'], K^{\beta\vee\gamma}[x/_r x'])$. Similar to the above case, we can prove that x' is replaceable by x in $X^{\beta[x/_r x']}.G^\alpha[x/_r x']$ and $Y^{\gamma[x/_r x']}.H^\alpha[x/_r x']$. By the induction hypothesis, x' is replaceable by x in $K^{\beta\vee\gamma}[x/_r x']$. Hence x' is replaceable by x in $F^\alpha[x/_r x']$.

Similarly for $ST(Q.X^\beta.G^\alpha, H^{\exists_2 Q\beta})$.

(v) $F^\alpha = \lambda y.G^\beta$.

If $x = y$, then $F^\alpha[x/_r x'] = F^\alpha$, so $x' \notin fv(F^\alpha[x/_r x'])$ since $x' \notin fv(F^\alpha)$ and hence x' is replaceable by x in $F^\alpha[x/_r x']$. Suppose $x \neq y$. Then $F^\alpha[x/_r x'] = \lambda y.G^\beta[x/_r x']$. By the induction hypothesis, x' is replaceable by x in $G^\beta[x/_r x']$ and hence in $F^\alpha[x/_r x']$.

(vi) $F^\alpha = ST(y.X^\beta.G^\alpha, H^{\exists y\beta})$.

By the induction hypothesis, x' is replaceable by x in $H^{\exists y\beta}[x/_r x']$.

If $x = y$, then $F^\alpha[x/_r x'] = ST(y.X^\beta.G^\alpha, H^{\exists y\beta}[x/_r x'])$ and $x' \notin fv(y.X^\beta.G^\alpha)$ since $x' \notin fv(F^\alpha)$, hence x' is replaceable by x in $y.X^\beta.G^\alpha$ and so in $F^\alpha[x/_r x']$.

Suppose $x \neq y$. Then $F^\alpha[x/_r x'] = ST(y.(X^\beta.G^\alpha)[x/_r x'], H^{\exists y\beta}[x/_r x'])$. We can show that x' is replaceable by x in $(X^\beta.G^\alpha)[x/_r x']$ as in (iii). Thus x' is replaceable by x in $F^\alpha[x/_r x']$.

Hence we have the claim.

To show that $\equiv$ is symmetric, it is enough to show this for a single change of bound variable. First, suppose an occurrence of a context $x.F^\alpha$ in a C-H term is replaced by $x'.F^\alpha[x/_r x']$, where x is replaceable by x', x' is free for x, and x' does not occur free in F^α. It is clear that x is free for x' and does not occur free in $F^\alpha[x/_r x']$. By the claim, x' is replaceable by x in $F^\alpha[x/_r x']$. Hence the change from $x'.F^\alpha[x/_r x']$ to $x.F^\alpha[x/_r x'][x'/_r x]$ which is $x.F^\alpha$ is legitimate. Similarly, if the replaced context is of the form $P.F^\alpha$.

Now, suppose a context $x.X^\beta.F^\alpha$ in a C-H term is replaced by $x'.X^{\beta[x/_r x']}.F^\alpha[x/_r x']$, where x' is an individual variable such that x is replaceable by x', x' is free for x, and x' does not occur free in $X^\beta.F^\alpha$. It is easy to see that x is free for x' and x does not occur free in $X^{\beta[x/_r x']}.F^\alpha[x/_r x']$. It remains to show that x' is replaceable by x in $X^{\beta[x/_r x']}.F^\alpha[x/_r x']$.

Since x is replaceable by x' in $X^\beta.F^\alpha$, x is replaceable by x' in F^α. Hence, by the above claim, x' is replaceable by x in $F^\alpha[x/_r x']$. We can show that $X^{\beta[x/_r x']}.F^\alpha[x/_r x']$ has no free occurrence of a term variable X^σ such that $\sigma[x'/x] \equiv \beta[x/_r x'][x'/x]$ in the same way as in (iii) of the above claim. Hence x' is replaceable by x in $X^{\beta[x/_r x']}.F^\alpha[x/_r x']$. Thus the change from $x'.X^{\beta[x/_r x']}.F^\alpha[x/_r x']$ to $x.X^{\beta[x/_r x'][x'/_r x]}.F^\alpha[x/_r x'][x'/_r x]$ which is $x.X^\beta.F^\alpha$ is also legitimate. Similarly, if the replaced context is of the form $P.X^\beta.F^\alpha$.

The proof for a change of bound term variable is similar to the proof of Lemma 2.9. □

The following three definitions interact and their terms are therefore defined simultaneously.

Definition 3.2.10. *Let F^α be a Curry-Howard term, $\underline{x} = x_1, \ldots, x_n$ be distinct* *individual variables, and* $\underline{t} = t_1, \ldots, t_n$ *be individual terms. The result of simultaneously substituting* $t_1, \ldots, t_n$ *for all free occurrences of* $x_1, \ldots, x_n$*, respectively, in* F^α*, denoted by* $F^\alpha[x_1/t_1, \ldots, x_n/t_n]$ *or* $F^\alpha[\underline{x}/\underline{t}]$*, is defined inductively as follows.*

i. $X^\alpha[\underline{x}/\underline{t}] = X^{\alpha[\underline{x}/\underline{t}]}$.

ii. $(G^\beta, H^\gamma)[\underline{x}/\underline{t}] = (G^\beta[\underline{x}/\underline{t}], H^\gamma[\underline{x}/\underline{t}])$.

Similarly for $(\pi_i G^{\beta\wedge\gamma})[\underline{x}/\underline{t}]$, $i = 1, 2$, $(G^{\beta\supset\alpha}(H^\beta))[\underline{x}/\underline{t}]$, $(G^{\forall y\beta}(s))[\underline{x}/\underline{t}]$, *and* $(G^\perp(\alpha))[\underline{x}/\underline{t}]$.

iii. $(\mu_1 G^\beta)^{\beta\vee\gamma}[\underline{x}/\underline{t}] = (\mu_1 G^\beta[\underline{x}/\underline{t}])^{(\beta\vee\gamma)[\underline{x}/\underline{t}]}$.

Similarly for $(\mu_2 G^\beta)^{\gamma\vee\beta}[\underline{x}/\underline{t}]$ *and* $I(s, G^{\beta(y/s)})^{\exists y\beta}[\underline{x}/\underline{t}]$.

iv. $(G^{\forall_2 P\beta}(T))[\underline{x}/\underline{t}] = G^{\forall_2 P\beta}[\underline{x}/\underline{t}](T)$.

v. $I(T, G^{\beta(P/T)})^{\exists_2 P\beta}[\underline{x}/\underline{t}] = I(T, G^{\beta(P/T)}[\underline{x}/\underline{t}])^{(\exists_2 P\beta)[\underline{x}/\underline{t}]}$.

vi. $(\lambda X^\beta.G^\gamma)[\underline{x}/\underline{t}] = \lambda(X^\beta.G^\gamma)[\underline{x}/\underline{t}]$*, where*

$(X^\beta.G^\gamma)[\underline{x}/\underline{t}] = Y^{\beta[\underline{x}/\underline{t}]}.(G^\gamma[X^\beta/Y^\beta][\underline{x}/\underline{t}])$*, where* Y^β *is* X^β *if* $X^\beta.G^\gamma$ *has no free term variable* X^σ *such that* $\sigma[\underline{x}/\underline{t}] \equiv \beta[\underline{x}/\underline{t}]$*, otherwise* Y^β *is the first term variable of type* $[\beta]$ *such that* $X^\beta.G^\gamma$ *has no free term variable* Y^σ *where* $\sigma[\underline{x}/\underline{t}] \equiv \beta[\underline{x}/\underline{t}]$.

vii. $\oplus(X^\beta.G^\alpha, Y^\gamma.H^\alpha, K^{\beta\vee\gamma})[\underline{x}/\underline{t}]$

$$= \oplus((X^\beta.G^\alpha)[\underline{x}/\underline{t}], (Y^\gamma.H^\alpha)[\underline{x}/\underline{t}], K^{\beta\vee\gamma}[\underline{x}/\underline{t}]).$$

viii. $(\lambda y.G^\beta)[\underline{x}/\underline{t}] = \lambda(y.G^\beta)[\underline{x}/\underline{t}]$*, where*

$(y.G^\beta)[\underline{x}/\underline{t}] = y'.(G^\beta[y/y'][\underline{x}^*/\underline{t}^*])$*, where* $\underline{x}^*$ *is the sublist of* $\underline{x}$ *consisting of those* x_i*'s which are in* $fv(y.G^\beta)$, $\underline{t}^*$ *is the corresponding sublist of* $\underline{t}$*, and* y' *is* y *if* $y \notin fv(\underline{t}^*)$*, otherwise* y' *is the first individual variable which is not in* $fv(G^\beta) \cup fv(\underline{t}^*)$.

ix. $ST(y.X^\beta.G^\alpha, H^{\exists y\beta})[\underline{x}/\underline{t}] = ST((y.X^\beta.G^\alpha)[\underline{x}/\underline{t}], H^{\exists y\beta}[\underline{x}/\underline{t}])$*, where*

$(y.X^\beta.G^\alpha)[\underline{x}/\underline{t}] = y'.((X^\beta.G^\alpha)[y/y'][\underline{x}^*/\underline{t}^*])$*, where* $\underline{x}^*$ *is the sublist of* $\underline{x}$ *consisting of those* x_i*'s which are in* $fv(y.X^\beta.G^\alpha)$, $\underline{t}^*$ *is the corresponding sublist of*

$\underline{t}$, *and* y' *is* y *if* $y \notin fv(\underline{t}^*)$, *otherwise* y' *is the first individual variable which is not in* $fv(X^\beta.G^\alpha) \cup fv(\underline{t}^*)$.

x. $(\lambda Q.G^\beta)[\underline{x}/\underline{t}] = \lambda(Q.G^\beta)[\underline{x}/\underline{t}]$, *where* $(Q.G^\beta)[\underline{x}/\underline{t}] = Q.(G^\beta[\underline{x}/\underline{t}])$.

xi. $ST(Q.X^\beta.G^\alpha, H^{\exists_2 Q\beta})[\underline{x}/\underline{t}] = ST((Q.X^\beta.G^\alpha)[\underline{x}/\underline{t}], H^{\exists_2 Q\beta}[\underline{x}/\underline{t}])$, *where* $(P.X^\beta.G^\alpha)[\underline{x}/\underline{t}] = P.((X^\beta.G^\alpha)[\underline{x}/\underline{t}])$.

Definition 3.2.11. *Let* F^α *be a Curry-Howard term,* $\underline{P} = P_1^{n_1}, \ldots, P_m^{n_m}$ *be distinct predicate variables, and* $\underline{T} = T_1, \ldots, T_m$, *where* $T_i = \lambda x_1^i, \ldots, x_{n_i}^i \delta_i$, $1 \leq i \leq m$, *be abstraction terms. We define* $F^\alpha[P_1/T_1, \ldots, P_m/T_m]$, *which can be written as* $F^\alpha[\underline{P}/\underline{T}]$, *inductively as follows.*

i. $X^\alpha[\underline{P}/\underline{T}] = X^{\alpha[\underline{P}/\underline{T}]}$.

ii. $(G^\beta, H^\gamma)[\underline{P}/\underline{T}] = (G^\beta[\underline{P}/\underline{T}], H^\gamma[\underline{P}/\underline{T}])$.

Similarly for $(\pi_i G^{\beta\wedge\gamma})[\underline{P}/\underline{T}]$, $i = 1, 2$, $(G^{\beta\supset\alpha}(H^\beta))[\underline{P}/\underline{T}]$, $(G^{\forall_2 Q\beta}(U))[\underline{P}/\underline{T}]$, *and* $(G^\perp(\alpha))[\underline{P}/\underline{T}]$.

iii. $(\mu_1 G^\beta)^{\beta\vee\gamma}[\underline{P}/\underline{T}] = (\mu_1 G^\beta[\underline{P}/\underline{T}])^{(\beta\vee\gamma)[\underline{P}/\underline{T}]}$.

Similarly for $(\mu_2 G^\beta)^{\gamma\vee\beta}[\underline{P}/\underline{T}]$ *and* $I(U, G^{\beta(Q/U)})^{\exists_2 Q\beta}[\underline{P}/\underline{T}]$.

iv. $(G^{\forall y\beta}(u))[\underline{P}/\underline{T}] = G^{\forall y\beta}[\underline{P}/\underline{T}](u)$.

v. $I(u, G^{\beta(y/t)})^{\exists y\beta}[\underline{P}/\underline{T}] = I(u, G^{\beta(y/u)}[\underline{P}/\underline{T}])^{(\exists y\beta)[\underline{P}/\underline{T}]}$.

vi. $(\lambda Y^\beta.G^\gamma)[\underline{P}/\underline{T}] = \lambda(Y^\beta.G^\gamma)[\underline{P}/\underline{T}]$, *where*

$(Y^\beta.G^\gamma)[\underline{P}/\underline{T}] = Z^{\beta[\underline{P}/\underline{T}]}.(G^\gamma[T^\beta/Z^\beta][\underline{P}/\underline{T}])$, *where* Z^β *is* Y^β *if* $Y^\beta.G^\gamma$ *has no free term variable* Y^σ *such that* $\sigma[\underline{P}/\underline{T}] \equiv \beta[\underline{P}/\underline{T}]$, *otherwise* Z^β *is the first term variable of type* $[\beta]$ *such that* $Y^\beta.G^\gamma$ *has no free term variable* Z^σ *where* $\sigma[\underline{P}/\underline{T}] \equiv \beta[\underline{P}/\underline{T}]$.

vii. $\oplus(Y^\beta.G^\alpha, Z^\gamma.H^\alpha, K^{\beta\vee\gamma})[\underline{P}/\underline{T}]$

$$= \oplus((Y^\beta.G^\alpha)[\underline{P}/\underline{T}], (Z^\gamma.H^\alpha)[\underline{P}/\underline{T}], K^{\beta\vee\gamma}[\underline{P}/\underline{T}]).$$

viii. $(\lambda y.G^\beta)[\underline{P}/\underline{T}] = \lambda(y.G^\beta)[\underline{P}/\underline{T}]$, *where* $(y.G^\beta)[\underline{P}/\underline{T}] = y.(G^\beta[\underline{P}/\underline{T}])$.

ix. $ST(y.Y^\beta.G^\alpha, H^{\exists y\beta})[\underline{P}/\underline{T}] = ST((y.Y^\beta.G^\alpha)[\underline{P}/\underline{T}], H^{\exists y\beta}[\underline{P}/\underline{T}])$, *where*

$(y.Y^{\beta}.G^{\alpha})[\underline{P}/\underline{T}] = y.((Y^{\beta}.G^{\alpha})[\underline{P}/\underline{T}])$.

x. $(\lambda Q.G^{\beta})[\underline{P}/\underline{T}] = \lambda(Q.G^{\beta})[\underline{P}/\underline{T}]$, *where*

$(Q.G^{\beta})[\underline{P}/\underline{T}] = Q'.(G^{\beta}[Q/Q'][\underline{P}^*/\underline{T}^*])$, *where* $\underline{P}^*$ *is the sublist of* $\underline{P}$ *consisting of those* P_i*'s which are in* $FV(Q.G^{\beta})$, $\underline{T}^*$ *is the corresponding sublist of* $\underline{T}$, *and* Q' *is* Q *if* $Q \notin FV(\underline{T}^*)$, *otherwise* Q' *is the first predicate variable with the same arity as* Q *which is not in* $FV(G^{\beta}) \cup FV(\underline{T}^*)$.

xi. $ST(Q.Y^{\beta}.G^{\alpha}, H^{\exists_2 Q\beta})[\underline{P}/\underline{T}] = ST((Q.Y^{\beta}.G^{\alpha})[\underline{P}/\underline{T}], H^{\exists_2 Q\beta}[\underline{P}/\underline{T}])$, *where*

$(Q.Y^{\beta}.G^{\alpha})[\underline{P}/\underline{T}] = Q'.((Y^{\beta}.G^{\alpha})[Q/Q'][\underline{P}^*/\underline{T}^*])$, *where* $\underline{P}^*$ *is the sublist of* $\underline{P}$ *consisting of those* P_i*'s which are in* $FV(Q.Y^{\beta}.G^{\alpha})$, $\underline{T}^*$ *is the corresponding sublist of* $\underline{T}$, *and* Q' *is* Q *if* $Q \notin FV(\underline{T}^*)$, *otherwise* Q' *is the first predicate variable with the same arity as* Q *which is not in* $FV(Y^{\beta}.G^{\alpha}) \cup FV(\underline{T}^*)$.

Definition 3.2.12. *Let* F^{α} *be a Curry-Howard term,* $\underline{X} = X_1^{\delta_1}, \ldots, X_n^{\delta_n}$ *be inequivalent term variables, and* $\underline{K} = K_1^{\delta'_1}, \ldots, K_n^{\delta'_n}$ *be Curry-Howard terms, where* $\delta_i \equiv \delta'_i$ *for all* $1 \leq i \leq n$*. The result of simultaneously substituting* $K_1^{\delta'_1}, \ldots, K_n^{\delta'_n}$ *for all free occurrences of term variables which are equivalent to* $X_1^{\delta_1}, \ldots, X_n^{\delta_n}$*, respectively, in* F^{α}*, denoted by* $F^{\alpha}[X_1^{\delta_1}/K_1^{\delta'_1}, \ldots, X_n^{\delta_n}/K_n^{\delta'_n}]$ *or* $F^{\alpha}[\underline{X}/\underline{K}]$*, is defined inductively as follows.*

i.

$$Y^{\alpha}[\underline{X}/\underline{K}] = \begin{cases} K_m^{\delta'_m} & \text{if } Y^{\alpha} \equiv X_m^{\delta_m} \text{ for some } 1 \leq m \leq n, \\ Y^{\alpha} & \text{otherwise.} \end{cases}$$

ii. $(G^{\beta}, H^{\gamma})[\underline{X}/\underline{K}] = (G^{\beta}[\underline{X}/\underline{K}], H^{\gamma}[\underline{X}/\underline{K}])$.

Similarly for $(\pi_i G^{\beta\wedge\gamma})[\underline{X}/\underline{K}]$, $(\mu_i G^{\beta})[\underline{X}/\underline{K}]$, $i = 1, 2$, *and* $(G^{\beta\supset\alpha}(H^{\beta}))[\underline{X}/\underline{K}]$.

iii. $(G^{\forall y\beta}(u))[\underline{X}/\underline{K}] = G^{\forall y\beta}[\underline{X}/\underline{K}](u)$.

Similarly for $I(u, G^{\beta(y/u)})[\underline{X}/\underline{K}], (G^{\forall_2 Q\beta}(U))[\underline{X}/\underline{K}]$, $I(U, G^{\beta(Q/U)})[\underline{X}/\underline{K}]$, *and* $(G^{\perp}(\alpha))[\underline{X}/\underline{K}]$.

iv. $(\lambda y.G^{\beta})[\underline{X}/\underline{K}] = \lambda(y.G^{\beta})[\underline{X}/\underline{K}]$, *where*

$(y.G^\beta)[\underline{X}/\underline{K}] = y'.(G^\beta[y/y'][\underline{X}^*/\underline{K}^*])$, *where* $\underline{X}^*$ *is the sublist of* $\underline{X}$ *consisting of those* $X_i^{\delta_i}$*'s which are equivalent to some free term variables of* G^β, $\underline{K}^*$ *is the corresponding sublist of* $\underline{K}$, *and* y' *is* y *if* $y \notin fv(\underline{K}^*)$, *otherwise* y' *is the first individual variable which is not in* $fv(G^\beta) \cup fv(\underline{K}^*)$.

v. $(\lambda Q.G^\beta)[\underline{X}/\underline{K}] = \lambda(Q.G^\beta)[\underline{X}/\underline{K}]$, *where*

$(Q.G^\beta)[\underline{X}/\underline{K}] = Q'.(G^\beta[Q/Q'][\underline{X}^*/\underline{K}^*])$, *where* $\underline{X}^*$ *is the sublist of* $\underline{X}$ *consisting of those* $X_i^{\delta_i}$*'s which are equivalent to some free term variables of* G^β, $\underline{K}^*$ *is the corresponding sublist of* $\underline{K}$, *and* Q' *is* Q *if* $Q \notin FV(\underline{K}^*)$, *otherwise* Q' *is the first predicate variable with the same arity as* Q *which is not in* $FV(G^\beta) \cup FV(\underline{K}^*)$.

vi. $(\lambda Y^\gamma.G^\beta)[\underline{X}/\underline{K}] = \lambda(Y^\gamma.G^\beta)[\underline{X}/\underline{K}]$, *where*

$(Y^\gamma.G^\beta)[\underline{X}/\underline{K}] = Z^\gamma.(G^\beta[Y^\gamma/Z^\gamma][\underline{X}^*/\underline{K}^*])$, *where* $\underline{X}^*$ *is the sublist of* $\underline{X}$ *consisting of those* $X_i^{\delta_i}$*'s which are equivalent to some free term variables of* $Y^\gamma.G^\beta$, $\underline{K}^*$ *is the corresponding sublist of* $\underline{K}$, *and* Z^γ *is* Y^γ *if* Y^γ *is not equivalent to any free term variable in* $\underline{K}^*$, *otherwise* Z^γ *is the first term variable of type* $[\gamma]$ *which is not equivalent to any free term variable in* $\underline{K}^*$ *or* G^β.

vii. $\oplus(Y^\beta.G^\alpha, Z^\gamma.H^\alpha, J^{\beta\vee\gamma})[\underline{X}/\underline{K}] =$

$$\oplus((Y^\beta.G^\alpha)[\underline{X}/\underline{K}], (Z^\gamma.H^\alpha)[\underline{X}/\underline{K}], J^{\beta\vee\gamma}[\underline{X}/\underline{K}]).$$

viii. $ST(y.Y^\beta.G^\alpha, H^{\exists y\beta})[\underline{X}/\underline{K}] = ST((y.Y^\beta.G^\alpha)[\underline{X}/\underline{K}], H^{\exists y\beta}[\underline{X}/\underline{K}])$, *where*

$(y.Y^\beta.G^\alpha)[\underline{X}/\underline{K}] = y'.((Y^\beta.G^\alpha)[y/y'][\underline{X}^*/\underline{K}^*])$, *where* $\underline{X}^*$ *is the sublist of* $\underline{X}$ *consisting of those* $X_i^{\delta_i}$*'s which are equivalent to some free term variables of* $Y^\beta.G^\alpha$, $\underline{K}^*$ *is the corresponding sublist of* $\underline{K}$, *and* y' *is* y *if* $y \notin fv(\underline{K}^*)$, *otherwise* y' *is the first individual variable which is not in* $fv(Y^\beta.G^\alpha) \cup fv(\underline{K}^*)$.

ix. $ST(Q.Y^\beta.G^\alpha, H^{\exists_2 Q\beta})[\underline{X}/\underline{K}] = ST((Q.Y^\beta.G^\alpha)[\underline{X}/\underline{K}], H^{\exists_2 Q\beta}[\underline{X}/\underline{K}])$, *where*

$(Q.Y^\beta.G^\alpha)[\underline{X}/\underline{K}] = Q'.((Y^\beta.G^\alpha)[Q/Q'][\underline{X}^*/\underline{K}^*])$, $\underline{X}^*$ *is the sublist of* $\underline{X}$ *consisting of those* $X_i^{\delta_i}$*'s which are equivalent to some free term variables of* $Y^\beta.G^\alpha$, $\underline{K}^*$ *is the corresponding sublist of* $\underline{K}$, *and* Q' *is* Q *if* $Q \notin FV(\underline{K}^*)$, *otherwise* Q' *is* *the first predicate variable with the same arity as* Q *which is not in* $FV(Y^\beta.G^\alpha)\cup FV(\underline{K}^*)$.

Notation. $Y^\beta.F^\alpha[\underline{x}/\underline{t}]$ will abbreviate $Y^\beta.(F^\alpha[\underline{x}/\underline{t}])$. Similarly for $Y^\beta.F^\alpha[\underline{P}/\underline{T}]$ and $Y^\beta.F^\alpha[\underline{X}/\underline{K}]$.

Y^β in the above statement can also be replaced by y, Q, $y.Y^\beta$ or $Q.Y^\beta$.

Note. From the above definitions, it can be proved by induction on F^α that

a. $F^\alpha[\underline{x}/\underline{t}] = F^\alpha[\underline{x}^*/\underline{t}^*]$, where $\underline{x}^*$ is the sublist of $\underline{x}$ consisting of those x_i's which are in $fv(F^\alpha)$ and $\underline{t}^*$ is the corresponding sublist of $\underline{t}$;

similarly for $F^\alpha[\underline{P}/\underline{T}]$;

b. $F^\alpha[\underline{X}/\underline{K}] = F^\alpha[\underline{X}^*/\underline{K}^*]$, where $\underline{X}^*$ is the sublist of $\underline{X}$ consisting of those $X_i^{\delta_i}$'s which are equivalent to some free term variables of F^α and $\underline{K}^*$ is the corresponding sublist of $\underline{K}$;

c. $F^\alpha[\underline{x}/\underline{x}] = F^\alpha$, $F^\alpha[\underline{P}/\underline{P}] = F^\alpha$, and $F^\alpha[\underline{X}/\underline{X}] = F^\alpha$;

d. $FV(F^\alpha[\underline{x}/\underline{t}]) = FV(F^\alpha)$ and $fv(F^\alpha[\underline{P}/\underline{T}]) = fv(F^\alpha)$;

e. if $\underline{x}^*$ is the sublist of $\underline{x}$ consisting of those variables which are in $fv(F^\alpha)$ and $\underline{t}^*$ is the corresponding sublist of $\underline{t}$, then $fv(F^\alpha[\underline{x}/\underline{t}]) = (fv(F^\alpha) - \{\underline{x}^*\}) \cup fv(\underline{t}^*)$;

similarly for $FV(F^\alpha[\underline{P}/\underline{T}])$;

f. $fv(F^\alpha) \subseteq fv(F^\alpha[\underline{X}/\underline{K}])$ and $FV(F^\alpha) \subseteq FV(F^\alpha[\underline{X}/\underline{K}])$;

g. if $\underline{Y} = Y_1^{\tau_1}, \ldots, Y_n^{\tau_n}$ are term variables such that $Y_i^{\tau_1} \equiv X_i^{\delta_i}$ for all $1 \le i \le n$, then $F[\underline{Y}/\underline{K}] = F[\underline{X}/\underline{K}]$.

Lemma 3.2.13. *Let* F^α *be a C-H term,* $\underline{x} = x_1, \ldots, x_n$ *be distinct individual variables, and* $\underline{t} = t_1, \ldots, t_n$ *be individual terms.*

Then $F^\alpha[\underline{x}/\underline{t}]$ *is a C-H term of type* $[\alpha[\underline{x}/\underline{t}]]$.

Lemma 3.2.14. *Let* F^α *be a C-H term,* $\underline{P} = P_1^{n_1}, \ldots, P_m^{n_m}$ *be distinct predicate variables, and* $\underline{T} = T_1, \ldots, T_m$*, where* $T_i = \lambda x_1^i, \ldots, x_{n_i}^i \delta_i$*,* $1 \le i \le m$*, be abstraction terms.*

Then $F^\alpha[\underline{P}/\underline{T}]$ *is a C-H term of type* $[\alpha[\underline{P}/\underline{T}]]$.

Lemma 3.2.15. *Let* F^α *be a C-H term,* $\underline{X} = X_1^{\delta_1}, \ldots, X_n^{\delta_n}$ *be inequivalent term variables, and* $\underline{K} = K_1^{\delta'_1}, \ldots, K_n^{\delta'_n}$ *be C-H terms, where* $\delta_i \equiv \delta'_i$ *for all* $1 \leq i \leq n$.

Then $F^\alpha[\underline{X}/\underline{K}]$ *is a C-H term of type* $[\alpha]$.

Proof. We will prove these three lemmas simultaneously by induction on F^α.

Proof of Lemma 3.2.13.

(i) $F^\alpha = X^\alpha$.

Then $F^\alpha[\underline{x}/\underline{t}] = X^{\alpha[\underline{x}/\underline{t}]}$ which is a term variable of type $[\alpha[\underline{x}/\underline{t}]]$.

(ii) $F^\alpha = (G^\beta, H^\gamma)$.

Then $F^\alpha[\underline{x}/\underline{t}] = (G^\beta[\underline{x}/\underline{t}], H^\gamma[\underline{x}/\underline{t}])$. By the induction hypothesis, $G^\beta[\underline{x}/\underline{t}]$ and $H^\gamma[\underline{x}/\underline{t}]$ are C-H terms of types $[\beta[\underline{x}/\underline{t}]]$ and $[\gamma[\underline{x}/\underline{t}]]$, respectively. Hence $F^\alpha[\underline{x}/\underline{t}]$ is a C-H term of type $[\beta[\underline{x}/\underline{t}] \wedge \gamma[\underline{x}/\underline{t}]] = [(\beta \wedge \gamma)[\underline{x}/\underline{t}]] = [\alpha[\underline{x}/\underline{t}]]$.

Similarly for $\pi_i G^{\beta\wedge\gamma}$, $\mu_i G^\beta, i = 1, 2$, $G^{\beta\supset\alpha}(H^\beta)$, and $G^\perp(\alpha)$.

(iii) $F^\alpha = G^{\forall y\beta}(u)$.

Then $[\alpha] = [\beta[y/u]]$ and $F^\alpha[\underline{x}/\underline{t}] = G^{\forall y\beta}[\underline{x}/\underline{t}](u[\underline{x}/\underline{t}])$. By the induction hypothesis, $G^{\forall y\beta}[\underline{x}/\underline{t}]$ is a C-H term of type $[(\forall y\beta)[\underline{x}/\underline{t}]]$.

We have $(\forall y\beta)[\underline{x}/\underline{t}] = \forall y'\beta[y/y'][\underline{x}^*/\underline{t}^*]$, where $\underline{x}^*$ is the sublist of $\underline{x}$ consisting of those x_i's which are in $fv(\forall y\beta)$, $\underline{t}^*$ is the corresponding sublist of $\underline{t}$, y' is y if $y \notin fv(\underline{t}^*)$, otherwise y' is the first individual variable which is not in $fv(\beta) \cup fv(\underline{t}^*)$. Hence $F^\alpha[\underline{x}/\underline{t}]$ is a C-H term of type $[\beta[y/y'][\underline{x}^*/\underline{t}^*][y'/u[\underline{x}/\underline{t}]]] = [\beta[y/y'][y'/u][\underline{x}/\underline{t}]] = [\beta[y/u][\underline{x}/\underline{t}]] = [\alpha[\underline{x}/\underline{t}]]$ by Lemmas 2.13 and 2.14.

(iv) $F^\alpha = I(u, G^{\beta(y/u)})^{\exists y\beta}$.

Then $F^\alpha[\underline{x}/\underline{t}] = I(u[\underline{x}/\underline{t}], G^{\beta(y/u)}[\underline{x}/\underline{t}])^{(\exists y\beta)[\underline{x}/\underline{t}]}$. By the induction hypothesis, $G^{\beta(y/u)}[\underline{x}/\underline{t}]$ is a C-H term of type $[\beta(y/u)[\underline{x}/\underline{t}]] = [\beta[y/u][\underline{x}/\underline{t}]]$.

We have $(\exists y\beta)[\underline{x}/\underline{t}] = \exists y'\beta[y/y'][\underline{x}^*/\underline{t}^*]$, where $\underline{x}^*$ is the sublist of $\underline{x}$ consisting of those x_i's which are in $fv(\exists y\beta)$, $\underline{t}^*$ is the corresponding sublist of $\underline{t}$,

y' is y if $y \notin fv(\underline{t}^*)$, otherwise y' is the first individual variable which is not in $fv(\beta) \cup fv(\underline{t}^*)$. Similar to (iii), $[\beta[y/u][\underline{x}/\underline{t}]] = [\beta[y/y'][\underline{x}^*/\underline{t}^*][y'/u[\underline{x}/\underline{t}]]]$, so $G^{\beta(y/u)}[\underline{x}/\underline{t}]$ is of type $[\beta[y/y'][\underline{x}^*/\underline{t}^*][y'/u[\underline{x}/\underline{t}]]]$. Hence $F^\alpha[\underline{x}/\underline{t}]$ is a C-H term of type $[\exists y' \beta[y/y'][\underline{x}^*/\underline{t}^*]] = [\alpha[\underline{x}/\underline{t}]]$.

(v) $F^\alpha = \lambda y.G^\beta$.

We have $F^\alpha[\underline{x}/\underline{t}] = \lambda y'.G^\beta[y/y'][\underline{x}^*/\underline{t}^*]$, where $\underline{x}^*$ is the sublist of $\underline{x}$ consisting of those x_i's which are in $fv(y.G^\beta)$, $\underline{t}^*$ is the corresponding sublist of $\underline{t}$, y' is y if $y \notin fv(\underline{t}^*)$, otherwise y' is the first individual variable which is not in $fv(G^\beta) \cup fv(\underline{t}^*)$. By the induction hypothesis, $G^\beta[y/y'][\underline{x}^*/\underline{t}^*]$ is a C-H term of type $[\beta[y/y'][\underline{x}^*/\underline{t}^*]]$. Thus $F^\alpha[\underline{x}/\underline{t}]$ is a C-H term of type $[\forall y' \beta[y/y'][\underline{x}^*/\underline{t}^*]] = [(\forall y' \beta[y/y'])[\underline{x}/\underline{t}]] = [(\forall y \beta)[\underline{x}/\underline{t}]] = [\alpha[\underline{x}/\underline{t}]]$ by Corollary 2.11 and Lemma 2.14.

(vi) $F^\alpha = \lambda Q.G^\beta$.

Then $F^\alpha[\underline{x}/\underline{t}] = \lambda Q.G^\beta[\underline{x}/\underline{t}]$. By the induction hypothesis, $G^\beta[\underline{x}/\underline{t}]$ is a C-H term of type $[\beta[\underline{x}/\underline{t}]]$. Thus $F^\alpha[\underline{x}/\underline{t}]$ is a C-H term of type $[\forall_2 Q \beta[\underline{x}/\underline{t}]] = [(\forall_2 Q \beta)[\underline{x}/\underline{t}]] = [\alpha[\underline{x}/\underline{t}]]$.

(vii) $F^\alpha = G^{\forall_2 Q\beta}(U)$.

Then $[\alpha] = [\beta[Q/U]]$ and $F^\alpha[\underline{x}/\underline{t}] = G^{\forall_2 Q\beta}[\underline{x}/\underline{t}](U)$. By the induction hypothesis, $G^{\forall_2 Q\beta}[\underline{x}/\underline{t}]$ is a C-H term of type $[(\forall_2 Q\beta)[\underline{x}/\underline{t}]] = [\forall_2 Q \beta[\underline{x}/\underline{t}]]$. Hence $F^\alpha[\underline{x}/\underline{t}]$ is a C-H term of type $[\beta[\underline{x}/\underline{t}][Q/U]] = [\beta[Q/U][\underline{x}/\underline{t}]] = [\alpha[\underline{x}/\underline{t}]]$ by Lemmas 2.14 and 2.15.

(viii) $F^\alpha = I(U, G^{\beta(Q/U)})^{\exists_2 Q\beta}$.

Then $F^\alpha[\underline{x}/\underline{t}] = I(U, G^{\beta(Q/U)}[\underline{x}/\underline{t}])$. By the induction hypothesis, $G^{\beta(Q/U)}[\underline{x}/\underline{t}]$ is a C-H term of type $[\beta(Q/U)[\underline{x}/\underline{t}]] = [\beta[Q/U][\underline{x}/\underline{t}]] = [\beta[\underline{x}/\underline{t}][Q/U]]$ by Lemmas 2.14 and 2.15. Hence $F^\alpha[\underline{x}/\underline{t}]$ is a C-H term of type $[\exists_2 Q \beta[\underline{x}/\underline{t}]] = [(\exists_2 Q\beta)[\underline{x}/\underline{t}]] = [\alpha[\underline{x}/\underline{t}]]$.

(ix) $F^\alpha = \lambda X^\beta.G^\gamma$.

We have $F^\alpha[\underline{x}/\underline{t}] = \lambda(X^\beta.G^\gamma)[\underline{x}/\underline{t}]$. By the following claim, $F^\alpha[\underline{x}/\underline{t}]$ is a C-H term of type $[\beta[\underline{x}/\underline{t}] \supset \gamma[\underline{x}/\underline{t}]] = [\alpha[\underline{x}/\underline{t}]]$.

Claim 1. $(X^\beta.G^\gamma)[\underline{x}/\underline{t}] = Y^{\beta^*}.H^{\gamma^*}$ for some term variable Y^{β^*} and some C-H term H^{γ^*} such that $[\beta^*] = [\beta[\underline{x}/\underline{t}]]$ and $[\gamma^*] = [\gamma[\underline{x}/\underline{t}]]$.

Proof of Claim 1. We have $(X^\beta.G^\gamma)[\underline{x}/\underline{t}] = Y^{\beta[\underline{x}/\underline{t}]}.G^\gamma[X^\beta/Y^\beta][\underline{x}/\underline{t}]$, where Y^β is X^β if $X^\beta.G^\gamma$ has no free term variable X^σ where $\sigma[\underline{x}/\underline{t}] \equiv \beta[\underline{x}/\underline{t}]$, otherwise Y^β is the first term variable of type $[\beta]$ such that $X^\beta.G^\gamma$ has no free term variable Y^σ where $\sigma[\underline{x}/\underline{t}] \equiv \beta[\underline{x}/\underline{t}]$.

By the induction hypothesis, $G^\gamma[X^\beta/Y^\beta][\underline{x}/\underline{t}]$ is a C-H term of type $[\gamma[\underline{x}/\underline{t}]]$.

Thus we have the claim.

(x) $F^\alpha = \oplus(X^\beta.G^\alpha, Y^\gamma.H^\alpha, K^{\beta\vee\gamma})$.

Then $F^\alpha[\underline{x}/\underline{t}] = \oplus((X^\beta.G^\alpha)[\underline{x}/\underline{t}], (Y^\gamma.H^\alpha)[\underline{x}/\underline{t}], K^{\beta\vee\gamma}[\underline{x}/\underline{t}])$.

By the induction hypothesis, $K^{\beta\vee\gamma}[\underline{x}/\underline{t}]$ is a C-H term of type $[(\beta\vee\gamma)[\underline{x}/\underline{t}]] = [\beta[\underline{x}/\underline{t}] \vee \gamma[\underline{x}/\underline{t}]]$. Hence, by the above claim, $F^\alpha[\underline{x}/\underline{t}]$ is a C-H term of type $[\alpha[\underline{x}/\underline{t}]]$.

(xi) $F^\alpha = ST(y.X^\beta.G^\alpha, H^{\exists y\beta})$.

We have $(y.X^\beta.G^\alpha)[\underline{x}/\underline{t}] = y'.((X^\beta.G^\alpha)[y/y'][\underline{x}^*/\underline{t}^*])$, where $\underline{x}^*$ is the sublist of $\underline{x}$ consisting of those x_i's which are in $fv(y.X^\beta.G^\alpha)$, $\underline{t}^*$ is the corresponding sublist of $\underline{t}$, y' is y if $y \notin fv(\underline{t}^*)$, otherwise y' is the first individual variable which is not in $fv(X^\beta.G^\alpha) \cup fv(\underline{t}^*)$.

By the claim, $(X^\beta.G^\alpha)[y/y'][\underline{x}^*/\underline{t}^*] = Y^{\beta^*}.K^{\alpha^*}$ for some term variable Y^{β^*} and some C-H term K^{α^*} such that $[\beta^*] = [\beta[y/y'][\underline{x}^*/\underline{t}^*]]$ and $[\alpha^*] = [\alpha[y/y'][\underline{x}^*/\underline{t}^*]] = \alpha[\underline{x}^*/\underline{t}^*]$ since $y \notin fv(\alpha)$.

Since y' is y or $y' \notin fv(X^\beta.G^\alpha)$ (so $y' \notin fv(\beta)$), by Corollary 2.11, $\exists y\beta \equiv \exists y'\beta[y/y']$. By the induction hypothesis, $H^{\exists y\beta}[\underline{x}/\underline{t}]$ is a C-H term of type $[(\exists y\beta)[\underline{x}/\underline{t}]] = [(\exists y'\beta[y/y'])[\underline{x}/\underline{t}]] = [\exists y'\beta[y/y'][\underline{x}^*/\underline{t}^*]]$ by Lemma 2.14. Hence

$ST(y'.(X^\beta.G^\alpha)[y/y'][\underline{x}^*/\underline{t}^*], H^{\exists y\beta}[\underline{x}/\underline{t}])$ is a C-H term of type $[\alpha[\underline{x}^*/\underline{t}^*]]$ i.e. $F^\alpha[\underline{x}/\underline{t}]$ is a C-H term of type $[\alpha[\underline{x}/\underline{t}]]$.

(xii) $F^\alpha = ST(Q.X^\beta.G^\alpha, H^{\exists_2 Q\beta})$.

Then $F^\alpha[\underline{x}/\underline{t}] = ST(Q.(X^\beta.G^\alpha)[\underline{x}/\underline{t}], H^{\exists_2 Q\beta}[\underline{x}/\underline{t}])$. By the induction hypothesis, $H^{\exists_2 Q\beta}[\underline{x}/\underline{t}]$ is a C-H term of type $[(\exists_2 Q\beta)[\underline{x}/\underline{t}]] = [\exists_2 Q\beta[\underline{x}/\underline{t}]]$. Hence, by the claim, $F^\alpha[\underline{x}/\underline{t}]$ is a C-H term of type $[\alpha[\underline{x}/\underline{t}]]$.

Lemma 3.2.14 can be proved in the same way as Lemma 3.2.13 by using Lemmas 2.16 and 2.17 instead of Lemmas 2.13 and 2.14, respectively.

Proof of Lemma 3.2.15.

(i) F^α is a term variable.

If $F^\alpha \equiv X_m^{\delta_m}$ for some $1 \leq m \leq n$, then $F^\alpha[\underline{X}/\underline{K}] = K_m^{\delta'_m}$ which is a C-H term of type $[\delta'_m] = [\delta_m] = [\alpha]$, otherwise $F^\alpha[\underline{X}/\underline{K}] = F^\alpha$ which is a C-H term of type $[\alpha]$.

(ii) $F^\alpha = (G^\beta, H^\gamma)$.

This case follows by the induction hypothesis.

Similarly for $\pi_i G^{\beta\wedge\gamma}$, $\mu_i G^\beta, i = 1, 2$, $G^{\beta\supset\alpha}(H^\beta)$, and $G^\perp(\alpha)$.

(iii) $F^\alpha = \lambda y.G^\beta$.

Then $F^\alpha[\underline{X}/\underline{K}] = \lambda y'.G^\beta[y/y'][\underline{X}^*/\underline{K}^*]$, where $\underline{X}^*$ is the sublist of $\underline{X}$ consisting of those $X_i^{\delta_i}$'s which are equivalent to some free term variables of G^β, $\underline{K}^*$ is the corresponding sublist of $\underline{K}$, and y' is y if $y \notin fv(\underline{K}^*)$, otherwise y' is the first individual variable which is not in $fv(G^\beta) \cup fv(\underline{K}^*)$, so $y' \notin fv(\beta)$.

By the induction hypothesis, $G^\beta[y/y'][\underline{X}^*/\underline{K}^*]$ is a C-H term of type $[\beta[y/y']]$. Hence $F^\alpha[\underline{X}/\underline{K}]$ is a C-H term of type $[\forall y'\beta[y/y']] = [\forall y\beta] = [\alpha]$ by Corollary 2.11.

Similarly for $\lambda Q.G^\beta$.

(iv) $F^\alpha = G^{\forall y\beta}(u)$.

Then $F^{\alpha}[\underline{X}/\underline{K}] = G^{\forall y\beta}[\underline{X}/\underline{K}](u)$. By the induction hypothesis, $G^{\forall y\beta}[\underline{X}/\underline{K}]$ is a C-H term of type $[\forall y\beta]$. Hence $F^{\alpha}[\underline{X}/\underline{K}]$ is a C-H term of type $[\beta(y/u)] = [\alpha]$.

Similarly for $G^{\forall_2 Q\beta}(U)$, $I(u, G^{\beta(y/u)})^{\exists y\beta}$, and $I(U, G^{\beta(Q/U)})^{\exists_2 Q\beta}$.

(vi) $\lambda Y^{\beta}.G^{\gamma}$.

This case follows straightforwardly by the following claim.

Claim 2. $(Y^{\beta}.G^{\gamma})[\underline{X}/\underline{K}] = Z^{\beta'}.H^{\gamma'}$ for some term variable $Z^{\beta'}$ and some C-H term $H^{\gamma'}$ such that $[\beta'] = [\beta]$ and $[\gamma'] = [\gamma]$.

Proof of Claim 2. We have $(Y^{\beta}.G^{\gamma})[\underline{X}/\underline{K}] = Z^{\beta}.G^{\gamma}[Y^{\beta}/Z^{\beta}][\underline{X}^*/\underline{K}^*]$, where $\underline{X}^*$ is the sublist of $\underline{X}$ consisting of those X_i's which are equivalent to some free term variables of $Y^{\beta}.G^{\gamma}$, $\underline{K}^*$ is the corresponding sublist of $\underline{K}$, Z^{β} is Y^{β} if Y^{β} is not equivalent to any free term variable in $\underline{K}^*$, otherwise Z^{β} is the first term variable of type $[\beta]$ which is not equivalent to any free term variable in $\underline{K}^*$ or G^{γ}. By the induction hypothesis, $G^{\gamma}[Y^{\beta}/Z^{\beta}][\underline{X}^*/\underline{K}^*]$ is a C-H term of type $[\gamma]$. Hence we have the claim.

(vii) $\oplus(Y^{\beta}.G^{\alpha}, Z^{\gamma}.H^{\alpha}, K^{\beta\vee\gamma})$

This case follows straightforwardly by the above claim and the induction hypothesis.

(viii) $F^{\alpha} = ST(y.Y^{\beta}.G^{\alpha}, H^{\exists y\beta})$.

We have $F^{\alpha}[\underline{X}/\underline{K}] = ST(y'.(Y^{\beta}.G^{\alpha})[y/y'][\underline{X}^*/\underline{K}^*], H^{\exists y\beta}[\underline{X}/\underline{K}])$, where $\underline{X}^*$ is the sublist of $\underline{X}$ consisting of those X_i's which are equivalent to some free term variables of $Y^{\beta}.G^{\alpha}$, $\underline{K}^*$ is the corresponding sublist of $\underline{K}$, y' is y if $y \notin fv(\underline{K}^*)$, otherwise y' is the first individual variable which is not in $fv(Y^{\beta}.G^{\alpha}) \cup fv(\underline{K}^*)$.

By the claims, $(Y^{\beta}.G^{\alpha})[y/y'][\underline{X}^*/\underline{K}^*] = Z^{\beta^*}.K^{\alpha^*}$ for some term variable Z^{β^*} and some C-H term K^{α^*} such that $[\beta^*] = [\beta[y/y']]$ and $[\alpha^*] = [\alpha[y/y']] = [\alpha]$ since $y \notin fv(\alpha)$. By the induction hypothesis, $H^{\exists y\beta}[\underline{X}/\underline{K}]$ is a C-H term of type $[\exists y\beta] = [\exists y'\beta[y/y']]$. Hence $F^{\alpha}[\underline{X}/\underline{K}]$ is a C-H term of type $[\alpha]$.

Similarly for $ST(Q.Y^\beta.G^\alpha, H^{\exists_2 Q\beta})$. $\square$

Lemma 3.2.16. *For any C-H term F,*

a. if $\underline{x} = x_1, \dots, x_n$ are distinct individual variables and $\underline{t} = t_1, \dots, t_n$ are individual terms, then $F[\underline{x}/\underline{t}] = F'[\underline{x}/\underline{t}]$ for some C-H term F' such that $F' \equiv F$, $\underline{x}$ is replaceable by $\underline{t}$, and $\underline{t}$ is free for $\underline{x}$ in F';

b. if $\underline{P} = P_1^{r_1}, \dots, P_n^{r_n}$ are distinct predicate variables and $\underline{T} = T_1, \dots, T_n$, where $T_i = \lambda z_1^i, \dots, z_{r_i}^i \delta_i$, $1 \le i \le n$, are abstraction terms, then $F[\underline{P}/\underline{T}] = F'[\underline{P}/\underline{T}]$ for some C-H term F' such that $F' \equiv F$, $\underline{P}$ is replaceable by $\underline{T}$, and $\underline{T}$ is free for $\underline{P}$ in F';

c. if $\underline{X} = X_1^{\delta_1}, \dots, X_n^{\delta_n}$ are inequivalent term variables and $\underline{K} = K_1^{\delta_1'}, \dots, K_n^{\delta_n'}$ are C-H terms, where $\delta_i \equiv \delta_i'$ for all $1 \le i \le n$, then $F[\underline{X}/\underline{K}] = F'[\underline{X}/\underline{K}]$ for some C-H term F' such that $F' \equiv F$ and $\underline{K}$ is free for $\underline{X}$ in F'.

Proof. Let F^α be a C-H term. We will prove a, b, and c simultaneously by induction on F^α.

a: Let $\underline{x} = x_1, \dots, x_n$ be distinct individual variables and $\underline{t} = t_1, \dots, t_n$ be individual terms.

(i) $F^\alpha = X^\alpha$.

By Lemma 2.10, $\alpha[\underline{x}/\underline{t}] = \alpha'[\underline{x}/\underline{t}]$ for some formula α' such that $\alpha' \equiv \alpha$ and $\underline{t}$ is free for $\underline{x}$ in α'. So we have $X^\alpha[\underline{x}/\underline{t}] = X^{\alpha[\underline{x}/\underline{t}]} = X^{\alpha'[\underline{x}/\underline{t}]} = X^{\alpha'}[\underline{x}/\underline{t}]$ where $X^\alpha \equiv X^{\alpha'}$, $\underline{x}$ is replaceable by $\underline{t}$, and $\underline{t}$ is free for $\underline{x}$ in $X^{\alpha'}$.

(ii) $F = \lambda X^\beta.G$.

This case follows by the following claim.

Claim 1. $(X^\beta.G)[\underline{x}/\underline{t}] = C[\underline{x}/\underline{t}]$ for some context C such that $C \equiv X^\beta.G$ and $\underline{x}$ is replaceable by $\underline{t}$ and $\underline{t}$ is free for $\underline{x}$ in C.

Proof of Claim 1. We have $(X^\beta.G)[\underline{x}/\underline{t}] = Y^{\beta[\underline{x}/\underline{t}]}.G[X^\beta/Y^\beta][\underline{x}/\underline{t}]$, where Y^β is X^β if $X^\beta.G$ has no free term variable X^σ such that $\sigma[\underline{x}/\underline{t}] \equiv \beta[\underline{x}/\underline{t}]$, otherwise Y^β is the first term variable of type $[\beta]$ such that $X^{\beta}.G$ has no free term variable Y^{σ} where $\sigma[\underline{x}/\underline{t}] \equiv \beta[\underline{x}/\underline{t}]$.

By the induction hypothesis, $G[X^{\beta}/Y^{\beta}] = G'[X^{\beta}/Y^{\beta}]$ and $G'[X^{\beta}/Y^{\beta}][\underline{x}/\underline{t}] = G^{*}[\underline{x}/\underline{t}]$ for some terms G' and G^{*} such that $G' \equiv G$, $G^{*} \equiv G'[X^{\beta}/Y^{\beta}]$, Y^{β} is free for X^{β} in G', and $\underline{x}$ is replaceable by $\underline{t}$ and $\underline{t}$ is free for $\underline{x}$ in G^{*}. By Lemma 2.10, $\beta[\underline{x}/\underline{t}] = \beta'[\underline{x}/\underline{t}]$ for some formula β' such that $\beta' \equiv \beta$ and $\underline{t}$ is free for $\underline{x}$ in β'. So we have $(X^{\beta}.G)[\underline{x}/\underline{t}] = Y^{\beta'[\underline{x}/\underline{t}]}.G^{*}[\underline{x}/\underline{t}] = (Y^{\beta'}.G^{*})[\underline{x}/\underline{t}]$ where $Y^{\beta'}.G^{*} \equiv Y^{\beta}.G'[X^{\beta}/Y^{\beta}] \equiv X^{\beta}.G' \equiv X^{\beta}.G$ and $\underline{t}$ is free for $\underline{x}$ in $Y^{\beta'}.G^{*}$. Since $G' \equiv G$ and $X^{\beta}.G$ has no free term variable Y^{σ} where $\sigma[\underline{x}/\underline{t}] \equiv \beta[\underline{x}/\underline{t}]$, $X^{\beta}.G'$ also has no such Y^{σ}. Since $G^{*} \equiv G'[X^{\beta}/Y^{\beta}]$, $Y^{\beta'}.G^{*}$ has no free term variable Y^{σ} such that $\sigma[\underline{x}/\underline{t}] \equiv \beta[\underline{x}/\underline{t}]$. Hence $\underline{x}$ is replaceable by $\underline{t}$ in $Y^{\beta'}.G^{*}$.

The case $\oplus(X.G, Y.H, K)$ also follows by the above claim and the induction hypothesis.

(iii) $F = \lambda y.G$.

This case follows by the following claim.

Claim 2. $(y.G)[\underline{x}/\underline{t}] = C[\underline{x}/\underline{t}]$ for some context C such that $C \equiv y.G$ and $\underline{x}$ is replaceable by $\underline{t}$ and $\underline{t}$ is free for $\underline{x}$ in C.

Proof of Claim 2. We have $(y.G)[\underline{x}/\underline{t}] = y'.(G[y/y'][\underline{x}^{*}/\underline{t}^{*}])$, where $\underline{x}^{*}$ is the sublist of $\underline{x}$ consisting of those x_i's which are in $fv(y.G)$, $\underline{t}^{*}$ is the corresponding sublist of $\underline{t}$, y' is y if $y \notin fv(\underline{t}^{*})$, otherwise y' is the first individual variable which is not in $fv(G) \cup fv(\underline{t}^{*})$.

By the induction hypothesis, $G[y/y'] = G'[y/y']$ and $G'[y/y'][\underline{x}^{*}/\underline{t}^{*}] = G^{*}[\underline{x}^{*}/\underline{t}^{*}]$ for some terms G' and G^{*} such that $G' \equiv G$, $G^{*} \equiv G'[y/y']$, y is replaceable by y' and y' is free for y in G', and $\underline{x}^{*}$ is replaceable by $\underline{t}^{*}$ and $\underline{t}^{*}$ is free for $\underline{x}^{*}$ in G^{*}. Hence $(y.G)[\underline{x}/\underline{t}] = y'.G^{*}[\underline{x}^{*}/\underline{t}^{*}] = (y'.G^{*})[\underline{x}/\underline{t}]$, where $y'.G^{*} \equiv y'.G'[y/y'] \equiv y.G' \equiv y.G$ and $\underline{x}$ is replaceable by $\underline{t}$ and $\underline{t}$ is free for $\underline{x}$ in $y'.G^{*}$.

(iv) $F = ST(y.X.G, K)$.

This case follows by the following claim and the induction hypothesis.

Claim 3. $(y.X.G)[\underline{x}/\underline{t}] = C[\underline{x}/\underline{t}]$ for some context C such that $C \equiv y.X.G$ and $\underline{x}$ is replaceable by $\underline{t}$ and $\underline{t}$ is free for $\underline{x}$ in C.

Proof of Claim 3. We have $(y.X.G)[\underline{x}/\underline{t}] = y'.(X.G)[y/y'][\underline{x}^*/\underline{t}^*]$, where $\underline{x}^*$ is the sublist of $\underline{x}$ consisting of those x_i's which are in $fv(y.X.G)$, $\underline{t}^*$ is the corresponding sublist of $\underline{t}$, y' is y if $y \notin fv(\underline{t}^*)$, otherwise y' is the first individual variable which is not in $fv(X.G) \cup fv(\underline{t}^*)$.

By Claim 1, $(X.G)[y/y'] = C[y/y']$ and $C[y/y'][\underline{x}^*/\underline{t}^*] = C^*[\underline{x}^*/\underline{t}^*]$ for some contexts C and C^* such that $C \equiv X.G$, $C^* \equiv C[y/y']$, y is replaceable by y' and y' is free for y in C and $\underline{x}^*$ is replaceable by $\underline{t}^*$ and $\underline{t}^*$ is free for $\underline{x}^*$ in C^*. So we have $(y.X.G)[\underline{x}/\underline{t}] = y'.C^*[\underline{x}^*/\underline{t}^*] = (y'.C^*)[\underline{x}/\underline{t}]$, where $y'.C^* \equiv y'.C[y/y'] \equiv y.C \equiv y.X.G$, and $\underline{x}$ is replaceable by $\underline{t}$ and $\underline{t}$ is free for $\underline{x}$ in $y'.C^*$.

Similarly for the case $ST(Q.X.G, K)$.

The remaining cases follow straightforwardly by the induction hypothesis.

b: The proof is similar to that for a.

c: Let $\underline{X} = X_1^{\delta_1}, \ldots, X_n^{\delta_n}$ be inequivalent term variables and $\underline{K} = K_1^{\delta'_1}, \ldots, K_n^{\delta'_n}$ be C-H terms, where $\delta_i \equiv \delta'_i$ for all $1 \le i \le n$.

If F is a term variable, then $\underline{K}$ is free for $\underline{X}$ in F.

(i) $F = \lambda Z^\beta.G$.

This case follows by the following claim which can be proved in the same way as Claim 2 in the proof of a.

Claim 1. $(Z^\beta.G)[\underline{X}/\underline{K}] = C[\underline{X}/\underline{K}]$ for some context C such that $C \equiv Z^\beta.G$ and $\underline{K}$ is free for $\underline{X}$ in C.

The case $\oplus(Y.G, Z.H, K)$ also follows by the above claim and the induction hypothesis.

(ii) $F = \lambda y.G$.

This case follows by the following claim.

Claim 2. $(y.G)[\underline{X}/\underline{K}] = C[\underline{X}/\underline{K}]$ for some context C such that $C \equiv y.G$ and $\underline{K}$ is free for $\underline{X}$ in C.

Proof of Claim 2. We have $(y.G)[\underline{X}/\underline{K}] = y'.G[y/y'][\underline{X}^*/\underline{K}^*]$, where $\underline{X}^*$ is the sublist of $\underline{X}$ consisting of those $X_i^{\delta_i}$'s which are equivalent to some free term variable of F, $\underline{K}^*$ is the corresponding sublist of $\underline{K}$, y' is y if $y \notin fv(\underline{K}^*)$, otherwise y' is the first individual variable which is not in $fv(G) \cup fv(\underline{K}^*)$.

By the induction hypothesis, $G[y/y'] = G'[y/y']$ and $G'[y/y'][\underline{X}^*/\underline{K}^*] = G^*[\underline{X}^*/\underline{K}^*]$ for some terms G' and G^* such that $G' \equiv G$, $G^* \equiv G'[y/y']$, y is replaceable by y' and y' is free for y in G', and $\underline{K}^*$ is free for $\underline{X}^*$ in G^*.

Hence $(y.G)[\underline{X}/\underline{K}] = y'.G^*[\underline{X}^*/\underline{K}^*] = (y'.G^*)[\underline{X}/\underline{K}]$, where $y'.G^* \equiv y'.G'[y/y']$ $\equiv y.G' \equiv y.G$ and $\underline{K}$ is free for $\underline{X}$ in $y'.G^*$.

Similarly for the case $\lambda Q.G$.

(iii) $F = ST(y.Z.G, H)$.

This case follows by the induction hypothesis and the following claim.

Claim 3. $(y.Z.G)[\underline{X}/\underline{K}] = C[\underline{X}/\underline{K}]$ for some context C such that $C \equiv y.Z.G$ and $\underline{K}$ is free for $\underline{X}$ in C.

Proof of Claim 3. We have $(y.Z.G)[\underline{X}/\underline{K}] = y'.(Z.G)[y/y'][\underline{X}^*/\underline{K}^*]$, where $\underline{X}^*$ is the sublist of $\underline{X}$ consisting of those $X_i^{\delta_i}$'s which are equivalent to some free term variable of $Z.G$, $\underline{K}^*$ is the corresponding sublist of $\underline{K}$, y' is y if $y \notin fv(\underline{K}^*)$, otherwise y' is the first individual variable which is not in $fv(Z.G) \cup fv(\underline{K}^*)$.

By Claim 1 of (a), $(Z.G)[y/y'] = C[y/y']$ for some context C such that $C \equiv Z.G$, y is replaceable by y' and y' is free for y in C. By Claim 1 of (c), $C[y/y'][\underline{X}^*/\underline{K}^*] = C^*[\underline{X}^*/\underline{K}^*]$ for some context C^* such that $C^* \equiv C[y/y']$ and $\underline{K}^*$ is free for $\underline{X}^*$ in C^*.

Hence $(y.Z.G)[\underline{X}/\underline{K}] = y'.C^*[\underline{X}^*/\underline{K}^*] = (y'.C^*)[\underline{X}/\underline{K}]$, where $y'.C^* \equiv y'.C[y/y'] \equiv y.C \equiv y.Z.G$ and $\underline{K}$ is free for $\underline{X}$ in $y'.C^*$.

Similarly for the case $ST(Q.Z.G, H)$.

The remaining cases follow straightforwardly by the induction hypothesis. □

Corollary 3.2.17. *Lemma* 3.2.16 *also holds for contexts.*

Proof. By using Lemma 3.2.16, this corollary can be proved in the same way as the claims in the lemma. □

Corollary 3.2.18.

a. For any context $x.F$ (respectively $x.X.F$), if $y \notin fv(F)$ (respectively $y \notin fv(X.F)$), then $x.F \equiv y.F[x/y]$ (respectively $x.X.F \equiv y.(X.F)[x/y]$).

b. For any context $P^n.F$ (respectively $P^n.X.F$), if $Q^n \notin FV(F)$ (respectively $Q^n \notin FV(X.F)$), then $P.F \equiv Q.F[P/Q]$ (respectively $P.X.F \equiv Q.(X.F)[P/Q]$).

c. For any context $X^\alpha.F$, if $Y^{\alpha'}$, where $\alpha' \equiv \alpha$, is a term variable which is not equivalent to any free term variable of F, then $X^\alpha.F \equiv Y^{\alpha'}.F[X/Y^{\alpha'}]$, and so for any individual variable x, $x.X^\alpha.F \equiv x.Y^{\alpha'}.F[X^\alpha/Y^{\alpha'}]$ and for any predicate variable P, $P.X^\alpha.F \equiv P.Y^{\alpha'}.F[X^\alpha/Y^{\alpha'}]$.

Proof. By using Lemma 3.2.16, the proof is similar to the proof of Corollary 2.11. □

Lemma 3.2.19. *Let F^α be a C-H term, $\underline{x} = x_1, \ldots, x_m$ and $\underline{y} = y_1, \ldots, y_n$ be sequences of distinct individual variables, and $\underline{t} = t_1, \ldots, t_m$ and $\underline{u} = u_1, \ldots, u_n$ be individual terms. Then*

$$F^\alpha[\underline{x}/\underline{t}][\underline{y}/\underline{u}] \equiv F^\alpha[x_1/t_1[\underline{y}/\underline{u}], \ldots, x_m/t_m[\underline{y}/\underline{u}], y_{i_1}/u_{i_1}, \ldots, y_{i_k}/u_{i_k}],$$

where $y_{i_1}, \ldots, y_{i_k}$ is the sublist of $\underline{y}$ consisting of those y_j's which are in $fv(F^\alpha) - \{\underline{x}\}$.

Lemma 3.2.20. *Let F^α be a C-H term, $\underline{P} = P_1^{r_1}, \ldots, P_m^{r_m}$ and $\underline{R} = R_1^{l_1}, \ldots, R_n^{l_n}$ be sequences of distinct predicate variables, $\underline{T} = T_1, \ldots, T_m$, where $T_i = \lambda x_1^i, \ldots, x_{r_i}^i \delta_i$, $1 \leq i \leq m$, and $\underline{U} = U_1, \ldots, U_n$, where $U_j = \lambda y_1^j, \ldots, y_{l_j}^j \sigma_j$, $1 \leq j \leq n$, be abstraction terms. Then*

$$F^\alpha[\underline{P}/\underline{T}][\underline{R}/\underline{U}] \equiv F^\alpha[P_1/T_1[\underline{R}/\underline{U}], \ldots, P_m/T_m[\underline{R}/\underline{U}], R_{i_1}/U_{i_1}, \ldots, R_{i_k}/U_{i_k}],$$

where $R_{i_1}, \ldots, R_{i_k}$ is the sublist of $\underline{R}$ consisting of those R_j's which are in $FV(F^\alpha) - \{\underline{P}\}$.

Lemma 3.2.21. *Let F^α be a C-H term, $\underline{X} = X_1^{\delta_1}, \ldots, X_m^{\delta_m}$ and $\underline{Y} = Y_1^{\tau_1}, \ldots, Y_n^{\tau_n}$ be sequences of inequivalent term variables, and $\underline{H} = H_1^{\delta'_1}, \ldots, H_m^{\delta'_m}$ and $\underline{K} = K_1^{\tau'_1}, \ldots, K_n^{\tau'_n}$ be C-H terms, where $\delta_i \equiv \delta'_i$ for all $1 \leq i \leq m$ and $\tau_j \equiv \tau'_j$ for all $1 \leq j \leq n$. Then*

$$F^\alpha[\underline{X}/\underline{H}][\underline{Y}/\underline{K}] \equiv F^\alpha[X_1/H_1[\underline{Y}/\underline{K}], \ldots, X_m/H_m[\underline{Y}/\underline{K}], Y_{i_1}/K_{i_1}, \ldots, Y_{i_k}/K_{i_k}],$$

where $Y_{i_1}, \ldots, Y_{i_k}$ is the sublist of $\underline{Y}$ consisting of those Y_j's which are equivalent to some free term variables of F^α but are not equivalent to any X_i in $\underline{X}$.

Lemma 3.2.22. *Let F^α be a C-H term, $\underline{x} = x_1, \ldots, x_n$ be distinct individual variables, $\underline{t} = t_1, \ldots, t_n$ be individual terms, $\underline{P} = P_1^{r_1}, \ldots, P_m^{r_m}$ be distinct predicate variables, and $\underline{T} = T_1, \ldots, T_m$, where $T_i = \lambda z_1^i, \ldots, z_{r_i}^i \delta_i$, $1 \leq i \leq m$, be abstraction terms.*

Then $F^\alpha[\underline{P}/\underline{T}][\underline{x}/\underline{t}] \equiv F^\alpha[\underline{x}/\underline{t}][\underline{P}/\underline{T}]$.

Lemma 3.2.23. *Let F^α be a C-H term, $\underline{X} = X_1^{\delta_1}, \ldots, X_m^{\delta_m}$ be inequivalent term variables, and $\underline{K} = K_1^{\delta'_1}, \ldots, K_m^{\delta'_m}$ be C-H terms such that $\delta_i \equiv \delta'_i$ for all $1 \leq i \leq m$.*

a. For any distinct individual variables $\underline{x} = x_1, \ldots, x_n$ and any individual terms $\underline{t} = t_1, \ldots, t_n$, if for all $1 \leq i \leq m$, F^α has no free term variable X_i^σ such that $\sigma \not\equiv \delta_i$ but $\sigma[\underline{x}/\underline{t}] \equiv \delta_i[\underline{x}/\underline{t}]$, then

$$F^\alpha[\underline{X}/\underline{K}][\underline{x}/\underline{t}] \equiv F^\alpha[\underline{x}/\underline{t}][X_1^{\delta_1[\underline{x}/\underline{t}]}/K_1^{\delta'_1}[\underline{x}/\underline{t}], \ldots, X_m^{\delta_m[\underline{x}/\underline{t}]}/K_m^{\delta'_m}[\underline{x}/\underline{t}]].$$

b. For any distinct predicate variables $\underline{P} = P_1^{l_1}, \ldots, P_n^{l_n}$ *and any abstraction terms* $\underline{T} = T_1, \ldots, T_n$*, where* $T_i = \lambda z_1^i, \ldots, z_{l_i}^i \delta_i$*,* $1 \leq i \leq n$*, if for all* $1 \leq i \leq m$*,* F^α *has no free term variable* X_i^σ *such that* $\sigma \not\equiv \delta_i$ *but* $\sigma[\underline{P}/\underline{T}] \equiv \delta_i[\underline{P}/\underline{T}]$*, then*

$$F^\alpha[\underline{X}/\underline{K}][\underline{P}/\underline{T}] \equiv F^\alpha[\underline{P}/\underline{T}][X_1^{\delta_1[\underline{P}/\underline{T}]}/K_1^{\delta'_1}[\underline{P}/\underline{T}], \ldots, X_m^{\delta_m[\underline{P}/\underline{T}]}/K_m^{\delta'_m}[\underline{P}/\underline{T}]].$$

Lemma 3.2.24. *Let* F *and* F' *be C-H terms,* $\underline{x} = x_1, \ldots, x_n$ *be distinct individual variables, and* $\underline{t} = t_1, \ldots, t_n$ *be individual terms.*

If $F \equiv F'$*, then* $F[\underline{x}/\underline{t}] \equiv F'[\underline{x}/\underline{t}]$*.*

Lemma 3.2.25. *Let* F *and* F' *be C-H terms,* $\underline{P} = P_1^{r_1}, \ldots, P_m^{r_m}$ *be distinct predicate variables, and* $\underline{T} = T_1, \ldots, T_m$*, where* $T_j = \lambda x_1^j, \ldots, x_{r_j}^j \delta_j$*,* $1 \leq j \leq m$*, be abstraction terms.*

If $F \equiv F'$*, then* $F[\underline{P}/\underline{T}] \equiv F'[\underline{P}/\underline{T}]$*.*

Lemma 3.2.26. *Let* F *and* F' *be C-H terms,* $\underline{X} = X_1^{\delta_1}, \ldots, X_n^{\delta_n}$ *be inequivalent term variables, and* $\underline{H} = H_1^{\delta'_1}, \ldots, H_n^{\delta'_n}$ *and* $\underline{K} = K_1^{\delta''_1}, \ldots, K_n^{\delta''_n}$ *be C-H terms, where* $\delta_i \equiv \delta'_i \equiv \delta''_i$ *and* $H_i^{\delta'_i} \equiv K_i^{\delta''_i}$ *for all* $1 \leq i \leq n$*.*

If $F \equiv F'$*, then* $F[\underline{X}/\underline{H}] \equiv F'[\underline{X}/\underline{K}]$*.*

Proof. We will prove all the above lemmas simultaneously by induction on F^α.

Proof of Lemma 3.2.19.

It follows by Lemma 2.13 if F^α is a term variable.

(i) $F^\alpha = \lambda X^\beta.G$.

This case follows by the following claim.

Claim 1. The context $X^\beta.G$ satisfies the lemma.

Proof of Claim 1. We have

$$\begin{aligned}(X^\beta.G)[\underline{x}/\underline{t}][\underline{y}/\underline{u}] &= (Y_1^{\beta[\underline{x}/\underline{t}]}.G[X^\beta/Y_1^\beta][\underline{x}/\underline{t}])[\underline{y}/\underline{u}] \\ &= Y_2^{\beta[\underline{x}/\underline{t}][\underline{y}/\underline{u}]}.G[X^\beta/Y_1^\beta][\underline{x}/\underline{t}][Y_1^{\beta[\underline{x}/\underline{t}]}/Y_2^{\beta[\underline{x}/\underline{t}]}][\underline{y}/\underline{u}],\end{aligned}$$

where Y_1^{β} is X^{β} (respectively $Y_2^{\beta[\underline{x}/\underline{t}]}$ is $Y_1^{\beta[\underline{x}/\underline{t}]}$) if $X^{\beta}.G$ (respectively $Y_1^{\beta[\underline{x}/\underline{t}]}.G[X^{\beta}/Y_1^{\beta}][\underline{x}/\underline{t}]$) has no free term variable X^{σ} such that $\sigma[\underline{x}/\underline{t}] \equiv \beta[\underline{x}/\underline{t}]$ (respectively Y_1^{σ} such that $\sigma[\underline{y}/\underline{u}] \equiv \beta[\underline{x}/\underline{t}][\underline{y}/\underline{u}]$), otherwise Y_1^{β} (respectively $Y_2^{\beta[\underline{x}/\underline{t}]}$) is the first term variable of type $[\beta]$ (respectively $[\beta[\underline{x}/\underline{t}]]$) such that there is no free term variable Y_1^{σ} of $X^{\beta}.G$ where $\sigma[\underline{x}/\underline{t}] \equiv \beta[\underline{x}/\underline{t}]$ (respectively Y_2^{σ} of $Y_1^{\beta[\underline{x}/\underline{t}]}.G[X^{\beta}/Y_1^{\beta}][\underline{x}/\underline{t}]$ where $\sigma[\underline{y}/\underline{u}] \equiv \beta[\underline{x}/\underline{t}][\underline{y}/\underline{u}]$).

Let β^* denote $\beta[x_1/t_1[\underline{y}/\underline{u}], \ldots, x_m/t_m[\underline{y}/\underline{u}], y_{i_1}/u_{i_1}, \ldots, y_{i_k}/u_{i_k}]$.

By Lemma 2.13, $\beta[\underline{x}/\underline{t}][\underline{y}/\underline{u}] \equiv \beta^*$.

We have $(X^{\beta}.G)[x_1/t_1[\underline{y}/\underline{u}], \ldots, x_m/t_m[\underline{y}/\underline{u}], y_{i_1}/u_{i_1}, \ldots, y_{i_k}/u_{i_k}] =$ $Y_3^{\beta^*}.G[X^{\beta}/Y_3^{\beta}][x_1/t_1[\underline{y}/\underline{u}], \ldots, x_m/t_m[\underline{y}/\underline{u}], y_{i_1}/u_{i_1}, \ldots, y_{i_k}/u_{i_k}]$, where Y_3^{β} is X^{β} if $X^{\beta}.G$ has no free term variable X^{σ} such that $\sigma[\underline{x}/\underline{t}] \equiv \beta[\underline{x}/\underline{t}]$, otherwise Y_3^{β} is the first term variable of type $[\beta]$ such that there is no free term variable Y_3^{σ} of $X^{\beta}.G$ where $\sigma[\underline{x}/\underline{t}] \equiv \beta[\underline{x}/\underline{t}]$.

Let $Y_*^{\beta[\underline{x}/\underline{t}][\underline{y}/\underline{u}]}$ be a term variable of type $[\beta[\underline{x}/\underline{t}][\underline{y}/\underline{u}]]$ which is not equivalent to any term variable occurring in $G[X^{\beta}/Y_1^{\beta}][\underline{x}/\underline{t}][Y_1^{\beta[\underline{x}/\underline{t}]}/Y_2^{\beta[\underline{x}/\underline{t}]}][\underline{y}/\underline{u}]$ or $G[X^{\beta}/Y_3^{\beta}][x_1/t_1[\underline{y}/\underline{u}], \ldots, x_m/t_m[\underline{y}/\underline{u}]]$.

By the induction hypothesis, we have

$(X^{\beta}.G)[\underline{x}/\underline{t}][\underline{y}/\underline{u}]$

$$
\begin{aligned}
&= Y_2^{\beta[\underline{x}/\underline{t}][\underline{y}/\underline{u}]}.G[X^{\beta}/Y_1^{\beta}][\underline{x}/\underline{t}][Y_1^{\beta[\underline{x}/\underline{t}]}/Y_2^{\beta[\underline{x}/\underline{t}]}][\underline{y}/\underline{u}] \\
&\equiv Y_*^{\beta[\underline{x}/\underline{t}][\underline{y}/\underline{u}]}.G[X^{\beta}/Y_1^{\beta}][\underline{x}/\underline{t}][Y_1^{\beta[\underline{x}/\underline{t}]}/Y_2^{\beta[\underline{x}/\underline{t}]}][\underline{y}/\underline{u}][Y_2^{\beta[\underline{x}/\underline{t}][\underline{y}/\underline{u}]}/Y_*^{\beta[\underline{x}/\underline{t}][\underline{y}/\underline{u}]}] \\
&\equiv Y_*^{\beta[\underline{x}/\underline{t}][\underline{y}/\underline{u}]}.G[X^{\beta}/Y_1^{\beta}][\underline{x}/\underline{t}][Y_1^{\beta[\underline{x}/\underline{t}]}/Y_2^{\beta[\underline{x}/\underline{t}]}][Y_2^{\beta[\underline{x}/\underline{t}]}/Y_*^{\beta[\underline{x}/\underline{t}]}][\underline{y}/\underline{u}] \\
&\equiv Y_*^{\beta[\underline{x}/\underline{t}][\underline{y}/\underline{u}]}.G[X^{\beta}/Y_1^{\beta}][\underline{x}/\underline{t}][Y_1^{\beta[\underline{x}/\underline{t}]}/Y_*^{\beta[\underline{x}/\underline{t}]}][\underline{y}/\underline{u}] \\
&\equiv Y_*^{\beta[\underline{x}/\underline{t}][\underline{y}/\underline{u}]}.G[X^{\beta}/Y_1^{\beta}][Y_1^{\beta}/Y_*^{\beta}][\underline{x}/\underline{t}][\underline{y}/\underline{u}] \\
&\equiv Y_*^{\beta[\underline{x}/\underline{t}][\underline{y}/\underline{u}]}.G[X^{\beta}/Y_*^{\beta}][\underline{x}/\underline{t}][\underline{y}/\underline{u}], \text{ and}
\end{aligned}
$$

$(X^\beta.G)[x_1/t_1[\underline{y}/\underline{u}],\ldots,x_m/t_m[\underline{y}/\underline{u}],y_{i_1}/u_{i_1},\ldots,y_{i_k}/u_{i_k}]$

$$
\begin{aligned}
&= Y_3^{\beta^*}.G[X^\beta/Y_3^\beta][x_1/t_1[\underline{y}/\underline{u}],\ldots,x_m/t_m[\underline{y}/\underline{u}],y_{i_1}/u_{i_1},\ldots,y_{i_k}/u_{i_k}]\\
&\equiv Y_*^{\beta^*}.G[X^\beta/Y_3^\beta][x_1/t_1[\underline{y}/\underline{u}],\ldots,x_m/t_m[\underline{y}/\underline{u}],y_{i_1}/u_{i_1},\ldots,y_{i_k}/u_{i_k}][Y_3^{\beta^*}/Y_*^{\beta^*}]\\
&\equiv Y_*^{\beta^*}.G[X^\beta/Y_3^\beta][Y_3^\beta/Y_*^\beta][x_1/t_1[\underline{y}/\underline{u}],\ldots,x_m/t_m[\underline{y}/\underline{u}],y_{i_1}/u_{i_1},\ldots,y_{i_k}/u_{i_k}]\\
&\equiv Y_*^{\beta^*}.G[X^\beta/Y_*^\beta][x_1/t_1[\underline{y}/\underline{u}],\ldots,x_m/t_m[\underline{y}/\underline{u}],y_{i_1}/u_{i_1},\ldots,y_{i_k}/u_{i_k}]\\
&\equiv Y_*^{\beta^*}.G[X^\beta/Y_*^\beta][\underline{x}/\underline{t}][\underline{y}/\underline{u}]\\
&\equiv Y_*^{\beta[\underline{x}/\underline{t}][\underline{y}/\underline{u}]}.G[X^\beta/Y_*^\beta][\underline{x}/\underline{t}][\underline{y}/\underline{u}].
\end{aligned}
$$

The cases $\oplus(X.G,Y.H,K)$ and $ST(P.X.G,H)$ also follow by the above claim and the induction hypothesis.

(ii) $F^\alpha = \lambda z.G$.

This case follows by the following claim which can be proved in the same way as the case $\forall z\beta$ in the proof of Lemma 2.13.

Claim 2. The context $z.G$ satisfies the lemma.

Similarly, for the case $ST(z.X.G,H)$ we prove the lemma for the context $z.X.G$ (by using Claim 1) and apply the induction hypothesis to H.

The remaining cases follow straightforwardly by the induction hypothesis.

Lemma 3.2.20 can be proved similarly (using Lemma 2.16 for the atomic case).

Proof of Lemma 3.2.21.

(i) F^α is a term variable.

If $F^\alpha \equiv Y_r^{\tau_r}$ for some $1 \le r \le n$ where $Y_r^{\tau_r} \not\equiv X_i$ for all $1 \le i \le m$, then $F^\alpha[\underline{X}/\underline{H}][\underline{Y}/\underline{K}] = K_r = F^\alpha[X_1/H_1[\underline{Y}/\underline{K}],\ldots,X_m/H_m[\underline{Y}/\underline{K}],Y_r/K_r]$, otherwise $F^\alpha[\underline{X}/\underline{H}][\underline{Y}/\underline{K}]$

$$
\begin{aligned}
&= \begin{cases} H_q^{\delta'_q}[\underline{Y}/\underline{K}] & \text{if } F^\alpha \equiv X_q^{\delta_q} \text{ for some } 1 \le q \le m,\\ F^\alpha & \text{otherwise,}\end{cases}\\
&= F^\alpha[X_1/H_1[\underline{Y}/\underline{K}],\ldots,X_m/H_m[\underline{Y}/\underline{K}]].
\end{aligned}
$$

(ii) $F^\alpha = \lambda Y^\beta.G$.

This case follows by the following claim which can be proved in the same way as the case $\forall z\beta$ in the proof of Lemma 2.13.

Claim 1. The context $Y^\beta.G$ satisfies the lemma.

The case $\oplus(Z_1.F_1, Z_2.F_2, G)$ also follows by the above claim and the induction hypothesis.

(iii) $F^\alpha = \lambda x.G$.

This case follows by the following claim.

Claim 2. The context $x.G$ satisfies the lemma.

Proof of Claim 2. We have $(x.G)[\underline{X}/\underline{H}][\underline{Y}/\underline{K}] = x''.G[x/x'][\underline{X}^*/\underline{H}^*][x'/x''][\underline{Y}^*/\underline{K}^*]$, where $\underline{X}^* = X_{j_1}, \ldots, X_{j_l}$ (respectively $\underline{Y}^*$) is the sublist of $\underline{X}$ (respectively $\underline{Y}$) consisting of those term variables which are equivalent to some free term variable of $x.G$ (respectively $(x.G)[\underline{X}/\underline{H}]$), $\underline{H}^*$ (respectively $\underline{K}^*$) is the corresponding sublist of $\underline{H}$ (respectively $\underline{K}$), x' and x'' are individual term variables such that x' is x (respectively x'' is x') if $x \notin fv(\underline{H}^*)$ (respectively $x' \notin fv(\underline{K}^*)$), otherwise x' (respectively x'') is the first individual variable which is not in $fv(G) \cup fv(\underline{H}^*)$ (respectively $fv(G[x/x'][\underline{X}^*/\underline{H}^*]) \cup fv(\underline{K}^*)$), and

$(x.G)[X_1/H_1[\underline{Y}/\underline{K}], \ldots, X_m/H_m[\underline{Y}/\underline{K}], Y_{i_1}/K_{i_1}, \ldots, Y_{i_k}/K_{i_k}]$

$= x'''.G[x/x'''][X_{j_1}/H_{j_1}[\underline{Y}/\underline{K}], \ldots, X_{j_l}/H_{j_l}[\underline{Y}/\underline{K}], Y_{i_1}/K_{i_1}, \ldots, Y_{i_k}/K_{i_k}]$, where x''' is x if x does not occur free in $H_{j_1}[\underline{Y}/\underline{K}], \ldots, H_{j_l}[\underline{Y}/\underline{K}], K_{i_1}, \ldots, K_{i_k}$, otherwise x''' is the first individual variable which does not occur free in G or $H_{j_1}[\underline{Y}/\underline{K}], \ldots, H_{j_l}[\underline{Y}/\underline{K}], K_{i_1}, \ldots, K_{i_k}$.

Let x^* be an individual variable which does not occur in

$G[x/x'][\underline{X}^*/\underline{H}^*][x'/x''][\underline{Y}^*/\underline{K}^*]$ or

$G[x/x'''][X_{j_1}/H_{j_1}[\underline{Y}/\underline{K}], \ldots, X_{j_l}/H_{j_l}[\underline{Y}/\underline{K}], Y_{i_1}/K_{i_1}, \ldots, Y_{i_k}/K_{i_k}]$.

Suppose for a contradiction that there is some term variable Y_r^σ, where $Y_r^{\tau_r}$

is in $\underline{Y}^*$, such that $\sigma \not\equiv \tau_r$ but $\sigma[x''/x^*] \equiv \tau_r[x''/x^*]$ ($= \tau_r$ since $x'' \notin fv(K_r^{\tau_r'})$ and so $x'' \notin fv(\tau_r)$). Then x^* must occur free in τ_r. This is a contradiction by the choice of x^*. Hence there is no such term variable. Similarly, there is no term variable $X_{j_q}^{\sigma}$, $1 \leq q \leq l$, such that $\sigma \not\equiv \delta_{j_q}$ but $\sigma[x'/x^*] \equiv \delta_{j_q}[x'/x^*]$ or $\sigma[x'''/x^*] \equiv \delta_{j_q}[x'''/x^*]$. Thus, by the induction hypothesis, we have

$$\begin{aligned}
(x.G)[\underline{X}/\underline{H}][\underline{Y}/\underline{K}] &= x''.G[x/x'][\underline{X}^*/\underline{H}^*][x'/x''][\underline{Y}^*/\underline{K}^*] \\
&\equiv x^*.G[x/x'][\underline{X}^*/\underline{H}^*][x'/x''][\underline{Y}^*/\underline{K}^*][x''/x^*] \\
&\equiv x^*.G[x/x'][\underline{X}^*/\underline{H}^*][x'/x''][x''/x^*][\underline{Y}^*/\underline{K}^*] \\
&\equiv x^*.G[x/x'][\underline{X}^*/\underline{H}^*][x'/x^*][\underline{Y}^*/\underline{K}^*] \\
&\equiv x^*.G[x/x'][x'/x^*][\underline{X}^*/\underline{H}^*][\underline{Y}^*/\underline{K}^*] \\
&\equiv x^*.G[x/x^*][\underline{X}^*/\underline{H}^*][\underline{Y}^*/\underline{K}^*], \text{ and}
\end{aligned}$$

$$(x.G)[X_1/H_1[\underline{Y}/\underline{K}], \ldots, X_m/H_m[\underline{Y}/\underline{K}], Y_{i_1}/K_{i_1}, \ldots, Y_{i_k}/K_{i_k}]$$

$$\begin{aligned}
&= x'''.G[x/x'''][X_{j_1}/H_{j_1}[\underline{Y}/\underline{K}], \ldots, X_{j_l}/H_{j_l}[\underline{Y}/\underline{K}], Y_{i_1}/K_{i_1}, \ldots, Y_{i_k}/K_{i_k}] \\
&= x'''.G[x/x'''][X_{j_1}/H_{j_1}[\underline{Y}^*/\underline{K}^*], \ldots, X_{j_l}/H_{j_l}[\underline{Y}^*/\underline{K}^*], Y_{i_1}/K_{i_1}, \ldots, Y_{i_k}/K_{i_k}] \\
&\equiv x^*.G[x/x'''][X_{j_1}/H_{j_1}[\underline{Y}^*/\underline{K}^*], \ldots, X_{j_l}/H_{j_l}[\underline{Y}^*/\underline{K}^*], Y_{i_1}/K_{i_1}, \ldots, Y_{i_k}/K_{i_k}] \\
&\quad [x'''/x^*] \\
&\equiv x^*.G[x/x'''][x'''/x^*] \\
&\quad [X_{j_1}/H_{j_1}[\underline{Y}^*/\underline{K}^*], \ldots, X_{j_l}/H_{j_l}[\underline{Y}^*/\underline{K}^*], Y_{i_1}/K_{i_1}, \ldots, Y_{i_k}/K_{i_k}] \\
&\equiv x^*.G[x/x^*][X_{j_1}/H_{j_1}[\underline{Y}^*/\underline{K}^*], \ldots, X_{j_l}/H_{j_l}[\underline{Y}^*/\underline{K}^*], Y_{i_1}/K_{i_1}, \ldots, Y_{i_k}/K_{i_k}] \\
&\equiv x^*.G[x/x^*][\underline{X}^*/\underline{H}^*][\underline{Y}^*/\underline{K}^*].
\end{aligned}$$

Similarly for $\lambda P.G$.

(iv) $F^{\alpha} = ST(x.Z.G_1, G_2)$.

Similar to the above case, by using the previous results for the context $Z.G_1$, we first show that the context $x.Z.G_1$ satisfies the lemma in the same way as in the above claim, and then apply the induction hypothesis to G_2.

Similarly for $ST(P.Z.G_1, G_2)$.

The remaining cases follow straightforwardly by the induction hypothesis.

Proof of Lemma 3.2.22. It follows by Lemma 2.15 if F^α is a term variable. The cases $\lambda y.G$ and $\lambda Q.G$ can be proved in the same way as the cases $\forall y\beta$ and $\forall_2 Q\beta$, respectively, in Lemma 2.15.

The following cases, each of which contains a context of the form $X^\beta.G$, need the claim below.

Claim 1. $X^\beta.G$ satisfies the lemma.

Proof of Claim 1. We have

$$\begin{aligned}(X^\beta.G)[\underline{P}/\underline{T}][\underline{x}/\underline{t}] &= (Y_1^{\beta[\underline{P}/\underline{T}]}.G[X^\beta/Y_1^\beta][\underline{P}/\underline{T}])[\underline{x}/\underline{t}] \\ &= Y_2^{\beta[\underline{P}/\underline{T}][\underline{x}/\underline{t}]}.G[X^\beta/Y_1^\beta][\underline{P}/\underline{T}][Y_1^{\beta[\underline{P}/\underline{T}]}/Y_2^{\beta[\underline{P}/\underline{T}]}][\underline{x}/\underline{t}],\end{aligned}$$

where Y_1^β is X^β (respectively $Y_2^{\beta[\underline{P}/\underline{T}]}$ is $Y_1^{\beta[\underline{P}/\underline{T}]}$) if $X^\beta.G$ (respectively $Y_1^{\beta[\underline{P}/\underline{T}]}.G[X^\beta/Y_1^\beta][\underline{P}/\underline{T}]$) has no free term variable X^σ (respectively Y_1^σ) such that $\sigma[\underline{P}/\underline{T}] \equiv \beta[\underline{P}/\underline{T}]$ (respectively $\sigma[\underline{x}/\underline{t}] \equiv \beta[\underline{P}/\underline{T}][\underline{x}/\underline{t}]$), otherwise Y_1^β (respectively $Y_2^{\beta[\underline{P}/\underline{T}]}$) is the first term variable of type $[\beta]$ (respectively $[\beta[\underline{P}/\underline{T}]]$) such that there is no free term variable Y_1^σ of $X^\beta.G$ where $\sigma[\underline{P}/\underline{T}] \equiv \beta[\underline{P}/\underline{T}]$ (respectively Y_2^σ of $Y_1^{\beta[\underline{P}/\underline{T}]}.G[X^\beta/Y_1^\beta][\underline{P}/\underline{T}]$ where $\sigma[\underline{x}/\underline{t}] \equiv \beta[\underline{P}/\underline{T}][\underline{x}/\underline{t}]$), and

$$\begin{aligned}(X^\beta.G)[\underline{x}/\underline{t}][\underline{P}/\underline{T}] &= (Y_3^{\beta[\underline{x}/\underline{t}]}.G[X^\beta/Y_3^\beta][\underline{x}/\underline{t}])[\underline{P}/\underline{T}] \\ &= Y_4^{\beta[\underline{x}/\underline{t}][\underline{P}/\underline{T}]}.G[X^\beta/Y_3^\beta][\underline{x}/\underline{t}][Y_3^{\beta[\underline{x}/\underline{t}]}/Y_4^{\beta[\underline{x}/\underline{t}]}][\underline{P}/\underline{T}],\end{aligned}$$

where Y_3^β is X^β (respectively $Y_4^{\beta[\underline{x}/\underline{t}]}$ is $Y_3^{\beta[\underline{x}/\underline{t}]}$) if $X^\beta.G$ (respectively $Y_3^{\beta[\underline{x}/\underline{t}]}.G[X^\beta/Y_3^\beta][\underline{x}/\underline{t}]$) has no free term variable X^σ (respectively Y_3^σ) such that $\sigma[\underline{x}/\underline{t}] \equiv \beta[\underline{x}/\underline{t}]$ (respectively $\sigma[\underline{P}/\underline{T}] \equiv \beta[\underline{x}/\underline{t}][\underline{P}/\underline{T}]$), otherwise Y_3^β (respectively $Y_4^{\beta[\underline{x}/\underline{t}]}$) is the first term variable of type $[\beta]$ (respectively $[\beta[\underline{x}/\underline{t}]]$) such that there is no free term variable Y_3^σ of $X^\beta.G$ where $\sigma[\underline{x}/\underline{t}] \equiv \beta[\underline{x}/\underline{t}]$ (respectively Y_4^σ of $Y_3^{\beta[\underline{x}/\underline{t}]}.G[X^\beta/Y_3^\beta][\underline{x}/\underline{t}]$ where $\sigma[\underline{P}/\underline{T}] \equiv \beta[\underline{x}/\underline{t}][\underline{P}/\underline{T}]$).

Let $Z^{\beta[\underline{P}/\underline{T}][\underline{x}/\underline{t}]}$ be a term variable of type $[\beta[\underline{P}/\underline{T}][\underline{x}/\underline{t}]]$ ($= [\beta[\underline{x}/\underline{t}][\underline{P}/\underline{T}]]$ by Lemma 2.15) which is not equivalent to any term variable occurring in $G[X^\beta/Y_1^\beta][\underline{P}/\underline{T}][Y_1^{\beta[\underline{P}/\underline{T}]}/Y_2^{\beta[\underline{P}/\underline{T}]}][\underline{x}/\underline{t}]$ or $G[X^\beta/Y_3^\beta][\underline{x}/\underline{t}][Y_3^{\beta[\underline{x}/\underline{t}]}/Y_4^{\beta[\underline{x}/\underline{t}]}][\underline{P}/\underline{T}]$.

By the induction hypothesis, we have

$$
\begin{aligned}
&(X^\beta.G)[\underline{P}/\underline{T}][\underline{x}/\underline{t}]\\
&= Y_2^{\beta[\underline{P}/\underline{T}][\underline{x}/\underline{t}]}.G[X^\beta/Y_1^\beta][\underline{P}/\underline{T}][Y_1^{\beta[\underline{P}/\underline{T}]}/Y_2^{\beta[\underline{P}/\underline{T}]}][\underline{x}/\underline{t}]\\
&\equiv Z^{\beta[\underline{P}/\underline{T}][\underline{x}/\underline{t}]}.G[X^\beta/Y_1^\beta][\underline{P}/\underline{T}][Y_1^{\beta[\underline{P}/\underline{T}]}/Y_2^{\beta[\underline{P}/\underline{T}]}][\underline{x}/\underline{t}][Y_2^{\beta[\underline{P}/\underline{T}][\underline{x}/\underline{t}]}/Z^{\beta[\underline{P}/\underline{T}][\underline{x}/\underline{t}]}]\\
&\equiv Z^{\beta[\underline{P}/\underline{T}][\underline{x}/\underline{t}]}.G[X^\beta/Y_1^\beta][\underline{P}/\underline{T}][Y_1^{\beta[\underline{P}/\underline{T}]}/Y_2^{\beta[\underline{P}/\underline{T}]}][Y_2^{\beta[\underline{P}/\underline{T}]}/Z^{\beta[\underline{P}/\underline{T}]}][\underline{x}/\underline{t}]\\
&\equiv Z^{\beta[\underline{P}/\underline{T}][\underline{x}/\underline{t}]}.G[X^\beta/Y_1^\beta][\underline{P}/\underline{T}][Y_1^{\beta[\underline{P}/\underline{T}]}/Z^{\beta[\underline{P}/\underline{T}]}][\underline{x}/\underline{t}]\\
&\equiv Z^{\beta[\underline{P}/\underline{T}][\underline{x}/\underline{t}]}.G[X^\beta/Y_1^\beta][Y_1^\beta/Z^\beta][\underline{P}/\underline{T}][\underline{x}/\underline{t}]\\
&\equiv Z^{\beta[\underline{P}/\underline{T}][\underline{x}/\underline{t}]}.G[X^\beta/Z^\beta][\underline{P}/\underline{T}][\underline{x}/\underline{t}]\\
&\equiv Z^{\beta[\underline{x}/\underline{t}][\underline{P}/\underline{T}]}.G[X^\beta/Z^\beta][\underline{x}/\underline{t}][\underline{P}/\underline{T}], \text{ and}
\end{aligned}
$$

$$
\begin{aligned}
&(X^\beta.G)[\underline{x}/\underline{t}][\underline{P}/\underline{T}]\\
&= Y_4^{\beta[\underline{x}/\underline{t}][\underline{P}/\underline{T}]}.G[X^\beta/Y_3^\beta][\underline{x}/\underline{t}][Y_3^{\beta[\underline{x}/\underline{t}]}/Y_4^{\beta[\underline{x}/\underline{t}]}][\underline{P}/\underline{T}]\\
&\equiv Z^{\beta[\underline{x}/\underline{t}][\underline{P}/\underline{T}]}.G[X^\beta/Y_3^\beta][\underline{x}/\underline{t}][Y_3^{\beta[\underline{x}/\underline{t}]}/Y_4^{\beta[\underline{x}/\underline{t}]}][\underline{P}/\underline{T}][Y_4^{\beta[\underline{x}/\underline{t}][\underline{P}/\underline{T}]}/Z^{\beta[\underline{x}/\underline{t}][\underline{P}/\underline{T}]}]\\
&\equiv Z^{\beta[\underline{x}/\underline{t}][\underline{P}/\underline{T}]}.G[X^\beta/Y_3^\beta][\underline{x}/\underline{t}][Y_3^{\beta[\underline{x}/\underline{t}]}/Y_4^{\beta[\underline{x}/\underline{t}]}][Y_4^{\beta[\underline{x}/\underline{t}]}/Z^{\beta[\underline{x}/\underline{t}]}][\underline{P}/\underline{T}]\\
&\equiv Z^{\beta[\underline{x}/\underline{t}][\underline{P}/\underline{T}]}.G[X^\beta/Y_3^\beta][\underline{x}/\underline{t}][Y_3^{\beta[\underline{x}/\underline{t}]}/Z^{\beta[\underline{x}/\underline{t}]}][\underline{P}/\underline{T}]\\
&\equiv Z^{\beta[\underline{x}/\underline{t}][\underline{P}/\underline{T}]}.G[X^\beta/Y_3^\beta][Y_3^\beta/Z^\beta][\underline{x}/\underline{t}][\underline{P}/\underline{T}]\\
&\equiv Z^{\beta[\underline{x}/\underline{t}][\underline{P}/\underline{T}]}.G[X^\beta/Z^\beta][\underline{x}/\underline{t}][\underline{P}/\underline{T}].
\end{aligned}
$$

Hence we have the claim.

The case $\lambda X.G$ follows by the claim. The case $\oplus(X.G, Y.H, K)$ follows by the claim and the induction hypothesis. By applying the induction hypothesis to H, the cases $ST(y.X.G,H)$ and $ST(Q.X.G,H)$ can be proved in the same way as the cases $\lambda y.G$ and $\lambda Q.G$, respectively, by using the above claim.

The remaining cases follow straightforwardly by the induction hypothesis.

Proof of Lemma 3.2.23.

a: Let $\underline{x} = x_1,\ldots,x_n$ be distinct individual variables and $\underline{t} = t_1,\ldots,t_n$ be individual terms. Suppose for all $1 \le i \le m$, F^α has no free term variable X_i^σ such that $\sigma \not\equiv \delta_i$ but $\sigma[\underline{x}/\underline{t}] \equiv \delta_i[\underline{x}/\underline{t}]$.

(i) F^α is a term variable.

By the assumption of F^α, we have

$$\begin{aligned} F^\alpha[\underline{X}/\underline{K}][\underline{x}/\underline{t}] &= \begin{cases} K^{\delta'_q}[\underline{x}/\underline{t}] & \text{if } F^\alpha \equiv X^{\delta_q} \text{ for some } 1 \le q \le m, \\ F^\alpha[\underline{x}/\underline{t}] & \text{otherwise,} \end{cases} \\ &= F^\alpha[\underline{x}/\underline{t}][X_1^{\delta_1[\underline{x}/\underline{t}]}/K_1^{\delta'_1}[\underline{x}/\underline{t}],\ldots,X_m^{\delta_m[\underline{x}/\underline{t}]}/K_m^{\delta'_m}[\underline{x}/\underline{t}]]. \end{aligned}$$

For the following cases, suppose $\underline{X}^* = X_{i_1},\ldots,X_{i_r}$ (respectively $\underline{x}^*$) is the sublist of $\underline{X}$ (respectively $\underline{x}$) consisting of those variables which are equivalent to some free term variables of F^α (respectively are in $fv(F^\alpha[\underline{X}/\underline{K}])$) and $\underline{K}^*$ (respectively $\underline{t}^*$) is the corresponding sublist of $\underline{K}$ (respectively $\underline{t}$).

(ii) $F^\alpha = \lambda y.G$.

This case follows by the following claim.

Claim 1. The context $y.G$ satisfies the lemma.

Proof of Claim 1. We have $(y.G)[\underline{X}/\underline{K}][\underline{x}/\underline{t}] = y''.G[y/y'][\underline{X}^*/\underline{K}^*][y'/y''][\underline{x}^*/\underline{t}^*]$, where y' is y (respectively y'' is y') if $y \notin fv(\underline{K}^*)$ (respectively $y' \notin fv(\underline{t}^*)$), otherwise y' (respectively y'') is the first individual variable which is not in $fv(G) \cup fv(\underline{K}^*)$ (respectively $fv(G[y/y'][\underline{X}^*/\underline{K}^*]) \cup fv(\underline{t}^*)$).

Suppose $\underline{x}^{**}$ is the sublist of $\underline{x}$ consisting of those x_i's which are in $fv(F^\alpha)$ and $\underline{t}^{**}$ is the corresponding sublist of $\underline{t}$. Note that $\underline{x}^{**}$is also a sublist of $\underline{x}^*$ since $fv(F^\alpha) \subseteq fv(F^\alpha[\underline{X}/\underline{K}])$ (Note (f) on page 60).

We have $(y.G)[\underline{x}/\underline{t}][X_1^{\delta_1[\underline{x}/\underline{t}]}/K_1^{\delta'_1}[\underline{x}/\underline{t}],\ldots,X_m^{\delta_m[\underline{x}/\underline{t}]}/K_m^{\delta'_m}[\underline{x}/\underline{t}]] =$
$y''''.G[y/y'''][\underline{x}^{**}/\underline{t}^{**}][y'''/y''''][X_{i_1}^{\delta_{i_1}[\underline{x}^*/\underline{t}^*]}/K_{i_1}^{\delta'_{i_1}}[\underline{x}^*/\underline{t}^*],\ldots,X_{i_r}^{\delta_{i_r}[\underline{x}^*/\underline{t}^*]}/K_{i_r}^{\delta'_{i_r}}[\underline{x}^*/\underline{t}^*]]$,
where y''' is y (respectively y'''' is y''') if $y \notin fv(\underline{t}^{**})$ (respectively $y''' \notin \bigcup_{j=1}^{r} fv(K_{i_j}^{\delta'_{i_j}}[\underline{x}^*/\underline{t}^*])$), otherwise y''' (respectively y'''') is the first individual variable which is not in $fv(G) \cup fv(\underline{t}^{**})$ (respectively $fv(G[y/y'''][\underline{x}^{**}/\underline{t}^{**}]) \cup \bigcup_{j=1}^{r} fv(K_{i_j}^{\delta'_{i_j}}[\underline{x}^*/\underline{t}^*])$).

Let z be an individual variable which does not occur in $\underline{x}^*$,
$G[y/y'][\underline{X}^*/\underline{K}^*][y'/y''][\underline{x}^*/\underline{t}^*]$ or
$G[y/y'''][\underline{x}^{**}/\underline{t}^{**}][y'''/y''''][X_{i_1}^{\delta_{i_1}}/K_{i_1}^{\delta'_{i_1}}[\underline{x}^*/\underline{t}^*],\ldots,X_{i_r}^{\delta_{i_r}}/K_{i_r}^{\delta'_{i_r}}[\underline{x}^*/\underline{t}^*]]$.

By the induction hypothesis, we have

$(y.G)[\underline{X}/\underline{K}][\underline{x}/\underline{t}]$

$$
\begin{aligned}
&= y''.G[y/y'][\underline{X}^*/\underline{K}^*][y'/y''][\underline{x}^*/\underline{t}^*]\\
&\equiv z.G[y/y'][\underline{X}^*/\underline{K}^*][y'/y''][\underline{x}^*/\underline{t}^*][y''/z]\\
&\equiv z.G[y/y'][\underline{X}^*/\underline{K}^*][y'/y''][y''/z][\underline{x}^*/\underline{t}^*]\\
&\equiv z.G[y/y'][\underline{X}^*/\underline{K}^*][y'/z][\underline{x}^*/\underline{t}^*]\\
&\equiv z.G[y/y'][y'/z][\underline{X}^*/\underline{K}^*][\underline{x}^*/\underline{t}^*]\\
&\equiv z.G[y/z][\underline{X}^*/\underline{K}^*][\underline{x}^*/\underline{t}^*]\\
&\equiv z.G[y/z][\underline{x}^*/\underline{t}^*][X_{i_1}^{\delta_{i_1}[\underline{x}^*/\underline{t}^*]}/K_{i_1}^{\delta'_{i_1}}[\underline{x}^*/\underline{t}^*],\ldots,X_{i_r}^{\delta_{i_r}[\underline{x}^*/\underline{t}^*]}/K_{i_r}^{\delta'_{i_r}}[\underline{x}^*/\underline{t}^*]]\\
&= z.G[y/z][\underline{x}^{**}/\underline{t}^{**}][X_{i_1}^{\delta_{i_1}[\underline{x}^*/\underline{t}^*]}/K_{i_1}^{\delta'_{i_1}}[\underline{x}^*/\underline{t}^*],\ldots,X_{i_r}^{\delta_{i_r}[\underline{x}^*/\underline{t}^*]}/K_{i_r}^{\delta'_{i_r}}[\underline{x}^*/\underline{t}^*]], \text{ and}
\end{aligned}
$$

$(y.G)[\underline{x}/\underline{t}][X_1^{\delta_1[\underline{x}/\underline{t}]}/K_1^{\delta'_1}[\underline{x}/\underline{t}],\ldots,X_m^{\delta_m[\underline{x}/\underline{t}]}/K_m^{\delta'_m}[\underline{x}/\underline{t}]]$

$$
\begin{aligned}
&= y'''.G[y/y'''][\underline{x}^{**}/\underline{t}^{**}][y'''/y''''][X_{i_1}^{\delta_{i_1}[\underline{x}^*/\underline{t}^*]}/K_{i_1}^{\delta'_{i_1}}[\underline{x}^*/\underline{t}^*],\ldots,X_{i_r}^{\delta_{i_r}[\underline{x}^*/\underline{t}^*]}/K_{i_r}^{\delta'_{i_r}}[\underline{x}^*/\underline{t}^*]]\\
&\equiv z.G[y/y'''][\underline{x}^{**}/\underline{t}^{**}][y'''/y''''][X_{i_1}^{\delta_{i_1}[\underline{x}^*/\underline{t}^*]}/K_{i_1}^{\delta'_{i_1}}[\underline{x}^*/\underline{t}^*],\ldots,X_{i_r}^{\delta_{i_r}[\underline{x}^*/\underline{t}^*]}/K_{i_r}^{\delta'_{i_r}}[\underline{x}^*/\underline{t}^*]]\\
&\quad [y''''/z]\\
&\equiv z.G[y/y'''][\underline{x}^{**}/\underline{t}^{**}][y'''/y''''][y''''/z]\\
&\quad [X_{i_1}^{\delta_{i_1}[\underline{x}^*/\underline{t}^*]}/K_{i_1}^{\delta'_{i_1}}[\underline{x}^*/\underline{t}^*],\ldots,X_{i_r}^{\delta_{i_r}[\underline{x}^*/\underline{t}^*]}/K_{i_r}^{\delta'_{i_r}}[\underline{x}^*/\underline{t}^*]]\\
&\equiv z.G[y/y'''][\underline{x}^{**}/\underline{t}^{**}][y'''/z][X_{i_1}^{\delta_{i_1}[\underline{x}^*/\underline{t}^*]}/K_{i_1}^{\delta'_{i_1}}[\underline{x}^*/\underline{t}^*],\ldots,X_{i_r}^{\delta_{i_r}[\underline{x}^*/\underline{t}^*]}/K_{i_r}^{\delta'_{i_r}}[\underline{x}^*/\underline{t}^*]]\\
&\equiv z.G[y/y'''][y'''/z][\underline{x}^{**}/\underline{t}^{**}][X_{i_1}^{\delta_{i_1}[\underline{x}^*/\underline{t}^*]}/K_{i_1}^{\delta'_{i_1}}[\underline{x}^*/\underline{t}^*],\ldots,X_{i_r}^{\delta_{i_r}[\underline{x}^*/\underline{t}^*]}/K_{i_r}^{\delta'_{i_r}}[\underline{x}^*/\underline{t}^*]]\\
&\equiv z.G[y/z][\underline{x}^{**}/\underline{t}^{**}][X_{i_1}^{\delta_{i_1}[\underline{x}^*/\underline{t}^*]}/K_{i_1}^{\delta'_{i_1}}[\underline{x}^*/\underline{t}^*],\ldots,X_{i_r}^{\delta_{i_r}[\underline{x}^*/\underline{t}^*]}/K_{i_r}^{\delta'_{i_r}}[\underline{x}^*/\underline{t}^*]].
\end{aligned}
$$

Thus we have the claim.

(iii) $F^\alpha = \lambda P.G$.

This case follows by the following claim.

Claim 2. The context $P.G$ satisfies the lemma.

Proof of Claim 2. We have $(P.G)[\underline{X}/\underline{K}][\underline{x}/\underline{t}] = P'.G[P/P'][\underline{X}^*/\underline{K}^*][\underline{x}/\underline{t}]$ and
$(P.G)[\underline{x}/\underline{t}][X_1^{\delta_1[\underline{x}/\underline{t}]}/K_1^{\delta'_1}[\underline{x}/\underline{t}],\ldots,X_m^{\delta_m[\underline{x}/\underline{t}]}/K_m^{\delta'_m}[\underline{x}/\underline{t}]]$

$= P'.G[\underline{x}/\underline{t}][P/P'][X_{i_1}^{\delta_{i_1}[\underline{x}/\underline{t}]}/K_{i_1}^{\delta'_{i_1}}[\underline{x}/\underline{t}],\ldots,X_{i_r}^{\delta_{i_r}[\underline{x}/\underline{t}]}/K_{i_r}^{\delta'_{i_r}}[\underline{x}/\underline{t}]]$,

where P' is P if $P \notin FV(\underline{K}^*)$ (so $P \notin \bigcup_{j=1}^{r} FV(K_{i_j}^{\delta'_{i_j}}[\underline{x}/\underline{t}])$), otherwise P' is the first predicate variable with the same arity as P which is not in $FV(G)\cup FV(\underline{K}^*)$ $(= FV(G[\underline{x}/\underline{t}]) \cup \bigcup_{j=1}^{r} FV(K_{i_j}^{\delta'_{i_j}}[\underline{x}/\underline{t}]))$.

By the induction hypothesis, we have

$$
\begin{aligned}
&(P.G)[\underline{x}/\underline{t}][X_1^{\delta_1[\underline{x}/\underline{t}]}/K_1^{\delta'_1}[\underline{x}/\underline{t}],\ldots,X_m^{\delta_m[\underline{x}/\underline{t}]}/K_m^{\delta'_m}[\underline{x}/\underline{t}]] \\
&\qquad = P'.G[\underline{x}/\underline{t}][P/P'][X_{i_1}^{\delta_{i_1}[\underline{x}/\underline{t}]}/K_{i_1}^{\delta'_{i_1}}[\underline{x}/\underline{t}],\ldots,X_{i_r}^{\delta_{i_r}[\underline{x}/\underline{t}]}/K_{i_r}^{\delta'_{i_r}}[\underline{x}/\underline{t}]] \\
&\qquad \equiv P'.G[P/P'][\underline{x}/\underline{t}][X_{i_1}^{\delta_{i_1}[\underline{x}/\underline{t}]}/K_{i_1}^{\delta'_{i_1}}[\underline{x}/\underline{t}],\ldots,X_{i_r}^{\delta_{i_r}[\underline{x}/\underline{t}]}/K_{i_r}^{\delta'_{i_r}}[\underline{x}/\underline{t}]] \\
&\qquad \equiv P'.G[P/P'][\underline{X}^*/\underline{K}^*][\underline{x}/\underline{t}] \\
&\qquad = (P.G)[\underline{X}/\underline{K}][\underline{x}/\underline{t}].
\end{aligned}
$$

(iv) $F^\alpha = \lambda Y^\beta.G$.

This case follows by the following claim.

Claim 3. The context $Y^\beta.G$ satisfies the lemma.

Proof of Claim 3. We have

$$
\begin{aligned}
(Y^\beta.G)[\underline{X}/\underline{K}][\underline{x}/\underline{t}] &= (Z_1^\beta.G[Y^\beta/Z_1^\beta][\underline{X}^*/\underline{K}^*])[\underline{x}/\underline{t}] \\
&= Z_2^{\beta[\underline{x}/\underline{t}]}.G[Y^\beta/Z_1^\beta][\underline{X}^*/\underline{K}^*][Z_1^\beta/Z_2^\beta][\underline{x}/\underline{t}],
\end{aligned}
$$

where Z_1^β is Y^β if Y^β is not equivalent to any free term variable in $\underline{K}^*$, otherwise Z_1^β is the first term variable of type $[\beta]$ which is not equivalent to any free term variable in $\underline{K}^*$ or G, and Z_2^β is Z_1^β if $Z_1^\beta.G[Y^\beta/Z_1^\beta][\underline{X}^*/\underline{K}^*]$ has no free term variable Z_1^σ such that $\sigma[\underline{x}/\underline{t}] \equiv \beta[\underline{x}/\underline{t}]$, otherwise Z_2^β is the first term variable of type $[\beta]$ such that there is no free term variable Z_2^σ of $Z_1^\beta.G[Y^\beta/Z_1^\beta][\underline{X}^*/\underline{K}^*]$ where $\sigma[\underline{x}/\underline{t}] \equiv \beta[\underline{x}/\underline{t}]$, and

$$
\begin{aligned}
&(Y^\beta.G)[\underline{x}/\underline{t}][X_1^{\delta_1[\underline{x}/\underline{t}]}/K_1^{\delta'_1}[\underline{x}/\underline{t}],\ldots,X_m^{\delta_m[\underline{x}/\underline{t}]}/K_m^{\delta'_m}[\underline{x}/\underline{t}]] \\
&\quad = (Z_3^{\beta[\underline{x}/\underline{t}]}.G[Y^\beta/Z_3^\beta][\underline{x}/\underline{t}])[X_1^{\delta_1[\underline{x}/\underline{t}]}/K_1^{\delta'_1}[\underline{x}/\underline{t}],\ldots,X_m^{\delta_m[\underline{x}/\underline{t}]}/K_m^{\delta'_m}[\underline{x}/\underline{t}]] \\
&\quad = Z_4^{\beta[\underline{x}/\underline{t}]}.G[Y^\beta/Z_3^\beta][\underline{x}/\underline{t}][Z_3^{\beta[\underline{x}/\underline{t}]}/Z_4^{\beta[\underline{x}/\underline{t}]}] \\
&\qquad\quad [X_{i_1}^{\delta_{i_1}[\underline{x}/\underline{t}]}/K_{i_1}^{\delta'_{i_1}}[\underline{x}/\underline{t}],\ldots,X_{i_r}^{\delta_{i_r}[\underline{x}/\underline{t}]}/K_{i_r}^{\delta'_{i_r}}[\underline{x}/\underline{t}]],
\end{aligned}
$$

where Z_3^β is Y^β if $Y^\beta.G$ has no free term variable Y^σ such that $\sigma[\underline{x}/\underline{t}] \equiv \beta[\underline{x}/\underline{t}]$, otherwise Z_3^β is the first term variable of type $[\beta]$ such that there is no free term variable Z_3^{σ} of $Y^{\beta}.G$ such that $\sigma[\underline{x}/\underline{t}] \equiv \beta[\underline{x}/\underline{t}]$, and $Z_4^{\beta[\underline{x}/\underline{t}]}$ is $Z_3^{\beta[\underline{x}/\underline{t}]}$ if $Z_3^{\beta[\underline{x}/\underline{t}]}$ is not equivalent to any free term variable in $K_{i_1}^{\delta'_{i_1}}[\underline{x}/\underline{t}], \ldots, K_{i_r}^{\delta'_{i_r}}[\underline{x}/\underline{t}]$, otherwise $Z_4^{\beta[\underline{x}/\underline{t}]}$ is the first term variable of type $[\beta[\underline{x}/\underline{t}]]$ which is not equivalent to any free term variable in $K_{i_1}^{\delta'_{i_1}}[\underline{x}/\underline{t}], \ldots, K_{i_r}^{\delta'_{i_r}}[\underline{x}/\underline{t}]$ or $G[Y^{\beta}/Z_3^{\beta}][\underline{x}/\underline{t}]$.

Let $Z^{\beta[\underline{x}/\underline{t}]}$ be a term variable which is not equivalent to any term variable occurring in $\underline{X}^*$, $X_{i_1}^{\delta_{i_1}[\underline{x}/\underline{t}]}, \ldots, X_{i_r}^{\delta_{i_r}[\underline{x}/\underline{t}]}$, $G[Y^{\beta}/Z_1^{\beta}][\underline{X}^*/\underline{K}^*][Z_1^{\beta}/Z_2^{\beta}][\underline{x}/\underline{t}]$, or $G[Y^{\beta}/Z_3^{\beta}][\underline{x}/\underline{t}][Z_3^{\beta[\underline{x}/\underline{t}]}/Z_4^{\beta[\underline{x}/\underline{t}]}][X_{i_1}^{\delta_{i_1}[\underline{x}/\underline{t}]}/K_{i_1}^{\delta'_{i_1}}[\underline{x}/\underline{t}], \ldots, X_{i_r}^{\delta_{i_r}[\underline{x}/\underline{t}]}/K_{i_r}^{\delta'_{i_r}}[\underline{x}/\underline{t}]]$.

By the induction hypothesis, we have

$$
\begin{aligned}
&(Y^{\beta}.G)[\underline{x}/\underline{t}][X_1^{\delta_1[\underline{x}/\underline{t}]}/K_1^{\delta'_1}[\underline{x}/\underline{t}], \ldots, X_m^{\delta_m[\underline{x}/\underline{t}]}/K_m^{\delta'_m}[\underline{x}/\underline{t}]] \\
&\quad = Z_4^{\beta[\underline{x}/\underline{t}]}.G[Y^{\beta}/Z_3^{\beta}][\underline{x}/\underline{t}][Z_3^{\beta[\underline{x}/\underline{t}]}/Z_4^{\beta[\underline{x}/\underline{t}]}] \\
&\qquad\qquad [X_{i_1}^{\delta_{i_1}[\underline{x}/\underline{t}]}/K_{i_1}^{\delta'_{i_1}}[\underline{x}/\underline{t}], \ldots, X_{i_r}^{\delta_{i_r}[\underline{x}/\underline{t}]}/K_{i_r}^{\delta'_{i_r}}[\underline{x}/\underline{t}]] \\
&\quad \equiv Z^{\beta[\underline{x}/\underline{t}]}.G[Y^{\beta}/Z_3^{\beta}][\underline{x}/\underline{t}][Z_3^{\beta[\underline{x}/\underline{t}]}/Z_4^{\beta[\underline{x}/\underline{t}]}] \\
&\qquad\qquad [X_{i_1}^{\delta_{i_1}[\underline{x}/\underline{t}]}/K_{i_1}^{\delta'_{i_1}}[\underline{x}/\underline{t}], \ldots, X_{i_r}^{\delta_{i_r}[\underline{x}/\underline{t}]}/K_{i_r}^{\delta'_{i_r}}[\underline{x}/\underline{t}]][Z_4^{\beta[\underline{x}/\underline{t}]}/Z^{\beta[\underline{x}/\underline{t}]}] \\
&\quad \equiv Z^{\beta[\underline{x}/\underline{t}]}.G[Y^{\beta}/Z_3^{\beta}][\underline{x}/\underline{t}][Z_3^{\beta[\underline{x}/\underline{t}]}/Z_4^{\beta[\underline{x}/\underline{t}]}][Z_4^{\beta[\underline{x}/\underline{t}]}/Z^{\beta[\underline{x}/\underline{t}]}] \\
&\qquad\qquad [X_{i_1}^{\delta_{i_1}[\underline{x}/\underline{t}]}/K_{i_1}^{\delta'_{i_1}}[\underline{x}/\underline{t}], \ldots, X_{i_r}^{\delta_{i_r}[\underline{x}/\underline{t}]}/K_{i_r}^{\delta'_{i_r}}[\underline{x}/\underline{t}]] \\
&\quad \equiv Z^{\beta[\underline{x}/\underline{t}]}.G[Y^{\beta}/Z_3^{\beta}][\underline{x}/\underline{t}][Z_3^{\beta[\underline{x}/\underline{t}]}/Z^{\beta[\underline{x}/\underline{t}]}] \\
&\qquad\qquad [X_{i_1}^{\delta_{i_1}[\underline{x}/\underline{t}]}/K_{i_1}^{\delta'_{i_1}}[\underline{x}/\underline{t}], \ldots, X_{i_r}^{\delta_{i_r}[\underline{x}/\underline{t}]}/K_{i_r}^{\delta'_{i_r}}[\underline{x}/\underline{t}]] \\
&\quad \equiv Z^{\beta[\underline{x}/\underline{t}]}.G[Y^{\beta}/Z_3^{\beta}][Z_3^{\beta}/Z^{\beta}][\underline{x}/\underline{t}] \\
&\qquad\qquad [X_{i_1}^{\delta_{i_1}[\underline{x}/\underline{t}]}/K_{i_1}^{\delta'_{i_1}}[\underline{x}/\underline{t}], \ldots, X_{i_r}^{\delta_{i_r}[\underline{x}/\underline{t}]}/K_{i_r}^{\delta'_{i_r}}[\underline{x}/\underline{t}]] \\
&\quad \equiv Z^{\beta[\underline{x}/\underline{t}]}.G[Y^{\beta}/Z^{\beta}][\underline{x}/\underline{t}][X_{i_1}^{\delta_{i_1}[\underline{x}/\underline{t}]}/K_{i_1}^{\delta'_{i_1}}[\underline{x}/\underline{t}], \ldots, X_{i_r}^{\delta_{i_r}[\underline{x}/\underline{t}]}/K_{i_r}^{\delta'_{i_r}}[\underline{x}/\underline{t}]] \\
&\quad \equiv Z^{\beta[\underline{x}/\underline{t}]}.G[Y^{\beta}/Z^{\beta}][\underline{X}^*/\underline{K}^*][\underline{x}/\underline{t}], \text{ and}
\end{aligned}
$$

$$
\begin{aligned}
(Y^{\beta}.G)[\underline{X}/\underline{K}][\underline{x}/\underline{t}] &= Z_2^{\beta[\underline{x}/\underline{t}]}.G[Y^{\beta}/Z_1^{\beta}][\underline{X}^{*}/\underline{K}^{*}][Z_1^{\beta}/Z_2^{\beta}][\underline{x}/\underline{t}] \\
&\equiv Z^{\beta[\underline{x}/\underline{t}]}.G[Y^{\beta}/Z_1^{\beta}][\underline{X}^{*}/\underline{K}^{*}][Z_1^{\beta}/Z_2^{\beta}][\underline{x}/\underline{t}][Z_2^{\beta[\underline{x}/\underline{t}]}/Z^{\beta[\underline{x}/\underline{t}]}] \\
&\equiv Z^{\beta[\underline{x}/\underline{t}]}.G[Y^{\beta}/Z_1^{\beta}][\underline{X}^{*}/\underline{K}^{*}][Z_1^{\beta}/Z_2^{\beta}][Z_2^{\beta}/Z^{\beta}][\underline{x}/\underline{t}] \\
&\equiv Z^{\beta[\underline{x}/\underline{t}]}.G[Y^{\beta}/Z_1^{\beta}][\underline{X}^{*}/\underline{K}^{*}][Z_1^{\beta}/Z^{\beta}][\underline{x}/\underline{t}] \\
&\equiv Z^{\beta[\underline{x}/\underline{t}]}.G[Y^{\beta}/Z_1^{\beta}][Z_1^{\beta}/Z^{\beta}][\underline{X}^{*}/\underline{K}^{*}][\underline{x}/\underline{t}] \\
&\equiv Z^{\beta[\underline{x}/\underline{t}]}.G[Y^{\beta}/Z^{\beta}][\underline{X}^{*}/\underline{K}^{*}][\underline{x}/\underline{t}].
\end{aligned}
$$

Thus we have the claim.

The case $\oplus(Y_1.G_1, Y_2.G_2, H)$ also follows by the above claim and the induction hypothesis. By applying the induction hypothesis to H, the cases $ST(y.Y.G, H)$ and $ST(P.Y.G, H)$ can be proved in the same way as the cases $\lambda y.G$ and $\lambda P.G$, respectively, by using the previous claims for the context $Y.G$.

For the proofs of Lemmas 3.2.24, 3.2.25, and 3.2.26, suppose $F \equiv F'$. Then there exists a sequence of C-H terms $F = F_0, \ldots, F_k = F'$, $k \geq 1$ such that for each $1 \leq i \leq k$, F_i is obtained from F_{i-1} either by replacing some occurrences of term variables by equivalent term variables or by a legitimate change of bound variable.

Proof of Lemma 3.2.24.

(i) $F = Z^{\alpha}$.

Then $F' = Z^{\alpha'}$ for some formula α' where $\alpha' \equiv \alpha$. By Lemma 2.14, $\alpha'[\underline{x}/\underline{t}] \equiv \alpha[\underline{x}/\underline{t}]$. Hence $F[\underline{x}/\underline{t}] = Z^{\alpha[\underline{x}/\underline{t}]} \equiv Z^{\alpha'[\underline{x}/\underline{t}]} = F'[\underline{x}/\underline{t}]$.

For the following cases, we will prove by induction on k. We will prove only the case $k = 1$ since the case $k > 1$ easily follows by the subsidiary induction hypothesis and the case $k = 1$.

(ii) $F = \lambda y.G$.

Then $F[\underline{x}/\underline{t}] = \lambda z.G[y/z][\underline{x}^*/\underline{t}^*]$, where $\underline{x}^*$ is the sublist of $\underline{x}$ consisting of those variables which are in $fv(F)$, $\underline{t}^*$ is the corresponding sublist of $\underline{t}$, z is y if $y \notin fv(\underline{t}^*)$, otherwise z is the first individual variable which is not in $fv(G) \cup fv(\underline{t}^*)$.

Case 1. $F' = \lambda y.G'$ where $G' \equiv G$.

Since $fv(G) = fv(G')$, $F'[\underline{x}/\underline{t}] = \lambda z.G'[y/z][\underline{x}^*/\underline{t}^*]$. By the main induction hypothesis, $G[y/z][\underline{x}^*/\underline{t}^*] \equiv G'[y/z][\underline{x}^*/\underline{t}^*]$. Hence $F[\underline{x}/\underline{t}] \equiv F'[\underline{x}/\underline{t}]$.

Case 2. $F' = \lambda y'.G[y/y']$ where y is replaceable by y', y' is free for y, and y' does not occur free in G.

Then $F'[\underline{x}/\underline{t}] = \lambda z'.G[y/y'][y'/z'][\underline{x}^*/\underline{t}^*]$ where z' is y' if $y' \notin fv(\underline{t}^*)$, otherwise z' is the first individual variable which is not in $fv(G[y/y']) \cup fv(\underline{t}^*)$.

Since $z \notin (fv(G) - \{y\}) \cup fv(\underline{t}^*)$, either $z \notin fv(G[y/z'][\underline{x}^*/\underline{t}^*])$ or $z = z'$.

Hence, by the induction hypothesis (Lemmas 3.2.19 and 3.2.24) and Corollary 3.2.18,

$$\begin{aligned} F'[\underline{x}/\underline{t}] &= \lambda z'.G[y/y'][y'/z'][\underline{x}^*/\underline{t}^*] \\ &\equiv \lambda z'.G[y/z'][\underline{x}^*/\underline{t}^*] \\ &\equiv \lambda z.G[y/z'][\underline{x}^*/\underline{t}^*][z'/z] \\ &\equiv \lambda z.G[y/z'][z'/z][\underline{x}^*/\underline{t}^*] \\ &\equiv \lambda z.G[y/z][\underline{x}^*/\underline{t}^*] = F[\underline{x}/\underline{t}]. \end{aligned}$$

(iii) $F = \lambda P.G$.

Case 1. $F' = \lambda P.G'$ where $G' \equiv G$.

This case follows by the main induction hypothesis.

Case 2. $F' = \lambda P'.G[P/P']$ where P' is of the same arity as P, P is replaceable by P', P' is free for P, and P' does not occur free in G.

Then $P' \notin FV(G[\underline{x}/\underline{t}])$. Hence, by the induction hypothesis (Lemma 3.2.20) and Corollary 3.2.18, we have $F[\underline{x}/\underline{t}] = \lambda P.G[\underline{x}/\underline{t}] \equiv \lambda P'.G[\underline{x}/\underline{t}][P/P'] \equiv$

$\lambda P'.G[P/P'][\underline{x}/\underline{t}] = F'[\underline{x}/\underline{t}]$.

(iv) $F = \lambda X^{\beta}.G$.

Then $F[\underline{x}/\underline{t}] = \lambda Y^{\beta[\underline{x}/\underline{t}]}.G[X^{\beta}/Y^{\beta}][\underline{x}/\underline{t}]$, where Y^{β} is X^{β} if $X^{\beta}.G$ has no free term variable X^{σ} such that $\sigma[\underline{x}/\underline{t}] \equiv \beta[\underline{x}/\underline{t}]$, otherwise Y^{β} is the first term variable of type $[\beta]$ such that $X^{\beta}.G$ has no free term variable Y^{σ} where $\sigma[\underline{x}/\underline{t}] \equiv \beta[\underline{x}/\underline{t}]$.

Case 1. $F' = \lambda X^{\beta'}.G'$ where $\beta \equiv \beta'$ and $G' \equiv G$.

Since $fv(G) = fv(G')$, we have $F'[\underline{x}/\underline{t}] = \lambda Y^{\beta'[\underline{x}/\underline{t}]}.G'[X^{\beta'}/Y^{\beta'}][\underline{x}/\underline{t}]$.

By the induction hypothesis (Lemmas 3.2.24 and 3.2.26), $G[X^{\beta}/Y^{\beta'}][\underline{x}/\underline{t}] \equiv G'[X^{\beta'}/Y^{\beta'}][\underline{x}/\underline{t}]$. Hence $F[\underline{x}/\underline{t}] \equiv F'[\underline{x}/\underline{t}]$.

Case 2. $F' = \lambda Z^{\beta'}.G[X^{\beta}/Z^{\beta'}]$ where $\beta' \equiv \beta$, $Z^{\beta'}$ is not equivalent to any free term variable of G and is free for X^{β} in G.

Then $F'[\underline{x}/\underline{t}] = \lambda Z_1^{\beta'[\underline{x}/\underline{t}]}.G[X^{\beta}/Z^{\beta'}][Z^{\beta'}/Z_1^{\beta'}][\underline{x}/\underline{t}]$, where $Z_1^{\beta'}$ is $Z^{\beta'}$ if $Z^{\beta'}.G[X^{\beta}/Z^{\beta'}]$ has no free term variable Z^{σ} such that $\sigma[\underline{x}/\underline{t}] \equiv \beta'[\underline{x}/\underline{t}]$, otherwise $Z_1^{\beta'}$ is the first term variable of type $[\beta']$ such that there is no free term variable Z_1^{σ} of $Z^{\beta'}.G[X^{\beta}/Z^{\beta'}]$ where $\sigma[\underline{x}/\underline{t}] \equiv \beta'[\underline{x}/\underline{t}]$.

Let $Z_*^{\beta[\underline{x}/\underline{t}]}$ be a term variable which is not equivalent to any term variable occurring in G, $G[X^{\beta}/Y^{\beta}][\underline{x}/\underline{t}]$ or $G[X^{\beta}/Z^{\beta'}][Z^{\beta'}/Z_1^{\beta'}][\underline{x}/\underline{t}]$.

By the induction hypothesis (Lemmas 3.2.21, 3.2.23, and 3.2.24), we have

$$\begin{aligned} F'[\underline{x}/\underline{t}] &= \lambda Z_1^{\beta'[\underline{x}/\underline{t}]}.G[X^{\beta}/Z^{\beta'}][Z^{\beta'}/Z_1^{\beta'}][\underline{x}/\underline{t}] \\ &\equiv \lambda Z_*^{\beta[\underline{x}/\underline{t}]}.G[X^{\beta}/Z^{\beta'}][Z^{\beta'}/Z_1^{\beta'}][\underline{x}/\underline{t}][Z_1^{\beta'[\underline{x}/\underline{t}]}/Z_*^{\beta[\underline{x}/\underline{t}]}] \\ &\equiv \lambda Z_*^{\beta[\underline{x}/\underline{t}]}.G[X^{\beta}/Z^{\beta'}][Z^{\beta'}/Z_1^{\beta'}][Z_1^{\beta'}/Z_*^{\beta}][\underline{x}/\underline{t}] \\ &\equiv \lambda Z_*^{\beta[\underline{x}/\underline{t}]}.G[X^{\beta}/Z^{\beta'}][Z^{\beta'}/Z_*^{\beta}][\underline{x}/\underline{t}] \\ &\equiv \lambda Z_*^{\beta[\underline{x}/\underline{t}]}.G[X^{\beta}/Z_*^{\beta}][\underline{x}/\underline{t}], \text{ and} \end{aligned}$$

$$\begin{aligned}F[\underline{x}/\underline{t}] &= \lambda Y^{\beta[\underline{x}/\underline{t}]}.G[X^\beta/Y^\beta][\underline{x}/\underline{t}] \\ &\equiv \lambda Z_*^{\beta[\underline{x}/\underline{t}]}.G[X^\beta/Y^\beta][\underline{x}/\underline{t}][Y^{\beta[\underline{x}/\underline{t}]}/Z_*^{\beta[\underline{x}/\underline{t}]}] \\ &\equiv \lambda Z_*^{\beta[\underline{x}/\underline{t}]}.G[X^\beta/Y^\beta][Y^\beta/Z_*^\beta][\underline{x}/\underline{t}] \\ &\equiv \lambda Z_*^{\beta[\underline{x}/\underline{t}]}.G[X^\beta/Z_*^\beta][\underline{x}/\underline{t}].\end{aligned}$$

From the proofs of cases (ii)-(iv) we can also conclude that:

if C is a context of the form $y.G$, $P.G$ or $X.G$ and C' is a context which is equivalent to C, then $C[\underline{x}/\underline{t}] \equiv C'[\underline{x}/\underline{t}]$.

By using the above result for contexts of the form $X.G$, we can prove in the same way as the cases $\lambda y.G$ and $\lambda P.G$, respectively, that this result also holds for contexts of the form $y.X.G$ and $P.X.G$.

The cases $\oplus(X.G, Y.H, K)$, $ST(y.X.G, K)$, and $ST(P.X.G, K)$ follow by applying the induction hypothesis to K and the above results to $X.G$, $Y.H$, $y.X.G$, and $P.X.G$.

Lemma 3.2.25 can be proved in the same way as Lemma 3.2.24 (using Lemma 2.17 when F is a term variable).

Proof of Lemma 3.2.26.

(i) F is a term variable.

If $F \equiv X^{\delta_q}$ for some $1 \leq q \leq n$, then $F' \equiv X^{\delta_q}$ and hence $F[\underline{X}/\underline{H}] = H_q^{\delta'_q} \equiv K_q^{\delta''_q} = F'[\underline{X}/\underline{K}]$, otherwise $F[\underline{X}/\underline{H}] = F \equiv F' = F'[\underline{X}/\underline{K}]$.

For the following cases, we will prove by induction on k. We will prove only the case $k = 1$ since the case $k > 1$ easily follows by the subsidiary induction hypothesis and the case $k = 1$.

Suppose $\underline{X}^* = X_{i_1}, \ldots, X_{i_l}$ is the sublist of $\underline{X}$ consisting of those term variables which are equivalent to some free term variables of F, and $\underline{H}^*$ and $\underline{K}^*$ are the corresponding sublists of $\underline{H}$ and $\underline{K}$, respectively.

(ii) $F = \lambda y.G$.

Then $F[\underline{X}/\underline{H}] = \lambda y'.G[y/y'][\underline{X}^*/\underline{H}^*]$, where y' is y if $y \notin fv(\underline{H}^*)$, otherwise y' is the first individual variable which is not in $fv(G) \cup fv(\underline{H}^*)$.

Case 1. $F' = \lambda y.G'$ where $G' \equiv G$.

Since $H_i \equiv K_i$ for all $1 \leq i \leq n$ and $G' \equiv G$, $fv(H_i) = fv(K_i)$ for all $1 \leq i \leq n$ and $fv(G') = fv(G)$. Hence $F'[\underline{X}/\underline{K}] = \lambda y'.G'[y/y'][\underline{X}^*/\underline{K}^*]$. By the induction hypothesis (Lemmas 3.2.24 and 3.2.26), we have $F[\underline{X}/\underline{H}] = \lambda y'.G[y/y'][\underline{X}^*/\underline{H}^*] \equiv \lambda y'.G'[y/y'][\underline{X}^*/\underline{K}^*] = F'[\underline{X}/\underline{K}]$.

Case 2. $F' = \lambda z.G[y/z]$ where y is replaceable by z, z is free for y, and z does not occur free in G.

Then $F'[\underline{X}/\underline{K}] = \lambda z'.G[y/z][z/z'][\underline{X}^*/\underline{K}^*]$, where z' is z if $z \notin fv(\underline{K}^*)$, otherwise z' is the first individual variable which is not in $fv(G[y/z]) \cup fv(\underline{K}^*)$.

Let z^* be an individual variable which does not occur in $G[y/y'][\underline{X}^*/\underline{H}^*]$ or $G[y/z][z/z'][\underline{X}^*/\underline{K}^*]$.

Since $y' \notin fv(G)$ and y does not occur free in the type superscript of any free term variable of G, for all $1 \leq s \leq l$, $G[y/y']$ has no free term variable X^σ such that $X^\sigma \not\equiv X_{i_s}^{\delta_{i_s}}$ but $X^{\sigma[y'/z^*]} \equiv X_{i_s}^{\delta_{i_s}[y'/z^*]}$ i.e. $X^\sigma \equiv X_{i_s}^{\delta_{i_s}}$.

Similarly, for all $1 \leq s \leq l$, $G[y/z][z/z']$ has no free term variable X^σ such that $X^\sigma \not\equiv X_{i_s}^{\delta_{i_s}}$ but $X^{\sigma[z/z^*]} \equiv X_{i_s}^{\delta_{i_s}[z'/z^*]}$.

By the induction hypothesis (Lemmas 3.2.19, 3.2.23, and 3.2.26), we have

$$\begin{aligned}
F'[\underline{X}/\underline{K}] &= \lambda z'.G[y/z][z/z'][\underline{X}^*/\underline{K}^*] \\
&\equiv \lambda z^*.G[y/z][z/z'][\underline{X}^*/\underline{K}^*][z'/z^*] \\
&\equiv \lambda z^*.G[y/z][z/z'][z'/z^*][\underline{X}^*/\underline{K}^*] \\
&\equiv \lambda z^*.G[y/z][z/z^*][\underline{X}^*/\underline{K}^*] \\
&\equiv \lambda z^*.G[y/z^*][\underline{X}^*/\underline{K}^*] \\
&\equiv \lambda z^*.G[y/z^*][\underline{X}^*/\underline{H}^*], \text{ and}
\end{aligned}$$

$$\begin{aligned} F[\underline{X}/\underline{H}] &= \lambda y'.G[y/y'][\underline{X}^*/\underline{H}^*] \\ &\equiv \lambda z^*.G[y/y'][\underline{X}^*/\underline{H}^*][y'/z^*] \\ &\equiv \lambda z^*.G[y/y'][y'/z^*][\underline{X}^*/\underline{H}^*] \\ &\equiv \lambda z^*.G[y/z^*][\underline{X}^*/\underline{H}^*]. \end{aligned}$$

Similarly for the case $\lambda P.G$.

(iii) $F = \lambda Y^\beta.G$.

Then $F[\underline{X}/\underline{H}] = \lambda Y^\beta_*.G[Y^\beta/Y^\beta_*][\underline{X}^*/\underline{H}^*]$, where Y^β_* is Y^β if Y^β is not equivalent to any free term variable in $\underline{H}^*$, otherwise Y^β_* is the first term variable of type $[\beta]$ which is not equivalent to any free term variable in $\underline{H}^*$ or G.

Case 1. $F' = \lambda Y^{\beta'}.G'$ where $\beta \equiv \beta'$ and $G' \equiv G$.

Since $H_i \equiv K_i$ for all $1 \le i \le n$, $F'[\underline{X}/\underline{K}] = \lambda Y^\beta_*.G'[Y^{\beta'}/Y^\beta_*][\underline{X}^*/\underline{K}^*]$.

By the induction hypothesis, $G[Y^\beta/Y^\beta_*][\underline{X}^*/\underline{H}^*] \equiv G'[Y^{\beta'}/Y^\beta_*][\underline{X}^*/\underline{K}^*]$. Hence $F[\underline{X}/\underline{H}] \equiv F'[\underline{X}/\underline{K}]$.

Case 2. $F' = \lambda Z^{\beta'}.G[Y^\beta/Z^{\beta'}]$ where $\beta' \equiv \beta$, $Z^{\beta'}$ is free for Y^β and is not equivalent to any free term variable in G.

Then $F'[\underline{X}/\underline{K}] = \lambda Z^{\beta'}_*.G[Y^\beta/Z^{\beta'}][Z^{\beta'}/Z^{\beta'}_*][\underline{X}^*/\underline{K}^*]$, where $Z^{\beta'}_*$ is $Z^{\beta'}$ if $Z^{\beta'}$ is not equivalent to any free term variable in $\underline{K}^*$, otherwise $Z^{\beta'}_*$ is the first term variable of type $[\beta']$ which is not equivalent to any free term variable in $\underline{K}^*$ or $G[Y^\beta/Z^{\beta'}]$.

Since Y^β_* is Y^β or Y^β_* is not equivalent to any free term variable of G, either Y^β_* is $Z^{\beta'}_*$ or Y^β_* is not equivalent to any free term variable of $G[Y^\beta/Z^{\beta'}_*]$. Since Y^β_* is not equivalent to any free term variable in $\underline{H}^*$ and $H_i \equiv K_i$ for all $1 \le i \le n$, either Y^β_* is $Z^{\beta'}_*$ or Y^β_* is not equivalent to any free term variable of $G[Y^\beta/Z^{\beta'}_*][\underline{X}^*/\underline{K}^*]$.

Hence, by the induction hypothesis (Lemmas 3.2.21 and 3.2.26) and Corollary

3.2.18,

$$\begin{aligned}
F'[\underline{X}/\underline{K}] &= \lambda Z_*^{\beta'}.G[Y^\beta/Z^{\beta'}][Z^{\beta'}/Z_*^{\beta'}][\underline{X}^*/\underline{K}^*] \\
&\equiv \lambda Z_*^{\beta'}.G[Y^\beta/Z_*^{\beta'}][\underline{X}^*/\underline{K}^*] \\
&\equiv \lambda Y_*^{\beta}.G[Y^\beta/Z_*^{\beta'}][\underline{X}^*/\underline{K}^*][Z_*^{\beta'}/Y_*^{\beta}] \\
&\equiv \lambda Y_*^{\beta}.G[Y^\beta/Z_*^{\beta'}][Z_*^{\beta'}/Y_*^{\beta}][\underline{X}^*/\underline{K}^*] \\
&\equiv \lambda Y_*^{\beta}.G[Y^\beta/Y_*^{\beta}][\underline{X}^*/\underline{K}^*] \\
&\equiv \lambda Y_*^{\beta}.G[Y^\beta/Y_*^{\beta}][\underline{X}^*/\underline{H}^*] = F[\underline{X}/\underline{H}].
\end{aligned}$$

As in the proof of Lemma 3.2.24, the lemma also holds for contexts and the cases $\oplus(Y_1.F_1, Y_2.F_2, G)$, $ST(y.Y.F_1, F_2)$, and $ST(P.Y.F_1, F_2)$ follow by these results and the induction hypothesis. □

3.3 Reduction rules

We now give reduction rules for terms corresponding to the reductions of proofs which are obtained by short cutting an introduction which is immediately followed by an elimination of the same symbol.

Definition 3.3.1. *We say that a term F* ***reduces*** *to a term F', and write $F \succ F'$, if F' is obtained from F by a finite sequence of replacements of subterms using the* ***reduction rules*** *below.*

(∧Intro, ∧Elim) $\pi_1(G^\alpha, H^\beta) \succ G^\alpha$, $\pi_2(G^\alpha, H^\beta) \succ H^\beta$.

(⊃Intro, ⊃Elim) $(\lambda X^\alpha.G^\beta)(H^\alpha) \succ G^\beta[X^\alpha/H^\alpha]$.

(∀Intro, ∀Elim) $(\lambda x.G^\alpha)(t) \succ G^\alpha[x/t]$.

($\vee$Intro, $\vee$Elim) $\oplus(X^\alpha.G^\gamma, Y^\beta.H^\gamma, (\mu_1 K^\alpha)^{\alpha\vee\beta}) \succ G^\gamma[X^\alpha/K^\alpha]$,

$$\oplus(X^\alpha.G^\gamma, Y^\beta.H^\gamma, (\mu_2 J^\beta)^{\alpha\vee\beta}) \succ H^\gamma[Y^\beta/J^\beta].$$

($\exists$Intro, $\exists$Elim) $ST(x.X^\alpha.H^\gamma, I(t, G^{\alpha(x/t)})) \succ H^\gamma[X^\alpha/Y^\alpha][x/t][Y^{\alpha[x/t]}/G^{\alpha(x/t)}]$, *where* Y^α *is* X^α *if* $X^\alpha.H^\gamma$ *has no free term variable equivalent to* $X^{\alpha[x/t]}$, *otherwise* Y^α *is the first term variable of type* $[\alpha]$ *such that* $Y^{\alpha[x/t]}$ *is not equivalent to any free term variable of* $X^\alpha.H^\gamma$.

($\forall_2$Intro, $\forall_2$Elim) $(\lambda P.G^\alpha)(T) \succ G^\alpha[P/T]$.

($\exists_2$Intro, $\exists_2$Elim)

$$ST(P.X^\alpha.H^\gamma, I(T, G^{\alpha(P/T)})) \succ H^\gamma[X^\alpha/Y^\alpha][P/T][Y^{\alpha[P/T]}/G^{\alpha(P/T)}],$$

where Y^α *is* X^α *if* $X^\alpha.G^\gamma$ *has no free term variable equivalent to* $X^{\alpha[P/T]}$, *otherwise* Y^α *is the first term variable of type* $[\alpha]$ *such that* $Y^{\alpha[P/T]}$ *is not equivalent to any free term variable of* $X^\alpha.G^\gamma$.

In the above rules the expression on the left of the symbol $\succ$ *is called a* ***redex*** *and the expression on the right its* ***contractum****.*

If $F \succ F'$, *we say* F' *is a* ***reduct*** *of* F, *and if* F' *is obtained from* F *by a single application of one of the above rules, denoted by* $F \succ_1 F'$, F' *is called an* ***immediate reduct*** *of* F.

A term is ***normal*** *if it contains no redex.*

Note. It can be easily proved by induction on F that if $F \succ F'$, then

a. F' is of the same type as F;

b. $fv(F') \subseteq fv(F)$, $FV(F') \subseteq FV(F)$, and every free term variable of F' is equivalent to some free term variable of F.

Lemma 3.3.2. *Let F, G, and F' be C-H terms.*

If $F \equiv F'$ and $F \succ_1 G$, then $F' \succ_1 G'$ for some C-H term G' such that $G \equiv G'$.

Proof. Suppose $F \equiv F'$ and $F \succ_1 G$.

Since $F \equiv F'$, there is a sequence of C-H terms $F = F_0, \ldots, F_m = F'$, $m \geq 1$, such that for each $1 \leq i \leq m$, F_i is obtained from F_{i-1} either by replacing some occurrences of term variables by equivalent term variables or by a legitimate change of bound variable. We will prove by induction on m.

$m = 1$: We will prove by induction on F.

Suppose F is the redex which is reduced to G.

(i) $F = \pi_1(F_1, F_2)$.

Then $G = F_1$ and $F' = \pi_1(F_1', F_2')$ for some terms F_1' and F_2' such that $F_1' \equiv F_1$ and $F_2' \equiv F_2$. We have $F' = \pi_1(F_1', F_2') \succ_1 F_1' \equiv F_1 = G$.

Similarly for $\pi_2(F_1, F_2)$.

(ii) $F = (\lambda X^\alpha.F_1)(F_2^\alpha)$.

Then $G = F_1[X^\alpha/F_2^\alpha]$.

Case 1. $F' = (\lambda X^{\alpha'}.F_1')(F_2')$ where $\alpha' \equiv \alpha$, $F_1' \equiv F_1$, and $F_2' \equiv F_2$.

Then $F' \succ_1 F_1'[X^{\alpha'}/F_2'] \equiv F_1[X^\alpha/F_2] = G$ by Lemma 3.2.26.

Case 2. $F' = (\lambda Y^{\alpha'}.F_1[X^\alpha/Y^{\alpha'}])(F_2)$ where $\alpha' \equiv \alpha$, $Y^{\alpha'}$ is free for X^α and is not equivalent to any free term variable in F_1.

Then $F' \succ_1 F_1[X^\alpha/Y^{\alpha'}][Y^{\alpha'}/F_2] \equiv F_1[X^\alpha/F_2] = G$ by Lemma 3.2.21.

(iii) $F = (\lambda x.H)(t)$.

Then $G = H[x/t]$.

Case 1. $F' = (\lambda x.H')(t)$ where $H' \equiv H$.

Then $F' \succ_1 H'[x/t] \equiv H[x/t]$ by Lemma 3.2.24.

Case 2. $F' = (\lambda y.H[x/y])(t)$ where x is replaceable by y, y is free for x, and y does not occur free in H.

Then $F' \succ_1 H[x/y][y/t] \equiv H[x/t] = G$ by Lemma 3.2.19.

(iv) $F = (\lambda P.H)(T)$.

Then $G = H[P/T]$.

Case 1. $F' = (\lambda P.H')(T)$ where $H' \equiv H$.

Then $F' \succ_1 H'[P/T] \equiv H[P/T]$ by Lemma 3.2.25.

Case 2. $F' = (\lambda Q.H[P/Q])(T)$ where Q is of the same arity as P, P is replaceable by Q, Q is free for P, and Q does not occur free in H.

Then $F' \succ_1 H[P/Q][Q/T] \equiv H[P/T] = G$ by Lemma 3.2.20.

(v) $F = \oplus(X^\alpha.F_1, Y^\beta.F_2, \mu_1 H^\alpha)$.

Then $G = F_1[X^\alpha/H]$.

If the change when F becomes F' occurs only in $Y^\beta.F_2$, then $F' \succ_1 G$.

The remaining cases are as follows.

Case 1. $F' = \oplus(X^{\alpha'}.F_1', Y^{\beta'}.F_2', \mu_1 H')$ where $\alpha' \equiv \alpha$, $\beta' \equiv \beta$, $F_1' \equiv F_1$, $F_2' \equiv F_2$, and $H' \equiv H$.

Then $F' \succ_1 F_1'[X^{\alpha'}/H'] \equiv F_1[X^\alpha/H]$ by Lemma 3.2.26.

Case 2. $F' = \oplus(Z^{\alpha'}.F_1[X^\alpha/Z^{\alpha'}], Y^\beta.F_2, \mu_1 H)$ where $\alpha' \equiv \alpha$, $Z^{\alpha'}$ is free for X^α and is not equivalent to any free term variable in F_1.

Then $F' \succ_1 F_1[X^\alpha/Z^{\alpha'}][Z^{\alpha'}/H] \equiv F_1[X^\alpha/H] = G$ by Lemma 3.2.21.

Similarly for $\oplus(X^\alpha.F_1, Y^\beta.F_2, \mu_2 H^\beta)$.

(vi) $F = ST(x.X^\alpha.K^\gamma, I(t, H^{\alpha(x/t)}))$.

Then $G = K[X^\alpha/Y^\alpha][x/t][Y^{\alpha[x/t]}/H]$ where Y^α is X^α if $X^\alpha.K$ has no free term variable equivalent to $X^{\alpha[x/t]}$, otherwise Y^α is the first term variable of type $[\alpha]$ such that $Y^{\alpha[x/t]}$ is not equivalent to any free term variable of $X^\alpha.K$.

Case 1. $F' = ST(x.X^{\alpha'}.K', I(t, H'))$ where $\alpha' \equiv \alpha$, $K' \equiv K$, and $H' \equiv H$.

Then, by Lemmas 3.2.24 and 3.2.26,

$$\begin{aligned} F' &\succ_1 K'[X^{\alpha'}/Y^{\alpha'}][x/t][Y^{\alpha'[x/t]}/H'] \\ &\equiv K[X^{\alpha}/Y^{\alpha}][x/t][Y^{\alpha[x/t]}/H] = G. \end{aligned}$$

Case 2. $F' = ST(x.Z^{\alpha'}.K^{\gamma}[X^{\alpha}/Z^{\alpha'}], I(t,H))$ where $\alpha' \equiv \alpha$, $Z^{\alpha'}$ is free for X^{α} and is not equivalent to any free term variable in K.

Then $F' \succ_1 K[X^{\alpha}/Z^{\alpha'}][Z^{\alpha'}/Z_*^{\alpha'}][x/t][Z_*^{\alpha'[x/t]}/H]$, where $Z_*^{\alpha'}$ is $Z^{\alpha'}$ if $Z^{\alpha'}.K[X^{\alpha'}/Z^{\alpha'}]$ has no free term variable equivalent to $Z^{\alpha'[x/t]}$, otherwise $Z_*^{\alpha'}$ is the first term variable of type $[\alpha']$ such that $Z_*^{\alpha'[x/t]}$ is not equivalent to any free term variable of $Z^{\alpha'}.K[X^{\alpha}/Z^{\alpha'}]$.

Since $Z_*^{\alpha'[x/t]}$ is not equivalent to any free term variable of $Z^{\alpha'}.K[X^{\alpha}/Z^{\alpha'}]$ and x does not occur free in the type superscript of any free term variable of $X^{\alpha}.K$, either $Z_*^{\alpha'[x/t]}$ is not equivalent to any free term variable of $K[X^{\alpha}/Y^{\alpha}][x/t]$ or $Z_*^{\alpha'[x/t]} \equiv Y^{\alpha[x/t]}$.

Then, by Lemmas 3.2.21, 3.2.24, and 3.2.26,

$$\begin{aligned} F' &\succ_1 K[X^{\alpha}/Z^{\alpha'}][Z^{\alpha'}/Z_*^{\alpha'}][x/t][Z_*^{\alpha'[x/t]}/H] \\ &\equiv K[X^{\alpha}/Z_*^{\alpha'}][x/t][Z_*^{\alpha'[x/t]}/H] \\ &\equiv K[X^{\alpha}/Y^{\alpha}][Y^{\alpha}/Z_*^{\alpha'}][x/t][Z_*^{\alpha'[x/t]}/H] \\ &\equiv K[X^{\alpha}/Y^{\alpha}][x/t][Y^{\alpha[x/t]}/Z_*^{\alpha'[x/t]}][Z_*^{\alpha'[x/t]}/H] \\ &\equiv K[X^{\alpha}/Y^{\alpha}][x/t][Y^{\alpha[x/t]}/H] = G. \end{aligned}$$

Case 3. $F' = ST(y.X^{\alpha[x/y]}.K^{\gamma}[x/y], I(t,H))$ where x is replaceable by y, y is free for x, and y does not occur free in $X^{\alpha}.K$.

Then $F' \succ_1 K[x/y][X^{\alpha[x/y]}/Z^{\alpha[x/y]}][y/t][Z^{\alpha[x/y][y/t]}/H]$, where $Z^{\alpha[x/y]}$ is $X^{\alpha[x/y]}$ if $X^{\alpha[x/y]}.K[x/y]$ has no free term variable equivalent to $X^{\alpha[x/y][y/t]}$, otherwise $Z^{\alpha[x/y]}$ is the first term variable of type $[\alpha[x/y]]$ such that $Z^{\alpha[x/y][y/t]}$ is not equivalent to any free term variable of $X^{\alpha[x/y]}.K[x/y]$.

Note that:

(1) since $X^\alpha.K$ has no free term variable which is equivalent to $Y^{\alpha[x/t]}$, x does not occur free in the type superscript of any free term variable of $X^\alpha.K$, and $y \notin fv(X^\alpha.K)$, either $Y^{\alpha[x/t]} \equiv Z^{\alpha[x/t]}$ or $Y^{\alpha[x/t]}$ is not equivalent to any free term variable of $K[x/y][X^{\alpha[x/y]}/Z^{\alpha[x/y]}][y/t]$;

(2) since $X^{\alpha[x/y]}.K[x/y]$ has no free term variable which is equivalent to $Z^{\alpha[x/y][y/t]}$ ($\equiv Z^{\alpha[x/t]}$), $Z^{\alpha[x/y]}.K[x/y][X^{\alpha[x/y]}/Z^{\alpha[x/y]}]$ also has no free term variable which is equivalent to $Z^{\alpha[x/t]}$;

(3) since $y \notin fv(K)$ and x does not occur free in the type superscript of any free term variable of $X^\alpha.K$, either $Z^{\alpha[x/y]}$ is $X^{\alpha[x/y]}$ or $Z^{\alpha[x/y]}$ is not equivalent to any free term variable of $K[x/y]$;

(4) since $y \notin fv(K)$, $X^\alpha.K$ has no free term variable which is equivalent to $X^{\alpha[x/y]}$.

By Lemmas 3.2.19, 3.2.21, 3.2.23, 3.2.24, and 3.2.26,

$$\begin{aligned}
F' &\succ_1 K[x/y][X^{\alpha[x/y]}/Z^{\alpha[x/y]}][y/t][Z^{\alpha[x/y][y/t]}/H] \\
&\equiv K[x/y][X^{\alpha[x/y]}/Z^{\alpha[x/y]}][y/t][Z^{\alpha[x/y][y/t]}/Y^{\alpha[x/y][y/t]}][Y^{\alpha[x/y][y/t]}/H] \\
& \qquad \text{(since (1))} \\
&\equiv K[x/y][X^{\alpha[x/y]}/Z^{\alpha[x/y]}][Z^{\alpha[x/y]}/Y^{\alpha[x/y]}][y/t][Y^{\alpha[x/t]}/H] \qquad \text{(since (2))} \\
&\equiv K[x/y][X^{\alpha[x/y]}/Y^{\alpha[x/y]}][y/t][Y^{\alpha[x/t]}/H] \qquad \text{(since (3))} \\
&\equiv K[X^\alpha/Y^\alpha][x/y][y/t][Y^{\alpha'[x/t]}/H] \qquad \text{(since (4))} \\
&\equiv K[X^\alpha/Y^\alpha][x/t][Y^{\alpha[x/t]}/H] = G.
\end{aligned}$$

Similarly for $ST(P.X^\alpha.K, I(T, H^{\alpha(P/T)}))$.

Now suppose F is not the redex which is reduced to G.

(vii) $F = (F_1, F_2)$.

Without loss of generality, we may assume that the reduction occurs in F_1.

Then $G = (G_1, F_2)$ for some term G_1 such that $F_1 \succ_1 G_1$ and $F' = (F_1', F_2')$ for some terms F_1' and F_2' such that $F_1' \equiv F_1$ and $F_2' \equiv F_2$. By the subsidiary induction hypothesis, $F_1' \succ_1 G_1'$ for some term G_1' such that $G_1' \equiv G_1$. Hence $F' = (F_1', F_2') \succ_1 (G_1', F_2') \equiv (G_1, F_2) = G$.

The remaining cases can be proved similarly.

$m > 1$: By the main induction hypothesis, $F_{m-1} \succ_1 H'$ for some term H' such that $H' \equiv G$. By the case $m = 1$, $F' \succ_1 H$ for some term H such that $H \equiv H'$, and so $H \equiv G$. □

Corollary 3.3.3. *For any terms F, G, and F', if $F \equiv F'$ and $F \succ G$, then $F' \succ G'$ for some term G' such that $G \equiv G'$.*

Lemma 3.3.4. *Let F and G be terms. If $F \succ_1 G$, then*

a. for any individual variable x and any individual term t, $F[x/t] \succ_1 H$ for some term H such that $H \equiv G[x/t]$;

b. for any n-ary predicate variable P and any abstraction term $T = \lambda x_1, \ldots, x_n \delta$, $F[P/T] \succ_1 H$ for some term H such that $H \equiv G[P/T]$;

c. for any term variable X^α and any term $K^{\alpha'}$, where $\alpha \equiv \alpha'$, $F[X^\alpha/K^{\alpha'}] \succ_1 H$ for some term H such that $H \equiv G[X^\alpha/K^{\alpha'}]$.

Proof. Suppose $F \succ_1 G$. We will prove by induction on F.

a: Let x be an individual variable and t be an individual term.

First we suppose F is the redex which is reduced to G. By Lemma 3.2.16, we may assume that x is replaceable by t and t is free for x in F.

(i) $F = \pi_1(F_1, F_2)$.

Then $G = F_1$. Hence $F[x/t] = \pi_1(F_1[x/t], F_2[x/t]) \succ_1 F_1[x/t] = G[x/t]$.

Similarly for $\pi_2(F_1, F_2)$.

(ii) $F = (\lambda Y^\beta.F_1)(F_2^\beta)$.

Then $G = F_1[Y^\beta/F_2]$. By Lemmas 3.2.23, we have

$$\begin{aligned} F[x/t] &= (\lambda Y^{\beta[x/t]}.F_1[x/t])(F_2^\beta[x/t]) \\ &\succ_1 F_1[x/t][Y^{\beta[x/t]}/F_2[x/t]] \\ &\equiv F_1[Y^\beta/F_2][x/t] = G[x/t]. \end{aligned}$$

Similarly for $\oplus(Y_1^\beta.F_1, Y_2^\gamma.F_2, \mu_i H^{\gamma'}), i = 1, 2$.

(iii) $F = (\lambda y.H)(u)$.

Suppose $x \in fv(\lambda y.H)$. The proof of the other case can be easily modified from this proof.

By Lemma 3.2.19, $F[x/t] = (\lambda y.H[x/t])(u[x/t]) \succ_1 H[x/t][y/u[x/t]] \equiv H[y/u][x/t] = G[x/t]$.

Similarly for $(\lambda P.H)(T)$ (using Lemma 3.2.22).

(iv) $F = ST(y.Y^\beta.J^\gamma, I(u, H^{\beta(y/u)}))$.

Then $G = J[Y^\beta/Z^\beta][y/u][Z^{\beta[y/u]}/H]$, where Z^β is Y^β if $Y^\beta.J$ has no free term variable equivalent to $Y^{\beta[y/u]}$, otherwise Z^β is the first term variable of type $[\beta]$ such that $Z^{\beta[y/u]}$ is not equivalent to any free term variable of $Y^\beta.J$.

Suppose $x \in fv(y.Y^\beta.J)$. The proof of the other case can be modified from this proof. Then

$$\begin{aligned} F[x/t] &= ST(y.Y^{\beta[x/t]}.J[x/t], I(u[x/t], H[x/t])) \\ &\succ_1 J[x/t][Y^{\beta[x/t]}/Z_*^{\beta[x/t]}][y/u[x/t]][Z_*^{\beta[x/t][y/u[x/t]]}/H[x/t]], \end{aligned}$$

where $Z_*^{\beta[x/t]}$ is $Y^{\beta[x/t]}$ if $Y^{\beta[x/t]}.J[x/t]$ has no free term variable equivalent to $Y^{\beta[x/t][y/u[x/t]]}$, otherwise $Z_*^{\beta[x/t]}$ is the first term variable of type $[\beta[x/t]]$ such that $Z_*^{\beta[x/t][y/u[x/t]]}$ is not equivalent to any free term variable of $Y^{\beta[x/t]}.J[x/t]$.

Suppose for a contradiction that $Z_*^{\beta[y/u]}.J[Y^\beta/Z^\beta][y/u][Z^{\beta[y/u]}/Z_*^{\beta[y/u]}]$ has a free term variable Z_*^σ such that $\sigma[x/t] \equiv \beta[y/u][x/t]$. Since y does not occur free in the type superscript of any free term variable of $Y^{\beta}.J$, Z_*^{σ} is equivalent to some free term variable of $Y^{\beta}.J$. Hence $Z_*^{\sigma[x/t]}$ is equivalent to some free term variable of $Y^{\beta[x/t]}.J[x/t]$, and so is $Z_*^{\beta[y/u][x/t]}$ ($\equiv Z_*^{\beta[x/t][y/u[x/t]]}$). This is a contradiction. Thus: (1) $J[Y^{\beta}/Z^{\beta}][y/u][Z^{\beta[y/u]}/Z_*^{\beta[y/u]}]$ has no free term variable Z_*^{σ} such that $\sigma \not\equiv \beta[y/u]$ but $\sigma[x/t] \equiv \beta[y/u][x/t]$.

Since $Z_*^{\beta[x/t][y/u[x/t]]}$ is not equivalent to any free term variable of $Y^{\beta[x/t]}.J[x/t]$ (neither is $Z_*^{\beta[y/u][x/t]}$), $Z_*^{\beta[y/u]}$ is not equivalent to any free term variable of $Y^{\beta}.J$. Since y does not occur free in the type superscript of any free term variable of $Y^{\beta}.J$, we have: (2) either $Z_*^{\beta[y/u]}$ is not equivalent to any free term variable of $J[Y^{\beta}/Z^{\beta}][y/u]$ or $Z_*^{\beta[y/u]} \equiv Z^{\beta[y/u]}$.

By Lemmas 3.2.19, 3.2.21, 3.2.23, 3.2.24, and 3.2.26,

$$\begin{aligned}
F[x/t] &\succ_1 J[x/t][Y^{\beta[x/t]}/Z_*^{\beta[x/t]}][y/u[x/t]][Z_*^{\beta[x/t][y/u[x/t]]}/H[x/t]] \\
&\equiv J[Y^{\beta}/Z_*^{\beta}][x/t][y/u[x/t]][Z_*^{\beta[x/t][y/u[x/t]]}/H[x/t]] \\
&\equiv J[Y^{\beta}/Z_*^{\beta}][y/u][x/t][Z_*^{\beta[y/u][x/t]}/H[x/t]] \\
&\equiv J[Y^{\beta}/Z^{\beta}][Z^{\beta}/Z_*^{\beta}][y/u][x/t][Z_*^{\beta[y/u][x/t]}/H[x/t]] \\
&\equiv J[Y^{\beta}/Z^{\beta}][y/u][Z^{\beta[y/u]}/Z_*^{\beta[y/u]}][x/t][Z_*^{\beta[y/u][x/t]}/H[x/t]] \\
&\equiv J[Y^{\beta}/Z^{\beta}][y/u][Z^{\beta[y/u]}/Z_*^{\beta[y/u]}][Z_*^{\beta[y/u]}/H][x/t] \qquad \text{(since (1))} \\
&\equiv J[Y^{\beta}/Z^{\beta}][y/u][Z^{\beta[y/u]}/H][x/t] = G[x/t] \qquad \text{(since (2))}.
\end{aligned}$$

Similarly for $ST(P.Y^{\beta}.J, I(T, H^{\beta(P/T)}))$.

Next we suppose F is not the redex which is reduced to G.

(v) $F = (F_1, F_2)$.

Without loss of generality, we may assume that the reduction occurs in F_1. Then $G = (G_1, F_2)$ for some term G_1 such that $F_1 \succ_1 G_1$. By the induction hypothesis, $F_1[x/t] \succ_1 H$ for some term H such that $H \equiv G_1[x/t]$. Hence

$$\begin{aligned} F[x/t] &= (F_1[x/t], F_2[x/t]) \\ &\succ_1 (H, F_2[x/t]) \\ &\equiv (G_1[x/t], F_2[x/t]) = G[x/t]. \end{aligned}$$

The remaining cases can be proved similarly.

b: The proof is similar to that for a.

c: Let X^α be a term variable and $K^{\alpha'}$ be a term, where $\alpha \equiv \alpha'$.

First, suppose F is the redex which is reduced to G. By Lemma 3.2.16, we may assume that $K^{\alpha'}$ is free for X^α in F.

(i) $F = \pi_i(F_1, F_2)$, $i = 1, 2$.

The proof is similar to (i) of a.

(ii) $F = (\lambda Y^\beta.F_1)(F_2^\beta)$.

By using Lemma 3.2.21, the proof is similar to (iii) of a.

Similarly for $\oplus(Y_1.F_1, Y_2.F_2, \mu_i H)$, $i = 1, 2$.

(iii) $F = (\lambda x.H)(t)$.

Then $G = H[x/t]$ and $(\lambda x.H)[X^\alpha/K^{\alpha'}] = \lambda y.H[x/y][X^\alpha/K^{\alpha'}]$, where y is x if $x \notin fv(K^{\alpha'})$, otherwise y is the first individual variable which is not in $fv(H) \cup fv(K^{\alpha'})$.

Since x does not occur free in the type superscript of any free term variable of H and $y \notin fv(H)$, y does not occur free in the type superscript of any free term variable of $H[x/y]$. Hence: $(*)$ $H[x/y]$ has no free term variable X^σ such that $\sigma \not\equiv \alpha$ but $\sigma[y/t] = \sigma \equiv \alpha = \alpha[y/t]$.

By Lemmas 3.2.19, 3.2.23, and 3.2.26, we have

$$\begin{aligned} F[X^\alpha/K^{\alpha'}] &= (\lambda y.H[x/y][X^\alpha/K^{\alpha'}])(t) \\ &\succ_1 H[x/y][X^\alpha/K^{\alpha'}][y/t] \\ &\equiv H[x/y][y/t][X^\alpha/K^{\alpha'}] \qquad \text{(since } (*)) \\ &\equiv H[x/t][X^\alpha/K^{\alpha'}] = G[X^\alpha/K^{\alpha'}]. \end{aligned}$$

Similarly for $(\lambda P.H)(T)$.

(iv) $F = ST(x.Y^\beta.J, I(t, H^{\beta(x/t)}))$.

Then $G = J[Y^\beta/Z^\beta][x/t][Z^{\beta[x/t]}/H]$, where Z^β is Y^β if $Y^\beta.J$ has no free term variable equivalent to $Y^{\beta[x/t]}$, otherwise Z^β is the first term variable of type $[\beta]$ such that $Z^{\beta[x/t]}$ is not equivalent to any free term variable of $Y^\beta.J$.

Suppose X^α is equivalent to some free term variable of $x.Y^\beta.J$ (so $Z^{\beta[x/t]} \not\equiv X^\alpha$). The proof of the other case can be modified from this proof.

We have $(x.Y^\beta.J)[X^\alpha/K^{\alpha'}] = y.Y^{\beta[x/y]}.J[x/y][X^\alpha/K^{\alpha'}]$, where y is x if $x \notin fv(K^{\alpha'})$, otherwise y is the first individual variable which is not in $fv(Y^\beta.J) \cup fv(K^{\alpha'})$. Then

$$\begin{aligned} F[X^\alpha/K^{\alpha'}] &= ST(y.Y^{\beta[x/y]}.J[x/y][X^\alpha/K^{\alpha'}], I(t, H[X^\alpha/K^{\alpha'}])) \\ &\succ_1 J[x/y][X^\alpha/K^{\alpha'}][Y^{\beta[x/y]}/Z_*^{\beta[x/y]}][y/t][Z_*^{\beta[x/y][y/t]}/H[X^\alpha/K^{\alpha'}]], \end{aligned}$$

where $Z_*^{\beta[x/y]}$ is $Y^{\beta[x/y]}$ if $Y^{\beta[x/y]}.J[x/y][X^\alpha/K^{\alpha'}]$ has no free term variable equivalent to $Y^{\beta[x/y][y/t]}$, otherwise $Z_*^{\beta[x/y]}$ is the first term variable of type $[\beta[x/y]]$ such that $Z_*^{\beta[x/y][y/t]}$ is not equivalent to any free term variable of $Y^{\beta[x/y]}.J[x/y][X^\alpha/K^{\alpha'}]$.

Let $Z_0^{\beta[x/t]}$ be a term variable such that Z_0^σ does not occur in J or K for any type superscript σ and Z_0 is neither X nor Z_*.

By Lemmas 3.2.19, 3.2.21, 3.2.23, 3.2.24, and 3.2.26, we have

$$\begin{aligned}
G[X^\alpha/K^{\alpha'}] &= J[Y^\beta/Z^\beta][x/t][Z^{\beta[x/t]}/H][X^\alpha/K^{\alpha'}] \\
&\equiv J[Y^\beta/Z^\beta][x/t][Z^{\beta[x/t]}/Z_0^{\beta[x/t]}][Z_0^{\beta[x/t]}/H][X^\alpha/K^{\alpha'}] \\
&\equiv J[Y^\beta/Z^\beta][Z^\beta/Z_0^\beta][x/t][Z_0^{\beta[x/t]}/H][X^\alpha/K^{\alpha'}] \\
&\equiv J[Y^\beta/Z_0^\beta][x/t][Z_0^{\beta[x/t]}/H][X^\alpha/K^{\alpha'}], \text{ and}
\end{aligned}$$

$$\begin{aligned}
F[X^\alpha/K^{\alpha'}] &\succ_1 J[x/y][X^\alpha/K^{\alpha'}][Y^{\beta[x/y]}/Z_*^{\beta[x/y]}][y/t][Z_*^{\beta[x/y][y/t]}/H[X^\alpha/K^{\alpha'}]] \\
&\equiv J[x/y][X^\alpha/K^{\alpha'}][Y^{\beta[x/y]}/Z_*^{\beta[x/y]}][y/t][Z_*^{\beta[x/y][y/t]}/Z_0^{\beta[x/y][y/t]}] \\
&\quad [Z_0^{\beta[x/y][y/t]}/H[X^\alpha/K^{\alpha'}]] \\
&\equiv J[x/y][X^\alpha/K^{\alpha'}][Y^{\beta[x/y]}/Z_*^{\beta[x/y]}][Z_*^{\beta[x/y]}/Z_0^{\beta[x/y]}][y/t] \\
&\quad [Z_0^{\beta[x/y][y/t]}/H[X^\alpha/K^{\alpha'}]] \\
&\equiv J[x/y][X^\alpha/K^{\alpha'}][Y^{\beta[x/y]}/Z_0^{\beta[x/y]}][y/t][Z_0^{\beta[x/y][y/t]}/H[X^\alpha/K^{\alpha'}]] \\
&\equiv J[x/y][Y^{\beta[x/y]}/Z_0^{\beta[x/y]}][X^\alpha/K^{\alpha'}][y/t][Z_0^{\beta[x/y][y/t]}/H[X^\alpha/K^{\alpha'}]] \\
&\equiv J[Y^\beta/Z_0^\beta][x/y][X^\alpha/K^{\alpha'}][y/t][Z_0^{\beta[x/y][y/t]}/H[X^\alpha/K^{\alpha'}]] \\
&\equiv J[Y^\beta/Z_0^\beta][x/y][y/t][X^\alpha/K^{\alpha'}][Z_0^{\beta[x/y][y/t]}/H[X^\alpha/K^{\alpha'}]] \\
&\equiv J[Y^\beta/Z_0^\beta][x/t][X^\alpha/K^{\alpha'}][Z_0^{\beta[x/t]}/H[X^\alpha/K^{\alpha'}]] \\
&\equiv J[Y^\beta/Z_0^\beta][x/t][Z_0^{\beta[x/t]}/H][X^\alpha/K^{\alpha'}].
\end{aligned}$$

Similarly for $ST(P.Y^\beta.J, I(T, H^{\beta(P/T)}))$.

Proofs for the cases when F is not the redex which is reduced to G are as in a. □

Lemma 3.3.5. *Let X^α be a term variable, F and G be terms of type $[\alpha]$, and K be a C-H term.*

If $F \succ_1 G$, then $K[X^\alpha/F] \succ H$ for some term H such that $H \equiv K[X^\alpha/G]$.

Proof. We will prove by induction on K.

Suppose $F \succ_1 G$. It is trivial if X^α is not equivalent to any free term variable of K. Suppose X^α is equivalent to some free term variable of K.

(i) $K \equiv X^\alpha$.

Then $K[X^\alpha/F] = F \succ_1 G = K[X^\alpha/G]$.

(ii) $K = \lambda x.J$.

Then $K[X^\alpha/F] = \lambda y.J[x/y][X^\alpha/F]$, where y is x if $x \notin fv(F)$, otherwise y is the first individual variable which is not in $fv(J) \cup fv(F)$.

By the induction hypothesis, $J[x/y][X^\alpha/F] \succ H$ for some term H such that $H \equiv J[x/y][X^\alpha/G]$.

Note that since $y \notin fv(F)$ and $F \succ_1 G$, $y \notin fv(G)$ (by Note (b) on page 91). Hence, by Lemma 3.2.26 and Corollary 3.2.18,

$$\begin{aligned} K[X^\alpha/F] = \lambda y.J[x/y][X^\alpha/F] &\succ \lambda y.H \\ &\equiv \lambda y.J[x/y][X^\alpha/G] \\ &= (\lambda y.J[x/y])[X^\alpha/G] \\ &\equiv (\lambda x.J)[X^\alpha/G] = K[X^\alpha/G]. \end{aligned}$$

Similarly for $\lambda P.J$, $\lambda Y^\beta.J$, $\oplus(Y_1.K_1, Y_2.K_2, J)$, $ST(x.Y.H, J)$, and $ST(P.Y.H, J)$.

(iii) $K = (K_1, K_2)$.

Then $K[X^\alpha/F] = (K_1[X^\alpha/F], K_2[X^\alpha/F])$. By the induction hypothesis, $K_i[X^\alpha/F] \succ K_i[X^\alpha/G]$ for all $i = 1, 2$. Hence

$$\begin{aligned} K[X^\alpha/F] &= (K_1[X^\alpha/F], K_2[X^\alpha/F]) \\ &\succ (K_1[X^\alpha/G], K_2[X^\alpha/G]) = (K_1, K_2)[X^\alpha/G] = K[X^\alpha/G]. \end{aligned}$$

Similarly for the remaining cases. $\square$

In this chapter we have created new Curry-Howard terms in order to correspond to proofs in the second-order natural deduction system, NJ_2. We have defined substitutions as well as reduction rules for these new Curry-Howard terms and have proved some basic lemmas. As mentioned in Chapter I, for computational purposes, it is necessary that every Curry-Howard term must be strongly normalizable. This is the *strong normalization theorem* which will be proved in the next chapter.

CHAPTER IV

STRONG NORMALIZATION

The goal of this chapter is to give a proof of the strong normalization theorem for the new Curry-Howard terms defined in the previous chapter. Before proving the theorem, we need to give some basic definitions as in the following section.

4.1 Some basic definitions

Definition 4.1.1. *A* ***reduction sequence*** *is a sequence of terms such that each term which is not the first term in the sequence is an immediate reduct of the previous term.*

A term F is ***strongly normalizable*** *if all reduction sequences beginning with F are finite.*

The ***length*** *of a finite reduction sequence is the number of terms in the sequence -1.*

Note. By *König's lemma*, if F is strongly normalizable, then there is a number which bounds the length of every reduction sequence beginning with F.(See [7], page 27, for a proof.)

Notation. For any strongly normalizable term F, let $N(F)$ denote the least upper bound of lengths of the reduction sequences beginning with F.

Notes. Let F be a strongly normalizable term.

a. If G is a subterm of F, then G is also strongly normalizable and $N(G) \leq N(F)$.

b. If G is an immediate reduct of F, then G is also strongly normalizable and $N(G) < N(F)$.

Definition 4.1.2. *A term is **neutral** if it is not of one of the following forms:* (F, G), $\mu_1 F$, $\mu_2 F$, $\lambda X.F$, $\lambda x.F$, $\lambda P.F$, $I(t, F)$, *and* $I(T, F)$.

Notes.

a. A neutral term is a term variable or is of one of the following forms: $\pi_1 F$, $\pi_2 F$, $F(t)$, $F(T)$, $\oplus(X.F, Y.G, H)$, $F(G)$, $ST(x.X.G, H)$, $ST(P.X.G, H)$, and $F^{\perp}(\alpha)$.

b. If F is a neutral term, then a term of the form $F(G)$ (respectively $\pi_1 F$, $\pi_2 F$, $F(t)$, $F(T)$, and $F^{\perp}(\alpha)$) is not a redex, and hence its immediate reduct is of the form $F'(G)$ or $F(G')$ (respectively $\pi_1 F'$, $\pi_2 F'$, $F'(t)$, $F'(T)$, and $F'(\alpha)$), where $F \succ_1 F'$ and $G \succ_1 G'$.

Similarly, if H is a neutral term, then every immediate reduct of a term of the form $\oplus(X.F, Y.G, H)$ (respectively $ST(x.X.G, H)$ and $ST(P.X.G, H)$) is of the form $\oplus(X.F', Y.G, H)$, $\oplus(X.F, Y.G', H)$, or $\oplus(X.F, Y.G, H')$ (respectively $ST(x.X.G', H)$ or $ST(x.X.G, H')$ and $ST(P.X.G', H)$ or $ST(P.X.G, H')$), where $F \succ_1 F'$, $G \succ_1 G'$, and $H \succ_1 H'$.

4.2 A proof of the strong normalization theorem

Following Girard (see [7]) and extending Crossley and Shepherdson (see [3]), we now give a proof of the strong normalization theorem for the second-order system.

Definition 4.2.1. *A **candidate for reducibility** (CR) of type* $[\alpha]$ *is a set* C *of Curry-Howard terms of type* $[\alpha]$ *such that:*

CR0: if F *is in* C *and* $F' \equiv F$*, then* F' *is in* C*;*

CR1: if F *is in* C*, then* F *is strongly normalizable;*

CR2: if F is in C and F' is an immediate reduct of F, then F' is in C;

CR3: if F is neutral and all immediate reducts of F are in C, then F is in C;

CR3 gives, in particular,

CR4: if F is neutral and normal, then F is in C.

Note. It is easy to see that CR2 also holds if we replace "an immediate reduct" by "a reduct".

Lemma 4.2.2. *The set of all strongly normalizable terms of type $[\alpha]$ is a CR of type $[\alpha]$.*

Proof. It is clear that the set satisfies CR1, CR2, and CR3. It follows by Lemma 3.3.2 that the set satisfies CR0. □

Notation. Let SN_α denote the set of all strongly normalizable terms of type $[\alpha]$.

Definition 4.2.3. *Suppose C_1 and C_2 are sets of terms of types $[\alpha_1]$ and $[\alpha_2]$, respectively. We define*

i. $C_1 \supset C_2$ as the set of all terms F of type $[\alpha_1 \supset \alpha_2]$ such that for all terms G in C_1, the term $F(G)$ is in C_2;

ii. $C_1 \wedge C_2$ as the set of all terms F of type $[\alpha_1 \wedge \alpha_2]$ such that $\pi_1 F$ is in C_1 and $\pi_2 F$ is in C_2;

iii. $C_1 \vee C_2$ as the set of all terms F of type $[\alpha_1 \vee \alpha_2]$ such that for all types $[\gamma]$, all CRs C of type $[\gamma]$, all terms F_1 and F_2 in $C_1 \supset C$ and $C_2 \supset C$, respectively, and all term variables $X_1^{\alpha_1}$ and $X_2^{\alpha_2}$ which are not equivalent to any free term variables of F_1 and F_2, respectively, the term $\oplus(X_1^{\alpha_1}.F_1(X_1^{\alpha_1}), X_2^{\alpha_2}.F_2(X_2^{\alpha_2}), F)$ is in C.

Lemma 4.2.4. *Let C_1 and C_2 be CRs of types $[\alpha_1]$ and $[\alpha_2]$, respectively.*

If $(\mu_1 F^{\alpha_1})^{\alpha_1 \vee \alpha_2}$ is in $C_1 \vee C_2$, then F is in C_1.

Similarly for $(\mu_2 F^{\alpha_2})^{\alpha_1 \vee \alpha_2}$.

Proof. Suppose $(\mu_1 F^{\alpha_1})^{\alpha_1 \vee \alpha_2}$ is in $C_1 \vee C_2$.

Let $X_1^{\alpha_1}$ and $X_2^{\alpha_2}$ be term variables such that X_1 and X_2 are distinct.

Claim. $\lambda X_1^{\alpha_1}.X_1^{\alpha_1}$ is in $C_1 \supset C_1$ and $\lambda X_2^{\alpha_2}.X_1^{\alpha_1}$ is in $C_2 \supset C_1$.

Proof of the claim. To show that $\lambda X_1^{\alpha_1}.X_1^{\alpha_1}$ is in $C_1 \supset C_1$, let G be in C_1. We will show that $(\lambda X_1^{\alpha_1}.X_1^{\alpha_1})(G)$ is in C_1 by induction on $N(G)$. Since $(\lambda X_1^{\alpha_1}.X_1^{\alpha_1})(G)$ is neutral, by CR3 for C_1, it is enough to show that all its immediate reducts are in C_1. Every immediate reduct of $(\lambda X_1^{\alpha_1}.X_1^{\alpha_1})(G)$ is of one of the following forms.

(i) $X_1^{\alpha_1}[X_1^{\alpha_1}/G]$.

This is G which is in C_1.

(ii) $(\lambda X_1^{\alpha_1}.X_1^{\alpha_1})(G')$ where $G \succ_1 G'$.

By CR2, G' is in C_1. Since $N(G') < N(G)$, $(\lambda X_1^{\alpha_1}.X_1^{\alpha_1})(G')$ is in C_1 by the subsidiary induction hypothesis.

To show that $\lambda X_2^{\alpha_2}.X_1^{\alpha_1}$ is in $C_2 \supset C_1$, let H be in C_2. We will show that $(\lambda X_2^{\alpha_2}.X_1^{\alpha_1})(H)$ is in C_1 by induction on $N(H)$. Similar to the above proof, it remains to show that all its immediate reducts are in C_1. Every immediate reduct of $(\lambda X_2^{\alpha_2}.X_1^{\alpha_1})(H)$ is of one of the following forms.

(i) $X_1^{\alpha_1}[X_2^{\alpha_1}/H]$.

This is $X_1^{\alpha_1}$ which is in C_1 by CR4.

(ii) $(\lambda X_2^{\alpha_2}.X_1^{\alpha_1})(H')$ where $H \succ_1 H'$.

By CR2, H' is in C_2. Since $N(H') < N(H)$, $(\lambda X_2^{\alpha_2}.X_1^{\alpha_1})(H')$ is in C_1 by the subsidiary induction hypothesis.

Thus we have the claim.

Since $\mu_1 F$ is in $C_1 \vee C_2$, $\oplus(X_1^{\alpha_1}.(\lambda X_1^{\alpha_1}.X_1^{\alpha_1})(X_1^{\alpha_1}), X_2^{\alpha_2}.(\lambda X_2^{\alpha_2}.X_1^{\alpha_1})(X_2^{\alpha_2}), \mu_1 F)$ is in C_1. Since $\oplus(X_1^{\alpha_1}.(\lambda X_1^{\alpha_1}.X_1^{\alpha_1})(X_1^{\alpha_1}), X_2^{\alpha_2}.(\lambda X_2^{\alpha_2}.X_1^{\alpha_1})(X_2^{\alpha_2}), \mu_1 F) \succ_1$ $(\lambda X_1^{\alpha_1}.X_1^{\alpha_1})(X_1^{\alpha_1})[X_1^{\alpha_1}/F] = (\lambda X_1^{\alpha_1}.X_1^{\alpha_1})(F) \succ_1 X_1^{\alpha_1}[X_1^{\alpha_1}/F] = F$, by applying CR2 twice, F is in C_1. $\square$

Lemma 4.2.5. *Let C_1 and C_2 be CRs of types $[\alpha_1]$ and $[\alpha_2]$, respectively. Then $C_1 \supset C_2$, $C_1 \wedge C_2$, and $C_1 \vee C_2$ are CRs.*

Proof.

$C_1 \supset C_2$:

Assume F is in $C_1 \supset C_2$ for the proofs of CR0, CR1, and CR2.

CR0: Suppose $F' \equiv F$. Let G be in C_1. Since F is in $C_1 \supset C_2$, $F(G)$ is in C_2. Since $F'(G) \equiv F(G)$, by CR0 for C_2, $F'(G)$ is in C_2. Since G is arbitrary, F' is in $C_1 \supset C_2$.

CR1: By CR4 for C_1, X^{α_1} is in C_1. Since F is in $C_1 \supset C_2$, $F(X^{\alpha_1})$ is in C_2 and so it is strongly normalizable by CR1 for C_2. Since F is a subterm of $F(X^{\alpha_1})$, by Note on page 104, F is strongly normalizable.

CR2: By using CR2 for C_2, the proof is similar to the case CR0.

CR3: Suppose F is neutral and all immediate reducts of F are in $C_1 \supset C_2$. Let G be in C_1. Since $F(G)$ is neutral, by CR3 for C_2, to show that it is in C_2, it is enough to show that all its immediate reducts are in C_2. We will show this by induction on $N(G)$. Since F is neutral, by Note on page 105, each immediate reduct of $F(G)$ is of one of the following forms.

(i) $F'(G)$ where $F \succ_1 F'$.

By the assumption, F' is in $C_1 \supset C_2$. Hence $F'(G)$ is in C_2.

(ii) $F(G')$ where $G \succ_1 G'$.

By CR2 for C_1, G' is in C_1. Since $N(G') < N(G)$, by the induction hypothesis, $F(G')$ is in C_2.

Thus $F(G)$ is in C_2 and hence F is in $C_1 \supset C_2$.

$C_1 \wedge C_2$:

Assume F is in $C_1 \wedge C_2$ for the proofs of CR0, CR1, and CR2.

CR0: Suppose $F' \equiv F$. Since F is in $C_1 \wedge C_2$, $\pi_1 F$ is in C_1. Since $\pi_1 F' \equiv \pi_1 F$, by CR0 for C_1, $\pi_1 F'$ is in C_1. Similarly, $\pi_2 F'$ is in C_2. Hence F' is in $C_1 \wedge C_2$.

CR1: Since F is in $C_1 \wedge C_2$, $\pi_1 F$ is in C_1, and so it is strongly normalizable. Since F is a subterm of $\pi_1 F$, F is also strongly normalizable.

CR2: By using CR2 for C_1 and C_2, the proof is similar to the case CR0.

CR3: Suppose F is neutral and all immediate reducts of F are in $C_1 \wedge C_2$. Since $\pi_1 F$ is neutral, to show that it is in C_1, it is enough to show that all its immediate reducts are in C_1. Since F is neutral, every immediate reducts of $\pi_1 F$ is of the form $\pi_1 F'$. By the assumption, $\pi_1 F'$ is in C_1. Similarly, we can show that $\pi_2 F$ is in C_2. Thus F is in $C_1 \wedge C_2$.

$C_1 \vee C_2$:

Assume F is in $C_1 \vee C_2$ for the proofs of CR0, CR1, and CR2. Let $[\gamma]$ be a type, C be a CR of type $[\gamma]$, F_1 and F_2 be in $C_1 \supset C$ and $C_2 \supset C$, respectively, and $X_1^{\alpha_1}$ and $X_2^{\alpha_2}$ be term variables which are not equivalent to any free term variables of F_1 and F_2, respectively.

CR0: Suppose $F' \equiv F$. Then $\oplus(X_1^{\alpha_1}.F_1(X_1^{\alpha_1}), X_2^{\alpha_2}.F_2(X_2^{\alpha_2}), F') \equiv$
$\oplus\,(X_1^{\alpha_1}.F_1(X_1^{\alpha_1}), X_2^{\alpha_2}.F_2(X_2^{\alpha_2}), F)$. Since F is in $C_1 \vee C_2$,
$\oplus(X_1^{\alpha_1}.F_1(X_1^{\alpha_1}), X_2^{\alpha_2}.F_2(X_2^{\alpha_2}), F)$ is in C and, by CR0 for C, so is
$\oplus(X_1^{\alpha_1}.F_1(X_1^{\alpha_1}), X_2^{\alpha_2}.F_2(X_2^{\alpha_2}), F')$. Hence F' is in $C_1 \vee C_2$.

CR1: Since $\oplus(X_1^{\alpha_1}.F_1(X_1^{\alpha_1}), X_2^{\alpha_2}.F_2(X_2^{\alpha_2}), F)$ is in C, it is strongly normalizable. Since F is a subterm of $\oplus(X_1^{\alpha_1}.F_1(X_1^{\alpha_1}), X_2^{\alpha_2}.F_2(X_2^{\alpha_2}), F))$, F is also strongly normalizable.

CR2: By using CR2 for C, the proof is similar to the case CR0.

CR3: We first prove the following claim.

Claim. If G_1 and G_2 are in C, $Y_1^{\alpha_1}$ and $Y_2^{\alpha_2}$ are term variables, and G is a neutral term of type $[\alpha_1 \vee \alpha_2]$ such that for every immediate reduct G' of G, $\oplus(Y_1^{\alpha_1}.G_1, Y_2^{\alpha_2}.G_2, G')$ is in C, then $\oplus(Y_1^{\alpha_1}.G_1, Y_2^{\alpha_2}.G_2, G)$ is in C.

Proof of the claim. Suppose G_1 and G_2 are in C, $Y_1^{\alpha_1}$ and $Y_2^{\alpha_2}$ are term variables, and G is a neutral term of type $[\alpha_1 \vee \alpha_2]$ such that for every immediate reduct G' of G, $\oplus(Y_1^{\alpha_1}.G_1, Y_2^{\alpha_2}.G_2, G')$ is in C.

We will show that $\oplus(Y_1^{\alpha_1}.G_1, Y_2^{\alpha_2}.G_2, G)$ is in C by induction on $N(G_1) + N(G_2)$. Since $\oplus(Y_1^{\alpha_1}.G_1, Y_2^{\alpha_2}.G_2, G)$ is neutral, by CR3 for C, it is enough to show that all its immediate reducts are in C.

Since G is neutral, every immediate reduct of $\oplus(Y_1^{\alpha_1}.G_1, Y_2^{\alpha_2}.G_2, G)$ is of one of the following forms.

(i) $\oplus(Y_1^{\alpha_1}.G_1, Y_2^{\alpha_2}.G_2, G')$ where $G \succ_1 G'$.

By the assumption, $\oplus(Y_1^{\alpha_1}.G_1, Y_2^{\alpha_2}.G_2, G')$ is in C.

(ii) $\oplus(Y_1^{\alpha_1}.G_1', Y_2^{\alpha_2}.G_2, G)$ where $G_1 \succ_1 G_1'$.

First we will show that G_1' satisfies the conditions as G_1 in the hypothesis.

By CR2 for C, G_1' is in C. Since for every immediate reduct G' of G, $\oplus(Y_1^{\alpha_1}.G_1, Y_2^{\alpha_2}.G_2, G')$ is in C and $\oplus(Y_1^{\alpha_1}.G_1, Y_2^{\alpha_2}.G_2, G') \succ_1$ $\oplus(Y_1^{\alpha_1}.G_1', Y_2^{\alpha_2}.G_2, G')$, by CR2 for C, $\oplus(Y_1^{\alpha_1}.G_1', Y_2^{\alpha_2}.G_2, G')$ is also in C for every immediate reduct G' of G. Since $N(G_1') < N(G_1)$, by the induction hypothesis, $\oplus(Y_1^{\alpha_1}.G_1', Y_2^{\alpha_2}.G_2, G)$ is in C.

(iii) $\oplus(Y_1^{\alpha_1}.G_1, Y_2^{\alpha_2}.G_2', G)$ where $G_2 \succ_1 G_2'$.

The proof is similar to (ii).

Thus we have the claim.

Now suppose F is neutral and all immediate reducts of F are in $C_1 \vee C_2$. Then for any immediate reduct F' of F, $\oplus(X_1^{\alpha_1}.F_1(X_1^{\alpha_1}), X_2^{\alpha_2}.F_2(X_2^{\alpha_2}), F')$ is in C. Hence, by the claim, $\oplus(X_1^{\alpha_1}.F_1(X_1^{\alpha_1}), X_2^{\alpha_2}.F_2(X_2^{\alpha_2}), F)$ is in C. Thus F is in $C_1 \vee C_2$. □

In [3] *canonical CR* C_α, where α is a first-order formula, is defined by induction on α. We have some problems when α is a second-order formula. For example, suppose $\alpha = \forall_2 Q^n \beta$ where $\beta = Q(y_1, \ldots, y_n)$ and $T = \lambda y_1, \ldots, y_n \alpha$. We can see that $\beta(Q/T) \equiv \alpha$, so we cannot claim that $C_{\beta(Q/T)}$ has already been defined before we define C_α.

In [7], page 117, Girard defines a relation called "*parametric reducibility*" for the *system F*. We will adapt the definition for our second-order systems. We cannot do that straightforwardly since, unlike the type system of Girard, we use ordinary formulae as types. In order to have the definition, we need to define a set called *collections of CRs* first.

Definition 4.2.6. *Let $T = \lambda x_1, \ldots, x_n \delta$ be an abstraction term. For each sequence of individual terms $\underline{t} = t_1, \ldots, t_n$, let $C_{\underline{t}}$ be a CR of type $[\delta[\underline{x}/\underline{t}]]$, where $\underline{x} = x_1, \ldots, x_n$.*

*We call the set $\mathcal{C} = \{C_{\underline{t}} \mid \underline{t} = t_1, \ldots, t_n$ are individual terms.$\}$ a **collection** of CRs **corresponding** to **T**.*

Definition 4.2.7. *Let α be a formula, $\underline{P} = P_1^{m_1}, \ldots, P_n^{m_n}$ be distinct predicate variables, $\underline{T} = T_1, \ldots, T_n$, where $T_i = \lambda z_1^i, \ldots, z_{m_i}^i \delta_i$, $1 \leq i \leq n$, be abstraction terms, and $\underline{\mathcal{C}} = \mathcal{C}_1, \ldots, \mathcal{C}_n$ be collections of CRs corresponding to $T_1, \ldots, T_n$, respectively.*

We define a set $C_\alpha[P_1/\mathcal{C}_1, \ldots, P_n/\mathcal{C}_n]$, which can be written as $C_\alpha[\underline{P}/\underline{\mathcal{C}}]$, of terms of type $[\alpha[\underline{P}/\underline{T}]]$ inductively as follows.

If α is an atomic formula,

$$C_\alpha[\underline{P}/\underline{\mathcal{C}}] = \begin{cases} C_{\underline{t}}^q & \text{if } \alpha = P_q(t_1, \ldots, t_{m_q}) \text{ for some } 1 \leq q \leq n \text{ and some} \\ & \text{individual terms } \underline{t} = t_1, \ldots, t_{m_q}, \text{ where } C_{\underline{t}}^q \text{ is the} \\ & \text{element of } \mathcal{C}_q \text{ which is of type } [\delta_q[z_1^q/t_1, \ldots, z_{m_q}^q/t_{m_q}]], \\ SN_\alpha & \text{otherwise.} \end{cases}$$

$C_{(\beta\supset\gamma)}[\underline{P}/\underline{\mathcal{C}}]$ *is* $C_\beta[\underline{P}/\underline{\mathcal{C}}] \supset C_\gamma[\underline{P}/\underline{\mathcal{C}}]$.

$C_{(\beta\wedge\gamma)}[\underline{P}/\underline{\mathcal{C}}]$ *is* $C_\beta[\underline{P}/\underline{\mathcal{C}}] \wedge C_\gamma[\underline{P}/\underline{\mathcal{C}}]$.

$C_{(\beta\vee\gamma)}[\underline{P}/\underline{\mathcal{C}}]$ *is* $C_\beta[\underline{P}/\underline{\mathcal{C}}] \vee C_\gamma[\underline{P}/\underline{\mathcal{C}}]$.

$C_{(\forall x\beta)}[\underline{P}/\underline{\mathcal{C}}]$ *is the set of all terms* F *of type* $[\forall x\beta[\underline{P}/\underline{T}]]$ *such that* $F(u)$ *is in* $C_{\beta[x/u]}[\underline{P}/\underline{\mathcal{C}}]$ *for all individual terms* u.

$C_{(\exists x\beta)}[\underline{P}/\underline{\mathcal{C}}]$ *is the set of all terms* F *of type* $[\exists x\beta[\underline{P}/\underline{T}]]$ *such that for all individual variables* y *where* $y \notin fv(\beta) - \{x\}$, *all types* $[\gamma]$ *with* $y \notin fv(\gamma)$, *all CRs* D *of type* $[\gamma]$, *all terms* G *of type* $[\beta[x/y][\underline{P}/\underline{T}] \supset \gamma]$ *such that for each individual term* u, $G[y/u]$ *is in* $C_{\beta[x/u]}[\underline{P}/\underline{\mathcal{C}}] \supset D$, *and* y *is not free in the type superscript of any free term variable of* G, *and all term variables* $X^{\beta[x/y][\underline{P}/\underline{T}]}$ *which is not equivalent to any free term variable of* G, *the term* $ST(y.X^{\beta[x/y][\underline{P}/\underline{T}]}.G(X^{\beta[x/y][\underline{P}/\underline{T}]}), F)$ *is in* D; *and if* F *reduces to a term of the form* $I(t, H)$, *then* H *is in* $C_{\beta[x/t]}[\underline{P}/\underline{\mathcal{C}}]$.

Notation. *In the following,* $\underline{P}^*$ *is the sublist of* $\underline{P}$ *consisting of all* P_i*'s which are in* $FV(\alpha)$, $\underline{T}^*$ *and* $\underline{\mathcal{C}}^*$ *are the corresponding sublists of* $\underline{T}$ *and* $\underline{\mathcal{C}}$, *respectively.*

$C_{\forall_2 Q^q\beta}[\underline{P}/\underline{\mathcal{C}}]$ *is the set of all terms* F *of type* $[(\forall_2 Q\beta)[\underline{P}/\underline{T}]]$ *such that for all abstraction terms* $U = \lambda y_1, \ldots, y_q\sigma$ *and all collections of CRs* $\mathcal{D}$ *corresponding to* U, $F(U)$ *is in* $C_\beta[Q/\mathcal{D}, \underline{P}^*/\underline{\mathcal{C}}^*]$.

$C_{\exists_2 Q^q\beta}[\underline{P}/\underline{\mathcal{C}}]$ *is the set of all terms* F *of type* $[(\exists_2 Q\beta)[\underline{P}/\underline{T}]]$ *such that for all q-ary predicate variables* R *where* $R \notin FV(\beta[\underline{P}^*/\underline{T}^*]) - \{Q\}$, *all types* $[\gamma]$ *with* $R \notin FV(\gamma)$, *all CRs* D *of type* $[\gamma]$, *all terms* G *of type* $[\beta[Q/R, \underline{P}^*/\underline{T}^*] \supset \gamma]$ *such that for each abstraction term* $U = \lambda y_1, \ldots, y_q\sigma$ *and all collections of CRs* $\mathcal{E}$ *corresponding to* U, $G[R/U]$ *is in* $C_\beta[Q/\mathcal{E}, \underline{P}^*/\underline{\mathcal{C}}^*] \supset D$, *and* R *is not free in the type superscript of any free term variable of* G, *and all term variables* $X^{\beta[Q/R,\underline{P}^*/\underline{T}^*]}$ *which is not equivalent to any free term variable of* G, *the term* $ST(R.X^{\beta[Q/R,\underline{P}^*/\underline{T}^*]}.G(X^{\beta[Q/R,\underline{P}^*/\underline{T}^*]}), F)$ *is in* D; *and if* F *reduces to a term of the form* $I(U, H)$, *then* H *is in* $C_\beta[Q/\mathcal{D}, \underline{P}^*/\underline{\mathcal{C}}^*]$ *for some collection of CRs* $\mathcal{D}$

corresponding to U.

Note. From the above definition, it can be easily proved by induction on α that for every r-ary predicate variable R which is not in $\{\underline{P}\}$, if $R \notin FV(\alpha)$, then $C_\alpha[R/\mathcal{D}, \underline{P}/\underline{\mathcal{C}}] = C_\alpha[\underline{P}/\underline{\mathcal{C}}]$ for every collection of CRs $\mathcal{D}$ which corresponds to some abstraction term $\lambda y_1, \ldots, y_r\sigma$.

Lemma 4.2.8. *Let α be a formula, $\underline{P} = P_1^{m_1}, \ldots, P_n^{m_n}$ be distinct predicate variables, $\underline{T} = T_1, \ldots, T_n$, where $T_i = \lambda z_1^i, \ldots, z_{m_i}^i \delta_i$, $1 \leq i \leq n$, be abstraction terms, and $\underline{\mathcal{C}} = \mathcal{C}_1, \ldots, \mathcal{C}_n$ be collections of CRs corresponding to $T_1, \ldots, T_n$, respectively.*

Then $C_\alpha[\underline{P}/\underline{\mathcal{C}}]$ is a CR of type $[\alpha[\underline{P}/\underline{T}]]$.

Proof. We will prove this by induction on α.

It is clear from the definition if α is an atomic formula. It follows by Lemma 4.2.5 and the induction hypothesis if α is of the form $\beta \supset \gamma$, $\beta \wedge \gamma$, or $\beta \vee \gamma$. The remaining cases are as follows.

$C_{\forall y\beta}[\underline{P}/\underline{\mathcal{C}}]$:

Assume F is in $C_{\forall y\beta}[\underline{P}/\underline{\mathcal{C}}]$ for the proofs of CR0, CR1, and CR2.

Let u be an individual term. Then for the proofs of CR0, CR1, and CR2, $F(u)$ is in $C_{\beta[y/u]}[\underline{P}/\underline{\mathcal{C}}]$ which is a CR by the induction hypothesis.

CR0: Suppose $F' \equiv F$. Then $F'(u) \equiv F(u)$. Hence $F'(u)$ is in $C_{\beta[y/u]}[\underline{P}/\underline{\mathcal{C}}]$ by CR0. Thus F' is in $C_{\forall y\beta}[\underline{P}/\underline{\mathcal{C}}]$.

CR1: Since $F(u)$ is in $C_{\beta[y/u]}[\underline{P}/\underline{\mathcal{C}}]$, by CR1, $F(u)$ is strongly normalizable and so is F since F is a subterm of $F(u)$.

CR2: By using CR2 for $C_{\beta[y/u]}[\underline{P}/\underline{\mathcal{C}}]$, the proof is similar to the case CR0.

CR3: Assume F is neutral and all immediate reducts of F are in $C_{\forall y\beta}[\underline{P}/\underline{\mathcal{C}}]$. Since $F(u)$ is neutral, by CR3, to show that it is in $C_{\beta[y/u]}[\underline{P}/\underline{\mathcal{C}}]$, it is enough to show that all its immediate reducts are in $C_{\beta[y/u]}[\underline{P}/\underline{\mathcal{C}}]$. Suppose F^*

is an immediate reduct of $F(u)$. Since F is neutral, $F^* = F'(u)$ for some term F' such that $F \succ_1 F'$. By the assumption, F' is in $C_{\forall y\beta}[\underline{P}/\underline{\mathcal{C}}]$. Hence F^* is in $C_{\beta[y/u]}[\underline{P}/\underline{\mathcal{C}}]$.

Similarly for $C_{\forall_2 Q\beta}[\underline{P}/\underline{\mathcal{C}}]$.

$C_{\exists x\beta}[\underline{P}/\underline{\mathcal{C}}]$:

Assume F is in $C_{\exists x\beta}[\underline{P}/\underline{\mathcal{C}}]$ for the proofs of CR0, CR1, and CR2.

Let y be an individual variable such that $y \notin fv(\beta) - \{x\}$, $[\gamma]$ be a type with $y \notin fv(\gamma)$, D be a CR of type $[\gamma]$, G be a term of type $[\beta[x/y][\underline{P}/\underline{T}] \supset \gamma]$ such that for each individual term u, $G[y/u]$ is in $C_{\beta[x/u]}[\underline{P}/\underline{\mathcal{C}}] \supset D$ and y is not free in the type superscript of any free term variable of G, and $X^{\beta[x/y][\underline{P}/\underline{T}]}$ be a term variable which is not equivalent to any free term variable of G.

For the proofs of CR0, CR1, and CR2, since F is in $C_{\exists x\beta}[\underline{P}/\underline{\mathcal{C}}]$, $ST(y.X^{\beta[x/y][\underline{P}/\underline{T}]}.G(X^{\beta[x/y][\underline{P}/\underline{T}]}), F)$ is in D and if $F \succ I(t, H)$, then H is in $C_{\beta[x/t]}[\underline{P}/\underline{\mathcal{C}}]$.

CR0: Suppose $F' \equiv F$. Suppose $F' \succ I(t, H')$. By Corollary 3.3.3, $F \succ H^*$ for some term H^* such that $H^* \equiv I(t, H')$. Then $H^* = I(t, H)$ for some term H such that $H \equiv H$. Since H is in $C_{\beta[x/t]}[\underline{P}/\underline{\mathcal{C}}]$, so is H' by CR0. Since $ST(y.X^{\beta[x/y][\underline{P}/\underline{T}]}.G(X^{\beta[x/y][\underline{P}/\underline{T}]}), F') \equiv ST(y.X^{\beta[x/y][\underline{P}/\underline{T}]}.G(X^{\beta[x/y][\underline{P}/\underline{T}]}), F)$, by CR0 for D, $ST(y.X^{\beta[x/y][\underline{P}/\underline{T}]}.G(X^{\beta[x/y][\underline{P}/\underline{T}]}), F')$ is in D. Hence F' is in $C_{\exists x\beta}[\underline{P}/\underline{\mathcal{C}}]$.

CR1: Since $ST(y.X^{\beta[x/y][\underline{P}/\underline{T}]}.G(X^{\beta[x/y][\underline{P}/\underline{T}]}), F)$ is in D, by CR1 for D, it is strongly normalizable, and so is F since F is one of its subterms.

CR2: Suppose $F \succ_1 F'$. If $F' \succ I(t, H)$, then $F \succ I(t, H)$, and so H is in $C_{\beta[x/t]}[\underline{P}/\underline{\mathcal{C}}]$. The rest of the proof is similar to the case CR0.

CR3: First we will prove the following claim.

Claim. If H is in D, K is a neutral term of type $[\exists x\beta[\underline{P}/\underline{T}]]$ such that for every immediate reduct K' of K, $ST(y.X^{\beta[x/y][\underline{P}/\underline{T}]}.H, K')$ is in D, then

$ST(y.X^{\beta[x/y][\underline{P}/\underline{T}]}.H,K)$ is in D.

Proof of the claim. Suppose H is in D, K is a neutral term of type $[\exists x\beta[\underline{P}/\underline{T}]]$ such that for every immediate reduct K' of K, $ST(y.X^{\beta[x/y][\underline{P}/\underline{T}]}.H,K')$ is in D. We will show that $ST(y.X^{\beta[x/y][\underline{P}/\underline{T}]}.H,K)$ is in D by induction on $N(H)$. Since $ST(y.X^{\beta[x/y][\underline{P}/\underline{T}]}.H,K)$ is neutral, it is enough to prove that all its immediate reducts are in D. Since K is neutral, every immediate reduct of $ST(y.X^{\beta[x/y][\underline{P}/\underline{T}]}.H,K)$ is of one of the following forms.

(i) $ST(y.X^{\beta[x/y][\underline{P}/\underline{T}]}.H,K')$ where $K \succ_1 K'$.

This reduct is in D by the assumption.

(ii) $ST(y.X^{\beta[x/y][\underline{P}/\underline{T}]}.H',K)$ where $H \succ_1 H'$.

By CR2 for D, H' is in D. Since for every immediate reduct K' of K, $ST(y.X^{\beta[x/y][\underline{P}/\underline{T}]}.H,K') \succ_1 ST(y.X^{\beta[x/y][\underline{P}/\underline{T}]}.H',K')$ and $ST(y.X^{\beta[x/y][\underline{P}/\underline{T}]}.H,K')$ is in D, by CR2 for D, $ST(y.X^{\beta[x/y][\underline{P}/\underline{T}]}.H',K')$ is in D for every immediate reduct K' of K. Hence H' satisfies the conditions of the hypothesis. Since $N(H') < N(H)$, $ST(y.X^{\beta[x/y][\underline{P}/\underline{T}]}.H',K)$ is in D.

Thus we have the claim.

Now suppose F is neutral and all immediate reducts of F are in $C_{\exists x\beta}[\underline{P}/\underline{\mathcal{C}}]$. Then for every immediate reduct F' of F, $ST(y.X^{\beta[x/y][\underline{P}/\underline{T}]}.G(X^{\beta[x/y][\underline{P}/\underline{T}]}),F')$ is in D. Since G is in $C_{\beta[x/y]}[\underline{P}/\underline{\mathcal{C}}] \supset D$, $G(X^{\beta[x/y][\underline{P}/\underline{T}]})$ is in D. By the claim, $ST(y.X^{\beta[x/y][\underline{P}/\underline{T}]}.G(X^{\beta[x/y][\underline{P}/\underline{T}]}),F)$ is in D.

Suppose $F \succ I(t,H)$. Since F is neutral, there is a finite reduction sequence from F to $I(t,H)$ with length ≥ 1. Let F' be the immediate reduct of F in the sequence. By the assumption, F' is in $C_{\exists x\beta}[\underline{P}/\underline{\mathcal{C}}]$. Since $F' \succ I(t,H)$, H is in $C_{\beta[x/t]}[\underline{P}/\underline{\mathcal{C}}]$. Hence F is in $C_{\exists x\beta}[\underline{P}/\underline{\mathcal{C}}]$.

Similarly for $C_{\exists_2 Q\beta}[\underline{P}/\underline{\mathcal{C}}]$. □

Lemma 4.2.9. *Let α be a formula, $\underline{P} = P_1^{m_1},\ldots,P_n^{m_n}$ be distinct predicate vari-*

ables, $\underline{T} = T_1, \ldots, T_n$*, where* $T_i = \lambda z_1^i, \ldots, z_{m_i}^i \delta_i$, $1 \leq i \leq n$*, be abstraction terms, and* $\underline{\mathcal{C}} = \mathcal{C}_1, \ldots, \mathcal{C}_n$ *be collections of* CR*s corresponding to* $T_1, \ldots, T_n$*, respectively.*

If R *and* R' *are* r*-ary predicate variables which are not in* $\{\underline{P}\}$ *and* $R' \notin FV(\alpha) - \{R\}$*,* $U = \lambda z_1, \ldots, z_r \sigma$ *is an abstraction term, and* $\mathcal{D}$ *is a collection of* CR*s corresponding to* U*, then* $C_\alpha[R/\mathcal{D}, \underline{P}/\underline{\mathcal{C}}] = C_{\alpha[R/R']}[R'/\mathcal{D}, \underline{P}/\underline{\mathcal{C}}]$.

Proof. Suppose R and R' are r-ary predicate variables which are not in $\{\underline{P}\}$ and $R' \notin FV(\alpha) - \{R\}$, $U = \lambda z_1, \ldots, z_r \sigma$ is an abstraction term, and $\mathcal{D}$ is a collection of CRs corresponding to U.

We proceed by induction on α. The cases where α is $\beta \supset \gamma$, $\beta \wedge \gamma$, or $\beta \vee \gamma$ easily follow by the induction hypothesis.

Suppose $R \notin FV(\alpha)$. Then $R' \notin FV(\alpha[R/R'])$ since $R' \notin FV(\alpha) - \{R\}$. Hence, by the Note on page 113, $C_\alpha[R/\mathcal{D}, \underline{P}/\underline{\mathcal{C}}] = C_\alpha[\underline{P}/\underline{\mathcal{C}}] = C_{\alpha[R/R']}[R'/\mathcal{D}, \underline{P}/\underline{\mathcal{C}}]$. Suppose $R \in FV(\alpha)$.

(i) $\alpha = R(t_1, \ldots, t_r)$ for some individual terms $t_1, \ldots, t_r$.

We have $C_\alpha[R/\mathcal{D}, \underline{P}/\underline{\mathcal{C}}] = D_{\underline{t}}$ where $D_{\underline{t}}$ is the element of $\mathcal{D}$ which is of type $[\sigma[z_1/t_1, \ldots, z_r/t_r]]$ and $C_{\alpha[R/R']}[R'/\mathcal{D}, \underline{P}/\underline{\mathcal{C}}] = C_{R'(t_1,\ldots,t_r)}[R'/\mathcal{D}, \underline{P}/\underline{\mathcal{C}}] = D_{\underline{t}}$.

For the remaining cases, we will show that $C_\alpha[R/\mathcal{D}, \underline{P}/\underline{\mathcal{C}}] \subseteq C_{\alpha[R/R']}[R'/\mathcal{D}, \underline{P}/\underline{\mathcal{C}}]$ and omit proofs of the converse which can be done similarly.

Let F be in $C_\alpha[R/\mathcal{D}, \underline{P}/\underline{\mathcal{C}}]$.

(ii) $\alpha = \forall x \beta$.

Let t be an individual term. Then $F(t)$ is in $C_{\beta[x/t]}[R/\mathcal{D}, \underline{P}/\underline{\mathcal{C}}]$. By Lemma 2.12 and the induction hypothesis, $C_{\beta[x/t]}[R/\mathcal{D}, \underline{P}/\underline{\mathcal{C}}] = C_{\beta[x/t][R/R']}[R'/\mathcal{D}, \underline{P}/\underline{\mathcal{C}}] = C_{\beta[R/R'][x/t]}[R'/\mathcal{D}, \underline{P}/\underline{\mathcal{C}}]$. Thus $F(t)$ is in $C_{\beta[R/R'][x/t]}[R'/\mathcal{D}, \underline{P}/\underline{\mathcal{C}}]$, and hence F is in $C_{\forall x \beta[R/R']}[R'/\mathcal{D}, \underline{P}/\underline{\mathcal{C}}]$ i.e. $C_{\alpha[R/R']}[R'/\mathcal{D}, \underline{P}/\underline{\mathcal{C}}]$.

(iii) $\alpha = \exists x \beta$.

We want to show that F is in $C_{\exists x \beta[R/R']}[R'/\mathcal{D}, \underline{P}/\underline{\mathcal{C}}]$.

Since F is in $C_{\exists x\beta}[R/\mathcal{D}, \underline{P}/\underline{\mathcal{C}}]$, if $F \succ I(t, H)$, then H is in $C_{\beta[x/t]}[R/\mathcal{D}, \underline{P}/\underline{\mathcal{C}}]$ which is $C_{\beta[R/R'][x/t]}[R'/\mathcal{D}, \underline{P}/\underline{\mathcal{C}}]$ by the induction hypothesis and Lemma 2.12.

Let y be an individual variable such that $y \notin fv(\beta[R/R']) - \{x\}$, $[\gamma]$ be a type with $y \notin fv(\gamma)$, E be a CR of type $[\gamma]$, G be a term of type $[\beta[R/R'][x/y][R'/U, \underline{P}/\underline{T}] \supset \gamma]$ such that for each individual term u, $G[y/u]$ is in $C_{\beta[R/R'][x/u]}[R'/\mathcal{D}, \underline{P}/\underline{\mathcal{C}}] \supset E$, and y is not free in the type superscript of any free term variable of G, and $X^{\beta[R/R'][x/y][R'/U,\underline{P}/\underline{T}]}$ be a term variable which is not equivalent to any free term variable of G.

By Lemmas 2.12 and 2.16, $\beta[R/R'][x/y][R'/U, \underline{P}/\underline{T}] \equiv \beta[x/y][R/U, \underline{P}/\underline{T}]$. By the induction hypothesis and Lemma 2.12, $C_{\beta[R/R'][x/u]}[R'/\mathcal{D}, \underline{P}/\underline{\mathcal{C}}] = C_{\beta[x/u][R/R']}[R'/\mathcal{D}, \underline{P}/\underline{\mathcal{C}}] = C_{\beta[x/u]}[R/\mathcal{D}, \underline{P}/\underline{\mathcal{C}}]$ for every individual term u.

Then we have $X^{\beta[R/R'][x/y][R'/U,\underline{P}/\underline{T}]} \equiv X^{\beta[x/y][R/U,\underline{P}/\underline{T}]}$, G is of type $[\beta[x/y][R/U, \underline{P}/\underline{T}] \supset \gamma]$ and for each individual term u, $G[y/u]$ is in $C_{\beta[x/u]}[R/\mathcal{D}, \underline{P}/\underline{\mathcal{C}}] \supset E$.

Since F is in $C_{\exists x\beta}[R/\mathcal{D}, \underline{P}/\underline{\mathcal{C}}]$, $ST(y.X^{\beta[x/y][R/U,\underline{P}/\underline{T}]}.G(X^{\beta[x/y][R/U,\underline{P}/\underline{T}]}), F)$ is in E, and so is $ST(y.X^{\beta[R/R'][x/y][R'/U,\underline{P}/\underline{T}]}.G(X^{\beta[R/R'][x/y][R'/U,\underline{P}/\underline{T}]}), F)$. Thus F is in $C_{\exists x\beta[R/R']}[R'/\mathcal{D}, \underline{P}/\underline{\mathcal{C}}]$.

Notation. In the following, $\underline{P}^*$ is the sublist of $\underline{P}$ consisting of all P_i's which are in $FV(\alpha)$, $\underline{T}^*$ and $\underline{\mathcal{C}}^*$ are the corresponding sublists of $\underline{T}$ and $\underline{\mathcal{C}}$, respectively.

(iv) $\alpha = \forall_2 Q^q \beta$.

Let $V = \lambda y_1, \ldots, y_q \tau$ be an abstraction term and $\mathcal{E}$ be a collection of CRs corresponding to V. Then $F(V)$ is in $C_\beta[Q/\mathcal{E}, R/\mathcal{D}, \underline{P}^*/\underline{\mathcal{C}}^*]$.

We have $(\forall_2 Q\beta)[R/R'] = \forall_2 Q'\beta[Q/Q'][R/R']$, where Q' is a q-ary predicate variable which is not in $(FV(\beta) - \{Q\}) \cup \{R'\}$. By the induction hypothesis, $C_{\beta[Q/Q'][R/R']}[Q'/\mathcal{E}, R'/\mathcal{D}, \underline{P}^*/\underline{\mathcal{C}}^*] = C_{\beta[Q/Q']}[Q'/\mathcal{E}, R/\mathcal{D}, \underline{P}^*/\underline{\mathcal{C}}^*] = C_\beta[R/\mathcal{D}, Q/\mathcal{E}, \underline{P}^*/\underline{\mathcal{C}}^*]$. Hence $F(V)$ is in $C_{\beta[Q/Q'][R/R']}[Q'/\mathcal{E}, R'/\mathcal{D}, \underline{P}^*/\underline{\mathcal{C}}^*]$. Thus

F is in $C_{\forall_2 Q'\beta[Q/Q'][R/R']}[R'/\mathcal{D}, \underline{P}/\underline{\mathcal{C}}]$ i.e. $C_{\alpha[R/R']}[R'/\mathcal{D}, \underline{P}/\underline{\mathcal{C}}]$.

(v) $\alpha = \exists_2 Q^q \beta$.

We have $(\exists_2 Q\beta)[R/R'] = \exists_2 Q'\beta[Q/Q'][R/R']$, where Q' is a q-ary predicate variable which is not in $(FV(\beta) - \{Q\}) \cup \{R'\}$.

We want to show that F is in $C_{\exists_2 Q'\beta[Q/Q'][R/R']}[R'/\mathcal{D}, \underline{P}/\underline{\mathcal{C}}]$.

Since F is in $C_{\exists_2 Q\beta}[R/\mathcal{D}, \underline{P}/\underline{\mathcal{C}}]$, if $F \succ I(V, H)$, then H is in $C_\beta[Q/\mathcal{E}, R/\mathcal{D}, \underline{P}^*/\underline{\mathcal{C}}^*]$ which is $C_{\beta[Q/Q'][R/R']}[Q'/\mathcal{E}, R'/\mathcal{D}, \underline{P}^*/\underline{\mathcal{C}}^*]$ by the induction hypothesis, where $\mathcal{E}$ is some collection of CRs corresponding to V.

Let Q^* be a q-ary predicate variable such that $Q^* \notin FV(\beta[Q/Q'][R/R'][R'/U, \underline{P}^*/\underline{T}^*]) - \{Q'\}$, $[\gamma]$ be a type with $Q^* \notin FV(\gamma)$, E be a CR of type $[\gamma]$, G be a term of type $[\beta[Q/Q'][R/R'][Q'/Q^*, R'/U, \underline{P}^*/\underline{T}^*] \supset \gamma]$ such that for each abstraction term $V = \lambda y_1, \ldots, y_q \tau$ and all collections of CRs $\mathcal{E}$ corresponding to V, $G[Q^*/V]$ is in $C_{\beta[Q/Q'][R/R']}[Q'/\mathcal{E}, R'/\mathcal{D}, \underline{P}^*/\underline{\mathcal{C}}^*] \supset E$, and Q^* is not free in the type superscript of any free term variable of G, and $X^{\beta[Q/Q'][R/R'][Q'/Q^*, R'/U, \underline{P}^*/\underline{T}^*]}$ be a term variable which is not equivalent to any free term variable of G.

By Lemma 2.16, $\beta[Q/Q'][R/R'][Q'/Q^*, R'/U, \underline{P}^*/\underline{T}^*] \equiv \beta[R/U, Q/Q^*, \underline{P}^*/\underline{T}^*]$. By the induction hypothesis, $C_{\beta[Q/Q'][R/R']}[Q'/\mathcal{E}, R'/\mathcal{D}, \underline{P}^*/\underline{\mathcal{C}}^*] = C_\beta[Q/\mathcal{E}, R/\mathcal{D}, \underline{P}^*/\underline{\mathcal{C}}^*]$.

Then we have $X^{\beta[Q/Q'][R/R'][Q'/Q^*, R'/U, \underline{P}^*/\underline{T}^*]} \equiv X^{\beta[Q/Q^*, R/U, \underline{P}^*/\underline{T}^*]}$, G is of type $[\beta[Q/Q^*, R/U, \underline{P}^*/\underline{T}^*] \supset \gamma]$ and for each abstraction term $V = \lambda y_1, \ldots, y_q \tau$ and all collections of CRs $\mathcal{E}$ corresponding to V, $G[Q^*/V]$ is in $C_\beta[Q/\mathcal{E}, R/\mathcal{D}, \underline{P}^*/\underline{\mathcal{C}}^*] \supset E$.

Since $Q^* \notin FV(\beta[Q/Q'][R/R'][R'/U, \underline{P}^*/\underline{T}^*]) - \{Q'\}$, $Q^* \notin FV(\beta[R/U, \underline{P}^*/\underline{T}^*]) - \{Q\}$.

Hence, since F is in $C_{\exists_2 Q\beta}[R/\mathcal{D}, \underline{P}/\underline{\mathcal{C}}]$, $ST(Q^*.X^{\beta[Q/Q^*, R/U, \underline{P}^*/\underline{T}^*]}.G(X^{\beta[Q/Q^*, R/U, \underline{P}^*/\underline{T}^*]}), F)$ is in E, and so is

$ST(Q^*.X^{\beta[Q/Q'][R/R'][Q'/Q^*,R'/U,\underline{P}^*/\underline{T}^*]}.G(X^{\beta[Q/Q'][R/R'][Q'/Q^*,R'/U,\underline{P}^*/\underline{T}^*]}),F)$. Thus F is in $C_{\exists_2 Q'\beta[Q/Q'][R/R']}[R'/\mathcal{D},\underline{P}/\underline{\mathcal{C}}]$ i.e. $C_{\alpha[R/R']}[R'/\mathcal{D},\underline{P}/\underline{\mathcal{C}}]$. □

Lemma 4.2.10. *Let α and α' be formulae, $\underline{P} = P_1^{m_1},\ldots,P_n^{m_n}$ be distinct predicate variables, $\underline{T} = T_1,\ldots,T_n$, where $T_i = \lambda x_1^i,\ldots,x_{m_i}^i\delta_i$, $1 \le i \le n$, be abstraction terms, and $\underline{\mathcal{C}} = \mathcal{C}_1,\ldots,\mathcal{C}_n$ be collections of CRs corresponding to $T_1,\ldots,T_n$, respectively.*

If $\alpha \equiv \alpha'$, then $C_\alpha[\underline{P}/\underline{\mathcal{C}}] = C_{\alpha'}[\underline{P}/\underline{\mathcal{C}}]$.

Proof. We proceed by induction on α. The cases where α is $\beta \supset \gamma$, $\beta\wedge\gamma$, or $\beta\vee\gamma$ easily follow by the induction hypothesis.

Suppose $\alpha \equiv \alpha'$. It is trivial if $\alpha = \alpha'$. Suppose there is a sequence of formulae $\alpha = \alpha_0,\alpha_1,\ldots,\alpha_r = \alpha'$, $r \ge 1$, such that for each $1 \le i \le r$, α_i is obtained from α_{i-1} by a single legitimate change of bound variable. For the remaining cases, we proceed by induction on r. We will prove only the case $r = 1$ since the case $r > 1$ follows straightforwardly by the subsidiary induction hypothesis and the case $r = 1$.

We will show that $C_\alpha[\underline{P}/\underline{\mathcal{C}}] \subseteq C_{\alpha'}[\underline{P}/\underline{\mathcal{C}}]$ and omit proofs of the converse which can be done similarly.

Let F be in $C_\alpha[\underline{P}/\underline{\mathcal{C}}]$. Then F is of type $[\alpha[\underline{P}/\underline{T}]] = [\alpha'[\underline{P}/\underline{T}]]$.

(i) $\alpha = \forall x\beta$.

Let t be an individual term. Then $F(t)$ is in $C_{\beta[x/t]}[\underline{P}/\underline{\mathcal{C}}]$.

Case 1. $\alpha' = \forall x\beta'$ where $\beta' \equiv \beta$.

By Lemma 2.14, $\beta[x/t] \equiv \beta'[x/t]$. By the main induction hypothesis, $C_{\beta[x/t]}[\underline{P}/\underline{\mathcal{C}}] = C_{\beta'[x/t]}[\underline{P}/\underline{\mathcal{C}}]$. Thus $F(t)$ is in $C_{\beta'[x/t]}[\underline{P}/\underline{\mathcal{C}}]$, and hence F is in $C_{\alpha'}[\underline{P}/\underline{\mathcal{C}}]$.

Case 2. $\alpha' = \forall y\beta[x/y]$ where y is free for x and does not occur free in β.

By Lemma 2.13, $\beta[x/y][y/t] \equiv \beta[x/t]$ and the rest of the proof is similar to Case 1.

(ii) $\alpha = \exists x\beta$.

Case 1. $\alpha' = \exists x\beta'$ where $\beta' \equiv \beta$.

This case can be proved in the same way as the following case by using the fact that $C_{\beta'[x/t]}[\underline{P}/\underline{\mathcal{C}}] = C_{\beta[x/t]}[\underline{P}/\underline{\mathcal{C}}]$ for every individual term t, which is obtained by Lemma 2.14 and the main induction hypothesis.

Case 2. $\alpha' = \exists y\beta[x/y]$ where y is free for x and does not occur free in β.

If $F \succ I(t, H)$, then H is in $C_{\beta[x/t]}[\underline{P}/\underline{\mathcal{C}}]$ which is $C_{\beta[x/y][y/t]}[\underline{P}/\underline{\mathcal{C}}]$ by the main induction hypothesis and Lemma 2.13.

Let z be an individual variable such that $z \notin fv(\beta[x/y]) - \{y\}$, $[\gamma]$ be a type with $z \notin fv(\gamma)$, D be a CR of type $[\gamma]$, G be a term of type $[\beta[x/y][y/z][\underline{P}/\underline{T}] \supset \gamma]$ such that for each individual term t, $G[z/t]$ is in $C_{\beta[x/y][y/t]}[\underline{P}/\underline{\mathcal{C}}] \supset D$, and z is not free in the type superscript of any free term variable of G, and $X^{\beta[x/y][y/z][\underline{P}/\underline{T}]}$ be a term variable which is not equivalent to any free term variable of G.

By Lemmas 2.13 and 2.17, $\beta[x/y][y/z][\underline{P}/\underline{T}] \equiv \beta[x/z][\underline{P}/\underline{T}]$. Then $X^{\beta[x/y][y/z][\underline{P}/\underline{T}]} \equiv X^{\beta[x/z][\underline{P}/\underline{T}]}$. By Lemma 2.13, for every individual term t, $\beta[x/y][y/t] \equiv \beta[x/t]$, and so $C_{\beta[x/y][y/t]}[\underline{P}/\underline{\mathcal{C}}] = C_{\beta[x/t]}[\underline{P}/\underline{\mathcal{C}}]$ by the main induction hypothesis. Hence G is of type $[\beta[x/z][\underline{P}/\underline{T}] \supset \gamma]$ and for each individual term t, $G[z/t]$ is in $C_{\beta[x/t]}[\underline{P}/\underline{\mathcal{C}}] \supset D$. We also have that $z \notin fv(\beta) - \{x\}$ and z is not free in the type superscript of any free term variable of G. Hence, since F is in $C_{\exists x\beta}[\underline{P}/\underline{\mathcal{C}}]$, $ST(z.X^{\beta[x/z][\underline{P}/\underline{T}]}.G(X^{\beta[x/z][\underline{P}/\underline{T}]}), F)$ is in D, and so is $ST(z.X^{\beta[x/y][y/z][\underline{P}/\underline{T}]}.G(X^{\beta[x/y][y/z][\underline{P}/\underline{T}]}), F)$. Hence F is in $C_{\exists y\beta[x/y]}[\underline{P}/\underline{\mathcal{C}}]$.

Notation. In the following, $\underline{P}^*$ is the sublist of $\underline{P}$ consisting of all P_i's which are in $FV(\alpha)$, $\underline{T}^*$ and $\underline{\mathcal{C}}^*$ are the corresponding sublists of $\underline{T}$ and $\underline{\mathcal{C}}$, respectively.

(iii) $\alpha = \forall_2 Q^q\beta$.

Let $U = \lambda y_1, \ldots, y_q\sigma$ be an abstraction term and $\mathcal{D}$ be a collection of CRs corresponding to U.

Then $F(U)$ is in $C_\beta[Q/\mathcal{D}, \underline{P}^*/\underline{\mathcal{C}}^*]$.

Case 1. $\alpha' = \forall_2 Q\beta'$ where $\beta' \equiv \beta$.

By the main induction hypothesis, $C_\beta[Q/\mathcal{D}, \underline{P}^*/\underline{\mathcal{C}}^*] = C_{\beta'}[Q/\mathcal{D}, \underline{P}^*/\underline{\mathcal{C}}^*]$. Thus $F(U)$ is in $C_{\beta'}[Q/\mathcal{D}, \underline{P}^*/\underline{\mathcal{C}}^*]$, and hence F is in $C_{\alpha'}[\underline{P}/\underline{\mathcal{C}}]$.

Case 2. $\alpha' = \forall_2 Q'\beta[Q/Q']$ where Q' is a q-ary predicate variable which is free for Q and does not occur free in β.

By Lemma 4.2.9, $C_\beta[Q/\mathcal{D}, \underline{P}^*/\underline{\mathcal{C}}^*] = C_{\beta[Q/Q']}[Q'/\mathcal{D}, \underline{P}^*/\underline{\mathcal{C}}^*]$ and the rest of the proof is similar to Case 1.

(iv) $\alpha = \exists_2 Q^q\beta$.

Case 1. $\alpha' = \exists_2 Q\beta'$ where $\beta' \equiv \beta$.

This case can be proved in the same way as the following case by using the fact, which is obtained by the main induction hypothesis, that $C_{\beta'}[Q/\mathcal{D}, \underline{P}^*/\underline{\mathcal{C}}^*] = C_\beta[Q/\mathcal{D}, \underline{P}^*/\underline{\mathcal{C}}^*]$ for every collection of CRs $\mathcal{D}$ which corresponds to some abstraction term $U = \lambda y_1, \ldots, y_q\sigma$.

Case 2. $\alpha' = \exists_2 Q'\beta[Q/Q']$ where Q' is a q-ary predicate variable which is free for Q and does not occur free in β.

If $F \succ I(U, H)$, then H is in $C_\beta[Q/\mathcal{D}, \underline{P}^*/\underline{\mathcal{C}}^*]$ which is $C_{\beta[Q/Q']}[Q'/\mathcal{D}, \underline{P}^*/\underline{\mathcal{C}}^*]$ by Lemma 4.2.9, where $\mathcal{D}$ is some collection of CRs corresponding to U.

Let R be a q-ary predicate variable such that $R \notin FV(\beta[Q/Q'][\underline{P}^*/\underline{T}^*]) - \{Q'\}$, $[\gamma]$ be a type with $R \notin FV(\gamma)$, D be a CR of type $[\gamma]$, G be a term of type $[\beta[Q/Q'][Q'/R, \underline{P}^*/\underline{T}^*] \supset \gamma]$ such that for each abstraction term $U = \lambda y_1, \ldots, y_q\sigma$ and all collections of CRs $\mathcal{E}$ corresponding to U, $G[R/U]$ is in $C_{\beta[Q/Q']}[\underline{P}^*/\underline{\mathcal{C}}^*, Q'/\mathcal{E}]$ $\supset D$, and R is not free in the type superscript of any free term variable of G, and $X^{\beta[Q/Q'][Q'/R, \underline{P}^*/\underline{T}^*]}$ be a term variable which is not equivalent to any free term variable of G.

By Lemma 2.16, $\beta[Q/Q'][Q'/R,\underline{P}^*/\underline{T}^*] \equiv \beta[Q/R,\underline{P}^*/\underline{T}^*]$. Then $X^{\beta[Q/Q'][Q'/R,\underline{P}^*/\underline{T}^*]} \equiv X^{\beta[Q/R,\underline{P}^*/\underline{T}^*]}$. By Lemma 4.2.9, for all collections of CRs $\mathcal{E}$ corresponding to some abstraction term $U = \lambda y_1,\ldots,y_q\sigma$, $C_{\beta[Q/Q']}[Q'/\mathcal{E},\underline{P}^*/\underline{\mathcal{C}}^*] = C_\beta[Q/\mathcal{E},\underline{P}^*/\underline{\mathcal{C}}^*]$. Then G is of type $[\beta[Q/R,\underline{P}^*/\underline{T}^*] \supset \gamma]$ and for each abstraction term $U = \lambda y_1,\ldots,y_q\sigma$ and all collections of CRs $\mathcal{E}$ corresponding to U, $G[R/U]$ is in $C_\beta[Q/\mathcal{E},\underline{P}^*/\underline{\mathcal{C}}^*] \supset D$. Since $R \notin FV(\beta[Q/Q'][\underline{P}^*/\underline{T}^*]) - \{Q'\}$, $R \notin FV(\beta[\underline{P}^*/\underline{T}^*]) - \{Q\}$. Hence, since F is in $C_{\exists_2 Q\beta}[\underline{P}/\underline{\mathcal{C}}]$,
$ST(R.X^{\beta[Q/R,\underline{P}^*/\underline{T}^*]}.G(X^{\beta[Q/R,\underline{P}^*/\underline{T}^*]}),F)$ is in D, and so is
$ST(R.X^{\beta[Q/Q'][Q'/R,\underline{P}^*/\underline{T}^*]}.G(X^{\beta[Q/Q'][Q'/R,\underline{P}^*/\underline{T}^*]}),F)$. Hence F is in
$C_{\exists_2 Q'\beta[Q/Q']}[\underline{P}/\underline{\mathcal{C}}]$ i.e. $C_{\alpha'}[\underline{P}/\underline{\mathcal{C}}]$. □

Lemma 4.2.11. *Let α be a formula, $\underline{P} = P_1^{m_1},\ldots,P_n^{m_n}$ be distinct predicate variables, $\underline{T} = T_1,\ldots,T_n$, where $T_i = \lambda x_1^i,\ldots,x_{m_i}^i\delta_i$, $1 \le i \le n$, be abstraction terms, and $\underline{\mathcal{C}} = \mathcal{C}_1,\ldots,\mathcal{C}_n$ be collections of CRs corresponding to $T_1,\ldots,T_n$, respectively.*

Let R be a k-ary predicate variable which is not in $\{\underline{P}\}$, $U = \lambda z_1,\ldots,z_k\sigma$ be an abstraction term, say $\underline{z} = z_1,\ldots,z_k$, and let
$\mathcal{D} = \{C_{\sigma[\underline{z}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}] \mid \underline{t} = t_1,\ldots,t_k$ are individual terms.$\}$, so $\mathcal{D}$ is a collection of CRs corresponding to $U[\underline{P}/\underline{T}]$.

Then $C_\alpha[R/\mathcal{D},\underline{P}/\underline{\mathcal{C}}] = C_{\alpha[R/U]}[\underline{P}/\underline{\mathcal{C}}]$.

Proof. If $R \notin FV(\alpha)$, then, by the Note on page 113, $C_\alpha[R/\mathcal{D},\underline{P}/\underline{\mathcal{C}}] = C_\alpha[\underline{P}/\underline{\mathcal{C}}] = C_{\alpha[R/U]}[\underline{P}/\underline{\mathcal{C}}]$. Suppose $R \in FV(\alpha)$.

We proceed by induction on α. The cases where α is $\beta \supset \gamma$, $\beta \vee \gamma$, or $\beta \wedge \gamma$ easily follow by the induction hypothesis. The remaining cases are as follows.

(i) $\alpha = R(t_1,\ldots,t_k)$ for some individual terms $\underline{t} = t_1,\ldots,t_k$.

Then $C_\alpha[R/\mathcal{D},\underline{P}/\underline{\mathcal{C}}] = C_{\sigma[\underline{z}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}] = C_{\alpha[R/U]}[\underline{P}/\underline{\mathcal{C}}]$.

For the remaining cases, we will show that $C_\alpha[R/\mathcal{D}, \underline{P}/\underline{\mathcal{C}}] \subseteq C_{\alpha[R/U]}[\underline{P}/\underline{\mathcal{C}}]$ and omit proofs of the converse which can be done similarly.

Let F be in $C_\alpha[R/\mathcal{D}, \underline{P}/\underline{\mathcal{C}}]$.

(ii) $\alpha = \forall x\beta$.

By the induction hypothesis and Lemmas 2.15 and 4.2.10, for every individual term t, $C_{\beta[x/t]}[R/\mathcal{D}, \underline{P}/\underline{\mathcal{C}}] = C_{\beta[x/t][R/U]}[\underline{P}/\underline{\mathcal{C}}] = C_{\beta[R/U][x/t]}[\underline{P}/\underline{\mathcal{C}}]$. Then this case easily follows by this fact.

(iii) $\alpha = \exists x\beta$.

We will show that F is in $C_{\exists x\beta[R/U]}[\underline{P}/\underline{\mathcal{C}}]$. If $F \succ I(t, H)$, then H is in $C_{\beta[x/t]}[R/\mathcal{D}, \underline{P}/\underline{\mathcal{C}}]$ which is $C_{\beta[R/U][x/t]}[\underline{P}/\underline{\mathcal{C}}]$ by the induction hypothesis, Lemmas 2.15 and 4.2.10.

Let y be an individual variable such that $y \notin fv(\beta[R/U]) - \{x\}$, $[\gamma]$ be a type with $y \notin fv(\gamma)$, E be a CR of type $[\gamma]$, G be a term of type $[\beta[R/U][x/y][\underline{P}/\underline{T}] \supset \gamma]$ such that for each individual term t, $G[y/t]$ is in $C_{\beta[R/U][x/t]}[\underline{P}/\underline{\mathcal{C}}] \supset E$, and y is not free in the type superscript of any free term variable of G, and $X^{\beta[R/U][x/y][\underline{P}/\underline{T}]}$ be a term variable which is not equivalent to any free term variable of G.

By Lemmas 2.15, 2.16, and 2.17, $\beta[R/U][x/y][\underline{P}/\underline{T}] \equiv \beta[x/y][\underline{P}/\underline{T}, R/U[\underline{P}/\underline{T}]]$. By the induction hypothesis and Lemma 4.2.10, for every individual term t, $C_{\beta[R/U][x/t]}[\underline{P}/\underline{\mathcal{C}}] = C_{\beta[x/t][R/U]}[\underline{P}/\underline{\mathcal{C}}] = C_{\beta[x/t]}[\underline{P}/\underline{\mathcal{C}}, R/\mathcal{D}]$. Hence we have $X^{\beta[R/U][x/y][\underline{P}/\underline{T}]} \equiv X^{\beta[x/y][\underline{P}/\underline{T}, R/U[\underline{P}/\underline{T}]]}$ and G is of type $[\beta[x/y][\underline{P}/\underline{T}, R/U[\underline{P}/\underline{T}]] \supset \gamma]$ and for each individual term t, $G[y/t]$ is in $C_{\beta[x/t]}[\underline{P}/\underline{\mathcal{C}}, R/\mathcal{D}] \supset E$ and y is not free in the type superscript of any free term variable of G.

Since F is in $C_{\exists x\beta}[\underline{P}/\underline{\mathcal{C}}, R/\mathcal{D}]$,
$ST(y.X^{\beta[x/y][\underline{P}/\underline{T}, R/U[\underline{P}/\underline{T}]]}.G(X^{\beta[x/y][\underline{P}/\underline{T}, R/U[\underline{P}/\underline{T}]]}), F)$ is in E, and so is
$ST(y.X^{\beta[R/U][x/y][\underline{P}/\underline{T}]}.G(X^{\beta[R/U][x/y][\underline{P}/\underline{T}]}), F)$. Thus F is in $C_{\exists x\beta[R/U]}[\underline{P}/\underline{\mathcal{C}}]$.

Notation. In the following, $\underline{P}^*$ and $\underline{P}^{**}$ are the sublists of $\underline{P}$ consisting of all P_i's which are in $FV(\alpha[R/U])$ and $FV(\alpha)$, respectively, $\underline{T}^*$ and $\underline{T}^{**}$, $\underline{\mathcal{C}}^*$ and $\underline{\mathcal{C}}^{**}$ are the corresponding sublists of $\underline{T}$ and $\underline{\mathcal{C}}$, respectively.

Note that $\underline{P}^{**}$ is a sublist of $\underline{P}^*$ since $R \notin \{\underline{P}\}$, similarly for $\underline{T}^{**}$ and $\underline{\mathcal{C}}^{**}$.

(iv) $\alpha = \forall_2 Q^q \beta$.

By Lemma 4.2.10, we may assume that $Q \notin FV(U) \cup FV(\underline{T}^*)$, so $Q \notin \{\underline{P}^*\}$. We want to show that F is in $C_{\forall_2 Q\beta[R/U]}[\underline{P}/\underline{\mathcal{C}}]$.

Let $V = \lambda y_1, \ldots, y_q\tau$ be an abstraction term and $\mathcal{E}$ be a collection of CRs corresponding to V. Since F is in $C_{\forall_2 Q\beta}[R/\mathcal{D}, \underline{P}/\underline{\mathcal{C}}]$, $F(V)$ is in $C_\beta[Q/\mathcal{E}, R/\mathcal{D}, \underline{P}^{**}/\underline{\mathcal{C}}^{**}]$.

Since $Q \notin FV(U)$ i.e $Q \notin FV(\sigma)$, $C_{\sigma[\underline{z}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}] = C_{\sigma[\underline{z}/\underline{t}]}[Q/\mathcal{E}, \underline{P}^*/\underline{\mathcal{C}}^*]$ for all individual terms $\underline{t} = t_1, \ldots, t_k$. Hence $\mathcal{D} =$
$\{C_{\sigma[\underline{z}/\underline{t}]}[Q/\mathcal{E}, \underline{P}^*/\underline{\mathcal{C}}^*] \mid \underline{t} = t_1, \ldots, t_k$ are individual terms.$\}$. By the induction hypothesis, $C_\beta[Q/\mathcal{E}, R/\mathcal{D}, \underline{P}^{**}/\underline{\mathcal{C}}^{**}] = C_\beta[Q/\mathcal{E}, R/\mathcal{D}, \underline{P}^*/\underline{\mathcal{C}}^*] = C_{\beta[R/U]}[Q/\mathcal{E}, \underline{P}^*/\underline{\mathcal{C}}^*]$. Thus $F(V)$ is in $C_{\beta[R/U]}[Q/\mathcal{E}, \underline{P}^*/\underline{\mathcal{C}}^*]$, and so F is in $C_{\forall_2 Q\beta[R/U]}[\underline{P}/\underline{\mathcal{C}}]$ i.e. $C_{\alpha[R/U]}[\underline{P}/\underline{\mathcal{C}}]$.

(v) $\alpha = \exists_2 Q^q \beta$.

By Lemma 4.2.10, we may assume that $Q \notin FV(U) \cup FV(\underline{T}^*)$, so $Q \notin \{\underline{P}^*\}$. We want to show that F is in $C_{\exists_2 Q\beta[R/U]}[\underline{P}/\underline{\mathcal{C}}]$.

Since F is in $C_{\exists_2 Q\beta}[R/\mathcal{D}, \underline{P}/\underline{\mathcal{C}}]$, if $F \succ I(V, H)$, then H is in $C_\beta[\underline{P}^{**}/\underline{\mathcal{C}}^{**}, R/\mathcal{D}, Q/\mathcal{E}]$ which is $C_{\beta[R/U]}[\underline{P}^*/\underline{\mathcal{C}}^*, Q/\mathcal{E}]$ (as shown in the above case), where $\mathcal{E}$ is some collection of CRs corresponding to V.

Let Q' be a q-ary predicate variable such that $Q' \notin FV(\beta[R/U][\underline{P}^*/\underline{T}^*]) - \{Q\}$, $[\gamma]$ be a type with $Q' \notin FV(\gamma)$, E be a CR of type $[\gamma]$, G be a term of type $[\beta[R/U][Q/Q', \underline{P}^*/\underline{T}^*] \supset \gamma]$ such that for each abstraction term $V = \lambda y_1, \ldots, y_q\tau$ and all collections of CRs $\mathcal{E}$ corresponding to V, $G[Q'/V]$ is in $C_{\beta[R/U]}[\underline{P}^*/\underline{\mathcal{C}}^*, Q/\mathcal{E}] \supset E$, and Q' is not free in the type superscript of any free term variable of G, and $X^{\beta[R/U][Q/Q', \underline{P}^*/\underline{T}^*]}$ be a term variable which is not equivalent to any free term variable of G.

By Lemma 2.16, $\beta[R/U][Q/Q', \underline{P}^*/\underline{T}^*] \equiv \beta[R/U[\underline{P}/\underline{T}], Q/Q', \underline{P}^*/\underline{T}^*]$. As shown in the above case, $C_{\beta[R/U]}[Q/\mathcal{E}, \underline{P}^*/\underline{\mathcal{C}}^*] = C_\beta[Q/\mathcal{E}, \underline{P}^{**}/\underline{\mathcal{C}}^{**}]$ for all collections of CRs $\mathcal{E}$ which corresponds to some abstraction term $V = \lambda y_1, \ldots, y_q\tau$.

Hence we have $X^{\beta[R/U][Q/Q', \underline{P}^*/\underline{T}^*]} \equiv X^{\beta[R/U[\underline{P}/\underline{T}], Q/Q', \underline{P}^*/\underline{T}^*]}$, G is of type $[\beta[R/U[\underline{P}/\underline{T}], Q/Q', \underline{P}^*/\underline{T}^*] \supset \gamma]$ and for each abstraction term $V = \lambda y_1, \ldots, y_q\tau$ and all collections of CRs $\mathcal{E}$ corresponding to V, $G[Q'/V]$ is in $C_\beta[Q/\mathcal{E}, \underline{P}^{**}/\underline{\mathcal{C}}^{**}] \supset E$ and Q' is not free in the type superscript of any free term variable of G.

Since F is in $C_{\exists_2 Q\beta}[R/\mathcal{D}, \underline{P}/\underline{\mathcal{C}}]$, $ST(Q'.X^{\beta[Q/Q', R/U[\underline{P}/\underline{T}], \underline{P}^*/\underline{T}^*]}.G(X^{\beta[Q/Q', R/U[\underline{P}/\underline{T}], \underline{P}^*/\underline{T}^*]}), F)$ is in E, and so is $ST(Q'.X^{\beta[R/U][Q/Q', \underline{P}^*/\underline{T}^*]}.G(X^{\beta[R/U][Q/Q', \underline{P}^*/\underline{T}^*]}), F)$. Thus F is in $C_{\exists_2 Q\beta[R/U]}[\underline{P}/\underline{\mathcal{C}}]$ i.e. $C_{\alpha[R/U]}[\underline{P}/\underline{\mathcal{C}}]$. □

Lemma 4.2.12. *Let F^α be a Curry-Howard term, $\underline{x} = x_1, \ldots, x_n$ be distinct individual variables, $\underline{t} = t_1, \ldots, t_n$ be individual terms, $\underline{P} = P_1^{m_1}, \ldots, P_k^{m_k}$ be distinct predicate variables, $\underline{T} = T_1, \ldots, T_k$, where $T_i = \lambda z_1^i, \ldots, z_{m_i}^i\tau_i$, $1 \le i \le k$, be abstraction terms, $\underline{\mathcal{C}} = \mathcal{C}_1, \ldots, \mathcal{C}_k$ be collections of CRs corresponding to $T_1, \ldots, T_k$, respectively, $X_1^{\delta_1}, \ldots, X_l^{\delta_l}$ be inequivalent term variables such that every free term variable of F^α is equivalent to $X_i^{\delta_i}$ for some $1 \le i \le l$, and $\underline{X} = X_1^{\delta_1'}, \ldots, X_l^{\delta_l'}$, where $\delta_i' = \delta_i[\underline{x}/\underline{t}][\underline{P}/\underline{T}]$, $1 \le i \le l$, are inequivalent term variables, and let $\underline{K} = K_1^{\delta_1'}, \ldots, K_l^{\delta_l'}$ be Curry-Howard terms in $C_{\delta_1[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}], \ldots, C_{\delta_l[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$, respectively.*

Then $F^\alpha[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]$ is in $C_{\alpha[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$.

Proof. We will prove by induction on F^α.

Notation. Throughout this proof, γ' denotes $\gamma[\underline{x}/\underline{t}][\underline{P}/\underline{T}]$ for any formula γ.

(Atomic) $F^\alpha \equiv X_q^{\delta_q}$ for some $1 \le q \le l$:

Then $F^\alpha[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}] = K_q^{\delta_q'}$ which is in $C_{\delta_q[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$ i.e. $C_{\alpha[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$ by

Lemmas 2.14 and 4.2.10.

($\supset$ Intro) $F^\alpha = \lambda Y^\beta . G^\gamma$:

We first prove the following claim.

Claim 1. For any terms J and H, if J is in $C_{\beta[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$, H and $H[Y^{\beta'}/J]$ are in $C_{\gamma[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$, then $(\lambda Y^{\beta'}.H)(J)$ is in $C_{\gamma[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$.

Proof of Claim 1. Suppose J is in $C_{\beta[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$, H and $H[Y^{\beta'}/J]$ are in $C_{\gamma[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$. We proceed by induction on $N(J)+N(H)$. Since $(\lambda Y^{\beta'}.H)(J)$ is neutral, to show that it is in $C_{\gamma[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$, it is enough to show that all its immediate reducts are in $C_{\gamma[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$. Every immediate reduct of $(\lambda Y^{\beta'}.H)(J)$ is of one of the following forms.

(i) $H[Y^{\beta'}/J]$.

This is in $C_{\gamma[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$ by the assumption.

(ii) $(\lambda Y^{\beta'}.H')(J)$ where $H \succ_1 H'$.

By CR2 for $C_{\gamma[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$, H' is in $C_{\gamma[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$. By Lemma 3.3.4, $H[Y^{\beta'}/J] \succ_1 H^*$ for some term H^* such that $H^* \equiv H'[Y^{\beta'}/J]$. Since $H[Y^{\beta'}/J]$ is in $C_{\gamma[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$, by CR2, H^* is in $C_{\gamma[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$, and so is $H'[Y^{\beta'}/J]$ by CR0. Thus H' satisfies the conditions of the hypothesis and $N(H') < N(H)$. By the subsidiary induction hypothesis, $(\lambda Y^{\beta'}.H')(J)$ is in $C_{\gamma[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$.

(iii) $(\lambda Y^{\beta'}.H)(J')$ where $J \succ_1 J'$.

By CR2 for $C_{\beta[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$, J' is in $C_{\beta[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$. By Lemma 3.3.5, $H[Y^{\beta'}/J] \succ H^*$ for some term H^* such that $H^* \equiv H[Y^{\beta'}/J']$. By CR2 and CR0, H^* is in $C_{\gamma[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$, and so is $H[Y^{\beta'}/J']$. Thus J' satisfies the conditions of the hypothesis and $N(J') < N(J)$. Hence $(\lambda Y^{\beta'}.H)(J')$ is in $C_{\gamma[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$ by the induction hypothesis.

Thus we have Claim 1.

We have to show that $M = (\lambda Y^{\beta}.G^{\gamma})[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]$ is in $C_{\beta[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}] \supset C_{\gamma[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$. By CR0, we may assume that $Y^{\beta'}$ is not equivalent to any free term variable in $\underline{X}$ or $\underline{K}$ and G has no free term variable Y^{σ} such that $\sigma \not\equiv \beta$. Then $M = \lambda Y^{\beta'}.G^{\gamma}[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]$.

Let J be a term in $C_{\beta[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$. By the induction hypothesis, $G^{\gamma}[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]$ $(= G^{\gamma}[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}, Y^{\beta'}/Y^{\beta'}])$ and $G^{\gamma}[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}, Y^{\beta'}/J]$ are in $C_{\gamma[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$. Since $G^{\gamma}[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}][Y^{\beta}/J]$ $\equiv G^{\gamma}[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}, Y^{\beta}/J]$, by CR0, $G^{\gamma}[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}][Y^{\beta}/J]$ is in $C_{\gamma[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$. Thus, by Claim 1, $M(J)$ is in $C_{\gamma[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$. Since J is arbitrary, M is in $C_{\beta[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}] \supset C_{\gamma[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$ i.e. $C_{\alpha[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$.

($\supset$ Elim) $F^{\alpha} = G^{\beta\supset\alpha}(H^{\beta})$:

By the induction hypothesis, $G^{\beta\supset\alpha}[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]$ is in $C_{(\beta\supset\alpha)[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$ i.e. $C_{\beta[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}] \supset C_{\alpha[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$ and $H^{\beta}[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]$ is in $C_{\beta[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$. Hence $F^{\alpha}[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]$ is in $C_{\alpha[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$.

($\wedge$ Intro) $F^{\alpha} = (G^{\beta}, H^{\gamma})$:

We have to show that $M = (G^{\beta}[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}], H^{\gamma}[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}])$ is in $C_{(\beta\wedge\gamma)[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$ i.e. $C_{\beta[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}] \wedge C_{\gamma[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$. We will show that $\pi_1 M$ is in $C_{\beta[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$ by induction on $N(G^{\beta}[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]) + N(H^{\gamma}[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}])$. Since $\pi_1 M$ is neutral, by CR3, it is enough to show that all its immediate reducts are in $C_{\beta[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$. Every immediate reduct of $\pi_1 M$ is of one of the following forms.

(i) $G^{\beta}[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]$.

It is in $C_{\beta[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$ by the main induction hypothesis.

(ii) $\pi_1(G', H^{\gamma}[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}])$ where $G^{\beta}[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}] \succ_1 G'$.

By the main induction hypothesis, $G^{\beta}[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]$ is in $C_{\beta[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$, and so is G' by CR2. Hence $\pi_1(G', H^{\gamma}[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}])$ is in $C_{\beta[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$ by the subsidiary induction hypothesis since $N(G') < N(G^{\beta}[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}])$.

(iii) $\pi_1(G^{\beta}[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}], H')$ where $H^{\gamma}[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}] \succ_1 H'$.

This case follows by the subsidiary induction hypothesis as in (ii).

Similarly, we can show that $\pi_2 M$ is in $C_{\gamma[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$. Thus M is in $C_{(\beta\wedge\gamma)[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$.

($\wedge$ Elim) $F^{\alpha} = \pi_1 G^{\alpha\wedge\beta}$:

By the induction hypothesis, $G^{\alpha\wedge\beta}[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]$ is in $C_{(\alpha\wedge\beta)[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$ i.e. $C_{\alpha[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]\wedge C_{\beta[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$. Hence $F^{\alpha}[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]$ i.e. $\pi_1 G^{\alpha\wedge\beta}[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]$ is in $C_{\alpha[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$.

Similarly for $\pi_2 G^{\beta\wedge\alpha}$.

($\forall$ Intro) $F^{\alpha} = \lambda y.G^{\beta}$:

We want to show that $(\lambda y.G^{\beta})[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]$ is in $C_{(\forall y\beta)[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$. By Lemma 4.2.10 and CR0 for $C_{(\forall y\beta)[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$, we may assume that $y \notin \{\underline{x}\} \cup fv(\underline{t}) \cup fv(\underline{K})$. We have to show that $M = \lambda y.G^{\beta}[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]$ is in $C_{\forall y\beta[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$.

Let u be an individual term. We will show that $M(u)$ is in $C_{\beta[\underline{x}/\underline{t}][y/u]}[\underline{P}/\underline{\mathcal{C}}]$ by induction on $N(G^{\beta}[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}])$. Since $M(u)$ is neutral, it remains to show that all its immediate reducts are in $C_{\beta[\underline{x}/\underline{t}][y/u]}[\underline{P}/\underline{\mathcal{C}}]$. Every immediate reduct of $M(u)$ is of one of the following forms.

(i) $G^{\beta}[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}][y/u]$.

Since $y \notin fv(\underline{t})$ and y does not occur free in the type superscript of any free term variable of G^{β}, if $X_i^{\delta'_i}$ is equivalent to some free term variable of $G^{\beta}[\underline{x}/\underline{t}][\underline{P}/\underline{T}]$ for some $1 \leq i \leq l$, then there is no free term variable X_i^{σ} of $G^{\beta}[\underline{x}/\underline{t}][\underline{P}/\underline{T}]$ such that $\sigma \not\equiv \delta_i'$ but $\sigma[y/u] \equiv \delta_i'[y/u]$ (i.e. $\sigma \equiv \delta_i'$). By Lemmas 2.13, 3.2.19, 3.2.22, 3.2.23, 3.2.25, and 3.2.26, we have $G^\beta[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}][y/u] \equiv$ $G^\beta[\underline{x}/\underline{t}, y/u][\underline{P}/\underline{T}][\underline{X}/\underline{K}]$ and $\beta[\underline{x}/\underline{t}][y/u] \equiv \beta[\underline{x}/\underline{t}, y/u]$.

Since $y \notin \{\underline{x}\} \cup fv(\underline{K})$, $y \notin fv(\delta_i)$ and so $C_{\delta_i[\underline{x}/\underline{t},y/u]}[\underline{P}/\underline{\mathcal{C}}] = C_{\delta_i[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$ for all $1 \leq i \leq l$. Hence, by the induction hypothesis, $G^\beta[\underline{x}/\underline{t}, y/u][\underline{P}/\underline{T}][\underline{X}/\underline{K}]$ is in $C_{\beta[\underline{x}/\underline{t},y/u]}[\underline{P}/\underline{\mathcal{C}}]$ i.e. $C_{\beta[\underline{x}/\underline{t}][y/u]}[\underline{P}/\underline{\mathcal{C}}]$ by Lemma 4.2.10.

(ii) $(\lambda y.G')(u)$ where $G^\beta[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}] \succ_1 G'$.

By the main induction hypothesis, $G^\beta[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]$ is in $C_{\beta[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$, and so is G' by CR2. Since $N(G') < N(G^\beta[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}])$, by the subsidiary induction hypothesis, $(\lambda y.G')(u)$ is in $C_{\beta[\underline{x}/\underline{t}][y/u]}[\underline{P}/\underline{\mathcal{C}}]$.

($\forall_2$ Intro) $F^\alpha = \lambda Q^q.G^\beta$:

As in the above case, we may assume that $Q \notin \{\underline{P}\} \cup FV(\underline{T}) \cup FV(\underline{K})$. We have to show that $M = \lambda Q.G^\beta[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]$ is in $C_{\forall_2 Q\beta[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$.

Let $U = \lambda y_1, \ldots, y_q\sigma$ be an abstraction term and $\mathcal{D}$ be a collection of CRs corresponding to U. We will show that $M(U)$ is in $C_{\beta[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}, Q/\mathcal{D}]$ by induction on $N(G^\beta[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}])$. Since $M(U)$ is neutral, it remains to show that all its immediate reducts are in $C_{\beta[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}, Q/\mathcal{D}]$. Every immediate reduct of $M(U)$ is of one of the following forms.

(i) $G^\beta[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}][Q/U]$.

Similar to the above case, by Lemmas 3.2.20, 3.2.23, and 3.2.26, we have $G^\beta[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}][Q/U] \equiv G^\beta[\underline{x}/\underline{t}][\underline{P}/\underline{T}, Q/U][\underline{X}/\underline{K}]$.

Since $Q \notin \{\underline{P}\} \cup FV(\underline{K})$, $Q \notin FV(\delta_i[\underline{x}/\underline{t}])$, and so $C_{\delta_i[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}, Q/\mathcal{D}] = C_{\delta_i[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$ for all $1 \leq i \leq l$. Hence, by the induction hypothesis, $G^\beta[\underline{x}/\underline{t}][\underline{P}/\underline{T}, Q/U][\underline{X}/\underline{K}]$ is in $C_{\beta[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}, Q/\mathcal{D}]$.

(ii) $(\lambda_2 Q.G')(U)$ where $G^\beta[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}] \succ_1 G'$.

Similar to (ii) in the above case, this case follows by the subsidiary induction hypothesis.

($\forall$ Elim) $F^\alpha = G^{\forall y\beta}(u)$:

By the induction hypothesis, $G^{\forall y\beta}[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]$ is in $C_{(\forall y\beta)[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$. By Lemma 4.2.10, we may assume that $y \notin \{\underline{x}\} \cup fv(\underline{t})$. Then $G^{\forall y\beta}[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]$ is in $C_{\forall y\beta[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$. Hence $F^\alpha[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}] = G^{\forall y\beta}[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}](u[\underline{x}/\underline{t}])$ is in $C_{\beta[\underline{x}/\underline{t}][y/u[\underline{x}/\underline{t}]]}[\underline{P}/\underline{\mathcal{C}}]$. By Lemmas 2.13 and 4.2.10, $C_{\beta[\underline{x}/\underline{t}][y/u[\underline{x}/\underline{t}]]}[\underline{P}/\underline{\mathcal{C}}] = C_{\beta[y/u][\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}] = C_{\alpha[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$. Hence $F^\alpha[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]$ is in $C_{\alpha[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$.

($\forall_2$ Elim) $F^\alpha = G^{\forall_2 Q\beta}(U)$:

By the induction hypothesis, $G^{\forall_2 Q\beta}[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]$ is in $C_{\forall_2 Q\beta[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$. By Lemma 4.2.10, we may assume that $Q \notin \{\underline{P}\}$. Hence $M =$ $G^{\forall_2 Q\beta}[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}](U[\underline{P}/\underline{T}])$ is in $C_{\beta[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}, Q/\mathcal{E}]$ for all collections of CRs $\mathcal{E}$ corresponding to $U[\underline{P}/\underline{T}]$.

Say $U = \lambda y_1, \ldots, y_q\sigma$, $\underline{y} = y_1, \ldots, y_q$, and let $\mathcal{D} = \{C_{\sigma[\underline{y}/\underline{u}]}[\underline{P}/\underline{\mathcal{C}}] \mid \underline{u} = u_1, \ldots, u_q$ are individual terms.$\}$. Then $\mathcal{D}$ is a collection of CRs corresponding to $U[\underline{P}/\underline{T}]$. By Lemmas 2.15, 4.2.10, and 4.2.11, $C_{\beta[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}, Q/\mathcal{D}] =$ $C_{\beta[\underline{x}/\underline{t}][Q/U]}[\underline{P}/\underline{\mathcal{C}}] = C_{\beta[Q/U][\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}] = C_{\alpha[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$. Hence M is in $C_{\alpha[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$.

($\vee$ Intro) $F^\alpha = (\mu_1 G^{\alpha_1})^{\alpha_1 \vee \alpha_2}$:

We want to show that $M = \mu_1 G^{\alpha_1}[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]$ is in $C_{(\alpha_1 \vee \alpha_2)[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$ i.e. $C_{\alpha_1[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}] \vee C_{\alpha_2[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$.

Let $[\gamma]$ be a type, D be a CR of type $[\gamma]$, F_1 and F_2 be terms in $C_{\alpha_1[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}] \supset D$ and $C_{\alpha_2[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}] \supset D$, respectively, and $Y_1^{\alpha_1'}$ and $Y_2^{\alpha_2'}$ be term variables which are not equivalent to any free term variables of F_1 and F_2, respectively. We have to show that $\oplus(Y_1^{\alpha_1'}.F_1(Y_1^{\alpha_1'}), Y_2^{\alpha_2'}.F_2(Y_2^{\alpha_2'}), M)$ is in D.

We first prove the following claim.

Claim 2. For any terms G_1, G_2, and G_3, if G_3 is in $C_{\alpha_1[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$ and G_1, G_2, and $G_1[Y_1^{\alpha_1'}/G_3]$ are in D, then $\oplus(Y_1^{\alpha_1'}.G_1, Y_2^{\alpha_2'}.G_2, \mu_1 G_3)$ is in D.

Proof of Claim 2. Suppose G_3 is in $C_{\alpha_1[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$ and G_1, G_2, and $G_1[Y_1^{\alpha_1'}/G_3]$ are in D. We will prove by induction on $N(G_1) + N(G_2) + N(G_3)$. Since $\oplus(Y_1^{\alpha_1'}.G_1, Y_2^{\alpha_2'}.G_2, \mu_1 G_3)$ is neutral, to show that it is in D, it is enough to show that all its immediate reducts are in D. Every immediate reduct of $\oplus(Y_1^{\alpha_1'}.G_1, Y_2^{\alpha_2'}.G_2, \mu_1 G_3)$ is of one of the following forms.

(i) $G_1[Y_1^{\alpha_1'}/G_3]$.

This is in D by the assumption.

(ii) $\oplus(Y_1^{\alpha_1'}.G_1, Y_2^{\alpha_2'}.G_2', \mu_1 G_3)$ where $G_2 \succ_1 G_2'$.

This is in D by the subsidiary induction hypothesis.

(iii) $\oplus(Y_1^{\alpha_1'}.G_1', Y_2^{\alpha_2'}.G_2, \mu_1 G_3)$ where $G_1 \succ_1 G_1'$.

By CR2, G_1' is in D. By Lemma 3.3.4, $G_1'[Y_1^{\alpha_1'}/G_3]$ is equivalent to some immediate reduct of $G_1[Y_1^{\alpha_1'}/G_3]$. Since $G_1[Y_1^{\alpha_1'}/G_3]$ is in D, by CR0 and CR2, so is $G_1'[Y_1^{\alpha_1'}/G_3]$. Thus G_1' satisfies the conditions of the hypothesis. Hence $\oplus(Y_1^{\alpha_1'}.G_1', Y_2^{\alpha_2'}.G_2, \mu_1 G_3)$ is in D by the subsidiary induction hypothesis.

(iv) $\oplus(Y_1^{\alpha_1'}.G_1, Y_2^{\alpha_2'}.G_2, \mu_1 G_3')$ where $G_3 \succ_1 G_3'$.

By CR2, G_3' is in $C_{\alpha_1[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$. By Lemma 3.3.5, $G_1[Y_1^{\alpha_1'}/G_3']$ is equivalent to some reduct of $G_1[Y_1^{\alpha_1'}/G_3]$. Hence, by CR0 and CR2, $G_1[Y_1^{\alpha_1'}/G_3']$ is in D. Thus G_3' satisfies the conditions of the hypothesis. Hence $\oplus(Y_1^{\alpha_1'}.G_1, Y_2^{\alpha_2'}.G_2, \mu_1 G_3')$ is in D by the subsidiary induction hypothesis.

Thus we have Claim 2.

By the main induction hypothesis, $G^{\alpha_1}[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]$ is in $C_{\alpha_1[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$. Since F_1 is in $C_{\alpha_1[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}] \supset D$, $F_1(Y_1^{\alpha_1'})$ and $F_1(Y_1^{\alpha_1'})[Y_1^{\alpha_1'}/G^{\alpha_1}[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]]$

(i.e. $F_1(G^{\alpha_1}[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}])$) are in D. Since F_2 is in $C_{\alpha_2[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}] \supset D$, $F_2(Y_2^{\alpha'_2})$ is in D. Hence, by the claim, $\oplus(Y_1^{\alpha'_1}.F_1(Y_1^{\alpha'_1}), Y_2^{\alpha'_2}.F_2(Y_2^{\alpha'_2}), M)$ is in D. Thus M is in $C_{\alpha_1[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}] \vee C_{\alpha_2[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$ i.e. $C_{\alpha[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$.

Similarly for the case $F^\alpha = (\mu_2 G^{\alpha_2})^{\alpha_1 \vee \alpha_2}$.

($\vee$ Elim) $F^\alpha = \oplus(Y_1^{\beta_1}.F_1^\alpha, Y_2^{\beta_2}.F_2^\alpha, G^{\beta_1 \vee \beta_2})$:

Since $C_{\alpha[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$ is closed under equivalence of terms, we may assume that F_i has no free term variable Y_i^σ such that $\sigma \not\equiv \beta_i$ for all $i = 1, 2$, and both $Y_1^{\beta'_1}$ and $Y_2^{\beta'_2}$ are not equivalent to any free term variables in $\underline{X}$ or $\underline{K}$. We want to show that $M = \oplus(Y_1^{\beta'_1}.F_1^\alpha[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}], Y_2^{\beta'_2}.F_2^\alpha[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}],$
$G^{\beta_1 \vee \beta_2}[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}])$ is in $C_{\alpha[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$. Since M is neutral, it is enough to show that all its immediate reducts are in $C_{\alpha[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$. We will show this by induction on $N(F_1^\alpha[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]) + N(F_2^\alpha[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]) +$
$N(G^{\beta_1 \vee \beta_2}[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}])$. If an immediate reduct of M is obtained by reducing $F_1^\alpha[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]$, $F_2^\alpha[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]$, or $G^{\beta_1 \vee \beta_2}[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]$, then it is in $C_{\alpha[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$ by the subsidiary induction hypothesis.

The remaining case is obtained when $G^{\beta_1 \vee \beta_2}[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]$ is of the form $\mu_1 H$ or $\mu_2 H$. Suppose $G^{\beta_1 \vee \beta_2}[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}] = \mu_1 H^{\beta^*}$, where $\beta^* \equiv \beta'_1$. Then the immediate reduct is $F_1^\alpha[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}][Y_1^{\beta'_1}/H]$.

By the main induction hypothesis, $\mu_1 H$ is in $C_{(\beta_1 \vee \beta_2)[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$ i.e. $C_{\beta_1[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}] \vee C_{\beta_2[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$. By Lemma 4.2.4, H is in $C_{\beta_1[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$.

Hence, by the main induction hypothesis, $F_1^\alpha[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}, Y_1^{\beta'_1}/H]$ is in $C_{\alpha[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$. By Lemma 3.2.21, $F_1^\alpha[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}][Y_1^{\beta'_1}/H] \equiv$
$F_1^\alpha[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}, Y_1^{\beta'_1}/H]$. Hence $F_1^\alpha[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}][Y_1^{\beta'_1}/H]$ is in $C_{\alpha[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$ by CR0.

Similarly if $G^{\beta_1 \vee \beta_2}[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}] = \mu_2 H$.

($\exists$ Intro) $F^{\alpha} = I(u, G^{\beta(y/u)})^{\exists y\beta}$:

We want to show that $I(u[\underline{x}/\underline{t}], M)$, where $M = G^{\beta(y/u)}[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]$, is in $C_{(\exists y\beta)[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$.

As usual, we may assume that $y \notin \{\underline{x}\} \cup fv(\underline{t})$.

By the induction hypothesis, M is in $C_{\beta[y/u][\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$ which is $C_{\beta[\underline{x}/\underline{t}][y/u[\underline{x}/\underline{t}]]}[\underline{P}/\underline{\mathcal{C}}]$ by Lemmas 2.13 and 4.2.10. If $I(u[\underline{x}/\underline{t}], M) \succ I(u[\underline{x}/\underline{t}], M')$, $M \succ M'$, and so M' is in $C_{\beta[\underline{x}/\underline{t}][y/u[\underline{x}/\underline{t}]]}[\underline{P}/\underline{\mathcal{C}}]$ by CR2.

Let z be an individual variable such that $z \notin fv(\beta[\underline{x}/\underline{t}]) - \{y\}$, $[\gamma]$ be a type such that $z \notin fv(\gamma)$, D be a CR of type $[\gamma]$, H be a term of type $[\beta[\underline{x}/\underline{t}][y/z][\underline{P}/\underline{T}] \supset \gamma]$ such that for each individual term v, $H[z/v]$ is in $C_{\beta[\underline{x}/\underline{t}][y/v]}[\underline{P}/\underline{\mathcal{C}}] \supset D$, and z is not free in the type superscript of any free term variable of H, and $Y^{\beta[\underline{x}/\underline{t}][y/z][\underline{P}/\underline{T}]}$ be a term variable which is not equivalent to any free term variable of H.

We want to show that $ST(z.Y^{\beta[\underline{x}/\underline{t}][y/z][\underline{P}/\underline{T}]}.H(Y^{\beta[\underline{x}/\underline{t}][y/z][\underline{P}/\underline{T}]}), I(u[\underline{x}/\underline{t}], M))$ is in D. Again, since D is closed under equivalence of terms, we may assume that $Y^{\beta[\underline{x}/\underline{t}][y/z][\underline{P}/\underline{T}]}.H$ has no free term variable equivalent to $Y^{\beta[\underline{x}/\underline{t}][y/z][\underline{P}/\underline{T}][z/u[\underline{x}/\underline{t}]]}$.

We first prove the following claim.

Claim 3. For any terms G_1 and H_1, if G_1 is in $C_{\beta[y/u][\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$, H_1 is in D, z does not occur free in the type superscript of any free term variable of $Y^{\beta[\underline{x}/\underline{t}][y/z][\underline{P}/\underline{T}]}.H_1$, $Y^{\beta[\underline{x}/\underline{t}][y/z][\underline{P}/\underline{T}]}.H_1$ has no free term variable equivalent to $Y^{\beta[\underline{x}/\underline{t}][y/z][\underline{P}/\underline{T}][z/u[\underline{x}/\underline{t}]]}$, and $H_1[z/u[\underline{x}/\underline{t}]][Y^{\beta[\underline{x}/\underline{t}][y/z][\underline{P}/\underline{T}][z/u[\underline{x}/\underline{t}]]}/G_1]$ is in D, then $ST(z.Y^{\beta[\underline{x}/\underline{t}][y/z][\underline{P}/\underline{T}]}.H_1, I(u[\underline{x}/\underline{t}], G_1))$ is in D.

Proof of Claim 3. Suppose G_1 is in $C_{\beta[y/u][\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$, H_1 is in D, z does not occur free in the type superscript of any free term variable of $Y^{\beta[\underline{x}/\underline{t}][y/z][\underline{P}/\underline{T}]}.H_1$, $Y^{\beta[\underline{x}/\underline{t}][y/z][\underline{P}/\underline{T}]}.H_1$ has no free term variable equivalent to $Y^{\beta[\underline{x}/\underline{t}][y/z][\underline{P}/\underline{T}][z/u[\underline{x}/\underline{t}]]}$, and $H_1[z/u[\underline{x}/\underline{t}]][Y^{\beta[\underline{x}/\underline{t}][y/z][\underline{P}/\underline{T}][z/u[\underline{x}/\underline{t}]]}/G_1]$ is in D. We will prove by induction on $N(G_1) + N(H_1)$. Since $ST(z.Y^{\beta[\underline{x}/\underline{t}][y/z][\underline{P}/\underline{T}]}.H_1, I(u[\underline{x}/\underline{t}], G_1))$ is neutral, to show that it is in D, it is enough to show that all its immediate reducts are in D. Every immediate reduct of $ST(z.Y^{\beta[\underline{x}/\underline{t}][y/z][\underline{P}/\underline{T}]}.H_1, I(u[\underline{x}/\underline{t}], G_1))$ is of one of the following forms.

(i) $H_1[z/u[\underline{x}/\underline{t}]][Y^{\beta[\underline{x}/\underline{t}][y/z][\underline{P}/\underline{T}][z/u[\underline{x}/\underline{t}]]}/G_1]$.

This is in D by the assumption.

(ii) $ST(z.Y^{\beta[\underline{x}/\underline{t}][y/z][\underline{P}/\underline{T}]}.H_1', I(u[\underline{x}/\underline{t}], G_1))$ where $H_1 \succ_1 H_1'$.

By CR2, H_1' is in D. By Note (b) on page 91, z does not occur free in the type superscript of any free term variable of $Y^{\beta[\underline{x}/\underline{t}][y/z][\underline{P}/\underline{T}]}.H_1'$ and $Y^{\beta[\underline{x}/\underline{t}][y/z][\underline{P}/\underline{T}]}.H_1'$ has no free term variable equivalent to $Y^{\beta[\underline{x}/\underline{t}][y/z][\underline{P}/\underline{T}][z/u[\underline{x}/\underline{t}]]}$.

By Lemma 3.3.4, $H_1[z/u[\underline{x}/\underline{t}]] \succ_1 H^*$ for some term H^* such that $H^* \equiv H_1'[z/u[\underline{x}/\underline{t}]]$ and $H_1[z/u[\underline{x}/\underline{t}]][Y^{\beta[\underline{x}/\underline{t}][y/z][\underline{P}/\underline{T}][z/u[\underline{x}/\underline{t}]]}/G_1] \succ_1 H^{**}$ for some term H^{**} such that $H^{**} \equiv H^*[Y^{\beta[\underline{x}/\underline{t}][y/z][\underline{P}/\underline{T}][z/u[\underline{x}/\underline{t}]]}/G_1]$, so, by Lemma 3.2.26, $H^{**} \equiv H_1'[z/u[\underline{x}/\underline{t}]][Y^{\beta[\underline{x}/\underline{t}][y/z][\underline{P}/\underline{T}][z/u[\underline{x}/\underline{t}]]}/G_1]$.

Since $H_1[z/u[\underline{x}/\underline{t}]][Y^{\beta[\underline{x}/\underline{t}][y/z][\underline{P}/\underline{T}][z/u[\underline{x}/\underline{t}]]}/G_1]$ is in D, by CR0 and CR2, $H_1'[z/u[\underline{x}/\underline{t}]][Y^{\beta[\underline{x}/\underline{t}][y/z][\underline{P}/\underline{T}][z/u[\underline{x}/\underline{t}]]}/G_1]$ is in D. Hence H_1' satisfies the conditions of the hypothesis. Thus $ST(z.Y^{\beta[\underline{x}/\underline{t}][y/z][\underline{P}/\underline{T}]}.H_1', I(u[\underline{x}/\underline{t}], G_1))$ is in D by the subsidiary induction hypothesis.

(iii) $ST(z.Y^{\beta[\underline{x}/\underline{t}][y/z][\underline{P}/\underline{T}]}.H_1, I(u[\underline{x}/\underline{t}], G_1'))$ where $G_1 \succ_1 G_1'$.

By CR2, G_1' is in $C_{\beta[y/u][\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$. By Lemma 3.3.5,
$H_1[z/u[\underline{x}/\underline{t}]][Y^{\beta[\underline{x}/\underline{t}][y/z][\underline{P}/\underline{T}][z/u[\underline{x}/\underline{t}]]}/G_1']$ is equivalent to some reduct of
$H_1[z/u[\underline{x}/\underline{t}]][Y^{\beta[\underline{x}/\underline{t}][y/z][\underline{P}/\underline{T}][z/u[\underline{x}/\underline{t}]]}/G_1]$, so it is in D by CR0 and CR2. Hence G_1' satisfies the conditions of the hypothesis. Thus
$ST(z.Y^{\beta[\underline{x}/\underline{t}][y/z][\underline{P}/\underline{T}]}.H_1, I(u[\underline{x}/\underline{t}], G_1'))$ is in D.

Hence we have Claim 3.

We have

(1) M is in $C_{\beta[y/u][\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$ by the induction hypothesis;

(2) $H(Y^{\beta[\underline{x}/\underline{t}][y/z][\underline{P}/\underline{T}]})$ is in D since H is in $C_{\beta[\underline{x}/\underline{t}][y/z]}[\underline{P}/\underline{\mathcal{C}}] \supset D$;

(3) z does not occur free in the type superscript of any free term variable of $Y^{\beta[\underline{x}/\underline{t}][y/z][\underline{P}/\underline{T}]}.H(Y^{\beta[\underline{x}/\underline{t}][y/z][\underline{P}/\underline{T}]})$ and $Y^{\beta[\underline{x}/\underline{t}][y/z][\underline{P}/\underline{T}]}.H(Y^{\beta[\underline{x}/\underline{t}][y/z][\underline{P}/\underline{T}]})$ has no free term variable equivalent to $Y^{\beta[\underline{x}/\underline{t}][y/z][\underline{P}/\underline{T}][z/u[\underline{x}/\underline{t}]]}$;

(4) $H(Y^{\beta[\underline{x}/\underline{t}][y/z][\underline{P}/\underline{T}]})[z/u[\underline{x}/\underline{t}]][Y^{\beta[\underline{x}/\underline{t}][y/z][\underline{P}/\underline{T}][z/u[\underline{x}/\underline{t}]]}/M]$ is in D, since it is $H[z/u[\underline{x}/\underline{t}]](M)$, where $H[z/u[\underline{x}/\underline{t}]]$ is in $C_{\beta[\underline{x}/\underline{t}][y/u[\underline{x}/\underline{t}]]}[\underline{P}/\underline{\mathcal{C}}] \supset D$ and M is in $C_{\beta[y/u][\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$ i.e. $C_{\beta[\underline{x}/\underline{t}][y/u[\underline{x}/\underline{t}]]}[\underline{P}/\underline{\mathcal{C}}]$.

Thus, by (1)-(4) and the claim,
$ST(z.Y^{\beta[\underline{x}/\underline{t}][y/z][\underline{P}/\underline{T}]}.H(Y^{\beta[\underline{x}/\underline{t}][y/z][\underline{P}/\underline{T}]}), I(u[\underline{x}/\underline{t}], M))$ is in D. Hence $I(u[\underline{x}/\underline{t}], M)$ is in $C_{\exists y\beta[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$ i.e. $C_{\alpha[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$.

($\exists_2$ Intro) $F^\alpha = I(U, G^{\beta(Q/U)})^{\exists_2 Q^q \beta}$:

We want to show that $I(U[\underline{P}/\underline{T}], M)$, where $M = G^{\beta(Q/U)}[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]$, is in $C_{\exists_2 Q\beta[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$.

As usual, we may assume that $Q \notin \{\underline{P}\} \cup FV(\underline{T})$.

By the induction hypothesis, M is in $C_{\beta[Q/U][\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$. By Lemmas 2.15, 4.2.10, and 4.2.11, $C_{\beta[Q/U][\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}] = C_{\beta[\underline{x}/\underline{t}][Q/U]}[\underline{P}/\underline{\mathcal{C}}] = C_{\beta[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}, Q/\mathcal{D}]$ for some collection of CRs $\mathcal{D}$ corresponding to $U[\underline{P}/\underline{T}]$. Hence M is in $C_{\beta[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}, Q/\mathcal{D}]$. If $I(U[\underline{P}/\underline{T}], M) \succ I(U[\underline{P}/\underline{T}], M')$, $M \succ M'$, so M' is in $C_{\beta[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}, Q/\mathcal{D}]$ by CR2.

Let R be a q-ary predicate variable such that $R \notin FV(\beta[\underline{x}/\underline{t}][\underline{P}/\underline{T}]) - \{Q\}$, $[\gamma]$ be a type such that $R \notin FV(\gamma)$, D be a CR of type $[\gamma]$, H be a term of type $[\beta[\underline{x}/\underline{t}][Q/R, \underline{P}/\underline{T}] \supset \gamma]$ such that for each abstraction term $V = \lambda y_1, \ldots, y_q\sigma$ and all collections of CRs $\mathcal{E}$ corresponding to V, $H[R/V]$ is in $C_{\beta[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}, Q/\mathcal{E}] \supset D$ and R is not free in the type superscript of any free term variable of H, and

$Y^{\beta[\underline{x}/\underline{t}][Q/R,\underline{P}/\underline{T}]}$ be a term variable which is not equivalent to any free term variable of H.

We want to show that $ST(R.Y^{\beta[\underline{x}/\underline{t}][Q/R,\underline{P}/\underline{T}]}.H(Y^{\beta[\underline{x}/\underline{t}][Q/R,\underline{P}/\underline{T}]}), I(U[\underline{P}/\underline{T}], M))$ is in D. Again, since D is closed under equivalence of terms, we may assume that $Y^{\beta[\underline{x}/\underline{t}][Q/R,\underline{P}/\underline{T}]}.H$ has no free term variable equivalent to $Y^{\beta[\underline{x}/\underline{t}][Q/R,\underline{P}/\underline{T}][R/U[\underline{P}/\underline{T}]]}$.

We first need the following claim which can be proved in the same way as Claim 3.

Claim 4. For any terms G_1 and H_1, if G_1 is in $C_{\beta[Q/U][\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$, H_1 is in D, R does not occur free in the type superscript of any free term variable of $Y^{\beta[\underline{x}/\underline{t}][Q/R,\underline{P}/\underline{T}]}.H_1$, $Y^{\beta[\underline{x}/\underline{t}][Q/R,\underline{P}/\underline{T}]}.H_1$ has no free term variable equivalent to $Y^{\beta[\underline{x}/\underline{t}][Q/R,\underline{P}/\underline{T}][R/U[\underline{P}/\underline{T}]]}$, and $H_1[R/U[\underline{P}/\underline{T}]][Y^{\beta[\underline{x}/\underline{t}][Q/R,\underline{P}/\underline{T}][R/U[\underline{P}/\underline{T}]]}/G_1]$ is in D, then $ST(R.Y^{\beta[\underline{x}/\underline{t}][Q/R,\underline{P}/\underline{T}]}.H_1, I(U[\underline{P}/\underline{T}], G_1))$ is in D.

We have

(1) M is in $C_{\beta[Q/U][\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$ by the induction hypothesis;

(2) $H(Y^{\beta[\underline{x}/\underline{t}][Q/R,\underline{P}/\underline{T}]})$ is in D since H is in $C_{\beta[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}, Q/\mathcal{E}_R] \supset D$, where $\mathcal{E}_R$ is a collection of CRs corresponding to $\lambda y_1, \ldots, y_q R(y_1, \ldots, y_q)$ and $Y^{\beta[\underline{x}/\underline{t}][Q/R,\underline{P}/\underline{T}]}$ is in $C_{\beta[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}, Q/\mathcal{E}_R]$;

(3) R does not occur free in the type superscript of any free term variable of $Y^{\beta[\underline{x}/\underline{t}][Q/R,\underline{P}/\underline{T}]}.H(Y^{\beta[\underline{x}/\underline{t}][Q/R,\underline{P}/\underline{T}]})$ and $Y^{\beta[\underline{x}/\underline{t}][Q/R,\underline{P}/\underline{T}]}.H(Y^{\beta[\underline{x}/\underline{t}][Q/R,\underline{P}/\underline{T}]})$ has no free term variable equivalent to $Y^{\beta[\underline{x}/\underline{t}][Q/R,\underline{P}/\underline{T}][R/U[\underline{P}/\underline{T}]]}$;

(4) $H(Y^{\beta[\underline{x}/\underline{t}][Q/R,\underline{P}/\underline{T}]})[R/U[\underline{P}/\underline{T}]][Y^{\beta[\underline{x}/\underline{t}][Q/R,\underline{P}/\underline{T}][R/U[\underline{P}/\underline{T}]]}/M]$ is in D, since it is $H[R/U[\underline{P}/\underline{T}]](M)$, where $H[R/U[\underline{P}/\underline{T}]]$ is in $C_{\beta[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}, Q/\mathcal{E}] \supset D$ and, by Lemma 4.2.11, M is in $C_{\beta[Q/U][\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$ i.e. $C_{\beta[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}, Q/\mathcal{E}]$, where $U = \lambda z_1, \ldots, z_q\sigma$, $\underline{z} = z_1, \ldots, z_q$, and $\mathcal{E} =$
$\{C_{\sigma[\underline{z}/\underline{v}]}[\underline{P}/\underline{\mathcal{C}}] \mid \underline{v} = v_1, \ldots, v_q$ are individual terms.$\}$.

Thus, by (1)-(4) and the claim,

$ST(R.Y^{\beta[\underline{x}/\underline{t}][Q/R,\underline{P}/\underline{T}]}.H(Y^{\beta[\underline{x}/\underline{t}][Q/R,\underline{P}/\underline{T}]}), I(U[\underline{P}/\underline{T}], M))$ is in D, and so $I(U[\underline{P}/\underline{T}], M)$ is in $C_{\exists_2 Q\beta[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$ i.e. $C_{\alpha[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$.

($\exists$ Elim) $F^\alpha = ST(y.Y^\beta.G^\alpha, H^{\exists y\beta})$:

As usual, we may assume that $y \notin \{\underline{x}\} \cup fv(\underline{t}) \cup fv(\underline{K})$, G^α has no free term variable Y^σ such that $\sigma \not\equiv \beta$, and $Y^{\beta'}$ is not equivalent to any free term variable in $\underline{X}$ or $\underline{K}$.

We will show that $M = ST(y.Y^{\beta'}.G^\alpha[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}], H^{\exists y\beta}[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}])$ is in $C_{\alpha[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$ by induction on $N(G^\alpha[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]) +$ $N(H^{\exists y\beta}[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}])$. Since M is neutral, it is enough to show that all its immediate reducts are in $C_{\alpha[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$.

If an immediate reduct of M is obtained by reducing $G^\alpha[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]$ or $H^{\exists y\beta}[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]$, then it is in $C_{\alpha[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$ by the subsidiary induction hypothesis.

The remaining case is when $H^{\exists y\beta}[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}] = I(u, H_1)$ and the immediate reduct is $G^\alpha[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}][y/u][Y^{\beta'[y/u]}/H_1]$.

By the main induction hypothesis, $I(u, H_1)$ is in $C_{(\exists y\beta)[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$. Hence H_1 is in $C_{\beta[\underline{x}/\underline{t}][y/u]}[\underline{P}/\underline{\mathcal{C}}]$ i.e. $C_{\beta[\underline{x}/\underline{t},y/u]}[\underline{P}/\underline{\mathcal{C}}]$ by Lemmas 2.13 and 4.2.10.

Since $y \notin \{\underline{x}\} \cup fv(\underline{K})$, $y \notin fv(\delta_i)$ and so $C_{\delta_i[\underline{x}/\underline{t},y/u]}[\underline{P}/\underline{\mathcal{C}}] = C_{\delta_i[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$ for all $1 \leq i \leq l$.

Hence, by the main induction hypothesis, $G^\alpha[\underline{x}/\underline{t}, y/u][\underline{P}/\underline{T}][\underline{X}/\underline{K}, Y^{\beta'[y/u]}/H_1]$ is in $C_{\alpha[\underline{x}/\underline{t},y/u]}[\underline{P}/\underline{\mathcal{C}}]$ which is $C_{\alpha[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$ since $y \notin fv(\alpha)$.

By Lemmas 3.2.19, 3.2.21, 3.2.22, 3.2.23, 3.2.25, and 3.2.26,
$G^\alpha[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}][y/u][Y^{\beta'[y/u]}/H_1] \equiv G^\alpha[\underline{x}/\underline{t}, y/u][\underline{P}/\underline{T}][\underline{X}/\underline{K}, Y^{\beta'[y/u]}/H_1]$.
Hence, by CR0, $G^\alpha[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}][y/u][Y^{\beta'[y/u]}/H_1]$ is in $C_{\alpha[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$.

($\exists_2$ Elim) $F^\alpha = ST(Q.Y^\beta.G^\alpha, H^{\exists_2 Q\beta})$:

As usual, we may assume that $Q \notin \{\underline{P}\} \cup FV(\underline{T}) \cup FV(\underline{K})$, G^α has no free term variable Y^σ such that $\sigma \not\equiv \beta$, and $Y^{\beta'}$ is not equivalent to any free term variable in $\underline{X}$ or $\underline{K}$.

We will show that $M =$ $ST(Q.Y^{\beta'}.G^\alpha[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}], H^{\exists_2 Q\beta}[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}])$ is in $C_{\alpha[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$ by induction on $N(G^\alpha[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]) + N(H^{\exists_2 Q\beta}[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}])$. Since M is neutral, it is enough to show that all its immediate reducts are in $C_{\alpha[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$. If an immediate reduct of M is obtained by reducing $G^\alpha[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]$ or $H^{\exists_2 Q\beta}[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]$, then it is in $C_{\alpha[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$ by the subsidiary induction hypothesis.

The remaining case is when $H^{\exists_2 Q\beta}[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}] = I(U, H_1)$ and the immediate reduct is $G^\alpha[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}][Q/U][Y^{\beta'[Q/U]}/H_1]$.

By the main induction hypothesis, $I(U, H_1)$ is in $C_{(\exists_2 Q\beta)[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$. Hence H_1 is in $C_{\beta[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}, Q/\mathcal{D}]$ for some collection of CRs $\mathcal{D}$ corresponding to U.

Since $Q \notin \{\underline{P}\} \cup FV(\underline{K})$, $Q \notin FV(\delta_i[\underline{x}/\underline{t}])$ and so $C_{\delta_i[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}, Q/\mathcal{D}] =$ $C_{\delta_i[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$ for all $1 \leq i \leq l$. Hence, by the main induction hypothesis, $G^\alpha[\underline{x}/\underline{t}][\underline{P}/\underline{T}, Q/U][\underline{X}/\underline{K}, Y^{\beta'[Q/U]}/H_1]$ is in $C_{\alpha[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}, Q/\mathcal{D}]$ which is $C_{\alpha[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$ since $Q \notin FV(\alpha)$.

By Lemmas 3.2.21, 3.2.22, 3.2.23, and 3.2.26, $G^\alpha[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}][Q/U][Y^{\beta'[Q/U]}/H_1] \equiv G^\alpha[\underline{x}/\underline{t}][\underline{P}/\underline{T}, Q/U][\underline{X}/\underline{K}, Y^{\beta'[Q/U]}/H_1]$. Hence, by CR0, $G^\alpha[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}][Q/U][Y^{\beta'[Q/U]}/H_1]$ is in $C_{\alpha[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$.

($\bot$ Elim) $F^\alpha = G^\bot(\alpha)$:

By the main induction hypothesis, $G^\bot[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]$ is in $C_\bot[\underline{P}/\underline{\mathcal{C}}]$. We will show that $M = G^\bot[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}](\alpha')$ is in $C_{\alpha[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$ by induction on

$N(G^{\perp}[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}])$.

$N(G^{\perp}[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]) = 0$:

Then $G^{\perp}[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]$ is normal and so is M. Since M is neutral, by CR4, M is in $C_{\alpha[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$.

$N(G^{\perp}[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]) > 0$:

Since M is neutral, we will show that all its immediate reducts are in $C_{\alpha[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$.

Suppose G^* is an immediate reduct of M. Then $G^* = G'(\alpha')$ for some term G' such that $G^{\perp}[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}] \succ_1 G'$. Hence G^* is in $C_{\alpha[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$ by the subsidiary induction hypothesis. □

Theorem 4.2.13. *Each Curry-Howard term is strongly normalizable.*

Proof. Let F^{α} be a Curry-Howard term, $\underline{x} = x_1, \ldots, x_n$ be distinct individual variables, $\underline{P} = P_1^{m_1}, \ldots, P_k^{m_k}$ be distinct predicate variables, $\underline{\mathcal{C}} = \mathcal{C}_1, \ldots, \mathcal{C}_k$ be collections of CRs where each $\mathcal{C}_i$ is corresponding to $\lambda z_1^i, \ldots, z_{m_i}^i P_i(z_1^i, \ldots, z_{m_i}^i)$, $\underline{X} = X_1^{\delta_1}, \ldots, X_l^{\delta_l}$ be inequivalent term variables such that every free term variable of F^{α} is equivalent to some $X_i^{\delta_i}$ in $\underline{X}$.

By the above lemma, $F^{\alpha}[\underline{x}/\underline{x}][\underline{P}/\underline{P}][\underline{X}/\underline{X}]$ is in $C_{\alpha[\underline{x}/\underline{x}]}[\underline{P}/\underline{\mathcal{C}}]$ i.e. F^{α} is in $C_{\alpha}[\underline{P}/\underline{\mathcal{C}}]$. Hence F^{α} is strongly normalizable by CR1. □

We have shown that every Curry-Howard term for second-order logic is strongly normalizable. This means that we can then take proofs in second-order logic and directly produce programs from them.

CHAPTER V

TEMPLATES

In carrying out mathematical proofs the same patterns frequently recur. What we want to do is to characterize what a pattern, or template, is and then add new rules to the formal system NJ_2. We then define new Curry-Howard terms formation rules corresponding to the new rules as well as new reduction rules corresponding to reductions of proofs. After these additions, the Curry-Howard terms will still satisfy all the basic properties including the *strong normalization theorem*. Therefore we can use such patterns in the formal system properly.

In this chapter, we introduce two kinds of *templates* namely *induction templates* and *abbreviation templates* which can be used for different purposes. The details will be presented in the following sections.

5.1 Induction templates

We often use *induction* in ordinary mathematical proofs. Now, we will add induction to the formal system NJ_2. We want the new induction to be more versatile so that it does not have to be used only on natural numbers but on predicates that are defined inductively from finite numbers of basic constants. We need to add axioms and rules to NJ_2 as follows.

Let

ϕ be a unary predicate symbol;

$a_1, \ldots, a_n$ be constant symbols;

$f_1, \ldots, f_m$ be function symbols with arities $p_1 + q_1, \ldots, p_m + q_m$, respectively, where $p_i \geq 1$ and $q_i \geq 0$ for all $1 \leq i \leq m$;

$\psi_{r_1}, \ldots, \psi_{r_l}$ be unary predicate symbols, where $r_1, \ldots, r_l$ is the sublist of $1, \ldots, m$ consisting of all i's such that $q_i \neq 0$.

Note.

a. The sequence $\psi_{r_1}, \ldots, \psi_{r_l}$ may be empty.

b. Some parentheses will be omitted by using association to the left.

Axioms:

$\phi(a_1), \ldots, \phi(a_n)$,

$$\forall x_1 \ldots \forall x_{p_i} \forall y_1 \ldots \forall y_{q_i} (\phi(x_1) \wedge \ldots \wedge \phi(x_{p_i}) \wedge \psi_i(y_1) \wedge \ldots \wedge \psi_i(y_{q_i}) \supset \phi(f_i(x_1, \ldots, x_{p_i}, y_1, \ldots, y_{q_i}))), \text{ where } q_i \neq 0,\ 1 \leq i \leq m,$$

$$\forall x_1 \ldots \forall x_{p_i} (\phi(x_1) \wedge \ldots \wedge \phi(x_{p_i}) \supset \phi(f_i(x_1, \ldots, x_{p_i}))), \text{ where } q_i = 0,\ 1 \leq i \leq m.$$

Rule:

$$\frac{\begin{matrix} \vdots & & \vdots & \vdots & & \vdots \\ \alpha(x/a_1) & \ldots & \alpha(x/a_n) & \alpha_1 & \ldots & \alpha_m \end{matrix}}{\forall x(\phi(x) \supset \alpha)} \text{ (Induction)}$$

where α is any formula and for all $1 \leq i \leq m$, α_i denotes

$$\forall x_1 \ldots \forall x_{p_i} \forall y_1 \ldots \forall y_{q_i} (\phi(x_1) \wedge \ldots \wedge \phi(x_{p_i}) \wedge \psi_i(y_1) \wedge \ldots \wedge \psi_i(y_{q_i}) \supset (\alpha(x/x_1) \wedge \ldots \wedge \alpha(x/x_{p_i}) \supset \alpha(x/f_i(x_1, \ldots, x_{p_i}, y_1, \ldots, y_{q_i})))),$$

where $q_i \neq 0$ and $fv(\alpha) \cap \{x_1, \ldots, x_{p_i}, y_1, \ldots, y_{q_i}\} = \varnothing$, or

$$\forall x_1 \ldots \forall x_{p_i} (\phi(x_1) \wedge \ldots \wedge \phi(x_{p_i}) \supset (\alpha(x/x_1) \wedge \ldots \wedge \alpha(x/x_{p_i}) \supset \alpha(x/f_i(x_1, \ldots, x_{p_i})))),$$

where $q_i = 0$ and $fv(\alpha) \cap \{x_1, \ldots, x_{p_i}\} = \varnothing$.

Notation. Throughout this section, when a formula α is given, we will use the notations α_i, $1 \leq i \leq m$, as above.

Examples.

I. Let $L = \langle S, 0\rangle$ be a language for natural numbers.

We extend L to $L' = \langle N, S, 0\rangle$, where N is a unary predicate symbol.

We introduce the following axioms.

(i) $N(0)$;

(ii) $\forall x(N(x) \supset N(S(x)))$.

We have the following induction rule.

$$\frac{\begin{array}{cc}\vdots & \vdots \\ \alpha(x/0) & \forall x(N(x) \supset (\alpha \supset \alpha(x/S(x))))\end{array}}{\forall x(N(x) \supset \alpha)}$$

Next, we extend L' to $L'' = \langle N, S, List, con, 0, [\ ]\rangle$, where

$List$ is a unary predicate symbol;

con is a binary function symbol; and

$[\ ]$ is a constant symbol.

We add the following axioms.

(i) $List([\ ])$;

(ii) $\forall x\forall y(List(x) \wedge N(y) \supset List(con(x, y)))$.

We have the list induction rule as follows.

$$\frac{\begin{array}{cc}\vdots & \vdots \\ \alpha(z/[\ ]) & \forall x\forall y(List(x) \wedge N(y) \supset (\alpha(z/x) \supset \alpha(z/con(x, y))))\end{array}}{\forall z(List(z) \supset \alpha)}$$

II. We consider a finitely generated algebraic system. For definiteness, we consider a group which is finitely generated by $a_1 \ldots, a_n$.

Let $\langle G, inv, *, a_1, \ldots, a_n\rangle$ be a language, where

G is a unary predicate symbol;

inv is a unary function symbol;

$*$ is a binary function symbol; and

$a_1, \ldots, a_n$ are constant symbols.

We include the following axioms.

(i) $G(a_1), \ldots, G(a_n)$;

(ii) $\forall x(G(x) \supset G(inv(x)))$;

(iii) $\forall x \forall y(G(x) \wedge G(y) \supset G(*(x, y)))$.

Then we have the following rule.

$$\dfrac{\begin{array}{cccc} \vdots & & \vdots & \vdots \\ \alpha(z/a_1) & \ldots & \alpha(z/a_n) & \forall x(G(x) \supset (\alpha(z/x) \supset \alpha(z/inv(x)))) \\ & & & \forall x \forall y(G(x) \wedge G(y) \supset (\alpha(z/x) \wedge \alpha(z/y) \supset \alpha(z/*(x,y)))) \end{array}}{\forall z(G(z) \supset \alpha)}$$

In the following, we will define new Curry-Howard terms, called *constant terms*, which correspond to some formulae that are obtained from the axioms.

For each axiom α, let A^{α} be a *constant term* and then introduce reduction rules and new constant terms as follows.

For each $1 \leq i \leq m$ such that $q_i \neq 0$,

$A^{\forall x_1 \ldots \forall x_{p_i} \forall y_1 \ldots \forall y_{q_i}(\phi(x_1) \wedge \ldots \wedge \phi(x_{p_i}) \wedge \psi_i(y_1) \wedge \ldots \wedge \psi_i(y_{q_i}) \supset \phi(f_i(x_1, \ldots, x_{p_i}, y_1, \ldots, y_{q_i})))}(t_1)$

$\succ A^{\forall x_2 \ldots \forall x_{p_i} \forall y_1 \ldots \forall y_{q_i}(\phi(t_1) \wedge \phi(x_2) \wedge \ldots \wedge \phi(x_{p_i}) \wedge \psi_i(y_1) \wedge \ldots \wedge \psi_i(y_{q_i}) \supset \phi(f_i(t_1, x_2, \ldots, x_{p_i}, y_1, \ldots, y_{q_i})))}$;

$\vdots$

$A^{\forall y_{q_i}(\phi(t_1) \wedge \ldots \wedge \phi(t_{p_i}) \wedge \psi_i(u_1) \wedge \ldots \wedge \psi_i(u_{q_i - 1}) \wedge \psi_i(y_{q_i}) \supset \phi(f_i(t_1, \ldots, t_{p_i}, u_1, \ldots, u_{q_i - 1}, y_{q_i})))}(u_{q_i})$

$\succ A^{\phi(t_1) \wedge \ldots \wedge \phi(t_{p_i}) \wedge \psi_i(u_1) \wedge \ldots \wedge \psi_i(u_{q_i}) \supset \phi(f_i(t_1, \ldots, t_{p_i}, u_1, \ldots, u_{q_i}))}$,

where $t_1, \ldots, t_{p_i}, u_1, \ldots, u_{q_i}$ are closed individual terms.

For each $1 \leq i \leq m$ such that $q_i = 0$,

$A^{\forall x_1 \ldots \forall x_{p_i}(\phi(x_1) \wedge \ldots \wedge \phi(x_{p_i}) \supset \phi(f_i(x_1, \ldots, x_{p_i})))}(t_1)$

$\succ A^{\forall x_2 \ldots \forall x_{p_i}(\phi(t_1) \wedge \phi(x_2) \wedge \ldots \wedge \phi(x_{p_i}) \supset \phi(f_i(t_1, x_2, \ldots, x_{p_i})))}$;

$\vdots$

$A^{\forall x_{p_i}(\phi(t_1) \wedge \ldots \wedge \phi(t_{p_i - 1}) \wedge \phi(x_{p_i}) \supset \phi(f_i(t_1, \ldots, t_{p_i - 1}, x_{p_i})))}(t_{p_i}) \succ A^{\phi(t_1) \wedge \ldots \wedge \phi(t_{p_i}) \supset \phi(f_i(t_1, \ldots, t_{p_i}))}$,

where $t_1, \ldots, t_{p_i}$ are closed individual terms.

Note. For every constant term A^α, α is a closed formula.

We define a term formation rule corresponding to the induction rule as follows.

(Induction) If $F_1^{\alpha(x/a_1)}, \ldots, F_n^{\alpha(x/a_n)}, G_1^{\alpha_1}, \ldots, G_m^{\alpha_m}$ are terms of types $[\alpha(x/a_1)]$, $\ldots, [\alpha(x/a_n)], [\alpha_1], \ldots, [\alpha_m]$, respectively, then $\rho([F_1, \ldots, F_n], [G_1, \ldots, G_m])$ is a term of type $[\forall x(\phi(x) \supset \alpha)]$.

We introduce reduction rules as follows.

For each $1 \leq j \leq n$,

$\rho([F_1, \ldots, F_n], [G_1, \ldots, G_m])(a_j)(H^{\phi(a_j)}) \succ F_j^{\alpha(x/a_j)}$,

where $H^{\phi(a_j)}$ contains no free term variable.

For each $1 \leq i \leq m$ such that $q_i \neq 0$,

$\rho([F_1, \ldots, F_n], [G_1, \ldots, G_m])(f_i(t_1, \ldots, t_{p_i}, u_1, \ldots, u_{q_i}))$

$(H((J_1^{\phi(t_1)}, \ldots, J_{p_i}^{\phi(t_{p_i})}, K_1^{\psi(u_1)}, \ldots, K_{q_i}^{\psi(u_{q_i})})))$

$\succ G_i(t_1) \ldots (t_{p_i})(u_1) \ldots (u_{q_i})((J_1, \ldots, J_{p_i}, K_1, \ldots, K_{q_i}))$

$((\rho([F_1, \ldots, F_n], [G_1, \ldots, G_m])(t_1)(J_1), \ldots, \rho([F_1, \ldots, F_n], [G_1, \ldots, G_m])(t_{p_i})(J_{p_i})))$,

where H is a constant term of type $[\phi(t_1) \wedge \ldots \wedge \phi(t_{p_i}) \wedge \psi_i(u_1) \wedge \ldots \wedge \psi_i(u_{q_i}) \supset \phi(f_i(t_1, \ldots, t_{p_i}, u_1, \ldots, u_{q_i}))]$ and $J_1, \ldots, J_{p_i}, K_1, \ldots, K_{q_i}$ contain no free term variable.

For each $1 \leq i \leq m$ such that $q_i = 0$,

$$\rho([F_1, \ldots, F_n], [G_1, \ldots, G_m])(f_i(t_1, \ldots, t_{p_i}))(H((J_1^{\phi(t_1)}, \ldots, J_{p_i}^{\phi(t_{p_i})})))$$
$$\succ G_i(t_1) \ldots (t_{p_i})((J_1, \ldots, J_{p_i}))$$
$$((\rho([F_1, \ldots, F_n], [G_1, \ldots, G_m])(t_1)(J_1), \ldots, \rho([F_1, \ldots, F_n], [G_1, \ldots, G_m])(t_{p_i})(J_{p_i}))),$$

where H is a constant term of type $[\phi(t_1) \wedge \ldots \wedge \phi(t_{p_i}) \supset \phi(f_i(t_1, \ldots, t_{p_i}))]$ and $J_1, \ldots, J_{p_i}$ contain no free term variable.

Now, we have new forms of Curry-Howard terms as well as new reduction rules. In the following, we will add the new cases to some definitions in Chapter III. All lemmas in Chapter III, which can be proved in the same way as the old ones by straightforwardly following the definitions for the new cases, still hold after these additions.

Definitions 3.2.2 and 3.2.3.

$fv(A^\alpha) = fv(\alpha)$;

$fv(\rho([F_1, \ldots, F_n], [G_1, \ldots, G_m])) = \bigcup_{j=1}^{n} fv(F_j) \cup \bigcup_{i=1}^{m} fv(G_i)$.

Similarly for $FV(A^\alpha)$ and $FV(\rho([F_1, \ldots, F_n], [G_1, \ldots, G_m]))$.

Note. $fv(A^\alpha) = FV(A^\alpha) = \varnothing$.

Definition 3.2.4.

*Constant terms A^α and $A^{\alpha'}$ are **equivalent**, denoted by $A^\alpha \equiv A^{\alpha'}$, if $\alpha \equiv \alpha'$.*

Definition 3.2.6.

*$\underline{x}$ is **replaceable** by $\underline{t}$ in A^α;*

*$\underline{x}$ is **replaceable** by $\underline{t}$ in $\rho([F_1, \ldots, F_n], [G_1, \ldots, G_m])$ if $\underline{x}$ is replaceable by $\underline{t}$ in F_j for all $1 \leq j \leq n$ and G_i for all $1 \leq i \leq m$.*

Similarly for replaceability of $\underline{P}$ by $\underline{T}$.

Definitions 3.2.10 and 3.2.11.

$A^\alpha[\underline{x}/\underline{t}] = A^{\alpha[\underline{x}/\underline{t}]}$;

$\rho([F_1, \ldots, F_n], [G_1, \ldots, G_m])[\underline{x}/\underline{t}] = \rho([F_1[\underline{x}/\underline{t}], \ldots, F_n[\underline{x}/\underline{t}]], [G_1[\underline{x}/\underline{t}], \ldots, G_m[\underline{x}/\underline{t}]])$.

Similarly for $A^\alpha[\underline{P}/\underline{T}]$ *and* $\rho([F_1, \ldots, F_n], [G_1, \ldots, G_m])[\underline{P}/\underline{T}]$.

Note. $A^\alpha[\underline{x}/\underline{t}] = A^\alpha[\underline{P}/\underline{T}] = A^\alpha$.

Definition 3.2.12.

$A^\alpha[\underline{X}/\underline{K}] = A^\alpha$;

$\rho([F_1, \ldots, F_n], [G_1, \ldots, G_m])[\underline{X}/\underline{K}]$

$= \rho([F_1[\underline{X}/\underline{K}], \ldots, F_n[\underline{X}/\underline{K}]], [G_1[\underline{X}/\underline{K}], \ldots, G_m[\underline{X}/\underline{K}]])$.

Next, we will show that the new Curry-Howard terms satisfy the *strong normalization theorem.* All lemmas in Chapter IV still hold after the additions. We will give only the proof for the additional cases of Lemma 4.2.12 and omit the others of which proofs are similar to the old ones. First, we will extend the definition of *neutral terms* in Chapter IV as follows.

Definition 4.1.2. *A constant term* A^α *is* ***neutral*** *if* α *is* $\phi(t)$ *for some (closed) individual term* t*, otherwise* A^α *is not neutral.*

A term of the form $\rho([F_1, \ldots, F_n], [G_1, \ldots, G_m])$ *is not neutral.*

Lemma 4.2.12.

$F^\alpha[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]$ *is in* $C_{\alpha[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$.

Proof. (Term constant) $F^\alpha = A^\alpha$:

We have $A^\alpha[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}] = A^\alpha$ since α is a closed formula.

We will prove by induction on the number s of all occurrences of the symbol $\forall$ in α.

$s = 0$:

Case 1. $\alpha = \phi(u)$ for some (closed) individual term u:

Then A^α is neutral and normal, and so it is in $C_{\alpha[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$ by CR4.

Case 2. $\alpha = \phi(u_1)\wedge\ldots\wedge\phi(u_{p_i})\wedge\psi_i(v_1)\wedge\ldots\wedge\psi_i(v_{q_i}) \supset \phi(f_i(u_1,\ldots,u_{p_i},v_1,\ldots,v_{q_i}))$ for some $1 \leq i \leq m$ and some (closed) individual terms $u_1,\ldots,u_{p_i},v_1,\ldots,v_{q_i}$:

Let G be in $C_{\phi(u_1)\wedge\ldots\wedge\phi(u_{p_i})\wedge\psi_i(v_1)\wedge\ldots\wedge\psi_i(v_{q_i})}[\underline{P}/\underline{\mathcal{C}}]$. We will prove that $A^\alpha(G)$ is in $C_{\phi(f_i(u_1,\ldots,u_{p_i},v_1,\ldots,v_{q_i}))}[\underline{P}/\underline{\mathcal{C}}]$ by induction on $N(G)$. Since $A^\alpha(G)$ is neutral, we will show that all its immediate reducts are in $C_{\phi(f_i(u_1,\ldots,u_{p_i},v_1,\ldots,v_{q_i}))}[\underline{P}/\underline{\mathcal{C}}]$. It follows by the subsidiary induction hypothesis since every immediate reduct is obtained by reducing G.

Similarly if $\alpha = \phi(u_1)\wedge\ldots\wedge\phi(u_{p_i}) \supset \phi(f_i(u_1,\ldots,u_{p_i}))$ for some $1 \leq i \leq m$ and some (closed) individual terms $u_1,\ldots,u_{p_i}$.

$s > 0$: Then α is of the form $\forall y\gamma$. Note that $fv(\gamma) \subseteq \{y\}$ since α is closed. As usual, we may assume that $y \notin \{\underline{x}\}\cup fv(\underline{t})$. Let u be an individual term. We want to show that $A^\alpha(u)$ is in $C_{\gamma[\underline{x}/\underline{t}][y/u]}[\underline{P}/\underline{\mathcal{C}}]$. Since the only immediate reduct of $A^\alpha(u)$ is $A^{\gamma[y/u]}$ which is in $C_{\gamma[y/u]}[\underline{P}/\underline{\mathcal{C}}]$ i.e. $C_{\gamma[\underline{x}/\underline{t}][y/u]}[\underline{P}/\underline{\mathcal{C}}]$ by the subsidiary induction hypothesis, $A^\alpha(u)$ is in $C_{\gamma[\underline{x}/\underline{t}][y/u]}[\underline{P}/\underline{\mathcal{C}}]$ by CR3. Thus A^α is in $C_{\forall y\gamma[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$ i.e. $C_{\alpha[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$.

(Induction) $F^\alpha = \rho([F_1,\ldots,F_n],[G_1,\ldots,G_m])^{\forall y(\phi(y)\supset\beta)}$:

As usual, we may assume that y is not in $\{\underline{x}\}\cup fv(\underline{t})$.

Then $F^\alpha[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}] = \rho([F_1[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}],\ldots,F_n[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]],$ $[G_1[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}],\ldots,G_m[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]])^{\forall y(\phi(y)\supset\beta[\underline{x}/\underline{t}][\underline{P}/\underline{T}])}$.

Let u be an individual term. We have to show that $F^\alpha[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}](u)$ is in $C_{\phi(u)\supset\beta[\underline{x}/\underline{t}][y/u]}[\underline{P}/\underline{\mathcal{C}}]$.

Let H be in $C_{\phi(u)}[\underline{P}/\underline{\mathcal{C}}]$. We will prove that $F^\alpha[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}](u)(H)$ is in $C_{\beta[\underline{x}/\underline{t}][y/u]}[\underline{P}/\underline{\mathcal{C}}]$ by induction on $\sum_{j=1}^{n} N(F_j[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]) +$ $\sum_{i=1}^{m} N(G_i[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]) + N(H)$. We will show that all its immediate reducts are in $C_{\beta[\underline{x}/\underline{t}][y/u]}[\underline{P}/\underline{\mathcal{C}}]$. It follows by the subsidiary induction hypothesis if an immediate reduct is obtained by reducing $F_j[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]$ for some $1 \leq j \leq n$, $G_i[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]$ for some $1 \leq i \leq m$, or H. For the remaining cases, which are obtained when $F^\alpha[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}](u)(H)$ is the redex being reduced, we will prove by induction on u.

$u = a_j$ for some $1 \leq j \leq n$:

Then the immediate reduct is $F_j[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]$ which is in $C_{\beta[\underline{x}/\underline{t}][y/u]}[\underline{P}/\underline{\mathcal{C}}]$ by the main induction hypothesis.

$u = f_i(u_1, \ldots, u_{p_i}, v_1, \ldots, v_{q_i})$ for some $1 \leq i \leq m$ and some individual terms $u_1, \ldots, u_{p_i}, v_1, \ldots, v_{q_i}$ and $H = H_*((J_1^{\phi(u_1)}, \ldots, J_{p_i}^{\phi(u_{p_i})}, K_1^{\psi_i(v_1)}, \ldots, K_{q_i}^{\psi_i(v_{q_i})}))$, where H_* is a constant term of type $[\phi(u_1) \wedge \ldots \wedge \phi(u_{p_i}) \wedge \psi_i(v_1) \wedge \ldots \wedge \psi_i(v_{q_i}) \supset \phi(f_i(u_1, \ldots, u_{p_i}, v_1, \ldots, v_{q_i}))]$:

Then the immediate reduct is $M =$

$G_i[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}](u_1)\ldots(u_{p_i})(v_1)\ldots(v_{q_i})((J_1, \ldots, J_{p_i}, K_1, \ldots, K_{q_i}))$
$((\rho([F_1[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}], \ldots, F_n[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]], [G_1[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}], \ldots,$
$G_m[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]])(u_1)(J_1), \ldots, \rho([F_1[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}], \ldots, F_n[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]],$
$[G_1[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}], \ldots, G_m[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]])(u_{p_i})(J_{p_i})))$.

It can be easily checked by induction on h that if $H_1, \ldots, H_h$ are terms in CR $C_1, \ldots, C_h$, respectively, then $(H_1, \ldots, H_h)$ is in $C_1 \wedge \ldots \wedge C_h$. Since for all $1 \leq j \leq p_i$, J_j is a subterm of H which is in $C_{\phi(u)}[\underline{P}/\underline{\mathcal{C}}]$ i.e. $SN_{\phi(u)}$, J_j is in $SN_{\phi(u_j)}$ i.e. $C_{\phi(u_j)}[\underline{P}/\underline{\mathcal{C}}]$ for all $1 \leq j \leq p_i$ and similarly, for all $1 \leq j \leq q_i$, K_j is in $C_{\psi(v_j)}[\underline{P}/\underline{\mathcal{C}}]$. Hence $(J_1, \ldots, J_{p_i}, K_1, \ldots, K_{q_i})$ is in $C_{\phi(u_1)}[\underline{P}/\underline{\mathcal{C}}] \wedge \ldots \wedge C_{\phi(u_{p_i})}[\underline{P}/\underline{\mathcal{C}}] \wedge C_{\psi_i(v_1)}[\underline{P}/\underline{\mathcal{C}}] \wedge \ldots \wedge C_{\psi_i(v_{q_i})}[\underline{P}/\underline{\mathcal{C}}]$ i.e. $C_{\phi(u_1)\wedge\ldots\wedge\phi(u_{p_i})\wedge\psi_i(v_1)\wedge\ldots\wedge\psi_i(v_{q_i})}[\underline{P}/\underline{\mathcal{C}}]$.

By the main induction hypothesis, $G_i[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]$ is in $C_{\beta_i[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$ (see page 141 for the notation of β_i). Hence $G_i[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}](u_1)\ldots(u_{p_i})(v_1)\ldots(v_{q_i})$ is in $C_{\phi(u_1)\wedge\ldots\wedge\phi(u_{p_i})\wedge\psi_i(v_1)\wedge\ldots\wedge\psi_i(v_{q_i})\supset(\beta[\underline{x}/\underline{t}](y/u_1)\wedge\ldots\wedge\beta[\underline{x}/\underline{t}](y/u_{p_i})\supset\beta[\underline{x}/\underline{t}](y/f_i(u_1,\ldots,u_{p_i},v_1,\ldots,v_{q_i})))}$

$[\underline{P}/\underline{\mathcal{C}}]$. Thus $G_i[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}](u_1)\ldots(u_{p_i})(v_1)\ldots(v_{q_i})((J_1,\ldots,J_{p_i},K_1,\ldots,K_{q_i}))$ is in $C_{\beta[\underline{x}/\underline{t}](y/u_1)\wedge\ldots\wedge\beta[\underline{x}/\underline{t}](y/u_{p_i})\supset\beta[\underline{x}/\underline{t}](y/f_i(u_1,\ldots,u_{p_i},v_1,\ldots,v_{q_i}))}[\underline{P}/\underline{\mathcal{C}}]$.

By the induction hypothesis (on u), $\rho([F_1[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}],\ldots,F_n[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]],[G_1[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}],\ldots,$ $G_m[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]])(u_j)(J_j)$ is in $C_{\beta[\underline{x}/\underline{t}][y/u_j]}[\underline{P}/\underline{\mathcal{C}}]$ for all $1\leq j\leq p_i$. Hence $(\rho([F_1[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}],\ldots,F_n[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]],[G_1[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}],\ldots,$ $G_m[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]])(u_1)(J_1),\ldots,\rho([F_1[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}],\ldots,F_n[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]],$ $[G_1[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}],\ldots,G_m[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]])(u_{p_i})(J_{p_i}))$ is in $C_{\beta[\underline{x}/\underline{t}](y/u_1)\wedge\ldots\wedge\beta[\underline{x}/\underline{t}](y/u_{p_i})}[\underline{P}/\underline{\mathcal{C}}]$. Thus M is in $C_{\beta[\underline{x}/\underline{t}](y/f_i(u_1,\ldots,u_{p_i},v_1,\ldots,v_{q_i}))}[\underline{P}/\underline{\mathcal{C}}]$ i.e. $C_{\beta[\underline{x}/\underline{t}][y/u]}[\underline{P}/\underline{\mathcal{C}}]$.

$u=f_i(u_1,\ldots,u_{p_i})$ for some $1\leq i\leq m$ and some individual terms $u_1,\ldots,u_{p_i}$ and $H=H_*((J_1^{\phi(u_1)},\ldots,J_{p_i}^{\phi(u_{p_i})}))$, where H_* is a constant term of type $[\phi(u_1)\wedge\ldots\wedge\phi(u_{p_i})\supset\phi(f_i(u_1,\ldots,u_{p_i}))]$:

This case can be proved as the above case. □

5.2 Abbreviation templates

In ordinary mathematics, we often abbreviate a formula by a predicate. We will introduce *Abbreviation Introduction* and *Abbreviation Elimination* rules to NJ_2 which will allow us to use such abbreviations in the formal system.

For each formula α with $fv(\alpha)=\{x_1,\ldots,x_{n_\alpha}\}$, $n_\alpha\geq 1$, and $FV(\alpha)=\varnothing$, let P_α be a new n_α-ary predicate symbol corresponding to α. We call these new predicate symbols *abbreviation predicates*.

Notes.

a. For each P_α, α does not contain any abbreviation predicates.

b. We restrict the definition of abstraction terms $T=\lambda x_1,\ldots,x_n\alpha$ so that α

does not contain any abbreviation predicates.

c. We define $\beta[P/P_\alpha]$ and $\beta(P/P_\alpha)$ in the same way as substitutions by predicate variables in Definition 2.5 and Part B of Definition 2.18, respectively.

Every lemma that holds for substitutions by predicate variables also holds for substitutions by abbreviation predicates and the proof is similar.

Since we do not allow abstraction terms to contain abbreviation predicates, we will modify some lemmas in Chapter II for substitutions by abbreviation predicates as follows.

Note. In the following R is an r-ary predicate variable and P_σ is an r-ary abbreviation predicate.

Lemma 2.16 If $R \notin \{\underline{P}\} \cup FV(\underline{T})$, then $\alpha[\underline{P}/\underline{T}][R/P_\sigma] \equiv \alpha[R/P_\sigma][\underline{P}/\underline{T}]$.

Lemma 2.24 If $R \notin \{\underline{P}\} \cup FV(\underline{T})$, then $\{\alpha(\underline{P}/\underline{T})(R/P_\sigma)\} = \{\alpha(R/P_\sigma)(\underline{P}/\underline{T})\}$.

Proofs of both lemmas are similar to the original ones.

We introduce the following rules.

(Abbr Intro)

$$\begin{array}{c} \vdots \\ \dfrac{\beta(P/T)}{\beta(P/P_\alpha)} \end{array}$$

(Abbr Elim)

$$\dfrac{\begin{array}{c} \\ \vdots \\ \beta(P/P_\alpha) \end{array} \qquad \begin{array}{c} [\beta(P/T)] \\ \vdots \\ \gamma \end{array}}{\gamma}$$

where P is an n-ary predicate variable and $T = \lambda x_1, \ldots, x_n \alpha$ with $FV(\alpha) = \varnothing$.

We define the corresponding term formation rules as follows.

(Abbr Intro) If $F^{\beta(P/T)}$ is a term of type $[\beta(P/T)]$, where $T = \lambda x_1, \ldots, x_n\alpha$ with $FV(\alpha) = \varnothing$, then $abbr(P_\alpha, F^{\beta(P/T)})$ is a term of type $[\beta(P/P_\alpha)]$.

(Abbr Elim) If $F^{\beta(P/P_\alpha)}$ is a term of type $[\beta(P/P_\alpha)]$, G^γ is a term of type $[\gamma]$, and $X^{\beta(P/T)}$ is a term variable of type $[\beta(P/T)]$, where $T = \lambda x_1, \ldots, x_n\alpha$, then $unabbr(X^{\beta(P/T)}.G^\gamma, F^{\beta(P/P_\alpha)})$ is a term of type $[\gamma]$.

Note. Every occurrence of X^σ, where $[\sigma] = [\beta(P/T)]$, in $X^{\beta(P/T)}.G^\gamma$ is bound.

We add reduction rules as follows.

Note. In the following, $T = \lambda x_1, \ldots, x_n\alpha$ where $FV(\alpha) = \varnothing$.

(Abbr Intro, Abbr Elim)

$$unabbr(X^{\beta(P/T)}.G^\gamma, abbr(P_\alpha, F^{\beta(P/T)})) \succ G[X^{\beta(P/T)}/F^{\beta(P/T)}],$$

provided $P \in FV(\beta)$; and

$$unabbr(X^\beta.G^\gamma, F^\beta) \succ G[X^\beta/F^\beta];$$

$$abbr(P_\alpha, F^\beta)^\beta \succ F^\beta.$$

Note. The above two reduction rules are obtained from the trivial case i.e. when $P \notin FV(\beta)$.

(Abbr Intro, $\wedge$ Elim)

$$\pi_i(abbr(P_\alpha, F^{(\beta_1\wedge\beta_2)(P/T)})) \succ abbr(P_\alpha, \pi_i F),\ i = 1, 2.$$

(Abbr Intro, $\supset$ Elim)

$$(abbr(P_\alpha, F^{(\beta_1\supset\beta_2)(P/T)}))(G^{\beta_1(P/P_\alpha)}) \succ abbr(P_\alpha, F(unabbr(X^{\beta_1(P/T)}.X^{\beta_1(P/T)}, G))),$$

where $X^{\beta_1(P/T)}$ is the first term variable of type $[\beta_1(P/T)]$.

(Abbr Intro, $\vee$ Elim)

$$\oplus(X_1^{\beta_1(P/P_\alpha)}.G_1^{\beta_1(P/P_\alpha)\supset\gamma}(X_1^{\beta_1(P/P_\alpha)}), X_2^{\beta_2(P/P_\alpha)}.G_2^{\beta_2(P/P_\alpha)\supset\gamma}(X_2^{\beta_2(P/P_\alpha)}),$$
$$abbr(P_\alpha, F^{(\beta_1\vee\beta_2)(P/T)})) \succ \oplus(Y_1^{\beta_1(P/T)}.unabbr(Y_1^{\beta_1(P/T)\supset\gamma}.Y_1^{\beta_1(P/T)\supset\gamma}, G_1)(Y_1^{\beta_1(P/T)}),$$
$$Y_2^{\beta_2(P/T)}.unabbr(Y_2^{\beta_2(P/T)\supset\gamma}.Y_2^{\beta_2(P/T)\supset\gamma}, G_2)(Y_2^{\beta_2(P/T)}), F),$$

where $X_i^{\beta_i(P/P_\alpha)}$ is not equivalent to any free term variable of G_i and $Y_i^{\beta_i(P/T)}$ is the first term variable of type $[\beta_i(P/T)]$ which is not equivalent to any free term variable of G_i, $i = 1, 2$.

(Abbr Intro, $\forall$ Elim)

$$abbr(P_\alpha, F^{(\forall x\gamma)(P/T)})(t) \succ abbr(P_\alpha, F(t)).$$

(Abbr Intro, $\forall_2$ Elim)

$$abbr(P_\alpha, F^{(\forall_2 Q\gamma)(P/T)})(U) \succ abbr(P_\alpha, F(U)).$$

(Abbr Intro, $\exists$ Elim)

$$ST(x.X^{\sigma(P/P_\alpha)}.G^{\sigma(P/P_\alpha)\supset\gamma}(X^{\sigma(P/P_\alpha)}), abbr(P_\alpha, F^{\exists x\sigma(P/T)}))$$
$$\succ ST(x.Y^{\sigma(P/T)}.unabbr(Y^{\sigma(P/T)\supset\gamma}.Y^{\sigma(P/T)\supset\gamma}, G)(Y^{\sigma(P/T)}), F),$$

where $X^{\sigma(P/P_\alpha)}$ is not equivalent to any free term variable of G and $Y^{\sigma(P/T)}$ is the first term variable of type $[\sigma(P/T)]$ which is not equivalent to any free term variable of G.

(Abbr Intro, $\exists_2$ Elim)

$$ST(Q.X^{\sigma(P/P_\alpha)}.G^{\sigma(P/P_\alpha)\supset\gamma}(X^{\sigma(P/P_\alpha)}), abbr(P_\alpha, F^{\exists_2 Q\sigma(P/T)}))$$
$$\succ ST(Q.Y^{\sigma(P/T)}.unabbr(Y^{\sigma(P/T)\supset\gamma}.Y^{\sigma(P/T)\supset\gamma}, G)(Y^{\sigma(P/T)}), F),$$

where $X^{\sigma(P/P_\alpha)}$ is not equivalent to any free term variable of G and $Y^{\sigma(P/T)}$ is the first term variable of type $[\sigma(P/T)]$ which is not equivalent to any free term variable of G.

Now, we have new forms of Curry-Howard terms as well as new reduction rules. In the following, we will add the new cases to some definitions in Chapter III. All lemmas in Chapter III still hold after these additions. The proofs for the additional cases of all lemmas in Section 3.2 are similar to those in the section. For Section 3.3, we will prove Lemmas 3.3.2 and 3.3.4 for the new cases and omit the proof of Lemma 3.3.5 since it is similar to the original one.

Definitions 3.2.2 and 3.2.3.

$fv(abbr(P_\alpha, F^{\beta(P/T)})) = fv(F^{\beta(P/T)})$;

$fv(unabbr(X^{\beta(P/T)}.G^\gamma, F^{\beta(P/P_\alpha)})) = fv(X^{\beta(P/T)}.G^\gamma) \cup fv(F^{\beta(P/P_\alpha)})$.

Similarly for $FV(abbr(P_\alpha, F^{\beta(P/T)}))$ *and* $FV(unabbr(X^{\beta(P/T)}.G^\gamma, F^{\beta(P/P_\alpha)}))$.

Definition 3.2.6.

$\underline{x}$ *is* ***replaceable*** *by* $\underline{t}$ *in* $abbr(P_\alpha, F^{\beta(P/T)})$ *if* $\underline{x}$ *is replaceable by* $\underline{t}$ *in* $F^{\beta(P/T)}$*;*

$\underline{x}$ *is* ***replaceable*** *by* $\underline{t}$ *in* $unabbr(X^{\beta(P/T)}.G^\gamma, F^{\beta(P/P_\alpha)})$ *if* $\underline{x}$ *is replaceable by* $\underline{t}$ *in* $X^{\beta(P/T)}.G^\gamma$ *and* $F^{\beta(P/P_\alpha)}$.

Similarly for replaceability of $\underline{P}$ *by* $\underline{T}$

Definitions 3.2.10, 3.2.11, and 3.2.12.

$abbr(P_\alpha, F^{\beta(P/T)})[\underline{x}/\underline{t}] = abbr(P_\alpha, F^{\beta(P/T)}[\underline{x}/\underline{t}])$;

$unabbr(X^{\beta(P/T)}.G^\gamma, F^{\beta(P/P_\alpha)})[\underline{x}/\underline{t}] = unabbr((X^{\beta(P/T)}.G^\gamma)[\underline{x}/\underline{t}], F^{\beta(P/P_\alpha)}[\underline{x}/\underline{t}])$.

Similarly for $abbr(P_\alpha, F^{\beta(P/T)})[\underline{P}/\underline{T}]$, $unabbr(X^{\beta(P/T)}.G^\gamma, F^{\beta(P/P_\alpha)})[\underline{P}/\underline{T}]$, $abbr(P_\alpha, F^{\beta(P/T)})[\underline{X}/\underline{K}]$, and $unabbr(X^{\beta(P/T)}.G^\gamma, F^{\beta(P/P_\alpha)})[\underline{X}/\underline{K}]$.

Lemma 3.3.2. *If* $F \equiv F'$ *and* $F \succ_1 G$*, then* $F' \succ_1 G'$ *for some C-H term* G' *such that* $G \equiv G'$.

Proof. If F is not the redex which is reduced to G, the proof is as in Chapter III. Suppose F is the redex which is reduced to G. As shown in Chapter III, we may assume that F' is obtained from F by a single legitimate change of bound variable.

(i) $F = unabbr(X^{\beta(P/T)}.K^\gamma, abbr(P_\alpha, H^{\beta(P/T)}))$, where $P \in FV(\beta)$.

Then $G = K[X^{\beta(P/T)}/H]$.

Case 1. $F' = unabbr(X^{\beta(P/T)}.K', abbr(P_\alpha, H'))$ for some terms H' and K' such that $H' \equiv H$ and $K' \equiv K$.

Then $F' \succ_1 K'[X^{\beta(P/T)}/H'] \equiv G$ by Lemma 3.2.26.

Case 2. $F' = unabbr(Y^{\beta(P/T)}.K^{\gamma}[X^{\beta(P/T)}/Y^{\beta(P/T)}], abbr(P_\alpha, H))$ where $Y^{\beta(P/T)}$ is free for $X^{\beta(P/T)}$ and is not equivalent to any free term variable in K.

Then $F' \succ_1 K[X^{\beta(P/T)}/Y^{\beta(P/T)}][Y^{\beta(P/T)}/H] \equiv G$ by Lemma 3.2.21.

Similarly for the case $F = unabbr(X^{\beta}.K^{\gamma}, H^{\beta})$.

(ii) $F = \oplus(X_1^{\beta_1(P/P_\alpha)}.G_1^{\beta_1(P/P_\alpha)\supset\gamma}(X_1^{\beta_1(P/P_\alpha)}), X_2^{\beta_2(P/P_\alpha)}.G_2^{\beta_2(P/P_\alpha)\supset\gamma}(X_2^{\beta_2(P/P_\alpha)}), abbr(P_\alpha, H^{(\beta_1\vee\beta_2)(P/T)}))$, where $X_i^{\beta_i(P/P_\alpha)}$ is not equivalent to any free term variable of G_i, $i = 1, 2$.

Then $G = \oplus(Y_1^{\beta_1(P/T)}.unabbr(Y_1^{\beta_1(P/T)\supset\gamma}.Y_1^{\beta_1(P/T)\supset\gamma}, G_1)(Y_1^{\beta_1(P/T)}), Y_2^{\beta_2(P/T)}.unabbr(Y_2^{\beta_2(P/T)\supset\gamma}.Y_2^{\beta_2(P/T)\supset\gamma}, G_2)(Y_2^{\beta_2(P/T)}), H)$, where $Y_i^{\beta_i(P/T)}$ is the first term variable of type $[\beta_i(P/T)]$ which is not equivalent to any free term variable of G_i, $i = 1, 2$.

Case 1. $F' = \oplus(X_1^{\beta_1(P/P_\alpha)}.G_1'(X_1^{\beta_1(P/P_\alpha)}), X_2^{\beta_2(P/P_\alpha)}.G_2'(X_2^{\beta_2(P/P_\alpha)}), abbr(P_\alpha, H'))$, where $G_i' \equiv G_i$, $i = 1, 2$, and $H' \equiv H$.

Since $G_i' \equiv G_i$, $Y_i^{\beta_i(P/T)}$ is also the first term variable of type $[\beta_i(P/T)]$ which is not equivalent to any free term variable of G_i', $i = 1, 2$. Hence

$$F' \succ_1 \oplus(Y_1^{\beta_1(P/T)}.unabbr(Y_1^{\beta_1(P/T)\supset\gamma}.Y_1^{\beta_1(P/T)\supset\gamma}, G_1')(Y_1^{\beta_1(P/T)}),$$
$$Y_2^{\beta_2(P/T)}.unabbr(Y_2^{\beta_2(P/T)\supset\gamma}.Y_2^{\beta_2(P/T)\supset\gamma}, G_2')(Y_2^{\beta_2(P/T)}), H') \equiv G.$$

Case 2. $F' = \oplus(Z_1^{\beta_1(P/P_\alpha)}.G_1(Z_1^{\beta_1(P/P_\alpha)}), X_2^{\beta_2(P/P_\alpha)}.G_2(X_2^{\beta_2(P/P_\alpha)}), abbr(P_\alpha, H))$, where $Z_1^{\beta_1(P/P_\alpha)}$ is free for $X_1^{\beta_1(P/P_\alpha)}$ and is not equivalent to any free term variable in $G_1(X_1^{\beta_1(P/P_\alpha)})$.

Then $F' \succ_1 G$.

Similarly if the changed bound term variable in F is $X_2^{\beta_2(P/P_\alpha)}$.

(iii) $F = ST(x.X^{\sigma(P/P_\alpha)}.K^{\sigma(P/P_\alpha)\supset\gamma}(X^{\sigma(P/P_\alpha)}), abbr(P_\alpha, H^{(\exists x\sigma)(P/T)}))$, where $X^{\sigma(P/P_\alpha)}$ is not equivalent to any free term variable of K.

Then $G = ST(x.Y^{\sigma(P/T)}.unabbr(Y^{\sigma(P/T)\supset\gamma}.Y^{\sigma(P/T)\supset\gamma}, K)(Y^{\sigma(P/T)}), H)$, where $Y^{\sigma(P/T)}$ is the first term variable of type $[\sigma(P/T)]$ which is not equivalent to any free term variable of K.

Every case can be proved in the same way as in (ii) except the case $F' = ST(y.X^{\sigma(P/P_\alpha)[x/y]}.K[x/y](X^{\sigma(P/P_\alpha)[x/y]}), abbr(P_\alpha, H))$, where x is replaceable by y, y is free for x, and y does not occur free in $X^{\sigma(P/P_\alpha)}.K(X^{\sigma(P/P_\alpha)})$. We have

$$\begin{aligned} F' \succ_1 \; & ST(y.Z^{\sigma(P/T)[x/y]}.unabbr(Z^{(\sigma(P/T)\supset\gamma)[x/y]}.Z^{(\sigma(P/T)\supset\gamma)[x/y]}, K[x/y]) \\ & (Z^{\sigma(P/T)[x/y]}), H) \\ \equiv \; & ST(x.Z^{\sigma(P/T)}.unabbr(Z^{(\sigma(P/T)\supset\gamma)}.Z^{(\sigma(P/T)\supset\gamma)}, K)(Z^{\sigma(P/T)}), H) \\ \equiv \; & G, \end{aligned}$$

where $Z^{\sigma(P/T)[x/y]}$ is the first term variable of type $[\sigma(P/T)[x/y]]$ which is not equivalent to any free term variable of $K[x/y]$, so $Z^{\sigma(P/T)}$ is not equivalent to any free term variable of K.

Similarly for $ST(Q.X^{\sigma(P/P_\alpha)}.K^{\sigma(P/P_\alpha)\supset\gamma}(X^{\sigma(P/P_\alpha)}), abbr(P_\alpha, H^{(\exists_2 Q\sigma)(P/T)}))$.

(iv) $F = \pi_1(abbr(P_\alpha, H^{(\beta_1\wedge\beta_2)(P/T)}))$.

Then $G = abbr(P_\alpha, \pi_1 H)$ and $F' = \pi_1(abbr(P_\alpha, H'))$ for some term H' such that $H' \equiv H$. Hence $F' \succ_1 abbr(P_\alpha, \pi_1 H') \equiv G$.

Similarly for the remaining cases. □

Lemma 3.3.4. *If $F \succ_1 G$, then*

a. $F[x/t] \succ_1 H$ for some term H such that $H \equiv G[x/t]$;

b. $F[R/U] \succ_1 H$ for some term H such that $H \equiv G[R/U]$;

c. $F[X^\delta/K^{\delta'}] \succ_1 H$ for some term H such that $H \equiv G[X^\delta/K^{\delta'}]$.

Note. For (b), we use R and U instead of P and T, respectively, in Chapter III to avoid confusion with P and T in the type superscripts of new terms.

Proof. If F is not the redex which is reduced to G, the proof is as in Chapter III. Suppose F is the redex which is reduced to G. As in the original proof, we will omit the proof of (b) since it is similar to (a) and for (a) we assume that x is replaceable by t and t is free for x in F and for (c) $K^{\delta'}$ is free for X^{δ} in F.

(i) $F = unabbr(Y^{\beta(P/T)}.J^{\gamma}, abbr(P_{\alpha}, H^{\beta(P/T)}))$, where $P \in FV(\beta)$.

Then $G = J[Y^{\beta(P/T)}/H]$.

a: By Lemma 3.2.23, we have

$$\begin{aligned} F[x/t] &= unabbr(Y^{\beta(P/T)[x/t]}.J[x/t], abbr(P_{\alpha}, H[x/t])) \\ &\succ_1 J[x/t][Y^{\beta(P/T)[x/t]}/H[x/t]] \\ &\equiv G[x/t]. \end{aligned}$$

c: Suppose X^{δ} is equivalent to some free term variable of $Y^{\beta(P/T)}.J$. The proof of the other case can be easily modified from this proof.

By Lemma 3.2.21,

$$\begin{aligned} F[X^{\delta}/K] &= unabbr(Y^{\beta(P/T)}.J[X^{\delta}/K], abbr(P_{\alpha}, H[X^{\delta}/K])) \\ &\succ_1 J[X^{\delta}/K][Y^{\beta(P/T)}/H[X^{\delta}/K]] \\ &\equiv G[X^{\delta}/K]. \end{aligned}$$

Similarly for the case $F = unabbr(Y^{\beta}.J^{\gamma}, H^{\beta})$.

(ii) $F = \oplus(X_1^{\beta_1(P/P_{\alpha})}.G_1^{\beta_1(P/P_{\alpha})\supset\gamma}(X_1^{\beta_1(P/P_{\alpha})}), X_2^{\beta_2(P/P_{\alpha})}.G_2^{\beta_2(P/P_{\alpha})\supset\gamma}(X_2^{\beta_2(P/P_{\alpha})}),$ $abbr(P_{\alpha}, H^{(\beta_1\vee\beta_2)(P/T)}))$, where $X_i^{\beta_i(P/P_{\alpha})}$ is not equivalent to any free term variable of G_i, $i = 1, 2$.

Then $G = \oplus(Y_1^{\beta_1(P/T)}.unabbr(Y_1^{\beta_1(P/T)\supset\gamma}.Y_1^{\beta_1(P/T)\supset\gamma}, G_1)(Y_1^{\beta_1(P/T)}), Y_2^{\beta_2(P/T)}.$ $unabbr(Y_2^{\beta_2(P/T)\supset\gamma}.Y_2^{\beta_2(P/T)\supset\gamma}, G_2)(Y_2^{\beta_2(P/T)}), H)$, where $Y_i^{\beta_i(P/T)}$ is the first term variable of type $[\beta_i(P/T)]$ which is not equivalent to any free term variable of G_i, $i = 1, 2$.

a: We have

$G[x/t] = \oplus(Y_3^{\beta_1(P/T)[x/t]}.unabbr(Y_1^{\beta_1(P/T)[x/t]\supset\gamma[x/t]}.Y_1^{\beta_1(P/T)[x/t]\supset\gamma[x/t]}, G_1[x/t])$

$(Y_3^{\beta_1(P/T)[x/t]}), Y_4^{\beta_2(P/T)[x/t]}.unabbr(Y_2^{\beta_2(P/T)[x/t]\supset\gamma[x/t]}.Y_2^{\beta_2(P/T)[x/t]\supset\gamma[x/t]}, G_2[x/t])$ $(Y_4^{\beta_2[x/t](P/T)}), H[x/t])$, where $Y_{i+2}^{\beta_i(P/T)}$ is $Y_i^{\beta_i(P/T)}$ if G_i has no free term variable Y_i^σ such that $\sigma \not\equiv \beta_i(P/T)$ but $\sigma[x/t] \equiv \beta_i(P/T)[x/t]$, otherwise $Y_{i+2}^{\beta_i(P/T)}$ is the first term variable such that G_i has no free term variable Y_{i+2}^σ where $\sigma \not\equiv \beta_i(P/T)$ but $\sigma[x/t] \equiv \beta_i(P/T)[x/t]$, $i = 1, 2$, and

$$\begin{aligned}
F[x/t] &= \oplus(X_1^{\beta_1(P/P_\alpha)[x/t]}.G_1[x/t](X_1^{\beta_1(P/P_\alpha)[x/t]}), X_2^{\beta_2(P/P_\alpha)[x/t]}.G_2[x/t] \\
&\quad (X_2^{\beta_2(P/P_\alpha)[x/t]}), abbr(P_\alpha, H[x/t])) \\
&\succ_1 \oplus(Z_1^{\beta_1[x/t](P/T)}.unabbr(Z_1^{\beta_1[x/t](P/T)\supset\gamma[x/t]}.Z_1^{\beta_1[x/t](P/T)\supset\gamma[x/t]}, G_1[x/t]) \\
&\quad (Z_1^{\beta_1[x/t](P/T)}), unabbr(Z_2^{\beta_2[x/t](P/T)\supset\gamma[x/t]}.Z_2^{\beta_2[x/t](P/T)\supset\gamma[x/t]}, G_2[x/t]) \\
&\quad (Z_2^{\beta_2[x/t](P/T)}), H[x/t]) \\
&\equiv G[x/t],
\end{aligned}$$

where $Z_i^{\beta_i[x/t](P/T)}$ is the first term variable of type $[\beta_i[x/t](P/T)]$ which is not equivalent to any free term variable of $G_i[x/t]$, $i = 1, 2$.

c: Suppose X^δ is equivalent to some free term variables of G_1 and G_2. Proofs of other cases can be easily modified from this proof. We have

$G[X^\delta/K] = \oplus(Y_3^{\beta_1(P/T)}.unabbr(Y_1^{\beta_1(P/T)\supset\gamma}.Y_1^{\beta_1(P/T)\supset\gamma}, G_1[X^\delta/K])(Y_3^{\beta_1(P/T)}),$ $Y_4^{\beta_2(P/T)}.unabbr(Y_2^{\beta_2(P/T)\supset\gamma}.Y_2^{\beta_2(P/T)\supset\gamma}, G_2[X^\delta/K])(Y_4^{\beta_2(P/T)}), H[X^\delta/K])$, where $Y_{i+2}^{\beta_i(P/T)}$ is $Y_i^{\beta_i(P/T)}$ if $Y_i^{\beta_i(P/T)}$ is not equivalent to any free term variable of K, otherwise $Y_{i+2}^{\beta_i(P/T)}$ is the first term variable which is not equivalent to any free term variable in K or G_i, $i = 1, 2$, and

$$\begin{aligned}
F[X^\delta/K] &= \oplus(X_1^{\beta_1(P/P_\alpha)}.G_1[X^\delta/K](X_1^{\beta_1(P/P_\alpha)}), X_2^{\beta_2(P/P_\alpha)}.G_2[X^\delta/K] \\
&\quad (X_2^{\beta_2(P/P_\alpha)}), abbr(P_\alpha, H[X^\delta/K])) \\
&\succ_1 \oplus(Z_1^{\beta_1(P/T)}.unabbr(Z_1^{\beta_1(P/T)\supset\gamma}.Z_1^{\beta_1(P/T)\supset\gamma}, G_1[X^\delta/K]) \\
&\quad (Z_1^{\beta_1(P/T)}), unabbr(Z_2^{\beta_2(P/T)\supset\gamma}.Z_2^{\beta_2(P/T)\supset\gamma}, G_2[X^\delta/K]) \\
&\quad (Z_2^{\beta_2(P/T)}), H[X^\delta/K]) \\
&\equiv G[X^\delta/K],
\end{aligned}$$

where $Z_i^{\beta_i(P/T)}$ is the first term variable of type $[\beta_i(P/T)]$ which is not equivalent to any free term variable of $G_i[X^\delta/K]$, $i = 1, 2$.

Similarly for the cases $ST(y.Y^{\sigma(P/P_\alpha)}.J^{\sigma(P/P_\alpha)\supset\gamma}(Y^{\sigma(P/P_\alpha)}), abbr(P_\alpha, F^{(\exists y\sigma)(P/T)}))$ and $ST(Q.Y^{\sigma(P/P_\alpha)}.J^{\sigma(P/P_\alpha)\supset\gamma}(Y^{\sigma(P/P_\alpha)}), abbr(P_\alpha, H^{(\exists_2 Q\sigma)(P/T)}))$.

The remaining cases follow straightforwardly by the induction hypothesis. □

The aim of the rest of this section is to show that the new Curry-Howard terms satisfy the *strong normalization theorem*. In order to do this, we will extend the definitions of CR to CR^+ as well as $C_\alpha[\underline{P}/\underline{\mathcal{C}}]$ to $C_\alpha^+[\underline{P}/\underline{\mathcal{C}}]$ and use these new definitions instead of the old ones in proving the theorem. First, we will extend the definition of *neutral terms* in Chapter IV for the new forms of Curry-Howard terms as follows.

Definition 4.1.2. *A term of the form* $unabbr(X^{\beta(P/T)}.G^\gamma, F^{\beta(P/P_\alpha)})$ *is* ***neutral*** *while a term of the form* $abbr(P_\alpha, F^{\beta(P/T)})$ *is not.*

Definition 5.2.1. *A candidate for reducibility* CR C *of type* $[\alpha]$ *is a* CR^+ *of type* $[\alpha]$ *if it satisfies the following.*

If F^α *is in* C*, so is* $abbr(P_\beta, F^\alpha)^\alpha$ *for every abbreviation predicate* P_β*.*

Note. It can be easily checked that if F^α is in a CR (also CR^+) C, then so is $unabbr(X^\alpha.X^\alpha, F^\alpha)$ for every term variable X^α.

In the following, we will state and prove lemmas for CR^+ which correspond to those for CR in Chapter IV.

Lemma 5.2.2. *The set of all strongly normalizable terms of type* $[\alpha]$ *is a* CR^+ *of type* $[\alpha]$.

Proof. By Lemma 4.2.2, SN_α is a CR.

Suppose F^α is strongly normalizable. Since for all P_β, every reduction sequence beginning with $abbr(P_\beta, F^\alpha)^\alpha$ gives a reduction sequence beginning with F^α, $abbr(P_\beta, F^\alpha)^\alpha$ is also strongly normalizable for all P_β. Hence SN_α is a CR^+. □

Lemma 5.2.3. *Let* C_1 *and* C_2 *be* CR^+*s of types* $[\alpha_1]$ *and* $[\alpha_2]$*, respectively.*

Then $C_1 \wedge C_2$ *and* $C_1 \supset C_2$ *are* CR^+*s.*

Proof. By Lemma 4.2.5, $C_1 \wedge C_2$ and $C_1 \supset C_2$ are CRs.

$C_1 \wedge C_2$: It remains to show that if $F^{\alpha_1 \wedge \alpha_2}$ is in $C_1 \wedge C_2$, then so is $abbr(P_\beta, F)^{\alpha_1 \wedge \alpha_2}$ for every P_β.

Let P_β be given and suppose $F^{\alpha_1 \wedge \alpha_2}$ is in $C_1 \wedge C_2$.

First, we will show that $\pi_1 abbr(P_\beta, F)$ is in C_1 by induction on $N(F)$.

Since $\pi_1 abbr(P_\beta, F)$ is neutral, we will show that all its immediate reducts are in C_1. It follows by the subsidiary induction hypothesis if an immediate reduct is obtained by reducing F. The other immediate reducts are as follows.

(i) $\pi_1 F$.

It is in C_1 since F is in $C_1 \wedge C_2$.

(ii) $abbr(P_\beta, \pi_1 F)$.

Since $\pi_1 F$ is in C_1 which is a CR^+, $abbr(P_\beta, \pi_1 F)$ is also in C_1.

Hence $\pi_1 abbr(P_\beta, F)$ is in C_1.

Similarly, we can show that $\pi_2 abbr(P_\beta, F)$ is in C_2. Thus $abbr(P_\beta, F)$ is in $C_1 \wedge C_2$.

$C_1 \supset C_2$: It remains to show that if $F^{\alpha_1 \supset \alpha_2}$ is in $C_1 \supset C_2$, then so is $abbr(P_\beta, F)^{\alpha_1 \supset \alpha_2}$ for every P_β.

Let P_β be given and suppose $F^{\alpha_1 \supset \alpha_2}$ is in $C_1 \supset C_2$.

Let G be in C_1. We will prove that $abbr(P_\beta, F)(G)$ is in C_2 by induction on $N(F) + N(G)$. As above, we will show that all its immediate reducts are in C_2. It follows by the subsidiary induction hypothesis if an immediate reduct is obtained by reducing F or G. The other immediate reducts are as follows.

(i) $F(G)$.

It is in C_2 since F is in $C_1 \supset C_2$.

(ii) $abbr(P_\beta, F(unabbr(X^{\alpha_1}.X^{\alpha_1}, G^{\alpha_1})))$.

By Note on page 158, $unabbr(X^{\alpha_1}.X^{\alpha_1}, G^{\alpha_1})$ is in C_1.

Since F is in $C_1 \supset C_2$, $F(unabbr(X^{\alpha_1}.X^{\alpha_1}, G))$ is in C_2 which is a CR^+. Hence $abbr(P_\beta, F(unabbr(X^{\alpha_1}.X^{\alpha_1}, G)))$ is in C_2. □

Definition 5.2.4. *Let $T = \lambda x_1, \ldots, x_n \delta$ be an abstraction term. For each sequence of individual terms $\underline{t} = t_1, \ldots, t_n$, let $C_{\underline{t}}$ be a CR^+ of type $[\delta[\underline{x}/\underline{t}]]$, where $\underline{x} = x_1, \ldots, x_n$.*

*We call the set $\mathcal{C} = \{C_{\underline{t}} \mid \underline{t} = t_1, \ldots, t_n$ are individual terms.$\}$ a **collection** of CR^+**s corresponding** to T.*

The following definition is needed for defining a set $C^+_\alpha[\underline{P}/\underline{\mathcal{C}}]$ later.

Definition 5.2.5. *Let α be a formula, $\underline{P} = P_1^{m_1}, \ldots, P_n^{m_n}$ be distinct predicate variables, $\underline{T} = T_1, \ldots, T_n$, where $T_i = \lambda z_1^i, \ldots, z_{m_i}^i \delta_i$, $1 \leq i \leq n$, be abstraction terms, and $\underline{\mathcal{C}} = \mathcal{C}_1, \ldots, \mathcal{C}_n$ be collections of CR^+s corresponding to $T_1, \ldots, T_n$, respectively.*

We define a set $C'_\alpha[P_1/\mathcal{C}_1,\ldots,P_n/\mathcal{C}_n]$*, which can be written as* $C'_\alpha[\underline{P}/\underline{\mathcal{C}}]$*, of terms of type* $[\alpha[\underline{P}/\underline{T}]]$ *in the same way as* $C_\alpha[\underline{P}/\underline{\mathcal{C}}]$ *in Definition 4.2.7 (by replacing every* CR *and* $C_\sigma[\underline{P}/\underline{\mathcal{C}}]$ *by* CR^+ *and* $C'_\sigma[\underline{P}/\underline{\mathcal{C}}]$*, respectively) except for the existence cases below.*

$C'_{\exists x\beta}[\underline{P}/\underline{\mathcal{C}}]$ *is the set of all terms* F *which satisfy the conditions for the corresponding existence case in Definition 4.2.7 (with the replacements of every* CR *and* $C_\sigma[\underline{P}/\underline{\mathcal{C}}]$ *by* CR^+ *and* $C'_\sigma[\underline{P}/\underline{\mathcal{C}}]$*, respectively); and*

if β *contains some abbreviation predicates and*
$F \succ abbr(P_{\sigma_1}, abbr(\ldots, abbr(P_{\sigma_k}, I(u,H)^{\exists x\beta^*[\underline{P}/\underline{T}]})))^{\exists x\beta[\underline{P}/\underline{T}]}$*,* $k \geq 1$*, for some formula* β^**, then* $abbr(P_{\sigma_1}, abbr(\ldots, abbr(P_{\sigma_k}, H)))$ *is in* $C'_{\beta[x/u]}[\underline{P}/\underline{\mathcal{C}}]$*.*

$C'_{\exists_2 Q\beta}[\underline{P}/\underline{\mathcal{C}}]$ *is the set of all terms* F *which satisfy the conditions for the corresponding existence case in Definition 4.2.7 (with the replacements of every* CR *and* $C_\sigma[\underline{P}/\underline{\mathcal{C}}]$ *by* CR^+ *and* $C'_\sigma[\underline{P}/\underline{\mathcal{C}}]$*, respectively); and*

if β *contains some abbreviation predicates and*
$F \succ abbr(P_{\sigma_1}, abbr(\ldots, abbr(P_{\sigma_k}, I(U,H)^{(\exists_2 Q\beta^*)[\underline{P}/\underline{T}]})))^{(\exists_2 Q\beta)[\underline{P}/\underline{T}]}$*,* $k \geq 1$*, for some formula* β^**, then* $abbr(P_{\sigma_1}, abbr(\ldots, abbr(P_{\sigma_k}, H)))$ *is in* $C'_\beta[\underline{P}^*/\underline{\mathcal{C}}^*, Q/\mathcal{D}]$ *for some collection of* CR^+*s* $\mathcal{D}$ *corresponding to* U*, where* $\underline{P}^*$ *is the sublist of* $\underline{P}$ *consisting of all* P_i*'s which are in* $FV(\exists_2 Q\beta)$ *and* $\underline{\mathcal{C}}^*$ *is the corresponding sublist of* $\underline{\mathcal{C}}$*.*

Lemma 5.2.6. *Let* α *be a formula,* $\underline{P} = P_1^{m_1},\ldots,P_n^{m_n}$ *be distinct predicate variables,* $\underline{T} = T_1,\ldots,T_n$*, where* $T_i = \lambda z_1^i,\ldots,z_{m_i}^i\delta_i$*,* $1 \leq i \leq n$*, be abstraction terms, and* $\underline{\mathcal{C}} = \mathcal{C}_1,\ldots,\mathcal{C}_n$ *be collections of* CR^+*s corresponding to* $T_1,\ldots,T_n$*, respectively.*

Then $C'_\alpha[\underline{P}/\underline{\mathcal{C}}]$ *is a* CR^+ *of type* $[\alpha[\underline{P}/\underline{T}]]$*.*

Proof. We will prove by induction on α. It follows by Lemma 5.2.2 if α is atomic. It follows by Lemma 5.2.3 and the induction hypothesis if α is $\alpha_1 \wedge \alpha_2$ or $\alpha_1 \supset \alpha_2$. It can be proved in the same way as Lemma 4.2.8, that $C'_\alpha[\underline{P}/\underline{\mathcal{C}}]$ is a CR. For the remaining cases, it remains to show that if $F^{\alpha[\underline{P}/\underline{T}]}$ is in $C'_\alpha[\underline{P}/\underline{\mathcal{C}}]$, then so is $abbr(P_\beta, F)^{\alpha[\underline{P}/\underline{T}]}$ for every P_β.

Let P_β be given and suppose $F^{\alpha[\underline{P}/\underline{T}]}$ is in $C'_\alpha[\underline{P}/\underline{\mathcal{C}}]$.

$\alpha = \alpha_1 \vee \alpha_2$:

Let $[\gamma]$ be a type, C be a CR^+ of type $[\gamma]$, F_1 and F_2 be terms in $C'_{\alpha_1}[\underline{P}/\underline{\mathcal{C}}] \supset C$ and $C'_{\alpha_2}[\underline{P}/\underline{\mathcal{C}}] \supset C$, respectively, and $X_1^{\alpha_1[\underline{P}/\underline{T}]}$ and $X_2^{\alpha_2[\underline{P}/\underline{T}]}$ be term variables which are not equivalent to any free term variables of F_1 and F_2, respectively.

We will prove that $\oplus(X_1^{\alpha_1[\underline{P}/\underline{T}]}.F_1(X_1^{\alpha_1[\underline{P}/\underline{T}]}), X_2^{\alpha_2[\underline{P}/\underline{T}]}.F_2(X_2^{\alpha_2[\underline{P}/\underline{T}]}), abbr(P_\beta, F))$ is in C by induction on $N(F_1(X_1^{\alpha_1[\underline{P}/\underline{T}]})) + N(F_2(X_2^{\alpha_2[\underline{P}/\underline{T}]})) + N(F)$. As usual, we will show that all its immediate reducts are in C. If an immediate reduct is obtained by reducing $F_1(X_1^{\alpha_1[\underline{P}/\underline{T}]})$, $F_2(X_2^{\alpha_2[\underline{P}/\underline{T}]})$, or F, then it is in C by the subsidiary induction hypothesis. The other immediate reducts are as follows.

(i) $\oplus(X_1^{\alpha_1[\underline{P}/\underline{T}]}.F_1(X_1^{\alpha_1[\underline{P}/\underline{T}]}), X_2^{\alpha_2[\underline{P}/\underline{T}]}.F_2(X_2^{\alpha_2[\underline{P}/\underline{T}]}), F)$.

It is in C since F is in $C'_{\alpha_1\vee\alpha_2}[\underline{P}/\underline{\mathcal{C}}]$.

(ii) $\oplus(Y_1^{\alpha_1[\underline{P}/\underline{T}]}.unabbr(Y_1^{\alpha_1[\underline{P}/\underline{T}]\supset\gamma}.Y_1^{\alpha_1[\underline{P}/\underline{T}]\supset\gamma}, F_1)(Y_1^{\alpha_1[\underline{P}/\underline{T}]}), Y_2^{\alpha_2[\underline{P}/\underline{T}]}.$ $unabbr(Y_2^{\alpha_2[\underline{P}/\underline{T}]\supset\gamma}.Y_2^{\alpha_2[\underline{P}/\underline{T}]\supset\gamma}, F_2)(Y_2^{\alpha_2[\underline{P}/\underline{T}]}), F)$, where $Y_i^{\alpha_i}$ is the first term variable which is not equivalent to any free term variable of F_i, $i = 1, 2$.

By Lemma 5.2.3 and the induction hypothesis, $C'_{\alpha_i}[\underline{P}/\underline{\mathcal{C}}] \supset C$ is a CR^+ for all $i = 1, 2$. By Note on page 158, $unabbr(Y_i^{\alpha_i[\underline{P}/\underline{T}]\supset\gamma}.Y_i^{\alpha_i[\underline{P}/\underline{T}]\supset\gamma}, F_i^{\alpha_i[\underline{P}/\underline{T}]\supset\gamma})$ is in $C'_{\alpha_i}[\underline{P}/\underline{\mathcal{C}}] \supset C$ for all $i = 1, 2$.

Since F is in $C'_{\alpha_1\vee\alpha_2}[\underline{P}/\underline{\mathcal{C}}]$, $\oplus(Y_1^{\alpha_1[\underline{P}/\underline{T}]}.unabbr(Y_1^{\alpha_1[\underline{P}/\underline{T}]\supset\gamma}.Y_1^{\alpha_1[\underline{P}/\underline{T}]\supset\gamma}, F_1)$ $(Y_1^{\alpha_1[\underline{P}/\underline{T}]}), Y_2^{\alpha_2[\underline{P}/\underline{T}]}.unabbr(Y_2^{\alpha_2[\underline{P}/\underline{T}]\supset\gamma}.Y_2^{\alpha_2[\underline{P}/\underline{T}]\supset\gamma}, F_2)(Y_2^{\alpha_2[\underline{P}/\underline{T}]}), F)$ is in C.

$\alpha = \forall x\sigma$:

Let t be an individual term. We will prove by induction on $N(F)$ that $abbr(P_\beta, F)(t)$ is in $C'_{\sigma[x/t]}[\underline{P}/\underline{\mathcal{C}}]$, which is a CR^+ by the induction hypothesis. We will show that all its immediate reducts are in $C'_{\sigma[x/t]}[\underline{P}/\underline{\mathcal{C}}]$. If an immediate reduct is obtained by reducing F, then it is in $C'_{\sigma[x/t]}[\underline{P}/\underline{\mathcal{C}}]$ by the subsidiary induction hypothesis. The other immediate reducts are as follows.

(i) $F(t)$.

It is in $C'_{\sigma[x/t]}[\underline{P}/\underline{\mathcal{C}}]$ since F is in $C'_{\forall x\sigma}[\underline{P}/\underline{\mathcal{C}}]$.

(ii) $abbr(P_\beta, F(t))$.

It is in $C'_{\sigma[x/t]}[\underline{P}/\underline{\mathcal{C}}]$ since $F(t)$ is in $C'_{\sigma[x/t]}[\underline{P}/\underline{\mathcal{C}}]$.

Similarly for $C'_{\forall_2 Q\sigma}[\underline{P}/\underline{\mathcal{C}}]$.

$\alpha = \exists x\sigma$:

Let y be an individual variable such that $y \notin fv(\sigma) - \{x\}$, $[\gamma]$ be a type with $y \notin fv(\gamma)$, D be a CR^+ of type $[\gamma]$, G be a term of type $[\sigma[x/y][\underline{P}/\underline{T}] \supset \gamma]$ such that for each individual term t, $G[y/t]$ is in $C'_{\sigma[x/t]}[\underline{P}/\underline{\mathcal{C}}] \supset D$, and y is not free in the type superscript of any free term variable of G, and $X^{\sigma[x/y][\underline{P}/\underline{T}]}$ be a term variable which is not equivalent to any free term variable of G.

We will prove that $ST(y.X^{\sigma[x/y][\underline{P}/\underline{T}]}.G(X^{\sigma[x/y][\underline{P}/\underline{T}]}), abbr(P_\beta, F))$ is in D by induction on $N(G(X^{\sigma[x/y][\underline{P}/\underline{T}]})) + N(F)$. We will show that all its immediate reducts are in D. If an immediate reduct is obtained by reducing $G(X^{\sigma[x/y][\underline{P}/\underline{T}]})$ or F, then it is in D by the subsidiary induction hypothesis. The other immediate reducts are as follows.

(i) $ST(y.X^{\sigma[x/y][\underline{P}/\underline{T}]}.G(X^{\sigma[x/y][\underline{P}/\underline{T}]}), F)$.

It is in D since F is in $C'_{\exists x\sigma}[\underline{P}/\underline{\mathcal{C}}]$.

(ii) $ST(y.Y^{\sigma[x/y][\underline{P}/\underline{T}]}.unabbr(Y^{\sigma[x/y][\underline{P}/\underline{T}]\supset\gamma}.Y^{\sigma[x/y][\underline{P}/\underline{T}]\supset\gamma}, G)(Y^{\sigma[x/y][\underline{P}/\underline{T}]}), F)$, where $Y^{\sigma[x/y][\underline{P}/\underline{T}]}$ is the first term variable which is not equivalent to any free term variable of G.

Let t be an individual term. By the main induction hypothesis, $C'_{\sigma[x/t]}[\underline{P}/\underline{\mathcal{C}}]$ is a CR^+ and so is $C'_{\sigma[x/t]}[\underline{P}/\underline{\mathcal{C}}] \supset D$ by Lemma 5.2.3. Since $G[y/t]$ is in $C'_{\sigma[x/t]}[\underline{P}/\underline{\mathcal{C}}] \supset D$, by Note on page 158, $unabbr(Y^{\sigma[x/y][\underline{P}/\underline{T}]\supset\gamma}.Y^{\sigma[x/y][\underline{P}/\underline{T}]\supset\gamma}, G)[y/t]$ i.e.

$unabbr(Y^{\sigma[x/t][\underline{P}/\underline{T}]\supset\gamma}.Y^{\sigma[x/t][\underline{P}/\underline{T}]\supset\gamma}, G[y/t])$ is in $C'_{\sigma[x/t]}[\underline{P}/\underline{\mathcal{C}}] \supset D$.

Since F is in $C'_{\exists x\sigma}[\underline{P}/\underline{\mathcal{C}}]$, $ST(y.Y^{\sigma[x/y][\underline{P}/\underline{T}]}.unabbr(Y^{\sigma[x/y][\underline{P}/\underline{T}]\supset\gamma}.Y^{\sigma[x/y][\underline{P}/\underline{T}]\supset\gamma}, G)(Y^{\sigma[x/y][\underline{P}/\underline{T}]}), F)$ is in D.

Next, suppose $abbr(P_\beta, F) \succ I(u, K)$. Then there is a finite reduction sequence from $abbr(P_\beta, F)$ to $I(u, K)$ with length $r \geq 1$. We will show that K is in $C'_{\sigma[x/u]}[\underline{P}/\underline{\mathcal{C}}]$ by induction on r.

$r = 1$: Since $abbr(P_\beta, F)$ and $I(u, K)$ are of different forms, $abbr(P_\beta, F)$ must be the redex which is reduced to $I(u, K)$. Hence $abbr(P_\beta, F) \succ_1 F = I(u, K)$. Since F is in $C'_{\exists x\sigma}[\underline{P}/\underline{\mathcal{C}}]$, K is in $C'_{\sigma[x/u]}[\underline{P}/\underline{\mathcal{C}}]$.

$r > 1$: Suppose F^* is the immediate reduct in the reduction sequence.

Case 1. $F^* = abbr(P_\beta, F')$ where $F \succ_1 F'$.

By CR2, F' is in $C'_{\exists x\sigma}[\underline{P}/\underline{\mathcal{C}}]$. Since $abbr(P_\beta, F') \succ I(u, K)$ with length $< r$, by the subsidiary induction hypothesis, K is in $C'_{\sigma[x/u]}[\underline{P}/\underline{\mathcal{C}}]$.

Case 2. $F^* = F$.

Then $F \succ I(u, K)$. Hence K is in $C'_{\sigma[x/u]}[\underline{P}/\underline{\mathcal{C}}]$ since F is in $C'_{\exists x\sigma}[\underline{P}/\underline{\mathcal{C}}]$.

Now, suppose $abbr(P_\beta, F) \succ abbr(P_{\beta_1}, abbr(\ldots, abbr(P_{\beta_k}, I(u, K)^{\exists x\sigma^*})))$. Then there is a finite reduction sequence from $abbr(P_\beta, F)$ to $abbr(P_{\beta_1}, abbr(\ldots, abbr(P_{\beta_k}, I(u, K)^{\exists x\sigma^*})))$ with length $r \geq 0$. We will show that $abbr(P_{\beta_1}, abbr(\ldots, abbr(P_{\beta_k}, K)))$ is in $C'_{\sigma[x/u]}[\underline{P}/\underline{\mathcal{C}}]$ by induction on r.

$r = 0$: Then $\beta_1 = \beta$ and $F = abbr(P_{\beta_2}, abbr(\ldots, abbr(P_{\beta_k}, I(u, K))))^{\exists x\sigma[\underline{P}/\underline{T}]}$. Since F is in $C'_{\exists x\sigma}[\underline{P}/\underline{\mathcal{C}}]$, $abbr(P_{\beta_2}, abbr(\ldots, abbr(P_{\beta_k}, K)))^{\sigma[x/u][\underline{P}/\underline{T}]}$ is in $C'_{\sigma[x/u]}[\underline{P}/\underline{\mathcal{C}}]$ which is a CR^+ by the induction hypothesis. Hence $abbr(P_{\beta_1}, abbr(\ldots, abbr(P_{\beta_k}, K)))^{\sigma[x/u][\underline{P}/\underline{T}]}$ is in $C'_{\sigma[x/u]}[\underline{P}/\underline{\mathcal{C}}]$.

$r > 0$: Suppose F^* is the immediate reduct in the reduction sequence.

Case 1. $F^* = abbr(P_\beta, F')$ where $F \succ_1 F'$.

This case follows by the subsidiary induction hypothesis.

Case 2. $F^* = F$.

Then $F \succ abbr(P_{\beta_1}, abbr(\ldots, abbr(P_{\beta_k}, I(u,K))))$ and so this case follows by the fact that F is in $C'_{\exists x\sigma}[\underline{P}/\underline{\mathcal{C}}]$.

Thus we can conclude that $abbr(P_\beta, F)$ is in $C'_{\exists x\sigma}[\underline{P}/\underline{\mathcal{C}}]$.

Similarly for $C'_{\exists_2 Q\sigma}[\underline{P}/\underline{\mathcal{C}}]$. □

Definition 5.2.7. *Let α be a formula, $\underline{P} = P_1^{m_1}, \ldots, P_n^{m_n}$ be distinct predicate variables, $\underline{T} = T_1, \ldots, T_n$, where $T_i = \lambda z_1^i, \ldots, z_{m_i}^i \delta_i$, $1 \le i \le n$, be abstraction terms, and $\underline{\mathcal{C}} = \mathcal{C}_1, \ldots, \mathcal{C}_n$ be collections of CR^+s corresponding to $T_1, \ldots, T_n$, respectively.*

We define a set $C^+_\alpha[P_1/\mathcal{C}_1, \ldots, P_n/\mathcal{C}_n]$, which can be written as $C^+_\alpha[\underline{P}/\underline{\mathcal{C}}]$, of terms of type $[\alpha[\underline{P}/\underline{T}]]$ inductively as follows.

α is an atomic formula:

If α is $P_\sigma(\underline{t})$ for some abbreviation predicate P_σ and some individual terms $\underline{t}$, $C^+_\alpha[\underline{P}/\underline{\mathcal{C}}]$ is the set of all terms F in $SN_{P_\sigma(\underline{t})}$ such that if $F \succ abbr(P_\sigma, G^{\sigma[\underline{x}/\underline{t}]})^{P_\sigma(\underline{t})}$, where $fv(\sigma) = \{\underline{x}\}$, then G is in $C'_{\sigma[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$;

otherwise $C^+_\alpha[\underline{P}/\underline{\mathcal{C}}] = C'_\alpha[\underline{P}/\underline{\mathcal{C}}]$.

α is not an atomic formula:

$C^+_\alpha[\underline{P}/\underline{\mathcal{C}}]$ is defined in the same way as $C'_\alpha[\underline{P}/\underline{\mathcal{C}}]$ in Definition 5.2.5 (by replacing every $C'_\sigma[\underline{P}/\underline{\mathcal{C}}]$ by $C^+_\sigma[\underline{P}/\underline{\mathcal{C}}]$).

Note. It can be easily checked by induction on α that if α does not contain any abbreviation predicates, then $C^+_\alpha[\underline{P}/\underline{\mathcal{C}}] = C'_\alpha[\underline{P}/\underline{\mathcal{C}}]$.

From the above definition, we can see that we need the set $C'_{\sigma[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$ in defining $C^+_{P_\sigma(\underline{t})}[\underline{P}/\underline{\mathcal{C}}]$. Actually, we want to use the set $C^+_{\sigma[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$ in the definition but we cannot do that since we use it in the basic case of an inductive definition, so the set must already exist. By the restriction that every formula which can be abbreviated must not contain any abbreviation predicates and the above Note, ultimately, we have what we want.

Lemma 5.2.8. *Let α be a formula, $\underline{P} = P_1^{m_1}, \ldots, P_n^{m_n}$ be distinct predicate variables, $\underline{T} = T_1, \ldots, T_n$, where $T_i = \lambda z_1^i, \ldots, z_{m_i}^i \delta_i$, $1 \leq i \leq n$, be abstraction terms, and $\underline{\mathcal{C}} = \mathcal{C}_1, \ldots, \mathcal{C}_n$ be collections of CR^+s corresponding to $T_1, \ldots, T_n$, respectively.*

Then $C_\alpha^+[\underline{P}/\underline{\mathcal{C}}]$ is a CR^+ of type $[\alpha[\underline{P}/\underline{T}]]$.

Proof. We will prove by induction on α. It follows by Lemma 5.2.3 and the induction hypothesis if α is $\alpha_1 \wedge \alpha_2$ or $\alpha_1 \supset \alpha_2$. If α is $\alpha_1 \vee \alpha_2$, $\forall x\sigma$, $\exists x\sigma$, $\forall_2 Q\sigma$, or $\exists_2 Q\sigma$, the proof is similar to the proof of Lemma 5.2.6.

Suppose α is an atomic formula. It follows by Lemma 5.2.6 if α is not of the form $P_\sigma(\underline{t})$.

Suppose $\alpha = P_\sigma(\underline{t})$, where $fv(\sigma) = \{\underline{x}\}$.

For the proofs of CR0, CR1, and CR2, we suppose F is in $C^+_{P_\sigma(\underline{t})}[\underline{P}/\underline{\mathcal{C}}]$, so F is strongly normalizable and if $F \succ abbr(P_\sigma, G^{\sigma[\underline{x}/\underline{t}]})^{P_\sigma(\underline{t})}$, then G is in $C'_{\sigma[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$.

CR0: Suppose $F' \equiv F$. Since F is strongly normalizable, so is F' by Lemma 3.3.2. Suppose $F' \succ abbr(P_\sigma, G^{\sigma[\underline{x}/\underline{t}]})^{P_\sigma(\underline{t})}$. By Corollary 3.3.3, $F \succ abbr(P_\sigma, G')$ for some term G' such that $G' \equiv G$. Then G' is in $C'_{\sigma[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$, which is a CR^+ by Lemma 5.2.6, and so is G by CR0. Thus F' is in $C^+_{P_\sigma(\underline{t})}[\underline{P}/\underline{\mathcal{C}}]$.

CR1: It is clear from the definition.

CR2: Suppose $F \succ_1 F'$. Since F is strongly normalizable, so is F'. Suppose $F' \succ abbr(P_\sigma, G^{\sigma[\underline{x}/\underline{t}]})^{P_\sigma(\underline{t})}$. Then $F \succ abbr(P_\sigma, G^{\sigma[\underline{x}/\underline{t}]})^{P_\sigma(\underline{t})}$. Hence G is in $C'_{\sigma[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$. Thus F' is in $C^+_{P_\sigma(\underline{t})}[\underline{P}/\underline{\mathcal{C}}]$.

CR3: Suppose F is neutral and all its immediate reducts are in $C^+_{P_\sigma(\underline{t})}[\underline{P}/\underline{\mathcal{C}}]$. Then every immediate reduct of F is strongly normalizable, and so is F. Suppose $F \succ abbr(P_\sigma, G^{\sigma[\underline{x}/\underline{t}]})^{P_\sigma(\underline{t})}$. Since F is neutral, F and $abbr(P_\sigma, G)$ are not of the same form, so there is a finite reduction sequence from F to $abbr(P_\sigma, G^{\sigma[\underline{x}/\underline{t}]})^{P_\sigma(\underline{t})}$ with length ≥ 1. Suppose F' is the immediate reduct of F in the sequence. Then $F' \succ abbr(P_\sigma, G^{\sigma[\underline{x}/\underline{t}]})^{P_\sigma(\underline{t})}$. Since F' is in $C^+_{P_\sigma(\underline{t})}[\underline{P}/\underline{\mathcal{C}}]$, G is in $C'_{\sigma[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$. Thus F is in $C^+_{P_\sigma(\underline{t})}[\underline{P}/\underline{\mathcal{C}}]$.

We have shown that $C^+_{P_\sigma(\underline{t})}[\underline{P}/\underline{\mathcal{C}}]$ is a CR. Now, suppose $F^{P_\sigma(\underline{t})}$ is in $C^+_{P_\sigma(\underline{t})}[\underline{P}/\underline{\mathcal{C}}]$ and let P_β be given. It remains to show that $abbr(P_\beta, F^{P_\sigma(\underline{t})})^{P_\sigma(\underline{t})}$ is in $C^+_{P_\sigma(\underline{t})}[\underline{P}/\underline{\mathcal{C}}]$.

Since F is strongly normalizable and every reduction sequence beginning with $abbr(P_\beta, F)$ gives a reduction sequence beginning with F, $abbr(P_\beta, F)$ is also strongly normalizable. Suppose $abbr(P_\beta, F^{P_\sigma(\underline{t})}) \succ abbr(P_\sigma, H^{\sigma[\underline{x}/\underline{t}]})^{P_\sigma(\underline{t})}$. Then there is a finite reduction sequence from $abbr(P_\beta, F)$ to $abbr(P_\sigma, H)$ with length $r \geq 1$. We will show that H is in $C'_{\sigma[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$ by induction on r.

$r = 1$: Then $abbr(P_\sigma, H)$ is an immediate reduct of $abbr(P_\beta, F)$. Since $F^{P_\sigma(\underline{t})}$ and $H^{\sigma[\underline{x}/\underline{t}]}$ are not of the same type, H is not a reduct of F. Hence $abbr(P_\beta, F) \succ_1 F = abbr(P_\sigma, H)$. Since F is in $C^+_{P_\sigma(\underline{t})}[\underline{P}/\underline{\mathcal{C}}]$, H is in $C'_{\sigma[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$.

$r > 1$: Suppose F^* is the immediate reduct of F in the sequence.

Case 1. $F^* = abbr(P_\beta, F')$ where $F \succ_1 F'$.

This case follows by the subsidiary induction hypothesis.

Case 2. $F^* = F$.

Then $F \succ abbr(P_\sigma, H)$. Since F is in $C^+_{P_\sigma(\underline{t})}[\underline{P}/\underline{\mathcal{C}}]$, H is in $C'_{\sigma[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$.

Hence $abbr(P_\beta, F)$ is in $C^+_{P_\sigma(\underline{t})}[\underline{P}/\underline{\mathcal{C}}]$.

Thus we can conclude that $C^+_{P_\sigma(\underline{t})}[\underline{P}/\underline{\mathcal{C}}]$ is a CR^+. □

Lemmas 4.2.4, 4.2.9, 4.2.10, and 4.2.11 also hold if we replace every CR by CR^+ and every $C_\alpha[\underline{P}/\underline{\mathcal{C}}]$ by $C^+_\alpha[\underline{P}/\underline{\mathcal{C}}]$ and the proofs are similar.

The following lemma is a new lemma which is needed for the proof of the *strong normalization theorem*.

Lemma 5.2.9. *Let $\underline{P} = P_1^{m_1}, \ldots, P_n^{m_n}$ be distinct predicate variables, $\underline{T} = T_1, \ldots, T_n$, where $T_i = \lambda z_1^i, \ldots, z_{m_i}^i \delta_i$, $1 \leq i \leq n$, be abstraction terms, $\underline{\mathcal{C}} = \mathcal{C}_1, \ldots, \mathcal{C}_n$ be collections of CR^+s corresponding to $T_1, \ldots, T_n$, respectively, $F^{\beta[\underline{P}/\underline{T}](R/U)}$ be a term of type $[\beta[\underline{P}/\underline{T}](R/U)]$, where R is an r-ary predicate variable which is not in $\{\underline{P}\} \cup FV(\underline{T})$ and $U = \lambda z_1, \ldots, z_r\alpha$ is an abstraction term with $FV(\alpha) = \varnothing$.*

Then $F^{\beta[\underline{P}/\underline{T}](R/U)}$ is in $C^+_{\beta(R/U)}[\underline{P}/\underline{\mathcal{C}}]$ if and only if $abbr(P_\alpha, F)^{\beta[\underline{P}/\underline{T}](R/P_\alpha)}$ is in $C^+_{\beta(R/P_\alpha)}[\underline{P}/\underline{\mathcal{C}}]$.

Proof. We will prove by induction on β.

Suppose $R \notin FV(\beta)$. Then $[\beta[\underline{P}/\underline{T}](R/U)] = [\beta[\underline{P}/\underline{T}](R/P_\alpha)] = [\beta[\underline{P}/\underline{T}]]$.

If $F^{\beta[\underline{P}/\underline{T}]}$ is in $C^+_\beta[\underline{P}/\underline{\mathcal{C}}]$, which is a CR^+ by Lemma 5.2.8, then $abbr(P_\alpha, F^{\beta[\underline{P}/\underline{T}]})^{\beta[\underline{P}/\underline{T}]}$ is in $C^+_\beta[\underline{P}/\underline{\mathcal{C}}]$.

If $abbr(P_\alpha, F^{\beta[\underline{P}/\underline{T}]})^{\beta[\underline{P}/\underline{T}]}$ is in $C^+_\beta[\underline{P}/\underline{\mathcal{C}}]$, then, by CR2, so is F since $abbr(P_\alpha, F) \succ_1 F$.

Now, suppose $R \in FV(\beta)$.

$\beta = R(\underline{t})$:

Suppose $F^{\beta[\underline{P}/\underline{T}](R/U)}$ is in $C^+_{\beta(R/U)}[\underline{P}/\underline{\mathcal{C}}]$ i.e. $F^{\alpha[\underline{z}/\underline{t}]}$ is in $C^+_{\alpha[\underline{z}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$ (= $C'_{\alpha[\underline{z}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$ by Note on page 165). Then F is strongly normalizable. Since every reduction sequence beginning with $abbr(P_\alpha, F)$ gives a reduction sequence beginning with F, $abbr(P_\alpha, F)$ is also strongly normalizable.

Suppose $abbr(P_\alpha, F) \succ abbr(P_\alpha, F'^{\alpha[\underline{z}/\underline{t}]})$. Then $F \succ F'$. Since F is in $C'_{\alpha[\underline{z}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$, so is F'.

Thus $abbr(P_\alpha, F)$ is in $C^+_{P_\alpha(\underline{t})}[\underline{P}/\underline{\mathcal{C}}]$.

The converse follows straightforwardly by the definition and Note on page 165.

$\beta = \beta_1 \wedge \beta_2$:

Suppose $F^{\beta[\underline{P}/\underline{T}](R/U)}$ is in $C^+_{\beta(R/U)}[\underline{P}/\underline{C}]$. We will show that $\pi_1 abbr(P_\alpha, F)$ is in $C^+_{\beta_1(R/P_\alpha)}[\underline{P}/\underline{\mathcal{C}}]$ by induction on $N(F)$. Since $\pi_1 abbr(P_\alpha, F)$ is neutral, we will show that all its immediate reducts are in $C^+_{\beta_1(R/P_\alpha)}[\underline{P}/\underline{\mathcal{C}}]$. It follows by the subsidiary induction hypothesis if an immediate reduct is obtained by reducing F.

The other immediate reduct is $abbr(P_\alpha, \pi_1 F)$. Since F is in $C^+_{\beta(R/U)}[\underline{P}/\underline{\mathcal{C}}]$, $\pi_1 F$ is in $C^+_{\beta_1(R/U)}[\underline{P}/\underline{\mathcal{C}}]$. By the main induction hypothesis, $abbr(P_\alpha, \pi_1 F)$ is in $C^+_{\beta_1(R/P_\alpha)}[\underline{P}/\underline{\mathcal{C}}]$.

Similarly, we can show that $\pi_2 abbr(P_\alpha, F)$ is in $C^+_{\beta_2(R/P_\alpha)}[\underline{P}/\underline{\mathcal{C}}]$. Thus $abbr(P_\alpha, F)$ is in $C^+_{\beta(R/P_\alpha)}[\underline{P}/\underline{\mathcal{C}}]$.

Now, suppose $abbr(P_\alpha, F)$ is in $C^+_{\beta(R/P_\alpha)}[\underline{P}/\underline{\mathcal{C}}]$. Then $\pi_1 abbr(P_\alpha, F)$ is in $C^+_{\beta_1(R/P_\alpha)}[\underline{P}/\underline{\mathcal{C}}]$. Since $\pi_1 abbr(P_\alpha, F) \succ_1 abbr(P_\alpha, \pi_1 F)$, $abbr(P_\alpha, \pi_1 F)$ is in $C^+_{\beta_1(R/P_\alpha)}[\underline{P}/\underline{\mathcal{C}}]$ by CR2. Hence $\pi_1 F$ is in $C^+_{\beta_1(R/U)}[\underline{P}/\underline{\mathcal{C}}]$ by the induction hypothesis. Similarly, $\pi_2 F$ is in $C^+_{\beta_2(R/U)}[\underline{P}/\underline{\mathcal{C}}]$. Hence F is in $C^+_{\beta(R/U)}[\underline{P}/\underline{\mathcal{C}}]$.

$\beta = \beta_1 \supset \beta_2$:

Suppose F is in $C^+_{\beta(R/U)}[\underline{P}/\underline{\mathcal{C}}]$. We want to show that $abbr(P_\alpha, F)$ is in $C^+_{(\beta_1 \supset \beta_2)(R/P_\alpha)}[\underline{P}/\underline{\mathcal{C}}]$.

Let G be in $C^+_{\beta_1(R/P_\alpha)}[\underline{P}/\underline{\mathcal{C}}]$. We will prove that $abbr(P_\alpha, F)(G)$ is in $C^+_{\beta_2(R/P_\alpha)}[\underline{P}/\underline{\mathcal{C}}]$ by induction on $N(F) + N(G)$. Since $abbr(P_\alpha, F)(G)$ is neutral, we will show that all its immediate reducts are in $C^+_{\beta_2(R/P_\alpha)}[\underline{P}/\underline{\mathcal{C}}]$. It follows by the subsidiary induction hypothesis if an immediate reduct is obtained by reducing F or G. The other immediate reduct is $abbr(P_\alpha, F(unabbr(X^{\beta_1[\underline{P}/\underline{T}](R/U)}.$ $X^{\beta_1[\underline{P}/\underline{T}](R/U)}, G)))$.

First, we will prove that $unabbr(X^{\beta_1[\underline{P}/\underline{T}](R/U)}.X^{\beta_1[\underline{P}/\underline{T}](R/U)}, G)$ is in $C^+_{\beta_1(R/U)}[\underline{P}/\underline{\mathcal{C}}]$ by induction on $N(G)$. It follows by Note on page 158 if $R \notin FV(\beta_1[\underline{P}/\underline{T}])$. Suppose $R \in FV(\beta_1[\underline{P}/\underline{T}])$. We will show that all its immediate reducts are in $C^+_{\beta_1(R/U)}[\underline{P}/\underline{\mathcal{C}}]$. It follows by the subsidiary induction hypothesis if an immediate reduct is obtained by reducing G. The remaining case is when $G = abbr(P_\alpha, H^{\beta_1[\underline{P}/\underline{T}](R/U)})$ and the immediate reduct is H. Since G is in $C^+_{\beta_1(R/P_\alpha)}[\underline{P}/\underline{\mathcal{C}}]$, by the main induction hypothesis, H is in $C^+_{\beta_1(R/U)}[\underline{P}/\underline{\mathcal{C}}]$.

Thus we can conclude that $unabbr(X^{\beta_1[\underline{P}/\underline{T}](R/U)}.X^{\beta_1[\underline{P}/\underline{T}](R/U)}, G)$ is in $C^+_{\beta_1(R/U)}[\underline{P}/\underline{\mathcal{C}}]$.

Since F is in $C^+_{(\beta_1 \supset \beta_2)(R/U)}[\underline{P}/\underline{\mathcal{C}}]$, $F(unabbr(X^{\beta_1[\underline{P}/\underline{T}](R/U)}.X^{\beta_1[\underline{P}/\underline{T}](R/U)}, G))$ is in $C^+_{\beta_2(R/U)}[\underline{P}/\underline{\mathcal{C}}]$. Thus $abbr(P_\alpha, F(unabbr(X^{\beta_1[\underline{P}/\underline{T}](R/U)}.X^{\beta_1[\underline{P}/\underline{T}](R/U)}, G)))$ is in $C^+_{\beta_2(R/P_\alpha)}[\underline{P}/\underline{\mathcal{C}}]$ by the induction hypothesis.

Now, suppose $abbr(P_\alpha, F)$ is in $C^+_{(\beta_1 \supset \beta_2)(R/P_\alpha)}[\underline{P}/\underline{\mathcal{C}}]$. We want to show that F is in $C^+_{(\beta_1 \supset \beta_2)(R/U)}[\underline{P}/\underline{\mathcal{C}}]$. Let G be in $C^+_{\beta_1(R/U)}[\underline{P}/\underline{\mathcal{C}}]$.

By the induction hypothesis, $abbr(P_\alpha, G)$ is in $C^+_{\beta_1(R/P_\alpha)}[\underline{P}/\underline{\mathcal{C}}]$. Hence $abbr(P_\alpha, F)(abbr(P_\alpha, G))$ is in $C^+_{\beta_2(R/P_\alpha)}[\underline{P}/\underline{\mathcal{C}}]$. Since

$abbr(P_\alpha, F)(abbr(P_\alpha, G))$

$$\begin{aligned} &\succ_1 \quad abbr(P_\alpha, F(unabbr(X^{\beta_1[\underline{P}/\underline{T}](R/U)}.X^{\beta_1[\underline{P}/\underline{T}](R/U)}, abbr(P_\alpha, G)))) \\ &\succ \quad abbr(P_\alpha, F(G)), \end{aligned}$$

$abbr(P_\alpha, F(G))$ is in $C^+_{\beta_2(R/P_\alpha)}[\underline{P}/\underline{\mathcal{C}}]$. By the induction hypothesis, $F(G)$ is in $C^+_{\beta_2(R/U)}[\underline{P}/\underline{\mathcal{C}}]$. Thus F is in $C^+_{\beta(R/U)}[\underline{P}/\underline{\mathcal{C}}]$.

$\beta = \beta_1 \vee \beta_2$:

Suppose F is in $C^+_{\beta(R/U)}[\underline{P}/\underline{\mathcal{C}}]$. We want to show that $abbr(P_\alpha, F)$ is in $C^+_{(\beta_1 \vee \beta_2)(R/P_\alpha)}[\underline{P}/\underline{\mathcal{C}}]$.

Let $[\gamma]$ be a type, C be a CR^+ of type $[\gamma]$, F_1 and F_2 be terms in $C^+_{\beta_1(R/P_\alpha)}[\underline{P}/\underline{\mathcal{C}}] \supset C$ and $C^+_{\beta_2(R/P_\alpha)}[\underline{P}/\underline{\mathcal{C}}] \supset C$, respectively, and $X_1^{\beta_1[\underline{P}/\underline{T}](R/P_\alpha)}$ and $X_2^{\beta_2[\underline{P}/\underline{T}](R/P_\alpha)}$ be term variables which are not equivalent to any free term variables of F_1 and F_2, respectively.

We will prove that $\oplus(X_1^{\beta_1[\underline{P}/\underline{T}](R/P_\alpha)}.F_1(X_1^{\beta_1[\underline{P}/\underline{T}](R/P_\alpha)}), X_2^{\beta_2[\underline{P}/\underline{T}](R/P_\alpha)}.$ $F_2(X_2^{\beta_2[\underline{P}/\underline{T}](R/P_\alpha)}), abbr(P_\alpha, F))$ is in C by induction on $N(F_1(X_1^{\beta_1[\underline{P}/\underline{T}](R/P_\alpha)})) + N(F_2(X_2^{\beta_2[\underline{P}/\underline{T}](R/P_\alpha)})) + N(F)$. As usual, we will show that all its immediate

reducts are in C. It follows by the subsidiary induction hypothesis if an immediate reduct is obtained by reducing $F_1(X_1^{\beta_1[\underline{P}/\underline{T}](R/P_\alpha)})$, $F_2(X_2^{\beta_2[\underline{P}/\underline{T}](R/P_\alpha)})$, or F. The other immediate reduct is $M = \oplus(Y_1^{\beta_1[\underline{P}/\underline{T}](R/U)}.$
$unabbr(Y_1^{\beta_1[\underline{P}/\underline{T}](R/U)\supset\gamma}.Y_1^{\beta_1[\underline{P}/\underline{T}](R/U)\supset\gamma}, F_1)(Y_1^{\beta_1[\underline{P}/\underline{T}](R/U)}), Y_2^{\beta_2[\underline{P}/\underline{T}](R/U)}.$
$unabbr(Y_2^{\beta_2[\underline{P}/\underline{T}](R/U)\supset\gamma}.Y_2^{\beta_2[\underline{P}/\underline{T}](R/U)\supset\gamma}, F_2)(Y_2^{\beta_2[\underline{P}/\underline{T}](R/U)}), F)$, where $Y_i^{\beta_i[\underline{P}/\underline{T}](R/U)}$ is the first term variable which is not equivalent to any free term variable of F_i, $i = 1, 2$.

First, we will prove by induction on $N(F_1)$ that $unabbr(Y_1^{\beta_1[\underline{P}/\underline{T}](R/U)\supset\gamma}.$ $Y_1^{\beta_1[\underline{P}/\underline{T}](R/U)\supset\gamma}, F_1)$ is in $C^+_{\beta_1(R/U)}[\underline{P}/\underline{\mathcal{C}}] \supset C$, which is a CR^+ by Lemmas 5.2.3 and 5.2.6. It follows by Note on page 158 if $R \notin FV(\beta_1)$. Suppose $R \in FV(\beta_1)$. We will show that all its immediate reducts are in $C^+_{\beta_1(R/U)}[\underline{P}/\underline{\mathcal{C}}] \supset C$. It follows by the subsidiary induction hypothesis if an immediate reduct is obtained by reducing F_1. The remaining case is when $F_1 = abbr(P_\alpha, K^{\beta_1[\underline{P}/\underline{T}](R/U)\supset\gamma})$ and the immediate reduct is K. To show that K is in $C^+_{\beta_1(R/U)}[\underline{P}/\underline{\mathcal{C}}] \supset C$, let H be in $C^+_{\beta_1(R/U)}[\underline{P}/\underline{\mathcal{C}}]$. By the main induction hypothesis, $abbr(P_\alpha, H)$ is in $C^+_{\beta_1(R/P_\alpha)}[\underline{P}/\underline{\mathcal{C}}]$. Since F_1 i.e. $abbr(P_\alpha, K)$ is in $C^+_{\beta_1(R/P_\alpha)}[\underline{P}/\underline{\mathcal{C}}] \supset C$, $abbr(P_\alpha, K)(abbr(P_\alpha, H))$ is in C. Since

$$\begin{aligned} abbr(P_\alpha, K)(abbr(P_\alpha, H)) &\succ_1 abbr(P_\alpha, K(unabbr(Z^{\beta_1[\underline{P}/\underline{T}](R/U)}.Z^{\beta_1[\underline{P}/\underline{T}](R/U)},\\ &\qquad abbr(P_\alpha, H))))\\ &\succ_1 abbr(P_\alpha, K(H)^\gamma)^\gamma\\ &\succ_1 K(H), \end{aligned}$$

where $Z^{\beta_1[\underline{P}/\underline{T}](R/U)}$ is the first term variable of type $[\beta_1[\underline{P}/\underline{T}](R/U)]$, $K(H)$ is in C. Hence K is in $C^+_{\beta_1(R/U)}[\underline{P}/\underline{\mathcal{C}}] \supset C$. Thus we can conclude that $unabbr(Y_1^{\beta_1[\underline{P}/\underline{T}](R/U)\supset\gamma}.Y_1^{\beta_1[\underline{P}/\underline{T}](R/U)\supset\gamma}, F_1)$ is in $C^+_{\beta_1(R/U)}[\underline{P}/\underline{\mathcal{C}}] \supset C$.

Similarly, we can show that $unabbr(Y_2^{\beta_2[\underline{P}/\underline{T}](R/U)\supset\gamma}.Y_2^{\beta_2[\underline{P}/\underline{T}](R/U)\supset\gamma}, F_2)$ is in $C^+_{\beta_2(R/U)}[\underline{P}/\underline{\mathcal{C}}] \supset C$.

Since F is in $C^+_{\beta(R/U)}[\underline{P}/\underline{\mathcal{C}}]$, M is in C.

Now, suppose $abbr(P_\alpha, F)$ is in $C^+_{(\beta_1\vee\beta_2)(R/P_\alpha)}[\underline{P}/\underline{\mathcal{C}}]$.

Let $[\gamma]$ be a type, C be a CR^+ of type $[\gamma]$, F_1 and F_2 be terms in $C^+_{\beta_1(R/U)}[\underline{P}/\underline{\mathcal{C}}]$ $\supset C$ and $C^+_{\beta_2(R/U)}[\underline{P}/\underline{\mathcal{C}}] \supset C$, respectively, and $X_1^{\beta_1[\underline{P}/\underline{T}](R/U)}$ and $X_2^{\beta_2[\underline{P}/\underline{T}](R/U)}$ be term variables which are not equivalent to any free term variables of F_1 and F_2, respectively.

We want to show that $M = \oplus(X_1^{\beta_1[\underline{P}/\underline{T}](R/U)}.F_1(X_1^{\beta_1[\underline{P}/\underline{T}](R/U)}), X_2^{\beta_2[\underline{P}/\underline{T}](R/U)}.$ $F_2(X_2^{\beta_2[\underline{P}/\underline{T}](R/U)}), F)$ is in C.

First, we will prove that $abbr(P_\alpha, F_1)$ is in $C^+_{\beta_1(R/P_\alpha)}[\underline{P}/\underline{\mathcal{C}}] \supset C$. Let G be in $C^+_{\beta_1(R/P_\alpha)}[\underline{P}/\underline{\mathcal{C}}]$. We will prove that $abbr(P_\alpha, F_1)(G)$ is in C by induction on $N(F_1)+N(G)$. We will show that all its immediate reducts are in C. It follows by the subsidiary induction hypothesis if an immediate reduct is obtained by reducing F_1 or G. The other immediate reduct is $abbr(P_\alpha, F_1(unabbr(Z^{\beta_1[\underline{P}/\underline{T}](R/U)}.$ $Z^{\beta_1[\underline{P}/\underline{T}](R/U)}, G)))^\gamma$.

As in the proof of the case $\beta = \beta_1 \supset \beta_2$ on page 169, we can prove by induction on $N(G)$ that $unabbr(Z^{\beta_1[\underline{P}/\underline{T}](R/U)}.Z^{\beta_1[\underline{P}/\underline{T}](R/U)}, G)$ is in $C^+_{\beta_1(R/U)}[\underline{P}/\underline{\mathcal{C}}]$. Since F_1 is in $C^+_{\beta_1(R/U)}[\underline{P}/\underline{\mathcal{C}}] \supset C$, $F_1(unabbr(Z^{\beta_1[\underline{P}/\underline{T}](R/U)}.Z^{\beta_1[\underline{P}/\underline{T}](R/U)}, G))$ is in C, and so is $abbr(P_\alpha, F_1(unabbr(Z^{\beta_1[\underline{P}/\underline{T}](R/U)}.Z^{\beta_1[\underline{P}/\underline{T}](R/U)}, G))^\gamma)^\gamma$ since C is a CR^+.

Thus $abbr(P_\alpha, F_1)(G)$ is in C and so $abbr(P_\alpha, F_1)$ is in $C^+_{\beta_1(R/P_\alpha)}[\underline{P}/\underline{\mathcal{C}}] \supset C$.

Similarly, we can show that $abbr(P_\alpha, F_2)$ is in $C^+_{\beta_2(R/P_\alpha)}[\underline{P}/\underline{\mathcal{C}}] \supset C$.

Let $Z_1^{\beta_1[\underline{P}/\underline{T}](R/P_\alpha)}$ and $Z_2^{\beta_2[\underline{P}/\underline{T}](R/P_\alpha)}$ be term variables which are not equivalent to any free term variables of F_1 and F_2, respectively. Since $abbr(P_\alpha, F)$ is in $C^+_{(\beta_1\vee\beta_2)(R/P_\alpha)}[\underline{P}/\underline{\mathcal{C}}]$, $N = \oplus(Z_1^{\beta_1[\underline{P}/\underline{T}](R/P_\alpha)}.abbr(P_\alpha, F_1)(Z_1^{\beta_1[\underline{P}/\underline{T}](R/P_\alpha)}),$ $Z_2^{\beta_2[\underline{P}/\underline{T}](R/P_\alpha)}.abbr(P_\alpha, F_2)(Z_2^{\beta_2[\underline{P}/\underline{T}](R/P_\alpha)}), abbr(P_\alpha, F))$ is in C. Since

$$
\begin{aligned}
N \quad \succ_1 \quad & \oplus(Y_1^{\beta_1[\underline{P}/\underline{T}](R/U)}.unabbr(Y_1^{\beta_1[\underline{P}/\underline{T}](R/U)\supset\gamma}.Y_1^{\beta_1[\underline{P}/\underline{T}](R/U)\supset\gamma}, abbr(P_\alpha, F_1)) \\
& (Y_1^{\beta_1[\underline{P}/\underline{T}](R/U)}), Y_2^{\beta_2[\underline{P}/\underline{T}](R/U)}.unabbr(Y_2^{\beta_1[\underline{P}/\underline{T}](R/U)\supset\gamma}.Y_2^{\beta_2[\underline{P}/\underline{T}](R/U)\supset\gamma}, \\
& abbr(P_\alpha, F_2))(Y_2^{\beta_2[\underline{P}/\underline{T}](R/U)}), F) \\
\succ \quad & \oplus(Y_1^{\beta_1[\underline{P}/\underline{T}](R/U)}.F_1(Y_1^{\beta_1[\underline{P}/\underline{T}](R/U)}), Y_2^{\beta_2[\underline{P}/\underline{T}](R/U)}.F_2(Y_2^{\beta_2[\underline{P}/\underline{T}](R/U)}), F) \\
\equiv \quad & M,
\end{aligned}
$$

where $Y_i^{\beta_i[\underline{P}/\underline{T}](R/U)}$ is the first term variable which is not equivalent to any free term variable of F_i, $i = 1, 2$, by CR0 and CR2, M is in C.

$\beta = \forall x\sigma$:

Suppose F is in $C^+_{\beta(R/U)}[\underline{P}/\underline{\mathcal{C}}]$. We want to show that $abbr(P_\alpha, F)$ is in $C^+_{\forall x\sigma(R/P_\alpha)}[\underline{P}/\underline{\mathcal{C}}]$. Let t be an individual term. We will prove that $abbr(P_\alpha, F)(t)$ is in $C^+_{\sigma(R/P_\alpha)[x/t]}[\underline{P}/\underline{\mathcal{C}}]$ by induction on $N(F)$. We will show that all its immediate reducts are in $C^+_{\sigma(R/P_\alpha)[x/t]}[\underline{P}/\underline{\mathcal{C}}]$. It follows by the subsidiary induction hypothesis if an immediate reduct is obtained by reducing F.

The other immediate reduct is $abbr(P_\alpha, F(t))$. Since F is in $C^+_{\forall x\sigma(R/U)}[\underline{P}/\underline{\mathcal{C}}]$, $F(t)$ is in $C^+_{\sigma(R/U)[x/t]}[\underline{P}/\underline{\mathcal{C}}]$ i.e. $C^+_{\sigma[x/t](R/U)}[\underline{P}/\underline{\mathcal{C}}]$. Hence, by the main induction hypothesis, $abbr(P_\alpha, F(t))$ is in $C^+_{\sigma[x/t](R/P_\alpha)}[\underline{P}/\underline{\mathcal{C}}]$ i.e. $C^+_{\sigma(R/P_\alpha)[x/t]}[\underline{P}/\underline{\mathcal{C}}]$.

Now, suppose $abbr(P_\alpha, F)$ is in $C^+_{\forall x\sigma(R/P_\alpha)}[\underline{P}/\underline{\mathcal{C}}]$. Let t be an individual term. We want to show that $F(t)$ is in $C^+_{\sigma(R/U)[x/t]}[\underline{P}/\underline{\mathcal{C}}]$. Since $abbr(P_\alpha, F)$ is in $C^+_{\forall x\sigma(R/P_\alpha)}[\underline{P}/\underline{\mathcal{C}}]$, $abbr(P_\alpha, F)(t)$ is in $C^+_{\sigma(R/P_\alpha)[x/t]}[\underline{P}/\underline{\mathcal{C}}]$ i.e. $C^+_{\sigma[x/t](R/P_\alpha)}[\underline{P}/\underline{\mathcal{C}}]$, and so is $abbr(P_\alpha, F(t))$ since it is an immediate reduct of $abbr(P_\alpha, F)(t)$. By the induction hypothesis, $F(t)$ is in $C^+_{\sigma[x/t](R/U)}[\underline{P}/\underline{\mathcal{C}}]$ i.e. $C^+_{\sigma(R/U)[x/t]}[\underline{P}/\underline{\mathcal{C}}]$.

Similarly for the case $\beta = \forall_2 Q\sigma$.

$\beta = \exists x\sigma$:

First, suppose F is in $C^+_{\exists x\sigma(R/U)}[\underline{P}/\underline{\mathcal{C}}]$. We want to show that $abbr(P_\alpha, F)$ is in

$C^+_{\exists x\sigma(R/P_\alpha)}[\underline{P}/\underline{\mathcal{C}}]$.

Note that since $R \in FV(\beta)$, every reduct of $abbr(P_\alpha, F)$ must be obtained by reducing F, and so it must be of the same form as $abbr(P_\alpha, F)$.

Suppose $abbr(P_\alpha, F) \succ abbr(P_{\alpha_1}, abbr(\ldots, abbr(P_{\alpha_k}, I(u,H)^{\exists x\sigma^*})))$. Then $\alpha = \alpha_1$ and $F \succ abbr(P_{\alpha_2}, abbr(\ldots, abbr(P_{\alpha_k}, I(u,H)^{\exists x\sigma^*})))$. Since F is in $C^+_{\exists x\sigma(R/U)}[\underline{P}/\underline{\mathcal{C}}]$, $abbr(P_{\alpha_2}, abbr(\ldots, abbr(P_{\alpha_k}, H)))$ is in $C^+_{\sigma(R/U)[x/u]}[\underline{P}/\underline{\mathcal{C}}]$. By the induction hypothesis, $abbr(P_{\alpha_1}, abbr(\ldots, abbr(P_{\alpha_k}, H)))$ is in $C^+_{\sigma(R/P_\alpha)[x/u]}[\underline{P}/\underline{\mathcal{C}}]$.

Let y be an individual variable such that $y \notin fv(\sigma) - \{x\}$, $[\gamma]$ be a type with $y \notin fv(\gamma)$, D be a CR^+ of type $[\gamma]$, G be a term of type $[\sigma(R/P_\alpha)[x/y][\underline{P}/\underline{T}] \supset \gamma]$ such that for each individual term t, $G[y/t]$ is in $C^+_{\sigma(R/P_\alpha)[x/t]}[\underline{P}/\underline{\mathcal{C}}] \supset D$, and y is not free in the type superscript of any free term variable of G, and $X^{\sigma(R/P_\alpha)[x/y][\underline{P}/\underline{T}]}$ be a term variable which is not equivalent to any free term variable of G.

We will prove that $M = ST(y.X^{\sigma(R/P_\alpha)[x/y][\underline{P}/\underline{T}]}.G(X^{\sigma(R/P_\alpha)[x/y][\underline{P}/\underline{T}]}),$ $abbr(P_\alpha, F))$ is in D by induction on $N(G(X^{\sigma(R/P_\alpha)[x/y][\underline{P}/\underline{T}]})) + N(F)$. We will show that all its immediate reducts are in D. It follows by the subsidiary induction hypothesis if an immediate reduct is obtained by reducing $G(X^{\sigma(R/P_\alpha)[x/y][\underline{P}/\underline{T}]})$ or F. The other immediate reduct is $ST(y.Y^{\sigma(R/U)[x/y][\underline{P}/\underline{T}]}.$ $unabbr(Y^{\sigma(R/U)[x/y][\underline{P}/\underline{T}]\supset\gamma}.Y^{\sigma(R/U)[x/y][\underline{P}/\underline{T}]\supset\gamma}, G)(Y^{\sigma(R/U)[x/y][\underline{P}/\underline{T}]}), F)$, where $Y^{\sigma(R/U)[x/y][\underline{P}/\underline{T}]}$ is the first term variable which is not equivalent to any free term variable of G.

As in the proof of the case $\beta = \beta_1 \vee \beta_2$ on page 171, we can prove that for every individual term t, $unabbr(Y^{\sigma(R/U)[x/y][\underline{P}/\underline{T}]\supset\gamma}.Y^{\sigma(R/U)[x/y][\underline{P}/\underline{T}]\supset\gamma}, G)[y/t]$ is in $C^+_{\sigma(R/U)[x/t]}[\underline{P}/\underline{\mathcal{C}}] \supset D$ by induction on $N(G[y/t])$.

Since F is in $C^+_{\exists x\sigma(R/U)}[\underline{P}/\underline{\mathcal{C}}]$, $ST(y.Y^{\sigma(R/U)[x/y][\underline{P}/\underline{T}]}.unabbr(Y^{\sigma(R/U)[x/y][\underline{P}/\underline{T}]\supset\gamma}.$ $Y^{\sigma(R/U)[x/y][\underline{P}/\underline{T}]\supset\gamma}, G)(Y^{\sigma(R/U)[x/y][\underline{P}/\underline{T}]}), F)$ is in D. Thus M is in D.

Now, suppose $abbr(P_\alpha, F)$ is in $C^+_{\exists x\sigma(R/P_\alpha)}[\underline{P}/\underline{\mathcal{C}}]$. We want to show that F is in $C^+_{\exists x\sigma(R/U)}[\underline{P}/\underline{\mathcal{C}}]$.

Suppose $F \succ I(u, H^{\sigma(R/U)[x/u][\underline{P}/\underline{T}]})$. Then $abbr(P_\alpha, F) \succ$ $abbr(P_\alpha, I(u,H)^{\exists x\sigma(R/U)[\underline{P}/\underline{T}]})$. Since $abbr(P_\alpha, F)$ is in $C^+_{\exists x\sigma(R/P_\alpha)}[\underline{P}/\underline{\mathcal{C}}]$, $abbr(P_\alpha, H)$ is in $C^+_{\sigma(R/P_\alpha)[x/u]}[\underline{P}/\underline{\mathcal{C}}]$. By the induction hypothesis, H is in $C^+_{\sigma(R/U)[x/u]}[\underline{P}/\underline{\mathcal{C}}]$.

Now, suppose $F \succ abbr(P_{\alpha_1}, abbr(\ldots, abbr(P_{\alpha_k}, I(u,H)^{\exists x\sigma^*})))$. Then $abbr(P_\alpha, F) \succ abbr(P_\alpha, abbr(P_{\alpha_1}, abbr(\ldots, abbr(P_{\alpha_k}, I(u,H)))))$. Since $abbr(P_\alpha, F)$ is in $C^+_{\exists x\sigma(R/P_\alpha)}[\underline{P}/\underline{\mathcal{C}}]$, $abbr(P_\alpha, abbr(P_{\alpha_1}, abbr(\ldots, abbr(P_{\alpha_k}, H))))$ is in $C^+_{\sigma(R/P_\alpha)[x/u]}[\underline{P}/\underline{\mathcal{C}}]$. By the induction hypothesis, $abbr(P_{\alpha_1}, abbr(\ldots, abbr(P_{\alpha_k}, H)))$ is in $C^+_{\sigma(R/U)[x/u]}[\underline{P}/\underline{\mathcal{C}}]$.

Next, let y be an individual variable such that $y \notin fv(\sigma) - \{x\}$, $[\gamma]$ be a type with $y \notin fv(\gamma)$, D be a CR^+ of type $[\gamma]$, G be a term of type $[\sigma(R/U)[x/y][\underline{P}/\underline{T}] \supset \gamma]$ such that for each individual term t, $G[y/t]$ is in $C^+_{\sigma(R/U)[x/t]}[\underline{P}/\underline{\mathcal{C}}] \supset D$, and y is not free in the type superscript of any free term variable of G, and $X^{\sigma(R/U)[x/y][\underline{P}/\underline{T}]}$ be a term variable which is not equivalent to any free term variable of G.

We have to show that $M = ST(y.X^{\sigma(R/U)[x/y][\underline{P}/\underline{T}]}.G(X^{\sigma(R/U)[x/y][\underline{P}/\underline{T}]}), F)$ is in D.

As in the proof of the case $\beta = \beta_1 \vee \beta_2$ on page 172, we can prove by induction on $N(G[y/t]) + N(H)$ that for every individual term t, $abbr(P_\alpha, G)[y/t]$ is in $C^+_{\sigma(R/P_\alpha)[x/t]}[\underline{P}/\underline{\mathcal{C}}] \supset D$.

Let $Z^{\sigma(R/P_\alpha)[x/y][\underline{P}/\underline{T}]}$ be a term variable which is not equivalent to any free term variable of G.

Since $abbr(P_\alpha, F)$ is in $C^+_{\exists x\sigma(R/P_\alpha)}[\underline{P}/\underline{\mathcal{C}}]$, $N = ST(y.Z^{\sigma(R/P_\alpha)[x/y][\underline{P}/\underline{T}]}$, $abbr(P_\alpha, G)(Z^{\sigma(R/P_\alpha)[x/y][\underline{P}/\underline{T}]}), abbr(P_\alpha, F))$ is in D. Since

$$
\begin{aligned}
N \ \succ_1 \ & ST(y.Y^{\sigma(R/U)[x/y][\underline{P}/\underline{T}]}, unabbr(Y^{\sigma(R/U)[x/y][\underline{P}/\underline{T}]\supset\gamma}.Y^{\sigma(R/U)[x/y][\underline{P}/\underline{T}]\supset\gamma}, \\
& abbr(P_\alpha, G))(Y^{\sigma(R/U)[x/y][\underline{P}/\underline{T}]}), F) \\
\succ \ & ST(y.Y^{\sigma(R/U)[x/y][\underline{P}/\underline{T}]}, G(Y^{\sigma(R/U)[x/y][\underline{P}/\underline{T}]}), F) \\
\equiv \ & M,
\end{aligned}
$$

where $Y^{\sigma(R/U)[x/y][\underline{P}/\underline{T}]}$ is the first term variable which is not equivalent to any free term of G, M is in D. Thus F is in $C^+_{\exists x\sigma(R/U)}[\underline{P}/\underline{\mathcal{C}}]$.

Similarly for the case $\beta = \exists_2 Q\sigma$. □

Lemma 5.2.10. *Let F^α be a Curry-Howard term, $\underline{x} = x_1, \ldots, x_n$ be distinct individual variables, $\underline{t} = t_1, \ldots, t_n$ be individual terms, $\underline{P} = P_1^{m_1}, \ldots, P_k^{m_k}$ be distinct predicate variables, $\underline{T} = T_1, \ldots, T_k$, where $T_i = \lambda z_1^i, \ldots, z_{m_i}^i \tau_i$, $1 \leq i \leq k$, be abstraction terms, $\underline{\mathcal{C}} = \mathcal{C}_1, \ldots, \mathcal{C}_k$ be collections of CR^+s corresponding to $T_1, \ldots, T_k$, respectively, $X_1^{\delta_1}, \ldots, X_l^{\delta_l}$ be inequivalent term variables such that every free term variable of F^α is equivalent to $X_i^{\delta_i}$ for some $1 \leq i \leq l$, and $\underline{X} = X_1^{\delta'_1}, \ldots, X_l^{\delta'_l}$, where $\delta'_i = \delta_i[\underline{x}/\underline{t}][\underline{P}/\underline{T}]$, $1 \leq i \leq l$, are inequivalent term variables, and let $\underline{K} = K_1^{\delta'_1}, \ldots, K_l^{\delta'_l}$ be Curry-Howard terms in $C^+_{\delta_1[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}], \ldots, C^+_{\delta_l[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$, respectively.*

Then $F^\alpha[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]$ is in $C^+_{\alpha[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$.

Proof. We will prove by induction on F^α.

Notation. Throughout this proof, γ' denotes $\gamma[\underline{x}/\underline{t}][\underline{P}/\underline{T}]$ for any formula γ.

Every case except the following can be proved in the same way as in Lemma 4.2.12.

For the cases (Abbr Intro) and (Abbr Elim), Q is a q-ary predicate variable and $U = \lambda z_1, \ldots, z_q\sigma$ where $FV(\sigma) = \varnothing$.

(Abbr Intro) $F^{\alpha} = abbr(P_{\sigma}, G^{\beta(Q/U)})^{\beta(Q/P_{\sigma})}$:

By the induction hypothesis, $G[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]$ is in $C^{+}_{\beta(Q/U)[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$ i.e. $C^{+}_{\beta[\underline{x}/\underline{t}](Q/U)}[\underline{P}/\underline{\mathcal{C}}]$.

Note that since we can choose Q' such that $Q' \notin \{\underline{P}\} \cup FV(\underline{T}) \cup FV(\beta)$, and so $\beta(Q/Q')(Q'/U) \equiv \beta(Q/U)$, we may assume that $Q \notin \{\underline{P}\} \cup FV(\underline{T})$. Hence, by Lemma 5.2.9, $abbr(P_{\sigma}, G[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}])$ is in $C^{+}_{\beta[\underline{x}/\underline{t}](Q/P_{\sigma})}[\underline{P}/\underline{\mathcal{C}}]$ i.e. $F[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]$ is in $C^{+}_{\alpha[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$.

(Abbr Elim) $F^{\alpha} = unabbr(Z^{\beta(Q/U)}.G^{\alpha}, H^{\beta(Q/P_{\sigma})})^{\alpha}$:

Since CR^{+} is closed under equivalence of terms, we may assume that $\underline{x}$ is replaceable by $\underline{t}$ in $Z.G$, $\underline{P}$ is replaceable by $\underline{T}$ in $(Z.G)[\underline{x}/\underline{t}]$, and $Z^{\beta'(Q/U)}$ is not equivalent any free term variable in $\underline{X}$ or $\underline{K}$. As in the above case, we may assume that $Q \notin \{\underline{P}\} \cup FV(\underline{T})$.

By the induction hypothesis, $G[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]$ and $H[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]$ are in $C^{+}_{\alpha[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$ and $C^{+}_{\beta(Q/P_{\sigma})[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$, respectively.

We will prove that $unabbr(Z^{\beta'(Q/U)}.G[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}], H[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}])$ is in $C^{+}_{\alpha[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$ by induction on $N(G[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]) + N(H[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}])$. We will show that all its immediate reducts are in $C^{+}_{\alpha[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$. It follows by the subsidiary induction hypothesis if an immediate reduct is obtained by reducing $G[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]$ or $H[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]$. The remaining cases are as follows.

Case 1. $Q \notin FV(\beta)$ and the immediate reduct is $G[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}][Z^{\beta'(Q/U)}/H[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]]$.

By Lemma 3.2.21, $G[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}][Z^{\beta'(Q/U)}/H[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]] \equiv G[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}, Z^{\beta'(Q/U)}/H[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]]$ which is in $C^{+}_{\alpha[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$ by the main induction hypothesis.

Case 2. $Q \in FV(\beta)$, $H[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}] = abbr(P_{\sigma}, J^{\beta'(Q/U)})$, and the immediate reduct is $G[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}][Z^{\beta'(Q/U)}/J]$

Since $H[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]$ is in $C^+_{\beta(Q/P_\sigma)[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$ i.e. $abbr(P_\sigma, J^{\beta'(Q/U)})$ is in $C^+_{\beta[\underline{x}/\underline{t}](Q/P_\sigma)}[\underline{P}/\underline{\mathcal{C}}]$, by Lemma 5.2.9, J is in $C^+_{\beta[\underline{x}/\underline{t}](Q/U)}[\underline{P}/\underline{\mathcal{C}}]$ i.e. $C^+_{\beta(Q/U)[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$. Hence, by the main induction hypothesis, $G[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}, Z^{\beta'(Q/U)}/J]$ is in $C^+_{\alpha[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$. By Lemma 3.2.21, $G[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}][Z^{\beta'(Q/U)}/J] \equiv$ $G[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}, Z^{\beta'(Q/U)}/J]$. Hence $G[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}][Z^{\beta'(Q/U)}/J]$ is in $C^+_{\alpha[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$ by CR0.

($\vee$ Elim) $F^\alpha = \oplus(Y_1^{\beta_1}.F_1^\alpha, Y_2^{\beta_2}.F_2^\alpha, G^{\beta_1\vee\beta_2})$:

As in the proof of Lemma 4.2.12, we assume that F_i has no free term variable Y_i^σ such that $\sigma \not\equiv \beta_i$ for all $i = 1, 2$, and both $Y_1^{\beta'_1}$ and $Y_2^{\beta'_2}$ are not equivalent to any free term variables in $\underline{X}$ or $\underline{K}$. We will prove that $M = \oplus(Y_1^{\beta'_1}.F_1^\alpha[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}], Y_2^{\beta'_2}.F_2^\alpha[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}], G^{\beta_1\vee\beta_2}[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}])$ is in $C^+_{\alpha[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$ by induction on $N(F_1[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]) + N(F_2[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]) +$ $N(G[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}])$. We will show that all its immediate reducts are in $C^+_{\alpha[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$. The proof for every case except the following new one is as in the proof of Lemma 4.2.12.

The new case is when $M = \oplus(Y_1^{\beta'_1}.H_1^{\beta'_1\supset\alpha'}(Y_1^{\beta'_1}), Y_2^{\beta'_2}.H_2^{\beta'_2\supset\alpha'}(Y_2^{\beta'_2}),$ $abbr(P_\sigma, H^{(\beta_1^*\vee\beta_2^*)(Q/U)[\underline{P}/\underline{T}]})^{(\beta_1^*\vee\beta_2^*)(Q/P_\sigma)[\underline{P}/\underline{T}]})$, where $\beta_i[\underline{x}/\underline{t}] = \beta_i^*(Q/P_\sigma)$, $i = 1, 2$, $Q \notin \{\underline{P}\} \cup FV(\underline{T})$, $U = \lambda z_1, \ldots, z_q\sigma$, and the immediate reduct is $N =$ $\oplus\,(Z_1^{\beta_1^*[\underline{P}/\underline{T}](Q/U)}.unabbr(Z_1^{\beta_1^*[\underline{P}/\underline{T}](Q/U)\supset\alpha'}.Z_1^{\beta_1^*[\underline{P}/\underline{T}](Q/U)\supset\alpha'}, H_1)(Z_1^{\beta_1^*[\underline{P}/\underline{T}](Q/U)}),$ $Z_2^{\beta_2^*[\underline{P}/\underline{T}](Q/U)}.unabbr(Z_2^{\beta_2^*[\underline{P}/\underline{T}](Q/U)\supset\alpha'}.Z_2^{\beta_2^*[\underline{P}/\underline{T}](Q/U)\supset\alpha'}, H_2)(Z_2^{\beta_2^*[\underline{P}/\underline{T}](Q/U)}), H),$ where $Z_i^{\beta_i^*[\underline{P}/\underline{T}](Q/U)}$ is the first term variable of type $[\beta_i^*[\underline{P}/\underline{T}](Q/U)]$ which is not equivalent to any free term variable of H_i, $i = 1, 2$.

By the assumption, $Y_1^{\beta'_1}$ is not equivalent to any free term variable in $\underline{K}$. Hence, since $F_1^\alpha[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}] = H_1^{\beta'_1\supset\alpha'}(Y_1^{\beta'_1})$, $F_1 = J_1^{\beta_1\supset\alpha}(Y_1^{\beta_1})$ for some term

$J_1^{\beta_1\supset\alpha}$ and $H_1 = J_1[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]$. Hence, by the induction hypothesis, H_1 is in $C^+_{(\beta_1\supset\alpha)[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$ i.e. $C^+_{\beta_1[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}] \supset C^+_{\alpha[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$ i.e. $C^+_{\beta_1^*(Q/P_\sigma)}[\underline{P}/\underline{\mathcal{C}}] \supset C^+_{\alpha[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$. Similarly, H_2 is in $C^+_{\beta_2^*(Q/P_\sigma)}[\underline{P}/\underline{\mathcal{C}}] \supset C^+_{\alpha[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$.

Next, we will prove that $unabbr(Z_1^{\beta_1^*[\underline{P}/\underline{T}](Q/U)\supset\alpha'}.Z_1^{\beta_1^*[\underline{P}/\underline{T}](Q/U)\supset\alpha'}, H_1)$ is in $C^+_{\beta_1^*(Q/U)}[\underline{P}/\underline{\mathcal{C}}] \supset C^+_{\alpha[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$ by induction on $N(H_1)$. It follows by Note on page 158 if $Q \notin FV(\beta_1^*)$. Suppose $Q \in FV(\beta_1^*)$. We will show that all its immediate reducts are in $C^+_{\beta_1^*(Q/U)}[\underline{P}/\underline{\mathcal{C}}] \supset C^+_{\alpha[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$. It follows by the subsidiary induction hypothesis if an immediate reduct is obtained by reducing H_1. The remaining case is when $H_1 = abbr(P_\sigma, H_*^{\beta_1^*[\underline{P}/\underline{T}](Q/U)\supset\alpha'})$ and the immediate reduct is H_*. Since H_1 is in $C^+_{\beta_1^*(Q/P_\sigma)\supset\alpha[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$, by Lemma 5.2.9, H_* is in $C^+_{\beta_1^*(Q/U)\supset\alpha[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$ i.e. $C^+_{\beta_1^*(Q/U)}[\underline{P}/\underline{\mathcal{C}}] \supset C^+_{\alpha[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$. Thus
$unabbr(Z_1^{\beta_1^*[\underline{P}/\underline{T}](Q/U)\supset\alpha'}.Z_1^{\beta_1^*[\underline{P}/\underline{T}](Q/U)\supset\alpha'}, H_1)$ is in $C^+_{\beta_1^*(Q/U)}[\underline{P}/\underline{\mathcal{C}}] \supset C^+_{\alpha[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$.

Similarly, we can show that $unabbr(Z_2^{\beta_2^*[\underline{P}/\underline{T}](Q/U)\supset\alpha'}.Z_2^{\beta_2^*[\underline{P}/\underline{T}](Q/U)\supset\alpha'}, H_2)$ is in $C^+_{\beta_2^*(Q/U)}[\underline{P}/\underline{\mathcal{C}}] \supset C^+_{\alpha[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$.

By the induction hypothesis, $abbr(P_\sigma, H)$ i.e. $G^{\beta_1\vee\beta_2}[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}]$ is in $C^+_{(\beta_1\vee\beta_2)[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$ i.e. $C^+_{(\beta_1^*\vee\beta_2^*)(Q/P_\sigma)}[\underline{P}/\underline{\mathcal{C}}]$. By Lemma 5.2.9, H is in $C^+_{(\beta_1^*\vee\beta_2^*)(Q/U)}[\underline{P}/\underline{\mathcal{C}}]$. Thus N is in $C^+_{\alpha[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$.

($\exists$ Elim) $F^\alpha = ST(y.Y^\beta.G^\alpha, H^{\exists y\beta})$:

As in the proof of Lemma 4.2.12, we assume that $y \notin \{\underline{x}\} \cup fv(\underline{t}) \cup fv(\underline{K})$, G^α has no free term variable Y^σ such that $\sigma \not\equiv \beta$, and $Y^{\beta'}$ is not equivalent to any free term variable in $\underline{X}$ or $\underline{K}$.

We will prove that $M = ST(y.Y^{\beta'}.G^\alpha[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}], H^{\exists y\beta}[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}])$ is in $C^+_{\alpha[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$ by induction on $N(G[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}])+N(H[\underline{x}/\underline{t}][\underline{P}/\underline{T}][\underline{X}/\underline{K}])$. Since M is neutral, we will show that all its immediate reducts are in $C_{\alpha[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$. The proof for every case except the following new one is as in the proof of Lemma

4.2.12.

The new case is when $M = ST(y.Y^{\beta'}.G_*(Y^{\beta'}),$
$abbr(P_\sigma, J^{\exists y\beta^*(Q/U)[\underline{P}/\underline{T}]})^{\exists y\beta^*(Q/P_\sigma)[\underline{P}/\underline{T}]})$, where $\beta[\underline{x}/\underline{t}] = \beta^*(Q/P_\sigma)$, $Q \notin \{\underline{P}\} \cup FV(\underline{T})$, $U = \lambda z_1, \ldots, z_q\sigma$, and the immediate reduct is $N = ST(y.Z^{\beta^*(Q/U)[\underline{P}/\underline{T}]}.$ $unabbr(Z^{\beta^*(Q/U)[\underline{P}/\underline{T}]\supset\alpha'}.Z^{\beta^*(Q/U)[\underline{P}/\underline{T}]\supset\alpha'}, G_*)(Z^{\beta^*(Q/U)[\underline{P}/\underline{T}]}), J)$.

Similar to the above case, we can show that $unabbr(Z^{\beta^*(Q/U)[\underline{P}/\underline{T}]\supset\alpha'}.$ $Z^{\beta^*(Q/U)[\underline{P}/\underline{T}]\supset\alpha'}, G_*)$ is in $C^+_{\beta^*(Q/U)}[\underline{P}/\underline{\mathcal{C}}] \supset C^+_{\alpha[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$ and J is in $C^+_{\exists y\beta^*(Q/U)}[\underline{P}/\underline{\mathcal{C}}]$. Hence N is in $C^+_{\alpha[\underline{x}/\underline{t}]}[\underline{P}/\underline{\mathcal{C}}]$.

Similarly for the case ($\exists_2$ Elim). □

The *strong normalization theorem* for the new C-H terms follows by the above lemma in the same way as Theorem 4.2.13 follows from Lemma 4.2.12 in Chapter IV.

CHAPTER VI

CONCLUSIONS AND FURTHER WORK

We have extended the system of extracting programs from proofs in the language of first-order predicate calculus (in [3]) to second-order logic. We have shown that Curry-Howard terms produced in the new system still satisfy the *strong normalization theorem* by extending Crossley and Shepherdson (see [3]) and adapting Girard's technique of *parametric reducibility* (see [7]). By using this technique, we do not have to put any restrictions on formulae or abstraction terms. In [2], Basin and Matthews extend a standard intuitionistic first-order sequent calculus to second-order and put the restriction on an *abstracted formula* (which is an *abstraction term* in this thesis) $\lambda x_1, \ldots, x_n \alpha$ so that α does not contain any second-order quantifiers. This restriction enables them to prove the *second-order cut elimination theorem* by induction on the construction of a formula. In [16], Takayama introduces the second-order constructive calculus QPC_2 where second-order formulae are restricted so that the second-order universal quantifier never occurs inside a formula (only occurs at the head part of a formula) in order to make the sequence of the second-order proof normalization simple. Our work here goes further than theirs.

This new system is intended for templates to be added to but it is also useful on its own since now programs can be obtained directly from proofs in second-order logic in which a large part of mathematics is actually formulated (see [18]).

Finally, we have introduced two kinds of templates: induction templates and abbreviation templates by adding new rules to the system and then defined the associated Curry-Howard terms as well as new reduction rules. For the induction templates, everything is straightforward. We just add proofs for the new cases to the proof of every lemma. We then get the strong normalization theorem for the new Curry-Howard terms. On the contrary, for the abbreviation templates, the proof of the strong normalization theorem seems to be much more complicated. In order to have Lemma 5.2.9, which is the key to the strong normalization theorem, we have to extend CR to CR^+ and $C_\alpha[\underline{P}/\underline{\mathcal{C}}]$ to $C_\alpha^+[\underline{P}/\underline{\mathcal{C}}]$, where $C'_\alpha[\underline{P}/\underline{\mathcal{C}}]$ is introduced in order to define $C_\alpha^+[\underline{P}/\underline{\mathcal{C}}]$. We restrict abstraction terms not to contain any abbreviation predicates. This restriction is needed for the proofs of ($\vee$ Elim), ($\exists$ Elim), and ($\exists_2$ Elim) cases in Lemma 5.2.10. We also restrict the formulae which can be abbreviated not to contain any abbreviation predicates or free predicate variables. We make the latter restriction in order to make some basic lemmas (e.g. Lemma 3.2.14) hold. The reason for the former restriction is below the note on page 165 of which the result is needed for the proof of Lemma 5.2.9.

The induction templates allow us to use induction in formal proofs without going through natural numbers. As a result of this, programs extracted from proofs using induction in the new system would become shorter. The abbreviation templates enable us to abbreviate formulae by predicates in formal proofs. We can see that the new system takes us closer to the actual practice of mathematicians and the way they write proofs.

Some of the things we have not done in this thesis are the following.

We did not prove the *Church-Rosser theorem*, which states that "if a Curry-Howard term reduces to two terms, these two terms must have a common reduct", for the new Curry-Howard terms. This should be a straightforward extension from the proof of this theorem for the first-order system in [3].

We did not give a definition of *verifier*, which may be regarded as a variant of Kleene's notion of *realizer* (see [10]). Verifiers for the first-order system in [3] give constructive evidence for the truth of a formula in a structure. Extending these to the second-order system would be a good subject for further work.

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