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FACTORIZATION AND INDEPENDENCE OF ARITHMETIC FUNCTIONS

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วิทยานิพนธ์นี้เกี่ยวข้องกับสมบัติสองประการของฟังก์ชันเลขคณิต คือ การแยกตัวประกอบ และความเป็นอิสระต่อกัน

ในปี 1984 เรียริค ได้แสดงให้เห็นว่าริงของฟังก์ชันเลขคณิตเป็นโดเมนที่มีการแยกตัวประกอบได้แบบเดียว แต่ไม่เป็นโดเมนไอดีลมุขสำคัญ จึงไม่เป็นโดเมนแบบยุคลิด เมื่อไม่สามารถใช้ขั้นตอนวิธีแบบยุคลิดได้ การแยกตัวประกอบของฟังก์ชันเลขคณิตจึงเป็นเรื่องยาก ในส่วนแรกของวิทยานิพนธ์นี้เราเสนอเทคนิคการแยกตัวประกอบของฟังก์ชันเลขคณิตบางประเภทและพิสูจน์ผลเกี่ยวกับการแยกตัวประกอบโดยใช้นอร์มของฟังก์ชันเลขคณิตเหล่านั้นเป็นหลัก เทคนิคเหล่านี้เป็นการขยายงานของเรียริคซึ่งเกี่ยวกับการหาผลเฉลยของระบบสมการอนุพันธ์เชิงเส้นซึ่งมีสัมประสิทธิ์เป็นพหุนามในฟังก์ชันที่ต้องการแยกตัวประกอบ ผลเฉลยของระบบสมการนี้พิสูจน์ได้ว่าเป็นตัวประกอบที่ต้องการซึ่งเรียงตามนอร์มที่เพิ่มขึ้น ในที่นี้เราได้ให้ตัวอย่างที่แสดงการใช้เทคนิคเหล่านี้ด้วย

ในปี 1986 ชาปีโรและสแปเรอ พิสูจน์ผลจำนวนมากเกี่ยวกับความเป็นอิสระต่อกันเชิงพีชคณิตของอนุกรม ดีริชเลต์ เนื่องจากริงของอนุกรมดีริชเลต์สมสัณฐานกับริงของฟังก์ชันเลขคณิต ดังนั้นการศึกษาในโครงสร้างหนึ่งจะสมมูลกับในอีกโครงสร้างหนึ่ง การศึกษาของชาปีโรและสแปเรอเริ่มต้นด้วยทฤษฎีบทที่ว่าอนุกรม ดีริชเลต์จะเป็นอิสระต่อกันเชิงพีชคณิตถ้าจาโคเบียนของอนุกรมเหล่านั้นไม่เป็นศูนย์ เมื่อพิจารณาฟังก์ชันซีตาเป็นหลัก พบว่า อนุกรมดีริชเลต์ที่ไม่เป็นอิสระเชิงพีชคณิตต่อฟังก์ชันซีตาจะเขียนได้ในรูปอนุกรมกำลังในลอการิทึมของฟังก์ชันซีตา และได้ผลเกี่ยวกับความไม่เป็นอิสระต่อกันเชิงพีชคณิตของอนุกรมดีริชเลต์ตามมา ชาปีโรและสแปเรอยังได้ศึกษาผลในทำนองเดียวกัน สำหรับอนุกรมดีริชเลต์ในรูปนัยทั่วไปด้วย ในส่วนที่สองของวิทยานิพนธ์นี้ได้ขยายผลบางอย่างของชาปีโรและสแปเรอหรือทำให้ง่ายขึ้น ซึ่งรวมถึงการแทนที่ฟังก์ชันซีตาด้วยอนุกรมดีริชเลต์ที่มีสัมประสิทธิ์เป็นฟังก์ชันการคูณบริบูรณ์ แล้วให้การกระจายในอนุกรมลอการิทึมเช่นกัน การไม่เป็นอิสระต่อกันของอนุกรมที่มีเซตค้ำจุนเป็นเซตอนันต์ และ การไม่เป็นอิสระต่อกันของฟังก์ชันที่ไม่เป็นยูนิทซึ่งนอร์มเฉพาะสัมพัทธ์ต่อกัน

ภาควิชา **คณิตศาสตร์** ลายมือชื่อนิสิต..

สาขาวิชา **คณิตศาสตร์** ลายมือชื่ออาจารย์ที่ปรึกษา..

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PATTIRA RUENGSINSUB : FACTORIZATION AND INDEPENDENCE OF ARITHMETIC FUNCTIONS. THESIS ADVISOR : ASST. PROF. PATANEE UDOMKAVANICH, Ph. D., THESIS COADVISOR : ASSOC.PROF. VICHIAN LAOHAKOSOL, Ph.D., 53 pp. ISBN 974-17-4745-4

This thesis deals with two properties of arithmetic functions, namely, factorization and independence.

In 1984, Rearick pointed out that the ring of arithmetic functions is a unique factorization domain but is not a principal ideal domain and so is not a Euclidean domain. Without the Euclidean algorithm, the problem of factorizing arithmetic functions becomes quite difficult. In the first part of this thesis, we propose a technique of factorizing certain classes of arithmetic functions and prove some results about factorization which are based mainly on the norms of such functions. The technique is a generalization of the original works of Rearick which consists of solving a special system of linear differential equations whose coefficients are polynomials in the function to be factorized. The solutions of this system are proved to be the sought after factors with increasing norms. Examples illustrating the technique are also given.

In 1986, Shapiro and Sparer made an extensive study of algebraic independence of Dirichlet series. Since the ring of Dirichlet series is isomorphic to that of arithmetic functions, the study in one setting is then equivalent to the other. Shapiro and Sparer's investigation began with a theorem asserting that Dirichlet series are algebraically independent if their Jacobian does not vanish, which is classical in the case of real-valued functions. Taking the Riemann zeta function as a building block, they discovered that Dirichlet series algebraically dependent on the zeta function can uniquely be represented as power series in the logarithms of zeta function. A number of algebraic dependence results of Dirichlet series were derived as consequences. Shapiro and Sparer then went on to investigate analogous results for formal generalized Dirichlet series. Results in the second part of this thesis either extend or simplify some of Shapiro and Sperer's results. These include, for example, replacing the zeta function by Dirichlet series with completely multiplicative coefficients to obtain similar log-series expansions, dependence of series with infinite support, and dependence of non-units whose norms are relatively prime.

Department **Mathematics**	Student's signature.....................................
Field of study **Mathematics**	Advisor's signature………................................
Academic year **2003**	Co-advisor's signature ....

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CONTENTS

CHAPTER I

INTRODUCTION

The set $\mathcal{A}$ of all arithmetic functions forms an integral domain under addition and convolution, see [1] and [9]. It was proved by Cashwell-Everett [2], see also [3], that $\mathcal{A}$ is indeed a unique factorization domain. In this thesis we consider two properties of arithmetic functions, factorization and independence.

Rearick [7] pointed out that since the set of non-units in $\mathcal{A}$ is an ideal which is not principal, $\mathcal{A}$ is not a principal ideal domain and so not a Euclidean domain. Without the Euclidean algorithm, the problem of factorizing arithmetic functions becomes quite difficult. The first real attempt was due to Rearick [7] who did so by introducing the notion of standard forms and devised methods to obtain factors of arithmetic functions whose norms are of simple shapes.

Rearick's technique made use of a derivative-like operator to set up differential-like equations whose roots are the sought after factors. Later in [5], these results were simplified and put under a more natural setting by replacing the derivative-like operator with a true derivation, called p-basic derivation ([8]).

In Chapter III, we carry on the investigations of [7] and [5]. In the first part, we derive two theorems providing sufficient primality criteria based on functional values. These conditions are more desirable than their counter-parts in [7] and [5], where the conditions there, despite being both necessary and sufficient based on the forms of the functions themselves, seem harder to check. The second part is the crux of this chapter. We prove our main factorization theorem which leads to an algorithm exhibiting certain differential technique of finding factors of arithmetic functions. The proof is conceptually similar to those in [7] and [5]. Finally, examples showing various possibilities are worked out.

In fact, the ring of arithmetic functions is isomorphic to the ring of Dirichlet series. In Shapiro-Sparer[10], a systematic investigation of algebraic independence of Dirichlet series is made. A thorough study of this paper leads us to results in Chapter IV which either extend or simplify certain results in [10]. These results include:

(i) A Dirichlet series $\Xi(s)$, with arithmetic function ξ non-vanishing at infinitely many prime values of n as coefficients, does not satisfy any algebraic differential difference equation.

(ii) For an arithmetic function ξ which is completely multiplicative and non-vanishing at all primes and Ξ being its corresponding Dirichlet series, if an arithmetic function f satisfies differential equation over $\mathbb{C}[\xi]$, then its corresponding Dirichlet series F is a power series in $\log \Xi$.

(iii) For a normalized Dirichlet series Ξ as in (ii), any polynomial in $\log \Xi$ is not algebraic over $\mathbb{C}[\Xi]$.

(i), (ii) and (iii) extend the case $\xi(n) = 1$, for all $n \in \mathbb{N}$, of Riemann zeta function, in [10] and (i) is indeed an old result of Ostrowski[6].

(iv) For a normalized Dirichlet series Z which is multiplicative at two distinct primes belonging to its support, if another Dirichlet series F is $\mathbb{C}$-algebraically dependent on Z, then F can be uniquely represented as a power series in $\log Z$.

(v) For an arithmetic function z with infinite support, $[supp(z)]$, if two arithmetic functions are multiplicative over an infinite subset of $[supp(z)]$ and are $\mathbb{C}$-algebraically dependent on z, then one is a rational power of the other.

Results (iv) and (v) are slight extensions of those in [10] where "multiplicative at primes" is replaced by "multiplicative", while their proofs clarify and simplify certain obscurities in [10].

The last result, (vi), involves two results : the former is the algebraic independence of most commonly encountered arithmetic functions, viz. units, while the latter reveals relationships between norms of two dependent arithmetic functions.

CHAPTER II

PRELIMINARIES

In this chapter notations, definitions and theorems to be used are collected. The following symbols will be standard :

$\mathbb{N}$ the set of all natural numbers,

$\mathbb{C}$ the complex field.

2.1 Arithmetic Functions

Definition 2.1. An *arithmetic function* is a function from $\mathbb{N}$ to $\mathbb{C}$. Let $\mathcal{A}$ denote the set of all arithmetic functions. Addition $(+)$ and multiplication $(*)$, usually called *Dirichlet multiplication* (or *convolution*) of two arithmetic functions f and g are defined respectively by

$$(f+g)(n) = f(n) + g(n),$$

$$(f*g)(n) = \sum_{ij=n} f(i)g(j).$$

The ring $(\mathcal{A}, +, *)$ is an integral domain ([1],[9]), with the function I defined by

$$I(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{otherwise} \end{cases}$$

being its convolution identity. Cashwell and Everett [2], see also [3], proved that $(\mathcal{A}, +, *)$ is indeed a unique factorization domain.

Furthermore, $\mathcal{A}$ contains $\mathbb{C}$ via the identification of a $c \in \mathbb{C}$ with the function

$$c(n) = \begin{cases} c & \text{if } n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.2. A function $f \in \mathcal{A}$ is called a *unit* if there exists a function $g \in \mathcal{A}$ such that $f * g = I$. It is easily verified that $f \in \mathcal{A}$ is a unit if and only if $f(1) \neq 0$. A nonzero function $f \in \mathcal{A}$ *divides* a function $h \in \mathcal{A}$, written $f \mid h$, if there exists $g \in \mathcal{A}$ such that $f * g = h$, and g is also denoted by $\frac{h}{f}$. A function $h \in \mathcal{A}$ is called a *prime* if it cannot be factored into a convolution of two non-unit functions. An $f \in \mathcal{A}$ is said to be *multiplicative* if $f(nm) = f(n)f(m)$ for $n, m \in \mathbb{N}$ which are relatively prime and is said to be *completely multiplicative* if $f(nm) = f(n)f(m)$ for all $n, m \in \mathbb{N}$.

Definition 2.3. The *norm*, Nf, of a function $f \in \mathcal{A}$ is defined as

$$Nf = \begin{cases} \min\{n \in \mathbb{N} \mid f(n) \neq 0\} & \text{if } f \neq 0, \\ \infty & \text{if } f = 0. \end{cases}$$

Clearly, $N(f * g) = (Nf)(Ng)$, $N(f + g) \geq \min\{Nf, Ng\}$, and the units of $\mathcal{A}$ are those functions whose norms are equal to 1.

Definition 2.4. A *derivation* d over $\mathcal{A}$ is a map of $\mathcal{A}$ into itself satisfying

$$d(f * g) = df * g + f * dg, \quad d(c_1 f + c_2 g) = c_1 df + c_2 dg,$$

where f,g are in $\mathcal{A}$,and c_1,c_2 are complex numbers.

Derivations of higher orders are defined in the usual manner.

Two typical examples of derivation are

(i) the *p-basic derivation* ,p prime, defined by

$$(d_p f)(n) = f(np)\upsilon_p(np) \qquad (\forall\ n \in \mathbb{N}),$$

where $v_p(m)$ denotes the exponent of the highest power of p dividing m,

(ii) the *log-derivation* defined by

$$(d_L f)(n) = f(n)logn \qquad (\forall\ n \in \mathbb{N}).$$

The derivation d is extended to the field of quotients of $\mathcal{A}$ by

$$d(\frac{h}{f}) = \frac{f * dh - h * df}{f * f} \quad \text{for all } f, h \in \mathcal{A},\ f \neq 0.$$

Remarks 2.5. 1. Each derivation annihilates all $c \in \mathbb{C}$ and all usual rules of differentiation hold.

2. For all distinct primes p, q, we have $d_p d_q = d_q d_p$.

2.2 Standard Form

For a function $f \in \mathcal{A}$ with $Nf = s$, Rearick [7] showed that there exists a unique unit function $u_f \in \mathcal{A}$ such that

$$S_f(ns) := (u_f * f)(ns) = I(n) \quad (\forall\ n \in \mathbb{N}).$$

The function $S_f := u_f * f$ is called the *standard form of* f, and f is said to be *in standard form* if and only if $f(ns) = I(n)$ for all $n \in \mathbb{N}$.

The first lemma confirms the uniqueness of standard form.

Lemma 2.6. Let f be in $\mathcal{A} - \{0\}$,and let S_f be its standard form.Then f is in standard form if and only if $f = S_f$.

Proof. See [5], Lemma 1. □

Clearly, to find factors (upto unit factors) of any arithmetic function, it suffices to assume that it is in standard form.

Remark 2.7. If $f \in \mathcal{A}$ is in standard form with $Nf = p^\alpha$, then $N(d_p^i f) = p^{\alpha - i}$, for all $i = 1, \ldots, \alpha$.

Lemma 2.8. Let $f, e_0, e_1, \ldots, e_m$ be in $\mathcal{A}$,with $e_m \neq 0$.Let d be a derivation on $\mathcal{A}$. If

$$\sum_{i=0}^{m} e_i * f^i = 0$$

and $de_i = 0$ $(i = 0, \ldots, m)$, then $df = 0$.
(Here f^i denotes $f * f * \cdots * f$ (i terms)).

Proof. See [5], Lemma 2. □

Lemma 2.9. Let $p_1, \ldots, p_r$ be distinct primes, and $d_{p_1}, \ldots, d_{p_r}$ be their corresponding p_i-basic derivations. Let f be in $\mathcal{A}$, having norm $Nf = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, with $\alpha_1, \ldots, \alpha_r$ positive integers. Then f is in standard form if and only if

$$d_{p_1}^{\alpha_1} \cdots d_{p_r}^{\alpha_r} f(n) = \alpha_1! \ldots \alpha_r! I(n) \quad (\forall\ n \in \mathbb{N}).$$

Proof. See [5], Lemma 3. □

Rearick [7] proved that for $h, f, g \in \mathcal{A}$ such that $h = f * g$ and the norms of f and g being powers of the same prime , if among h, f, g two are in standard form, then so is the third. This result does not hold if the norms involved are not powers of the same prime. The next lemma shows a necessary condition for the convolution of two functions whose norms are not powers of the same prime to be in standard form.

Lemma 2.10. Let $f, g \in \mathcal{A}$ be in standard form with $Nf = p^\alpha, Ng = q^\beta$, where p, q are distinct primes, and α, β are positive integers. If $d_q^i f = 0$ $(i = 1, \ldots, \beta)$ or $d_p^j g = 0$ $(j = 1, \ldots, \alpha)$,then $h = f * g$ is in standard form with $Nh = p^\alpha q^\beta$.

Proof. Let $h = f * g$. Then $Nh = p^\alpha q^\beta$. Since

$$d_p^\alpha d_q^\beta h = \sum_{j=0}^{\alpha} \sum_{i=0}^{\beta} \binom{\alpha}{j} \binom{\beta}{i} d_p^j d_q^i f * d_p^{\alpha-j} d_q^{\beta-i} g,$$

and $d_q^i f = 0$ for all $i = 1, \dots, \beta$, then

$$\begin{aligned} d_p^\alpha d_q^\beta h &= \sum_{j=0}^{\alpha} \binom{\alpha}{j} d_p^j f * d_p^{\alpha-j} d_q^\beta g \\ &= \sum_{j=0}^{\alpha} \binom{\alpha}{j} d_p^j f * \beta! d_p^{\alpha-j} I \\ &= d_p^\alpha f * \beta! d_p^0 I = \alpha!\beta! I \end{aligned}$$

Similarly, if $d_p^j g = 0$ for all $j = 1, \dots, \alpha$, we have $d_q^\beta d_p^\alpha h = \alpha!\beta! I$, and the result follows from Lemma 2.9. □

2.3 Independence

Definition 2.11. Let $\mathcal{E}$ be a subring of $\mathcal{A}$. For $r > 1$, we say that $f_1, f_2, \dots, f_r \in \mathcal{A}$ are *algebraically dependent* over $\mathcal{E}$ if there exists $P \in \mathcal{E}[x_1, \dots, x_r] \backslash \{0\}$ such that

$$P(f_1, \dots, f_r) = \sum_{(i)} a_{(i)} * f_1^{i_1} * \cdots * f_r^{i_r} = 0,$$

and is said to be *algebraically independent* over $\mathcal{E}$ otherwise.

We say that f_1 is *algebraic* over $\mathcal{E}[f_2, \dots, f_r]$ if $f_1, f_2, \dots, f_r$ are algebraically dependent over $\mathcal{E}$.

An infinite subset $\mathcal{B}$ of $\mathcal{A}$ is said to be algebraically independent over a subring $\mathcal{E}$ of $\mathcal{A}$ if for any $r \geq 1$, $f_1, \dots, f_r \in \mathcal{B}$ are algebraically independent over $\mathcal{E}$.

We shall make use of the following results from [10]

Lemma 2.12. Let $\mathcal{E}$ be a subring of $\mathcal{A}$. If $f \in \mathcal{A}$ is such that there exists a derivation d over $\mathcal{A}$ which annihilates all of $\mathcal{E}$ and $d(f) \neq 0$, then f is not algebraic over $\mathcal{E}$.

Definition 2.13. Given $f_1, \dots, f_r \in \mathcal{A}$ and derivations $d_1, \dots, d_r$ over $\mathcal{A}$, the

Jacobian of the f_i relative to the d_i is the determinant

$$J(f_1, \ldots, f_r/d_1, \ldots, d_r) = det(d_i(f_j)).$$

For ease of writing , when the derivations d_i $(i = 1, \ldots, n)$ are the p_i-basic derivations instead of $J(f_1, \ldots, f_r/d_{p_1}, \ldots, d_{p_r})$ we write $J(f_1, \ldots, f_r/p_1, \ldots, p_r)$.

Theorem 2.14. Let $f_1, \ldots, f_r \in \mathcal{A}$ and $d_1, \ldots, d_r$ be distinct derivations over $\mathcal{A}$ which annihilate all elements of the subring $\mathcal{E}$. If $J(f_1, \ldots, f_r/d_1, \ldots, d_r) \neq 0$, then $f_1, \ldots, f_r$ are algebraically independent over $\mathcal{E}$.

The condition of this theorem is not sufficient as seen in the following example.

Example 2.15. Let $u(n) = 1$ for all $n \in \mathbb{N}$.
Then I and u are algebraically independent over $\mathbb{C}$, see [10], but for any primes $p \neq q$,

$$J(I, u/p, q)(n) = \begin{vmatrix} I(np)\nu_p(np) & u(np)\nu_p(np) \\ I(nq)\nu_q(nq) & u(nq)\nu_q(nq) \end{vmatrix} = \begin{vmatrix} 0 & \nu_p(np) \\ 0 & \nu_q(nq) \end{vmatrix} = 0.$$

□

CHAPTER III

FACTORIZING ARITHMETIC FUNCTIONS

In this chapter we carry on the investigations of factorizing arithmetic functions in [7] and [5]. In Section 3.1, we derive two theorems providing sufficient primality criteria based on functional values. Section 3.2 is the crux of this chapter. We prove our main factorization theorem (Theorem 3.6) which leads to an algorithm exhibiting certain differential technique of finding factors of arithmetic functions.

3.1 Some Prime Characterizations

Note first that if Nf = prime p, then f is a prime. Rearick ([7], see also [5]) derived the following necessary and sufficient condition for a function h, in standard form with norm p^2, p prime , to be a prime arithmetic function.

Proposition 3.1. Let $h \in \mathcal{A}$ be in standard form with $Nh = p^2$. Then h is a prime if and only if $(dh)^2 - 4h$ is not a square.

Evidently, this proposition is not easy to use. We propose simpler sufficiency tests in the next two results.

Theorem 3.2. Let $h \in \mathcal{A}$ be in standard form with $Nh = p^2$, p a prime. Assume that $(d_p h)^2 - 4h \neq 0$. If $[(d_p h)^2 - 4h](n^2) = 0$ for all $n > p$, then h is a prime.

Proof. Assume that h is not a prime. Let $g = (d_p h)^2 - 4h$. By Proposition 3.1, g is a square, i.e. $g = f * f$ for some $f \in \mathcal{A}$. Since h is in standard form and

$Nh = p^2$, then $N(d_p h) = p$. Thus $N(g) \geq \min\{N(d_p h)^2, N(-4h)\} = p^2$.

But $g(p^2) = \sum_{ij=p^2} d_p h(i) d_p h(j) \; - 4h(p^2) = 0$, so $Ng > p^2$ yielding $Nf > p$, say $Nf = p + k$ for some $k \geq 1$.

Thus $g((p+k)^2) = \sum_{ij=(p+k)^2} f(i)f(j) = f(p+k)f(p+k) \neq 0$, which is a contradiction. $\square$

The condition $[(dh)^2 - 4h](n^2) = 0$ for all $n > p$ cannot be improved, as seen in the next example.

Example 3.3. Let p and q be prime numbers such that $p < q$.

Define $f(n) = \begin{cases} 1 & \text{if } n = p, \\ 0 & \text{otherwise} \end{cases}$ and $g(n) = \begin{cases} 1 & \text{if } n = p \text{ or } q, \\ 0 & \text{otherwise.} \end{cases}$

Let $h = f * g$. Then $Nh = p^2$, $h(n) = \begin{cases} 1 & \text{if } n = p^2 \text{ or } pq, \\ 0 & \text{otherwise,} \end{cases}$

$$d_p h(n) = h(np)\upsilon_p(np) = \begin{cases} 2 & \text{if } n = p, \\ 1 & \text{if } n = q, \\ 0 & \text{otherwise,} \end{cases}$$

and $d_p^2 h(n) = h(np^2)\upsilon_p(np)\upsilon_p(np^2) = \begin{cases} 2 & \text{if } n = 1, \\ 0 & \text{otherwise.} \end{cases}$

Thus $d_p^2 h(n) = 2!I(n)$. By Lemma 2.9, h is in standard form and not a prime, but $[(d_p h)^2 - 4h](q^2) = 1 \neq 0$.

Lemma 3.4. Let $h \in \mathcal{A}$ with $h = f * g$, where $f, g \in \mathcal{A}$ are non-units. If there exist $m, p \in \mathbb{N}$ with $m < \min(Nf, Ng)$, p prime not dividing m, then $h(mp) = 0$.

Proof. $h(mp) = \sum_{ij=m} f(i)g(pj) + \sum_{ij=m} f(pi)g(j) = 0.$ $\square$

Theorem 3.5. Let $h \in \mathcal{A}$ be a non-unit with $Nh = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, where $p_1 < \ldots < p_r$ are primes and $\alpha_1, \ldots, \alpha_r \in \mathbb{N}$.

(i) If there exists a prime $q \neq p_1, ..., p_r$ such that $h(q) \neq 0$, then h is a prime.

(ii) If there exist primes $q_1 \neq q_2$ such that $q_1 < p_1$ and $h(q_1 q_2) \neq 0$, then h is a prime.

Proof. Suppose that $h = f * g$ is a nontrivial factorization. Then $Nf, Ng > 1$. (i) follows from Lemma 3.4 by taking $m = 1$, $p = q$ and (ii) follows from Lemma 3.4 by taking $m = q_1$, $p = q_2$. □

3.2 Factorization Theorems

Theorem 3.6. Let $h \in \mathcal{A}$ and p be the smallest prime divisor of Nh with highest exponent α. Assume that

(i) there is an integer $b \geq \alpha$ such that $d_p^b h \neq 0$, and $d_p^{b+1} h = 0$, and

(ii) the polynomial $P(f; h) = \displaystyle\sum_{k=0}^{b} \frac{(-1)^k}{k!} f^k * d_p^k h$ has a non-unit root $f_1 \in \mathcal{A}$.

Then f_1 is a divisor of h of norm p, in standard form.

Proof. Assume that $P(f_1; h) = 0$. Define the arithmetic function

$$r_p(n) = \begin{cases} 1 & \text{if } n = p \\ 0 & \text{otherwise.} \end{cases}$$

Then r_p is in standard form with $d_p r_p = I$. Writing $P(f_1; h)$ as a polynomial in $f_1 - r_p$, we get

$$0 = P(f_1; h) = \sum_{k=0}^{b} \frac{(-1)^k}{k!} d_p^k h * (f_1 - r_p + r_p)^k = \sum_{i=0}^{b} \frac{(-1)^i}{i!} C_i * (f_1 - r_p)^i,$$

where $C_i = \displaystyle\sum_{k=0}^{b-i} \frac{(-1)^k}{k!} r_p^k * d_p^{k+i} h$. We have $C_b = d_p^b h \neq 0$, $d_p C_b = d_p^{b+1} h = 0$ and for all $0 \leq i \leq b-1$, $d_p C_i = 0$. By Lemma 2.8, $d_p(f_1 - r_p) = 0$, so $d_p f_1 = I$.

Since $P(f_1;h) = 0$, f_1 divides the constant term of $P(f_1;h)$, which is h. Then Nf_1 divides Nh. Since $1 = I(1) = d_p f_1(1) = f_1(p)$ and p is the smallest prime divisor of Nh, then $Nf_1 \leq p$, so $Nf_1 = 1$ or p. As f_1 is a non-unit, it follows that $Nf_1 = p$, so f_1 is in standard form. □

Definition 3.7. Let p be a prime and $\alpha \in \mathbb{N}$. An arithmetic function h is said to have the *factorizable condition with respect to* p^α (F.C. wrt. p^α) if it satisfies the two conditions (i) and (ii) of Theorem 3.6.

The following theorem gives an algorithm for factorizing an arithmetic function h under certain condition via Theorem 3.6.

Theorem 3.8. Let $h_1 \in \mathcal{A}$ with $Nh_1 = p_1^{\alpha_1} \cdots p_m^{\alpha_m}$, where $p_1 < \ldots < p_m$ are primes and $\alpha_1, \ldots, \alpha_m \in \mathbb{N}$.

Step 1 : Assume that h_1 satisfies F.C. wrt. $p_1^{\alpha_1}$ with a non-unit root f_{11}.
If $\dfrac{h_1}{f_{11}}$ satisfies F.C. wrt. $p_1^{\alpha_1-1}$, determine whether $\dfrac{h_1}{f_{11} * f_{12}}$, where f_{12} is a root of $P(f, \dfrac{h_1}{f_{11}})$, satisfies F.C. wrt. $p_1^{\alpha_1-2}$. If so, continuing this process, we recursively obtain

$$h_1 \ , \ \frac{h_1}{f_{11}} \ , \ \frac{h_1}{f_{11} * f_{12}} \ , \ \ldots \ , \ \frac{h_1}{f_{11} * \cdots * f_{1\alpha_1}} \ ,$$

where $f_{1,i+1}$ is a non-unit root of $P(f, \dfrac{h_1}{f_{11} * \cdots * f_{1i}})$. Then proceed to step 2.

Step 2 : Let $h_2 = \dfrac{h_1}{f_{11} * \cdots * f_{1\alpha_1}}$. Assume that h_2 satisfies F.C. wrt. $p_2^{\alpha_2}$ with a non-unit root f_{21}. If $\dfrac{h_2}{f_{21}}$ satisfies F.C. wrt. $p_2^{\alpha_2-1}$, determine whether $\dfrac{h_2}{f_{21} * f_{22}}$, where f_{22} is a root of $P(f, \dfrac{h_2}{f_{21}})$, satisfies F.C. wrt. $p_2^{\alpha_2-2}$. If so, continuing this process, we recursively obtain

$$h_2 \ , \ \frac{h_2}{f_{21}} \ , \ \frac{h_2}{f_{21} * f_{22}} \ , \ \ldots \ , \ \frac{h_2}{f_{21} * \cdots * f_{2\alpha_2}} \ ,$$

where $f_{2,i+1}$ is a non-unit root of $P(f, \dfrac{h_2}{f_{21} * \cdots * f_{2i}})$. Then proceed the next step.

In general, to start step $j+1$,

$$h_j \ , \ \frac{h_j}{f_{j1}} \ , \ \dots \ , \ \frac{h_j}{f_{j1} * \cdots * f_{j\alpha_j}}$$

must be recursively obtainable.

Finally at the last step m, we need only determine divisors upto the one before last. Let $h_m = \dfrac{h_{m-1}}{f_{m-1,1} * \cdots * f_{m-1,\alpha_{m-1}}}$. Assume that h_m satisfies F.C. wrt. $p_m^{\alpha_m}$ with a non-unit root f_{m1}. If $\dfrac{h_m}{f_{m1}}$ satisfies F.C. wrt. $p_m^{\alpha_m-1}$, determine whether $\dfrac{h_m}{f_{m1} * f_{m2}}$, where f_{m2} is a root of $P(f, \dfrac{h_m}{f_{m1}})$, satisfies F.C. wrt. $p_m^{\alpha_m-2}$. If so, continuing this process, we recursively obtain

$$h_m \ , \ \frac{h_m}{f_{m1}} \ , \ \frac{h_m}{f_{m1} * f_{m2}} \ , \ \dots \ , \ \frac{h_m}{f_{m1} * \cdots * f_{m,\alpha_m-1}} \ ,$$

where $f_{m,i+1}$ is a non-unit root of $P(f, \dfrac{h_m}{f_{m1} * \cdots * f_{mi}})$ and
$H = \dfrac{h_m}{f_{m1} * \cdots * f_{m,\alpha_m-1}}$ is the last divisor of h_m of norm p_m.
After step m, then $h_1 = f_{11} * \cdots * f_{1\alpha_1} * \cdots * f_{m_1} * \cdots * f_{m,\alpha_m-1} * H$ is the prime factorization of h_1.

Example 3.9. Define $h(n) = \begin{cases} 1 & \text{if } n = 2^2 3^2 5^2, \\ 2 & \text{if } n = 2 \cdot 3^3 5^2, \\ 0 & \text{otherwise.} \end{cases}$

Then $Nh = 2^2 3^2 5^2$. First we will find divisors of h of norm 2. We have

$$d_2h(n) = \begin{cases} 2 & \text{if } n = 2 \cdot 3^2 5^2, \\ 2 & \text{if } n = 3^3 5^2, \\ 0 & \text{otherwise,} \end{cases} \qquad d_2^2h(n) = \begin{cases} 2 & \text{if } n = 3^2 5^2, \\ 0 & \text{otherwise,} \end{cases}$$

and $d_2^3h(n) = 0$ for all $n \in \mathbb{N}$.
Consider $P(f;h) = \displaystyle\sum_{k=0}^{2} \frac{(-1)^k}{k!} f^k * d_2^k h = 0$. Then $h = f * d_2h - \dfrac{1}{2} f^2 * d_2^2 h$.
To find a divisor of h via Theorem 3.6, it suffices to determine a root f_1 of $P(f, h) = 0$. We begin this process by investigating for each n, possible values of

$f_1(n)$.

From $0 = h(3^2 5^2) = -\frac{1}{2} f_1(1)^2 d_2^2 h(3^2 5^2) = -f_1(1)^2$, we get $f_1(1) = 0$.

From $1 = h(2^2 3^2 5^2) = f_1(2) d_2 h(2 \cdot 3^2 5^2) - \frac{1}{2} d_2^2 h(3^2 5^2) f_1(2)^2$

$$= 2f_1(2) - f_1(2)^2,$$

we get $f_1(2) = 1$.

From $0 = h(3^4 5^2) = f_1(3) d_2 h(3^3 5^2) - \frac{1}{2} d_2^2 h(3^2 5^2) f_1(3)^2$

$$= 2f_1(3) - f_1(3)^2,$$

we get $f_1(3) = 0$ or 2.

$$\begin{aligned} \text{From } 0 = & h(4 \cdot 3^3 5^2) = f_1(6) d_2 h(2 \cdot 3^2 5^2) + f_1(4) d_2 h(3^3 5^2) \\ & - \frac{1}{2} d_2^2 h(3^2 5^2)[2f_1(3) f_1(4) + 2f_1(2) f_1(6)] \\ & = 2f_1(6) + 2f_1(4) - 2f_1(6) - 2f_1(3) f_1(4), \end{aligned}$$

we get $f_1(4) = 0$.

For $n > 4$, assume that $f_1(k) = 0$, when $1 \leq k \leq n-1$, $k \neq 2, 3$.

From $0 = h(n 3^3 5^2) = 2f_1(n) - 2f_1(3) f_1(n)$, we get $f_1(n) = 0$.

Thus $f_1(n) = \begin{cases} 1 & \text{if } n = 2, \\ 0 & \text{otherwise} \end{cases}$ or $f_1(n) = \begin{cases} 1 & \text{if } n = 2, \\ 2 & \text{if } n = 3, \\ 0 & \text{otherwise.} \end{cases}$

Clearly, both functions are roots of $P(f, h)$. By Theorem 3.6, f_1 is a factor of h.

Case 1 : $f_1(n) = \begin{cases} 1 & \text{if } n = 2, \\ 0 & \text{otherwise.} \end{cases}$

Let $H = \frac{h}{f_1}$. Then $H(n) = \begin{cases} 1 & \text{if } n = 2 \cdot 3^2 5^2, \\ 2 & \text{if } n = 3^3 5^2, \\ 0 & \text{otherwise,} \end{cases}$

$d_2 H(n) = \begin{cases} 1 & \text{if } n = 3^2 5^2, \\ 0 & \text{otherwise,} \end{cases}$ and $d_2^2 H(n) = 0$ for all $n \in \mathbb{N}$.

Consider $P(f;H)=\sum_{k=0}^{1}\frac{(-1)^k}{k!}f^k * d_2^k H=0$. Then $H=f*d_2H$.
By using the same procedure, another divisor f_2 of H, of norm 2 and in standard form is defined by $f_2(n)=\begin{cases}1 & \text{if } n=2,\\ 2 & \text{if } n=3,\\ 0 & \text{otherwise.}\end{cases}$

Case 2 : $f_1(n)=\begin{cases}1 & \text{if } n=2,\\ 2 & \text{if } n=3,\\ 0 & \text{otherwise.}\end{cases}$

Let $H_1=\dfrac{h}{f_1}$. Then $H_1(n)=\begin{cases}1 & \text{if } n=2\cdot 3^2 5^2,\\ 0 & \text{otherwise,}\end{cases}$

$d_2H_1(n)=\begin{cases}1 & \text{if } n=3^2 5^2\\ 0 & \text{otherwise,}\end{cases}$ and $d_2^2H_1(n)=0$ for all $n\in\mathbb{N}$.

Consider $P(f;H_1)=\sum_{k=0}^{1}\frac{(-1)^k}{k!}f^k * d_2^k H_1=0$. Then $H_1=f*d_2H_1$.
As before, if a non-unit root f_2 exists, then f_2 is a divisor of H_1, of norm 2, in standard form and $f_2(n)=\begin{cases}1 & \text{if } n=2,\\ 0 & \text{otherwise.}\end{cases}$

In any case , h has two factors of norm 2, in standard forms, namely,

$f_1(n)=\begin{cases}1 & \text{if } n=2,\\ 0 & \text{otherwise}\end{cases}$ and $f_2(n)=\begin{cases}1 & \text{if } n=2,\\ 2 & \text{if } n=3,\\ 0 & \text{otherwise.}\end{cases}$

Next we find divisors of h of norm 3.

Let $G=\dfrac{h}{f_1*f_2}$. Then $G(n)=\begin{cases}1 & \text{if } n=3^2 5^2,\\ 0 & \text{otherwise,}\end{cases}$ $d_3G(n)=\begin{cases}2 & \text{if } n=3\cdot 5^2,\\ 0 & \text{otherwise,}\end{cases}$

$$d_3^2G(n) = \begin{cases} 2 & \text{if } n = 5^2, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad d_3^3h(n) = 0 \quad \text{for all } n \in \mathbb{N}.$$

Consider $P(g; G) = \sum_{k=0}^{2} \frac{(-1)^k}{k!} g^k * d_3^k G = 0$. Then $G = g * d_3G - \frac{1}{2}g^2 * d_3^2G$.
As before, if a non-unit root g_1 exists, then g_1 is a divisor of G, of norm 3, in standard form and $g_1(n) = \begin{cases} 1 & \text{if } n = 3, \\ 0 & \text{otherwise.} \end{cases}$

Let $G_1 = \dfrac{G}{g_1}$. Then $G_1(n) = \begin{cases} 1 & \text{if } n = 3 \cdot 5^2, \\ 0 & \text{otherwise,} \end{cases}$

$$d_3G_1(n) = \begin{cases} 1 & \text{if } n = 5^2, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and } d_3^2G_1(n) = 0 \quad \text{for all } n \in \mathbb{N}.$$

Consider $P(g; G_1) = \sum_{k=0}^{1} \frac{(-1)^k}{k!} g^k * d_3^k G_1 = 0$. Then $G_1 = g * d_3G_1$.
As before, if a non-unit root g_2 exists, then g_2 is a divisor of G_1, of norm 3, in standard form and $g_2(n) = \begin{cases} 1 & \text{if } n = 3, \\ 0 & \text{otherwise.} \end{cases}$
We see that $g_1 = g_2$ are two factors of norm 3 of G, and so of h.

It remains to find factors of h of norm 5.
Let $T = \dfrac{G}{g_1 * g_2}$. Then $T(n) = \begin{cases} 1 & \text{if } n = 5^2, \\ 0 & \text{otherwise,} \end{cases}$ $d_5T(n) = \begin{cases} 2 & \text{if } n = 5, \\ 0 & \text{otherwise,} \end{cases}$

$$d_5^2T(n) = \begin{cases} 2 & \text{if } n = 1, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and } d_5^3T(n) = 0 \quad \text{for all } n \in \mathbb{N}.$$

Consider $P(t; T) = \sum_{k=0}^{2} \frac{(-1)^k}{k!} t^k * d_5^k T = 0$. Then $T = t * d_5T - \frac{1}{2}t^2 * d_5^2T$.
As before, if a non-unit root t exists, then t is a divisor of T of norm 5, in

standard form and $t(n) = \begin{cases} 1 & \text{if } n = 5, \\ 0 & \text{otherwise.} \end{cases}$

Let $v = \dfrac{T}{t}$. Then $v(n) = \begin{cases} 1 & \text{if } n = 5, \\ 0 & \text{otherwise.} \end{cases}$

We get $t = v$ as two factors of norm 5 of T, and so of h. Since the norms of f_1, f_2, g_1, g_2, t, and v are primes, all of them are primes in $\mathcal{A}$. It can be directly checked that $h = f_1 * f_2 * g_1 * g_2 * t * v$ is the unique prime factorization of h.

The next example gives the case where the hypothesis of Theorem 3.6 fails but we can show directly that h is a prime.

Example 3.10. Let $h(n) = \begin{cases} 1 & \text{if } n = 6, 10, 35, \\ n & \text{if } n > 35 : n \neq \text{prime and } 2 \nmid n, \\ 0 & \text{otherwise.} \end{cases}$

Then $d_2h(n) = \begin{cases} 1 & \text{if } n = 3, 5, \\ 0 & \text{otherwise} \end{cases}$ and $d_2^2 h(n) = 0$ for all $n \in \mathbb{N}$.

Suppose f_1 were a root of $P(f,h) = \displaystyle\sum_{k=0}^{1} \frac{(-1)^k}{k!} f^k * d_2^k h = 0$. Then $h = f_1 * d_2h$.

$0 = h(3 \cdot 7) = f_1(7) d_2h(3) = f_1(7)$ and $1 = h(5 \cdot 7) = f_1(7) d_2h(5) = f_1(7)$, which is a contradiction. Thus $P(f,h)$ has no root.

To show that h has no divisor of norm 2, suppose on the contrary that f is a divisor of norm 2 of h in $\mathcal{A}$. Then $h = f * g$ for some $g \in \mathcal{A}$, with $Ng = 3$, so $f(2) \neq 0$, $g(3) \neq 0$ and $f(1) = g(1) = g(2) = 0$.

Since $0 = h(2 \cdot 7) = f(2)g(7)$ and $f(2) \neq 0$, we get $g(7) = 0$.

$1 = h(5 \cdot 7) = f(7)g(5)$ implies $f(7) \neq 0$.

Thus $0 \neq f(7)g(3) = h(3 \cdot 7) = 0$ which is a contradiction. Therefore h has no divisor of norm 2, which immediately implies that h has no divisor of norm 3 either, and hence h must be a prime.

The last example illustrates the case where Theorem 3.8 is not applicable at the first step. But if we ignore it, and skip to the next prime, the technique in Theorem 3.8 might enable us to determine a factor whose norm is the next prime.

Example 3.11. Let $h(n) = \begin{cases} 2 & \text{if } n = 12, \\ 1 & \text{if } n = 15, \\ 0 & \text{otherwise.} \end{cases}$ Then $Nh = 2^2 3$,

$d_2h(n) = \begin{cases} 4 & \text{if } n = 6, \\ 0 & \text{otherwise,} \end{cases}$ $\quad d_2^2h(n) = \begin{cases} 4 & \text{if } n = 3, \\ 0 & \text{otherwise,} \end{cases}$

and $d_2^3h(n) = 0$ for all $n \in \mathbb{N}$.

Suppose that f_1 were a root of $P(f,h) = \sum_{k=0}^{2} \frac{(-1)^k}{k!} f^k * d_2^k h = 0$.

Then $h = f_1 * d_2h - \frac{1}{2} f_1^2 * d_2^2 h$.

From $0 = h(3) = -2f_1(1)^2$, we get $f_1(1) = 0$.

But $1 = h(15) = -\frac{1}{2} d_2^2 h(3)[2f_1(1)f_1(5)] = 0$, so it is a contradiction. Thus $P(f,h)$ has no root. This shows that the algorithm in Theorem 3.8 cannot be applied in searching for a divisor of h of norm 2. Ignoring the smallest prime, we find

$d_3h(n) = \begin{cases} 2 & \text{if } n = 4, \\ 1 & \text{if } n = 5, \\ 0 & \text{otherwise} \end{cases}$ $\quad$ and $\quad d_3^2h(n) = 0$ for all $n \in \mathbb{N}$.

Consider $P(g,h) = \sum_{k=0}^{1} \frac{(-1)^k}{k!} g^k * d_3^k h = 0$. Then $h = g * d_3h$.

As before, we get $g(n) = \begin{cases} 1 & \text{if } n = 3, \\ 0 & \text{otherwise.} \end{cases}$

Let $G = \dfrac{h}{g}$. Then $G(n) = \begin{cases} 2 & \text{if } n = 4, \\ 1 & \text{if } n = 5, \\ 0 & \text{otherwise.} \end{cases}$

Since $Ng = 3$, g is a prime. Since $NG = 4$ and $G(5) \neq 0$, then G is a prime by Theorem 3.5. Hence $h = g * G$ is its prime factorization.

Example 3.11 leads to the following immediate consequence whose proof is a slight modification of that of Theorem 3.6 and so it is omitted.

Proposition 3.12. Let $h \in \mathcal{A}$ with $Nh = p_1^{\alpha_1} \cdots p_m^{\alpha_m}$, where $p_1 < \ldots < p_m$ are primes and $\alpha_1, \ldots, \alpha_m \in \mathbb{N}$. Suppose that $p_2 < p_1^2$, h has no factor of norm p_1, and there is an integer $b \geq \alpha_2$ such that $d_{p_2}^b h \neq 0$, and $d_{p_2}^{b+1} h = 0$. If $P(f;h) = \sum_{k=0}^{b} \frac{(-1)^k}{k!} f^k * d_{p_2}^k h$ has a non-unit root $f_1 \in \mathcal{A}$, then f_1 is a divisor of h of norm p_2, in standard form.

It is natural to ask whether the converse of Theorem 3.6 holds. The last theorem shows that it does with an extra condition, which also reveals that our proposed factorization technique applies to a particularly large class.

Theorem 3.13. Let $h \in \mathcal{A}$. If $f_1 \in \mathcal{A}$ is a divisor of h of norm p, in standard form, and $d_p^b(\frac{h}{f_1}) = 0$ for some positive integer b, then f_1 is a root of the polynomial

$$P(f, h) = \sum_{k=0}^{b} \frac{(-1)^k}{k!} f^k * d_p^k h.$$

Proof. Writing $h = f_1 * g$ for some $g \in \mathcal{A}$ i.e. $g = \dfrac{h}{f_1}$, we get $d_p^b g = 0$. Since f_1 is in standard form with norm p, then

$$d_p^k h = f_1 * d_p^k g + k d_p^{k-1} g \quad (k \in \mathbb{N}).$$

Thus $\frac{1}{k!} f_1^k * d_p^k h = \frac{1}{k!} f_1^{k+1} d_p^k g + \frac{1}{(k+1)!} f_1^k * d_p^{k-1} g$. Summing from $k = 1$ till $k = b$, the result is obtained. □

CHAPTER IV

INDEPENDENCE OF ARITHMETIC FUNCTIONS

In Shapiro-Sparer [10], a systematic investigation of algebraic independence of Dirichlet series is made. A thorough study of this paper leads us to results in this chapter which either extend or simplify certain results in sections 3,4,5 and 7 of [10].

4.1 Differential Difference Equations over $\mathbb{C}$

We first recall some definitions.

Definition 4.1. A (formal) *Dirichlet series* is an expression of the form

$$F(s) = \sum_{\text{n}=1}^{\infty} \frac{f(n)}{n^s}, \quad f(n) \in \mathbb{C}.$$

The set $(\mathcal{D}, +, \cdot)$ of all Dirichlet series equipped with addition and multiplication is isomorphic to $(\mathcal{A}, +, *)$, through the map

$$F = \sum_{\text{n}=1}^{\infty} \frac{f(n)}{n^s} \longleftrightarrow f$$

(see [2],[3]). Through this isomorphism, any algebraic relations from one setting have corresponding counterparts in the other, which allows us to refer to both interchangably, and we often do so without further ado.

Definition 4.2. Let Z be a Dirichlet series. A Dirichlet series F is $\mathbb{C}$-*algebraically dependent on* Z, written $F \in \overline{\mathbb{C}[Z]}$, if F and Z are algebraically dependent over $\mathbb{C}$, and F is *properly* $\mathbb{C}$-*algebraically dependent on* Z if $F \in \overline{\mathbb{C}[Z]} \backslash \mathbb{C} := \overline{\mathbb{C}[Z]}^*$.

If we define a derivation over $\mathcal{D}$ in the same way as $\mathcal{A}$, then we may also regard the derivation d over $\mathcal{A}$, also as a derivation over $\mathcal{D}$ via

$$dF = \sum_{\mathrm{n}=1}^{\infty} \frac{(df)(n)}{n^s}.$$

The results in this section are based on the study of the third section of Shapiro-Sparer [10].

Theorem 4.3. Let $\xi \in \mathcal{A}$ be such that $\xi(p) \neq 0$ for infinitely many primes p. Let $\mathcal{E}$ be a subring of $\mathcal{A}$ having the property that given any finite subset $\mathcal{E}^* \subseteq \mathcal{E}$, for all sufficiently large primes p, the derivations d_p annihilate all of $\mathcal{E}^*$. Then for any sequence of complex numbers $(r_i)_{i\geq 1}$, with distinct real parts, and any sequence of integers $(t_j)_{j\geq 1}$ (not necessarily distinct), the functions

$$f_{ij}(n) = \xi(n) n^{r_i} (\log n)^{t_j}$$

are algebraically independent over $\mathcal{E}$.

Proof. Suppose that the assertion is false, i.e. there is a finite subset of $\{f_{ij}\}$ which are algebraically dependent over $\mathcal{E}$. For ease of writing, we may assume that this set is $\{f_{11}, \ldots, f_{kl}\}$. Let $\mathcal{E}^*$ $(\subset \mathcal{E})$ be the finite set of all coefficients in this algebraic relation. By hypothesis, for all sufficiently large primes p, each d_p annihilates all of $\mathcal{E}^*$, and so each d_p annihilates all of $\mathcal{E}' = \langle \mathcal{E}^* \rangle$, the subring of $\mathcal{E}$ generated by $\mathcal{E}^*$. Thus $f_{11}, \ldots, f_{kl}$ are algebraically dependent over $\mathcal{E}'$. If we can choose primes p_{ij} among these so that

$$J(f_{11}, \ldots, f_{kl}/p_{11}, \ldots, p_{kl}) \neq 0,$$

then Theorem 2.14 implies that $f_{11}, \ldots, f_{kl}$ are algebraically independent over $\mathcal{E}'$, which is a contradiction and the desired result will follow.

We may assume without loss of generality that $-s \leq t_j \leq s$ for all $j \in \{1, \ldots, l\}$, where s is a fixed positive integer, and rewrite the above set as

$\{f_{ij} \mid i \in \{1,\ldots,k\}, j \in \{-s,\ldots,s\}\}$ instead of $\{f_{11},\ldots,f_{kl}\}$.

Let $T = (2s+1)k$. For any sequence of sufficiently large primes, $p_1 > p_2 > \ldots > p_T$, each $\xi(p_i) \neq 0$, we have

$$\begin{aligned} J(n) &:= J(f_{1,-s},\ldots,f_{1,s},\ldots,f_{k,-s},\ldots,f_{k,s}/p_1,\ldots,p_T)(n) \\ &= \det(d_{p_m}(f_{ij}))(n) \\ &= \det(f_{ij}(np_m)\upsilon_{p_m}(np_m)) \\ &= \det(\xi(np_m)(np_m)^{r_i}(\log np_m)^j \upsilon_{p_m}(np_m)), \end{aligned}$$

where $m = 1,\ldots,T; i \in \{1,\ldots,k\}; j \in \{-s,\ldots,s\}$.

Putting $n = 1$, we have

$$J(1) = \det(\xi(p_m)p_m^{r_i}(\log p_m)^j)) = \xi(p_1)\cdots\xi(p_T)\det(p_m^{r_i}(\log p_m)^j),$$

and consider

$$J^* = \frac{J(1)}{\xi(p_1)\cdots\xi(p_T)} = \det(p_m^{r_i}(\log p_m)^j).$$

Note that a typical term in the expansion of the determinant defining J^* is of the form

$$t(\vec{p},\vec{r},\vec{j}) := \pm p_1^{r_{\mu_1}}(\log p_1)^{j_1} p_2^{r_{\mu_2}}(\log p_2)^{j_2} \cdots p_T^{r_{\mu_T}}(\log p_T)^{j_T},$$

where $\mu_1,\ldots,\mu_T \in \{1,\ldots,k\}; j_1,\ldots,j_T \in \{-s,\ldots,s\}$.

We may assume that $Re(r_1) > Re(r_2) > \ldots > Re(r_k)$. In the first row , the column which has the unique largest absolute value is $p_1^{r_1}(\log p_1)^s$, so we exchange the first column with this column. In the second row, we consider the column which has the next unique largest absolute value (after the first column) and exchange the second column with this column. Continue this process. We claim that in the final determinant, by choosing $p_1 > p_2 > \ldots > p_T$ sufficiently large the term with largest absolute value is the main diagonal term

$$Y := a_{11}a_{22}\cdots a_{TT} = p_1^{r_1}(\log p_1)^s p_2^{(r)_2}(\log p_2)^{(s)_2}\cdots p_T^{(r)_T}(\log p_T)^{(s)_T},$$

where $(r)_i, (s)_i$ denote the diagonal exponents. Let

$$a_{\underline{\mathbf{j}}} := a_{1j_1}a_{2j_2}\cdots a_{Tj_T} = p_1^{\alpha_1}(\log p_1)^{\beta_1}\cdots p_T^{\alpha_T}(\log p_T)^{\beta_T}$$

be any term in the determinant expansion. There are three possibilities.

(i) If $r_1 \neq \alpha_1$ $(Re(r_1) > Re(\alpha_1))$, then choosing p_1 sufficiently large in comparison with other p_i's, we see that $p_1^{r_1} >> p_1^{\alpha_1}$ which leads to $|Y| > |a_{\underline{\mathbf{j}}}|$.

(ii) If $r_1 = \alpha_1, s > \beta_1$, then as in (i), $(\log p_1)^s >> (\log p_1)^{\beta_1}$ and so $|Y| > |a_{\underline{\mathbf{j}}}|$.

(iii) If $r_1 = \alpha_1, s = \beta_1$ (i.e. both terms arise from the expansion of the (1,1) term), repeating the same arguments as above we see that the next largest term must come from the main diagonal.

Furthermore, we can even choose the primes $p_1 > \ldots > p_T$ so large that

$$\left|\frac{t(\vec{p},\vec{i},\vec{j})}{Y}\right| < \frac{1}{T!} \quad \text{for each} \;\; t(\vec{p},\vec{i},\vec{j}) \neq Y.$$

Thus $\quad \dfrac{J^*}{Y} = 1 + ((T!-1)\text{terms each with absolute value } < \dfrac{1}{T!}) \neq 0.$

This shows that there are sets of primes such that $J^* \neq 0$, yielding $J(1) \neq 0$, as required. □

Theorem 4.3 reduces to Theorem 3.3 of [10] when $\xi(n) = u(n) = 1$ for all $n \in \mathbb{N}$. By the same proof as in Theorem 4.3 we also have the following result :

Theorem 4.4. Let $\xi \in \mathcal{A}$ be such that $\xi(p) \neq 0$ for all sufficiently large primes p. Let $\mathcal{E}$ be a subring of $\mathcal{A}$ having the property that given any finite subset $\mathcal{E}^* \subseteq \mathcal{E}$, there are infinitely many primes p, whose derivations d_p annihilate all of $\mathcal{E}^*$. Then for any sequence of complex numbers $(r_i)_{i\geq 1}$, with distinct real parts, and any sequence of integers $(t_j)_{j\geq 1}$ (not necessarily distinct), the functions

$$f_{ij}(n) = \xi(n)n^{r_i}(\log n)^{t_j}$$

are algebraically independent over $\mathcal{E}$.

Since for each prime p, d_p annihilates all elements of $\mathbb{C}$, from Theorem 4.3, we easily deduce

Corollary 4.5. Let $\xi \in \mathcal{A}$ be such that $\xi(p) \neq 0$ for infinitely many primes p. Let $(r_i)_{i\geq 1}$ be a sequence of complex numbers with distinct real parts, and $(t_j)_{j\geq 1}$ a sequence of integers (not necessarily distinct). Then the functions

$$f_{ij}(n) = \xi(n)n^{r_i}(\log n)^{t_j},$$

for all distinct (r_i, t_j), are algebraically independent over $\mathbb{C}$.

Corollary 4.6. Let $\Xi(s) = \sum_{n=1}^{\infty} \frac{\xi(n)}{n^s}$, where $\xi \in \mathcal{A}$ is such that $\xi(p) \neq 0$ for infinitely many primes p. Let $r_i, i = 1, \ldots, L$ be complex numbers with distinct real parts, and $m_j, j = 1, \ldots, L$ any nonnegative integers. Then the functions

$$\Xi^{(m_j)}(s - r_i), \quad i, j \in \{1, \ldots, L\}$$

are algebraically independent over $\mathbb{C}$.

Proof. This follows readily from Corollary 4.5, noting that

$$\Xi^{(m)}(s - r) = \sum_{n=1}^{\infty} \frac{(-1)^m \xi(n)}{n^{s-r}} (\log n)^m.$$

□

A rephrasing of Corollary 4.6 is :

Corollary 4.7. Let $\Xi(s) = \sum_{n=1}^{\infty} \frac{\xi(n)}{n^s}$, where $\xi \in \mathcal{A}$ is such that $\xi(p) \neq 0$ for infinitely many primes p. Then $\Xi(s)$ does not satisfy any nontrivial algebraic differential difference equation over $\mathbb{C}$.

4.2 Functions Which Are Algebraic Over $\mathbb{C}[\Xi]$

The results in this section are based on the study of the forth section of Shapiro-Sparer. In this section we assume $\xi \in \mathcal{A}$ to be completely multiplicative, with $\xi(p) \neq 0$ for all primes p and $\Xi(s) = \sum_{\text{n=1}}^{\infty} \frac{\xi(n)}{n^s}$ being its corresponding Dirichlet series. Let $f \in \mathcal{A}$ be algebraic over $\mathbb{C}[\xi]$. By Theorem 2.14, for every pair of primes $p \neq q$,

$$J(f, \xi/p, q) = \begin{vmatrix} d_p f & d_p \xi \\ d_q f & d_q \xi \end{vmatrix} = 0$$

i.e.

$$d_p f * d_q \xi = d_q f * d_p \xi. \tag{4.1}$$

Let $\mathcal{S}$ be the set of solutions of equation (4.1) and $\overline{\mathbb{C}[\xi]}$ denote the set of elements of $\mathcal{A}$ algebraic over $\mathbb{C}[\xi]$. Then $\overline{\mathbb{C}[\xi]} \subseteq \mathcal{S}$.

Theorem 4.8. The functions in $\mathcal{S}$ are precisely those functions $f \in \mathcal{A}$ whose corresponding Dirichlet series are of the form

$$\sum_{\text{n=1}}^{\infty} \frac{f(n)}{n^s} = \sum_{\nu=0}^{\infty} \frac{\phi_\nu}{\nu!} (\log \Xi)^\nu, \tag{4.2}$$

where

$$\phi_\nu = \frac{f(p_1 \cdots p_\nu)}{\xi(p_1 \cdots p_\nu)} \tag{4.3}$$

is independent of the choice of the ν distinct primes $p_1, \ldots, p_\nu$.
(ϕ_ν is called the *ν-value* of $\frac{f}{\xi}$.)

Proof. We note first that (see [1],Theorem 11.14)

$$\log \Xi(s) = \sum_{n=2}^{\infty} \frac{\xi(n)\Lambda(n)}{n^s \log n},$$

where Λ is the Mangoldt function defined by

$$\Lambda(n) = \begin{cases} \log p & \text{if} \quad n = p^m \text{ for some prime } p \text{ and } m \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned}
\sum_{\nu=0}^{\infty} \frac{\phi_\nu}{\nu!}(\log \Xi)^\nu &= \sum_{\nu=0}^{\infty} \frac{\phi_\nu}{\nu!} \left(\sum_{n=2}^{\infty} \frac{\xi(n)\Lambda(n)}{n^s \log n} \right)^\nu \\
&= \sum_{\nu=0}^{\infty} \frac{\phi_\nu}{\nu!} \left(\sum_{n=2^\nu}^{\infty} \frac{\xi(n)}{n^s} \sum_{\substack{n_1 \cdots n_\nu = n \\ n_i \geq 2}} \frac{\Lambda(n_1) \cdots \Lambda(n_\nu)}{(\log n_1) \cdots (\log n_\nu)} \right) \\
&= \frac{\phi_0}{1^s} + \sum_{n=2}^{\infty} \frac{\xi(n)}{n^s} \sum_{1 \leq \nu \leq \frac{\log n}{\log 2}} \frac{\phi_\nu}{\nu!} \left(\sum_{\substack{n_1 \cdots n_\nu = n \\ n_i \geq 2}} \frac{\Lambda(n_1) \cdots \Lambda(n_\nu)}{(\log n_1) \cdots (\log n_\nu)} \right) \\
&= \sum_{\mathrm{n}=1}^{\infty} \frac{f(n)}{n^s}
\end{aligned}$$

is always a Dirichlet series. Thus, for any primes $p \neq q$,

$$\begin{aligned}
d_p \left(\sum_{\nu=0}^{\infty} \frac{\phi_\nu}{\nu!}(\log \Xi)^\nu \right) &= \sum_{\nu=1}^{\infty} \frac{\phi_\nu}{\nu!} \nu (\log \Xi)^{\nu-1} d_p(\log \Xi) \\
&= \sum_{\nu=1}^{\infty} \frac{\phi_\nu}{(\nu-1)!}(\log \Xi)^{\nu-1} \left(\frac{d_p \Xi}{\Xi} \right) \\
&= \left(\frac{d_p \Xi}{\Xi} \right) \sum_{\nu=1}^{\infty} \frac{\phi_\nu}{(\nu-1)!}(\log \Xi)^{\nu-1}
\end{aligned}$$

and so,

$$\begin{aligned}
(d_q \Xi) d_p \left(\sum_{\nu=0}^{\infty} \frac{\phi_\nu}{\nu!}(\log \Xi)^\nu \right) &= (d_q \Xi) \left(\frac{d_p \Xi}{\Xi} \right) \sum_{\nu=1}^{\infty} \frac{\phi_\nu}{(\nu-1)!}(\log \Xi)^{\nu-1} \\
&= (d_p \Xi) d_q \left(\sum_{\nu=0}^{\infty} \frac{\phi_\nu}{\nu!}(\log \Xi)^\nu \right).
\end{aligned}$$

Hence, we have

$$\sum_{\nu=0}^{\infty} \frac{\phi_\nu}{\nu!}(\log \Xi)^\nu \in \mathcal{S}.$$

Since

$$\sum_{\mathrm{n}=1}^{\infty} \frac{f(n)}{n^s} = \frac{\phi_0}{1^s} + \sum_{n=2}^{\infty} \frac{\xi(n)}{n^s} \sum_{1 \leq \nu \leq \frac{\log n}{\log 2}} \frac{\phi_\nu}{\nu!} \left(\sum_{\substack{n_1 \cdots n_\nu = n \\ n_i \geq 2}} \frac{\Lambda(n_1) \cdots \Lambda(n_\nu)}{(\log n_1) \cdots (\log n_\nu)} \right),$$

for any $k \geq 1$ and primes $p_1, \ldots, p_k$, the coefficients of $(p_1 \cdots p_k)^{-s}$ in both sides are

$$f(p_1\cdots p_k)=\xi(p_1\cdots p_k)\frac{\phi_k}{k!}\sum_{p_{i_1}\cdots p_{i_k}=p_1\cdots p_k}\frac{\Lambda(p_{i_1})\cdots\Lambda(p_{i_k})}{(\log p_{i_1})\cdots(\log p_{i_k})}$$

$$=\xi(p_1\cdots p_k)\frac{\phi_k}{k!}k!=\xi(p_1\cdots p_k)\phi_k.$$

Then $\quad \phi_k=\dfrac{f(p_1\cdots p_k)}{\xi(p_1\cdots p_k)}\quad$ depends only on k.

Conversely, we show that (4.2) and (4.3) hold for all $f\in\mathcal{S}$.

Step 1. $\quad f\in\mathcal{S}$ is equivalent to the assertion that

$$\frac{f(np)\upsilon_p(np)}{\xi(p)}-f(n)\upsilon(n) \tag{4.4}$$

is independent of the prime p.

First we write (4.1) in a Dirichlet series representation as follows,

$$\begin{aligned}(d_pF)(d_q\Xi)&=\left(\sum_{n=1}^{\infty}\frac{f(np)\upsilon_p(np)}{n^s}\right)\left(\sum_{n=1}^{\infty}\frac{\xi(nq)\upsilon_q(nq)}{n^s}\right)\\&=\left(\sum_{n=1}^{\infty}\frac{f(np)\upsilon_p(np)}{n^s}\right)\xi(q)\left(\sum_{n=1}^{\infty}\frac{\xi(n)\upsilon_q(nq)}{n^s}\right)\\&=\left(\sum_{n=1}^{\infty}\frac{f(np)\upsilon_p(np)}{n^s}\right)\xi(q)\left(\frac{\Xi(s)}{1-\frac{\xi(q)}{q^s}}\right).\end{aligned}$$

Then

$$f\in\mathcal{S}\iff(d_pF)(d_q\Xi)=(d_qF)(d_p\Xi)$$

$$\iff\left(\sum_{n=1}^{\infty}\frac{f(np)\upsilon_p(np)}{n^s}\right)\xi(q)\left(\frac{\Xi(s)}{1-\frac{\xi(q)}{q^s}}\right)=\left(\sum_{n=1}^{\infty}\frac{f(nq)\upsilon_q(nq)}{n^s}\right)\xi(p)\left(\frac{\Xi(s)}{1-\frac{\xi(p)}{p^s}}\right)$$

$$\iff\frac{1}{\xi(p)}\left(\sum_{n=1}^{\infty}\frac{f(np)\upsilon_p(np)}{n^s}\right)\left(1-\frac{\xi(p)}{p^s}\right)=\frac{1}{\xi(q)}\left(\sum_{n=1}^{\infty}\frac{f(nq)\upsilon_q(nq)}{n^s}\right)\left(1-\frac{\xi(q)}{q^s}\right)$$

$$\iff\sum_{n=1}^{\infty}\frac{f(np)\upsilon_p(np)}{\xi(p)n^s}-\sum_{n=1}^{\infty}\frac{f(np)\upsilon_p(np)}{(np)^s}=\sum_{n=1}^{\infty}\frac{f(nq)\upsilon_q(nq)}{\xi(q)n^s}-\sum_{n=1}^{\infty}\frac{f(nq)\upsilon_q(nq)}{(nq)^s}$$

$$\iff \sum_{n=1}^{\infty} \frac{f(np)\upsilon_p(np)}{\xi(p)n^s} - \left(\sum_{n=1}^{\infty} \frac{f(np)\upsilon_p(np)}{(np)^s} + \sum_{(n,p)=1} \frac{f(n)\upsilon_p(n)}{n^s} \right)$$

$$= \sum_{n=1}^{\infty} \frac{f(nq)\upsilon_q(nq)}{\xi(q)n^s} - \left(\sum_{n=1}^{\infty} \frac{f(nq)\upsilon_q(nq)}{(nq)^s} + \sum_{(n,q)=1} \frac{f(n)\upsilon_q(n)}{n^s} \right)$$

$$\iff \sum_{n=1}^{\infty} \frac{f(np)\upsilon_p(np)}{\xi(p)n^s} - \sum_{n=1}^{\infty} \frac{f(n)\upsilon_p(n)}{n^s} = \sum_{n=1}^{\infty} \frac{f(nq)\upsilon_q(nq)}{\xi(q)n^s} - \sum_{n=1}^{\infty} \frac{f(n)\upsilon_q(n)}{n^s}$$

$$\iff \frac{f(np)\upsilon_p(np)}{\xi(p)} - f(n)\upsilon_p(n) = \frac{f(nq)\upsilon_q(nq)}{\xi(q)} - f(n)\upsilon_q(n)$$

$$\iff \frac{f(np)\upsilon_p(np)}{\xi(p)} - f(n)\upsilon_p(n) \text{ is independent of } p.$$

Step 2. For any prime p and $\alpha \geq 1$,

$$\frac{f(p^\alpha)}{\xi(p^\alpha)} = \sum_{\nu=1}^{\alpha} H_{p^\alpha,\nu} \frac{f(q_1 \cdots q_\nu)}{\xi(q_1 \cdots q_\nu)},$$

where $H_{p^\alpha,\nu}$ is a constant depending only on α,ν, and $q_1, \ldots, q_\nu$ are distinct primes all unequal to p.

Let q_1 be a prime not equal to p. Taking $n = p^{\alpha-1}, q = q_1$ in (4.4), we obtain

$$\frac{f(p^\alpha)\upsilon_p(p^\alpha)}{\xi(p)} - f(p^{\alpha-1})\upsilon_p(p^{\alpha-1}) = \frac{f(p^{\alpha-1}q_1)\upsilon_p(p^{\alpha-1}q_1)}{\xi(q_1)} - f(p^{\alpha-1})\upsilon_{q_1}(p^{\alpha-1}).$$

Then

$$\frac{\alpha f(p^\alpha)}{\xi(p)} - (\alpha - 1)f(p^{\alpha-1}) = \frac{f(p^{\alpha-1}q_1)}{\xi(q_1)}$$

or

$$f(p^\alpha) = (1 - \frac{1}{\alpha})\xi(p)f(p^{\alpha-1}) + \frac{1}{\alpha}\frac{\xi(p)}{\xi(q_1)}f(p^{\alpha-1}q_1). \tag{4.5}$$

Thus

$$f(p) = \frac{\xi(p)}{\xi(q_1)}f(q_1).$$

By (4.5), we have,

$$f(p^2) = \frac{1}{2}\xi(p)f(p) + \frac{1}{2}\frac{\xi(p)}{\xi(q_1)}f(pq_1).$$

Let $q_2 \neq p, q_1$ be a prime. Taking $n = q_1, q = q_2$ in (4.4), we obtain

$$\frac{f(pq_1)\upsilon_p(pq_1)}{\xi(p)} - f(q_1)\upsilon_p(q_1) = \frac{f(q_1q_2)\upsilon_{q_2}(q_1q_2)}{\xi(q_2)} - f(q_1)\upsilon_{q_2}(q_1)$$

and so,

$$f(pq_1) = \frac{\xi(p)}{\xi(q_2)} f(q_1q_2).$$

Thus

$$\begin{aligned} f(p^2) &= \frac{1}{2}\xi(p)\left(\frac{\xi(p)}{\xi(q_1)}f(q_1)\right) + \frac{1}{2}\frac{\xi(p)}{\xi(q_1)}\left(\frac{\xi(p)}{\xi(q_2)}f(q_1q_2)\right) \\ &= \xi(p^2)\left(\frac{1}{2}\frac{f(q_1)}{\xi(q_1)} + \frac{1}{2}\frac{f(q_1q_2)}{\xi(q_1q_2)}\right). \end{aligned}$$

Assume that

$$\frac{f(p^{\alpha-1})}{\xi(p^{\alpha-1})} = \sum_{\nu=1}^{\alpha-1} H'_{p^{\alpha-1},\nu} \frac{f(q_1\cdots q_\nu)}{\xi(q_1\cdots q_\nu)}$$

and

$$\frac{f(p^{\alpha-1}q_1)}{\xi(p^{\alpha-1})} = \sum_{i=1}^{\alpha-1} \frac{c_i f(q_1\cdots q_{i+1})}{\xi(q_2\cdots q_{i+1})},$$

where $H'_{p^{\alpha-1},\nu}$ and c_i are constants depending only on α, ν, and $q_1, \ldots, q_\nu$ are distinct primes all unequal to p. We have

$$\begin{aligned} f(p^\alpha) &= (1 - \frac{1}{\alpha})\xi(p)f(p^{\alpha-1}) + \frac{1}{\alpha}\frac{\xi(p)}{\xi(q_1)}f(p^{\alpha-1}q_1) \\ &= (1 - \frac{1}{\alpha})\xi(p)\xi(p^{\alpha-1})\sum_{\nu=1}^{\alpha-1} H'_{p^{\alpha-1},\nu}\frac{f(q_1\cdots q_\nu)}{\xi(q_1\cdots q_\nu)} \\ &\qquad + \frac{1}{\alpha}\frac{\xi(p)}{\xi(q_1}) \xi(p^{\alpha-1})\sum_{i=1}^{\alpha-1}\frac{c_i f(q_1\cdots q_{i+1})}{\xi(q_2\cdots q_{i+1})} \\ &= \xi(p^\alpha)\sum_{\nu=1}^{\alpha} H_{p^\alpha,\nu}\frac{f(q_1\cdots q_\nu)}{\xi(q_1\cdots q_\nu)}, \end{aligned}$$

where $H_{p^\alpha,\nu}$ is a constant depending only on α, ν, and $q_1, \ldots, q_\nu$ are distinct primes all unequal to p.

Step 3. If $q_1, \ldots, q_\nu$ are distinct primes and $q'_1, \ldots, q'_\nu$ are distinct primes, then

$$\frac{f(q_1\cdots q_\nu)}{\xi(q_1\cdots q_\nu)} = \frac{f(q'_1\cdots q'_\nu)}{\xi(q'_1\cdots q'_\nu)}.$$

First taking $n = q_1' \cdots q_{\nu-1}', p = q_\nu', q = q_\nu$ in (4.4), we obtain

$$\frac{f(q_1' \cdots q_\nu')}{\xi(q_\nu')} = \frac{f(q_1' \cdots q_{\nu-1}' q_\nu)}{\xi(q_\nu)}$$

and so

$$\frac{f(q_1' \cdots q_\nu')}{\xi(q_1' \cdots q_\nu')} = \frac{f(q_1' \cdots q_\nu')}{\xi(q_1') \cdots \xi(q_\nu')} = \frac{f(q_1' \cdots q_{\nu-1}' q_\nu)}{\xi(q_1') \cdots \xi(q_{\nu-1}')\xi(q_\nu)}.$$

Next, taking $n = q_1' \cdots q_{\nu-2}' q_\nu, p = q_{\nu-1}', q = q_{\nu-1}$ in (4.4), we obtain

$$\frac{f(q_1' \cdots q_{\nu-1}' q_\nu)}{\xi(q_{\nu-1}')} = \frac{f(q_1' \cdots q_{\nu-2}' q_{\nu-1} q_\nu)}{\xi(q_{\nu-1})}$$

and so

$$\frac{f(q_1' \cdots q_\nu')}{\xi(q_1' \cdots q_\nu')} = \frac{f(q_1' \cdots q_{\nu-1}' q_\nu)}{\xi(q_1') \cdots \xi(q_{\nu-1}')\xi(q_\nu)} = \frac{f(q_1' \cdots q_{\nu-2}' q_{\nu-1} q_\nu)}{\xi(q_1') \cdots \xi(q_{\nu-2}')\xi(q_{\nu-1})\xi(q_\nu)}.$$

Repeating this process, we have $\dfrac{f(q_1 \cdots q_\nu)}{\xi(q_1 \cdots q_\nu)} = \dfrac{f(q_1' \cdots q_\nu')}{\xi(q_1' \cdots q_\nu')}$.

By steps 2 and 3, we have

$$\frac{f(p^\alpha)}{\xi(p^\alpha)} = \sum_{\nu=1}^{\alpha} H_{p^\alpha,\nu}\phi_\nu, \tag{4.6}$$

where ϕ_ν is the ν-values of $\dfrac{f}{\xi}$, depending only on ν.

Step 4. We wish to extend (4.6) and prove that for all $n \geq 1$,

$$f(n) = \xi(n) \sum_{\nu=\omega(n)}^{\Omega(n)} H_{n,\nu}\phi_\nu,$$

where $\omega(n)$ is the number of distinct prime factors of n , $\Omega(n)$ the number of prime factors of n counting multiplicities , ϕ_ν is the ν-values of $\dfrac{f}{\xi}$ depending only on ν, and $H_{n,\nu}$ a constant depending on n, ν.

Let $1 < n = mp^{\alpha-1}$ be such that $(m, p) = 1$, $\alpha \geq 1$. Let $q_1 \neq p$ be a prime such that $(q_1, m) = 1$. By (4.4),

$$\frac{f(mp^\alpha)\upsilon_p(mp^\alpha)}{\xi(p)} - f(mp^{\alpha-1})\upsilon_p(mp^{\alpha-1})$$
$$= \frac{f(mp^{\alpha-1}q_1)\upsilon_{q_1}(mp^{\alpha-1}q_1)}{\xi(q_1)} - f(mp^{\alpha-1})\upsilon_{q_1}(mp^{\alpha-1})$$

and so

$$f(mp^{\alpha}) = (1 - \frac{1}{\alpha})\xi(p)f(mp^{\alpha-1}) + \frac{1}{\alpha}\frac{\xi(p)}{\xi(q_1)}f(mp^{\alpha-1}q_1). \tag{4.7}$$

Then

$$f(mp) = \frac{\xi(p)}{\xi(q_1)}f(mq_1).$$

By (4.7), we have

$$f(mp^2) = \frac{1}{2}\xi(p)f(mp) + \frac{1}{2}\frac{\xi(p)}{\xi(q_1)}f(mpq_1) = \frac{1}{2}\frac{\xi(p^2)}{\xi(q_1)}f(mq_1) + \frac{1}{2}\frac{\xi(p)}{\xi(q_1)}f(mpq_1).$$

Let q_2 be a prime such that $q_2 \neq p, q_1$ and $(q, m) = 1$. By (4.4),

$$\frac{f(mpq_1)\upsilon_p(mpq_1)}{\xi(p)} - f(mq_1)\upsilon_p(mq_1) = \frac{f(mq_1q_2)\upsilon_{q_2}(mq_1q_2)}{\xi(q_2)} - f(mq_1)\upsilon_{q_2}(mq_1)$$

and so

$$f(mpq_1) = \frac{\xi(p)}{\xi(q_2)}f(mq_1q_2).$$

Then

$$f(mp^2) = \xi(p^2)\left(\frac{1}{2}\frac{f(mq_1)}{\xi(q_1)} + \frac{1}{2}\frac{f(mq_1q_2)}{\xi(q_1q_2)}\right).$$

Assume that

$$\frac{f(mp^{\alpha-1})}{\xi(p^{\alpha-1})} = \sum_{\nu=1}^{\alpha-1} H'_{p^{\alpha-1},\nu}\frac{f(mq_1\cdots q_\nu)}{\xi(q_1\cdots q_\nu)}$$

and

$$\frac{f(mp^{\alpha-1}q_1)}{\xi(p^{\alpha-1})} = \sum_{i=1}^{\alpha-1} c_i\frac{f(mq_1\cdots q_{i+1})}{\xi(q_2\cdots q_{i+1})},$$

where $H'_{p^{\alpha-1},\nu}$ and c_i are constants depending on $\alpha-1, \nu$ and $q_1, \ldots, q_\nu$ are distinct primes all unequal to p such that $(m, q_i) = 1$ for all i. We have,

$$\begin{aligned} f(mp^{\alpha}) &= (1 - \frac{1}{\alpha})\xi(p)f(mp^{\alpha-1}) + \frac{1}{\alpha}\frac{\xi(p)}{\xi(q_1)}f(mp^{\alpha-1}q_1) \\ &= (1 - \frac{1}{\alpha})\xi(p^{\alpha})\sum_{\nu=1}^{\alpha-1} H'_{p^{\alpha-1},\nu}\frac{f(mq_1\cdots q_\nu)}{\xi(q_1\cdots q_\nu)} + \frac{1}{\alpha}\frac{\xi(p^{\alpha})}{\xi(q_1)}\sum_{i=1}^{\alpha-1} c_i\frac{f(mq_1\cdots q_{i+1})}{\xi(q_2\cdots q_{i+1})} \\ &= \xi(p^{\alpha})\sum_{\nu=1}^{\alpha} H_{p^{\alpha},\nu}\frac{f(mq_1\cdots q_\nu)}{\xi(q_1\cdots q_\nu)}, \end{aligned}$$

where $H_{p^\alpha,\nu}$ is a constant depending on α,ν.

If $n = p_1^{\alpha_1}\cdots p_k^{\alpha_k}$,where $p_1,\ldots,p_k$ are primes, then

$$\begin{aligned}
f(p_1^{\alpha_1}\cdot p_k^{\alpha_k}) &= \xi(p_1^{\alpha_1})\sum_{\nu_1=1}^{\alpha_1} H_{p_1^{\alpha_1},\nu_1}\frac{f(p_2^{\alpha_2}\cdots p_k^{\alpha_k}q_1^{(1)}\cdots q_{\nu_1}^{(1)})}{\xi(q_1^{(1)}\cdots q_{\nu_1}^{(1)})}\\
&= \xi(p_1^{\alpha_1})\xi(p_2^{\alpha_2})\sum_{\nu_1=1}^{\alpha_1} H_{p_1^{\alpha_1},\nu_1}\sum_{\nu_2=1}^{\alpha_2} H_{p_2^{\alpha_2},\nu_2}\frac{f(p_3^{\alpha_3}\cdots p_k^{\alpha_k}q_1^{(1)}\cdots q_{\nu_1}^{(1)}q_1^{(2)}\cdots q_{\nu_2}^{(2)})}{\xi(q_1^{(1)}\cdots q_{\nu_1}^{(1)})\xi(q_1^{(2)}\cdots q_{\nu_2}^{(2)})}\\
&= \xi(p_1^{\alpha_1}p_2^{\alpha_2})\sum_{\nu_1=1}^{\alpha_1}\sum_{\nu_2=1}^{\alpha_2} H_{p_1^{\alpha_1},\nu_1}H_{p_2^{\alpha_2},\nu_2}\frac{f(p_3^{\alpha_3}\cdots p_k^{\alpha_k}q_1^{(1)}\cdots q_{\nu_1}^{(1)}q_1^{(2)}\cdots q_{\nu_2}^{(2)})}{\xi(q_1^{(1)}\cdots q_{\nu_1}^{(1)}q_1^{(2)}\cdots q_{\nu_2}^{(2)})}\\
&\vdots\\
&= \xi(p_1^{\alpha_1}\cdots p_k^{\alpha_k})\sum_{\nu_1=1}^{\alpha_1}\cdots\sum_{\nu_k=1}^{\alpha_k} H_{p_1^{\alpha_1},\nu_1}\cdots H_{p_k^{\alpha_k},\nu_k}\frac{f(q_1^{(1)}\cdots q_{\nu_1}^{(1)}\cdots q_1^{(k)}\cdots q_{\nu_k}^{(k)})}{\xi(q_1^{(1)}\cdots q_{\nu_1}^{(1)}\cdots q_1^{(k)}\cdots q_{\nu_k}^{(k)})}.
\end{aligned}$$

Let $\nu = \nu_1 + \cdots + \nu_k$ and $q_1^{(1)}\cdots q_{\nu_1}^{(1)}\cdots q_1^{(k)}\cdots q_{\nu_k}^{(k)} = q_1\cdots q_\nu$. From step 3, we have that $\dfrac{f(q_1\cdots q_\nu)}{\xi(q_1\cdots q_\nu)}$ depends only on ν, and so the coefficients of this term in the above equation is

$$H_{n,\nu} = \sum_{\substack{\nu_1+\cdots+\nu_k=\nu\\ \nu_i\geq 1}} H_{p_1^{\alpha_1},\nu_1}\cdots H_{p_k^{\alpha_k},\nu_k},$$

a constant depending only on ν and n. Then for $n \geq 1$,

$$f(n) = \xi(n)\sum_{\nu=\omega(n)}^{\Omega(n)} H_{n,\nu}\phi_\nu,$$

where $\omega(1) = 0 = \Omega(1)$ and $H_{1,0} = 1$.

Step 5. From step 4, we have that

$$f(n) = \xi(n)\sum_{\nu=\omega(n)}^{\Omega(n)} H_{n,\nu}\phi_\nu = \xi(n)\sum_{\nu=0}^{\infty} H_{n,\nu}\phi_\nu,$$

where $H_{n,\nu} = 0$ if $\nu < \omega(n)$ or $\nu > \Omega(n)$. Then

$$\sum_{n=1}^{\infty}\frac{f(n)}{n^s} = \sum_{n=1}^{\infty}\frac{\xi(n)}{n^s}\sum_{\nu=0}^{\infty} H_{n,\nu}\phi_\nu = \sum_{\nu=0}^{\infty}\phi_\nu\sum_{n=1}^{\infty}\frac{\xi(n)}{n^s}H_{n,\nu}. \tag{4.8}$$

To calculate the Dirichlet series $\sum_{n=1}^{\infty}\dfrac{\xi(n)}{n^s}H_{n,\nu}$, $\nu = 0, 1, \ldots$, which are independent of the ϕ_ν (in fact independent of f), it suffices to calculate them for special f. Taking $f(p) = y\xi(p)$ and f multiplicative will suffice for this propose.

Lemma 4.9. If for all primes p, $\dfrac{f(p)}{\xi(p)} = y$, a constant, f is multiplicative, and $f \in \mathcal{S}$, then

$$\sum_{\text{n=1}}^{\infty} \frac{f(n)}{n^s} = (\Xi(s))^y.$$

Proof. From (4.5),

$$\begin{aligned} f(p^\alpha) &= \xi(p)\left((1-\frac{1}{\alpha})f(p^{\alpha-1}) + \frac{1}{\alpha}\frac{f(p^{\alpha-1}q_1)}{\xi(q_1)}\right) \\ &= \xi(p)\left(\frac{(\alpha-1)}{\alpha}f(p^{\alpha-1}) + \frac{1}{\alpha}\frac{f(p^{\alpha-1})f(q_1)}{\xi(q_1)}\right) \\ &= \xi(p)\frac{(\alpha+y-1)}{\alpha}f(p^{\alpha-1}). \end{aligned}$$

Using (4.5) repeatedly, and continuing this process, lead to

$$\begin{aligned} f(p^\alpha) &= \xi(p^\alpha)\left(\frac{\alpha+y-1}{\alpha}\right)\left(\frac{\alpha-1+y-1}{\alpha-1}\right)\cdots\left(\frac{1+y-1}{1}\right)f(1) \\ &= \xi(p^\alpha)\prod_{j=1}^{\alpha}\left(\frac{j+y-1}{j}\right) = \xi(p^\alpha)\binom{\alpha+y-1}{\alpha}. \end{aligned}$$

Then

$$\begin{aligned} \sum_{\text{n=1}}^{\infty}\frac{f(n)}{n^s} &= \prod_{\text{prime } p}\sum_{j=0}^{\infty}\frac{f(p^j)}{p^{js}} \qquad \text{(see [1], Theorem 11.7)} \\ &= \prod_{\text{prime } p}\sum_{j=0}^{\infty}\frac{\xi(p^j)}{p^{js}}\binom{j+y-1}{j} \\ &= \prod_{\text{prime } p}\sum_{j=0}^{\infty}\frac{\xi(p^j)}{p^{js}}(-1)^j\binom{-y}{j} \\ &= \prod_{\text{prime } p}\left(\frac{1}{1-\frac{\xi(p)}{p^s}}\right)^y = \left(\sum_{\text{n=1}}^{\infty}\frac{\xi(n)}{n^s}\right)^y = (\Xi(s))^y. \end{aligned}$$

□

By Lemma 4.9,

$$\sum_{\text{n=1}}^{\infty}\frac{f(n)}{n^s} = e^{y\log\Xi(s)} = \sum_{\nu=0}^{\infty}\frac{y^\nu}{\nu!}(\log\Xi(s))^\nu.$$

Since $\dfrac{f(p)}{\xi(p)} = y$ for all primes p and f multiplicative, then $\phi_\nu = \dfrac{f(p_1\cdots p_\nu)}{\xi(p_1\cdots p_\nu)} = y^\nu$. Thus

$$\sum_{\nu=0}^{\infty} y^\nu \sum_{n=1}^{\infty} \frac{\xi(n)}{n^s} H_{n,\nu} = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \sum_{\nu=0}^{\infty} \frac{y^\nu}{\nu!}(\log \Xi(s))^\nu.$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{\xi(n)}{n^s} H_{n,\nu} = \frac{1}{\nu!}(\log \Xi(s))^\nu.$$

Hence for any $f \in \mathcal{S}$,

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \sum_{\nu=0}^{\infty} \frac{\phi_\nu}{\nu!}(\log \Xi(s))^\nu,$$

where $\phi_\nu = \dfrac{f(p_1\cdots p_\nu)}{\xi(p_1\cdots p_\nu)}$ is independent of the choice of the ν distinct primes $p_1, \ldots, p_\nu$. $\square$

The next corollary follows from the proof of Theorem 4.8.

Corollary 4.10. If $f \in \mathcal{A}$ is algebraic over $\mathbb{C}[\xi]$, then $f(1) = \phi_0$ a constant and for $n \geq 2$,

$$f(n) = \xi(n) \sum_{1 \leq \nu \leq \frac{\log n}{\log 2}} \frac{\phi_\nu}{\nu!} \left(\sum_{\substack{n_1\cdots n_\nu = n \\ n_i \geq 2}} \frac{\Lambda(n_1)\cdots\Lambda(n_\nu)}{(\log n_1)\cdots(\log n_\nu)} \right),$$

where ϕ_ν is the ν-value of $\dfrac{f}{\xi}$ and Λ is the Mangoldt function.

Theorem 4.11. Let f be a multiplicative function. The following assertions are equivalent:

1. f and ξ are algebraically dependent over $\mathbb{C}$.
2. $\dfrac{f(p)}{\xi(p)} = c$ is a constant for all primes p, c is rational and

$$\frac{f(n)}{\xi(n)} = \prod_{p^\alpha || n} \binom{\alpha + c - 1}{\alpha}.$$

Proof. First we show that (1) $\Rightarrow$ (2). Assume that f and ξ are algebraically dependent over $\mathbb{C}$. By Theorem 4.8,

$$\frac{f(p)}{\xi(p)} = \phi_1 = c$$

a constant for all primes p and by the proof of Lemma 4.9,

$$\frac{f(p^\alpha)}{\xi(p^\alpha)} = \binom{\alpha + c - 1}{\alpha}$$

and $F(s) = (\Xi(s))^c$,where F is the corresponding Dirichlet series of f. By multiplicativity of f and ξ,

$$\frac{f(n)}{\xi(n)} = \prod_{p^\alpha || n} \binom{\alpha + c - 1}{\alpha}.$$

It remains to show that c is rational. Since F and Ξ are algebraically dependent over $\mathbb{C}$, then

$$0 = \sum_{k=0}^{K} \sum_{j=0}^{J} a_{kj}\ \Xi^k F^j = \sum_{k=0}^{K} \sum_{j=0}^{J} a_{kj}\ \Xi^{k+cj}, \tag{4.9}$$

where $a_{kj} \in \mathbb{C}$, not all zero. Consider for all k, j,

$$\begin{aligned}(\Xi(s))^{k+cj} &= \prod_{\text{prime } p} \left(\frac{1}{1 - \frac{\xi(p)}{p^s}} \right)^{k+cj} = \prod_{\text{prime } p} \sum_{l=0}^{\infty} \binom{-(k+cj)}{l} (-1)^l \frac{\xi(p^l)}{p^{ls}} \\ &= \left(1 + \frac{(k+cj)\xi(p_1)}{p_1^s} + \frac{(k+cj)(k+cj+1)\xi(p_1^2)}{2! p_1^{2s}} + \cdots \right) \\ &\quad \times \left(1 + \frac{(k+cj)\xi(p_2)}{p_2^s} + \frac{(k+cj)(k+cj+1)\xi(p_2^2)}{2! p_2^{2s}} + \cdots \right) \times \cdots .\end{aligned}$$

For primes $p_1, \ldots, p_l$, the coefficient of $(p_1 \cdots p_l)^{-s}$ in $(\Xi(s))^{k+cj}$ is

$$(k+cj)^l \xi(p_1) \cdots \xi(p_l) = (k+cj)^l \xi(p_1 \cdots p_l).$$

Then the coefficient of $(p_1 \cdots p_l)$ in (4.9) is

$$\sum_{\substack{k,j \\ (k,j) \neq (0,0)}} a_{kj} (k+cj)^l \xi(p_1 \cdots p_l) = 0.$$

Since $\xi(p_1 \cdots p_l) \neq 0$, then $\sum_{\substack{k,j \\ (k,j) \neq (0,0)}} a_{kj}(k+cj)^l = 0$. If c is not rational , then the $k + cj$ are all distinct and via the non-vanishing of the Vandermonde determinant all $a_{kj} = 0$, which is a contradiction. Hence c is rational.

(2)⇒(1) : Since

$$f(n) = \xi(n) \prod_{p^\alpha \| n} \binom{\alpha + c - 1}{\alpha},$$

then

$$f(p^\alpha) = \xi(p^\alpha) \binom{\alpha + c - 1}{\alpha}$$

for all primes p and $\alpha \geq 1$. By the proof of Lemma 4.9, we have $F(s) = (\Xi(s))^c$. Assume that $c = \frac{r}{t}$, where $r, t \in \mathbb{Z}$. Since $(\Xi^c)^t - \Xi^r = 0$, then Ξ and Ξ^c $(= F)$ are algebraically dependent over $\mathbb{C}$, so f and ξ are algebraically dependent over $\mathbb{C}$. □

Since $\xi(1) = 1$ and $\xi(2) \neq 0$, then $N(\xi - 1) = 2$, so $\xi - 1$ is a prime in $\mathcal{A}$. Then the principal ideal $\Phi = (\Xi - 1)$ is a prime ideal in $\mathcal{D}$. Since

$$\begin{aligned} \log \Xi &= -(1 - \Xi) - \frac{(1-\Xi)^2}{2} - \frac{(1-\Xi)^3}{3} - \cdots \\ &= -(1 - \Xi)\left(1 + \frac{(1-\Xi)}{2} + \frac{(1-\Xi)^2}{3} + \cdots\right) = (\Xi - 1) \cdot U, \end{aligned}$$

where U is a unit in $\mathcal{D}$, then $\log \Xi$ is associated to $\Xi - 1$ (in the arithmetic of $\mathcal{D}$ but not in that of $\mathbb{C}[\Xi]$).

Theorem 4.12. The set $\mathcal{S}$ consists of the local integers in $(\mathbb{C}[\Xi])_\Phi$, the Φ-adic completion of $\mathbb{C}[\Xi]$, $\Phi = (\Xi - 1)$

Proof. (⇒) An element of $\mathcal{S}$ has the corresponding Dirichlet series of the form

$$\begin{aligned} \sum_{\nu=0}^{\infty} \frac{\phi_\nu}{\nu!} (\log \Xi)^\nu &= \sum_{\nu=0}^{\infty} \frac{\phi_\nu}{\nu!} \left(\sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} (\Xi - 1)^j \right)^\nu \\ &= \sum_{\nu=0}^{\infty} \frac{\phi_\nu}{\nu!} \sum_{l \geq \nu} c_{l\nu} (\Xi - 1)^l \\ &= \sum_{l=0}^{\infty} (\Xi - 1)^l \sum_{\nu \leq l} \frac{\phi_\nu}{\nu!} c_{l\nu}, \end{aligned}$$

where the calculations are carried out in the Φ-adic norm, and the last expansion is clearly a local integer of $(\mathbb{C}[\Xi])_\Phi$.

($\Leftarrow$) Conversely starting with an integral element of $(\mathbb{C}[\Xi])_\Phi$, we have

$$\begin{aligned}\sum_{l=0}^{\infty} a_l(\Xi-1)^l = \sum_{l=0}^{\infty} a_l(e^{\log\Xi}-1)^l &= \sum_{l=0}^{\infty}\left(\sum_{\nu=0}^{\infty}\frac{(\log\Xi)^\nu}{\nu!}-1\right)^l \\ &= \sum_{l=0}^{\infty} a_l\left(\sum_{\nu=1}^{\infty}\frac{(\log\Xi)^\nu}{\nu!}\right)^l \\ &= \sum_{l=0}^{\infty} a_l \sum_{\nu_i\geq 1}\frac{(\log\Xi)^{\nu_1+\cdots+\nu_l}}{\nu_1!\cdots\nu_l!} \\ &= a_0 + \sum_{\lambda=1}^{\infty}\frac{(\log\Xi)^\lambda}{\lambda!}\sum_{l=1}^{\infty} a_l \sum_{\nu_1+\cdots+\nu_l=\lambda}\frac{\lambda!}{\nu_1!\cdots\nu_l!} \\ &= \sum_{\lambda=0}^{\infty} b_\lambda(\log\Xi)^\lambda\end{aligned}$$

which is in $\mathcal{S}$ $\square$

4.3 Functions Which Are Not Algebraic over $\mathbb{C}[\Xi]$

In this section we assume $\xi \in \mathcal{A}$ to be completely multiplicative, with $\xi(p) \neq 0$ for all primes p and $\Xi(s) = \sum_{\text{n}=1}^{\infty}\frac{\xi(n)}{n^s}$ being its corresponding Dirichlet series. From the beginning of section 2, we know that every element of $\mathcal{A}$ which is algebraic over $\mathbb{C}[\xi]$ is in $\mathcal{S}$, yet the converse is not true as we now show that there are elements of $\mathcal{S}$ which are not algebraic over $\mathbb{C}[\xi]$. We begin with

Theorem 4.13. $\log\Xi$ is not algebraic over $\mathbb{C}[\Xi]$.

Proof. Suppose that $\log\Xi$ is algebraic over $\mathbb{C}[\Xi]$. Then

$$\sum_{i=0}^{I}\sum_{j=0}^{J} a_{ij}\ \Xi^i(\log\Xi)^j = 0, \tag{4.10}$$

where $a_{ij} \in \mathbb{C}, a_{IJ} \neq 0$. Consider

$$\Xi^i(\log\Xi)^j = \left(\sum_{l=1}^{\infty}\frac{\xi(l)}{l^s}\right)^i\left(\sum_{m=2}^{\infty}\frac{\xi(m)\Lambda(m)}{m^s\log m}\right)^j$$

$$= \left(\sum_{l=1}^{\infty} \frac{\xi(l)}{l^s} \sum_{\substack{l_1 \cdots l_i = l \\ l_1, \ldots, l_i \geq 1}} 1 \right) \left(\sum_{m=2}^{\infty} \frac{\xi(m)}{m^s} \sum_{\substack{m_1 \cdots m_j = m \\ m_1, \ldots, m_j > 1}} \frac{\Lambda(m_1) \cdots \Lambda(m_j)}{(\log m_1) \cdots (\log m_j)} \right)$$

$$= \left(\sum_{l=1}^{\infty} \frac{\xi(l)}{l^s} D_l \right) \left(\sum_{m=2}^{\infty} \frac{\xi(m)}{m^s} A_m \right)$$

$$= \sum_{n=2}^{\infty} \frac{1}{n^s} \sum_{lm=n} \xi(l)\xi(m) D_l A_m$$

$$= \sum_{n=2}^{\infty} \frac{\xi(n)}{n^s} \sum_{lm=n} D_l A_m,$$

where $D_l = \sum_{\substack{l_1 \cdots l_i = l \\ l_1, \ldots, l_i \geq 1}} 1$ and $A_m = \sum_{\substack{m_1 \cdots m_j = m \\ m_1, \ldots, m_j > 1}} \frac{\Lambda(m_1) \cdots \Lambda(m_j)}{(\log m_1) \cdots (\log m_j)}$.

For k sufficiently large and $p_1, \ldots, p_k$ primes, the coefficient of $n^{-s} = (p_1 \cdots p_k)^{-s}$ in $\Xi^i (\log \Xi)^j$ is

$$\begin{aligned} \xi(p_1 \cdots p_k) \sum_{lm = p_1 \cdots p_k} D_l A_m &= \xi(p_1 \cdots p_k) \sum_{l(p_{\nu_1} \cdots p_{\nu_j}) = p_1 \cdots p_k} D_l \sum_{q_1 \cdots q_j = p_{\nu_1} \cdots p_{\nu_j}} \frac{\Lambda(q_1) \cdots \Lambda(q_j)}{(\log q_1) \cdots (\log q_j)} \\ &= \xi(p_1 \cdots p_k) \sum_{l(p_{\nu_1} \cdots p_{\nu_j}) = p_1 \cdots p_k} \left(\sum_{\substack{l_1 \cdots l_i = l \\ l_1, \ldots, l_i \geq 1}} 1 \right) \left(\sum_{q_1 \cdots q_j = p_{\nu_1} \cdots p_{\nu_j}} 1 \right) \\ &= \xi(p_1 \cdots p_k) \binom{k}{j} j! i^{(k-j)}. \end{aligned} \tag{4.11}$$

Case(i) $I = 0$. Now (4.10) reduces to

$$\sum_{j=0}^{J} a_{0j} \ (\log \Xi)^j = 0,$$

where $a_{0J} \neq 0$. Then

$$\begin{aligned} 0 = \sum_{j=0}^{J} a_{0j} \ (\log \Xi)^j &= \sum_{j=0}^{J} a_{0j} \left(\sum_{n=2}^{\infty} \frac{\xi(n)\Lambda(n)}{n^s \log n} \right)^j \\ &= \sum_{j=0}^{J} a_{0j} \sum_{n=2}^{\infty} \frac{\xi(n)}{n^s} \sum_{\substack{n_1 \cdots n_j = n \\ n_1, \ldots, n_j > 1}} \frac{\Lambda(n_1) \cdots \Lambda(n_j)}{(\log n_1) \cdots (\log n_j)}. \end{aligned} \tag{4.12}$$

Let $p_1, \ldots, p_J$ be primes. The coefficients of $(p_1 \cdots p_J)^{-s}$ in (4.12) are

$$0 = a_{0J} \xi(p_1 \cdots p_J) \sum_{\substack{q_1 \cdots q_J = p_1 \cdots p_J \\ q_\nu \in \{p_1, \ldots, p_J\}}} \frac{\Lambda(q_1) \cdots \Lambda(q_J)}{(\log q_1) \cdots (\log q_J)} = a_{0J} \xi(p_1 \cdots p_J) J!.$$

Since $J! \neq 0$ and $\xi(p_1 \cdots p_J) \neq 0$, then $a_{0J} = 0$, which is a contradiction.

Case(ii) $I \neq 0$. The coefficient of $(p_1 \cdots p_k)^{-s}$ in (4.10) is

$$0 = \sum_{(i,j)\neq(0,0)}^{I,J} a_{ij}\xi(p_1 \cdots p_k) i^{(k-j)} \binom{k}{j} j! = \xi(p_1 \cdots p_k) \sum_{(i,j)\neq(0,0)}^{I,J} a_{ij} i^{(k-j)} \binom{k}{j} j!,$$

and since $\xi(p_1 \cdots p_k) \neq 0$, then

$$\sum_{(i,j)\neq(0,0)}^{I,J} a_{ij} i^{(k-j)} \binom{k}{j} j! = 0. \tag{4.13}$$

In (4.13), the coefficient of a_{ij} equals $i^{(k-j)} \binom{k}{j} j! \approx i^{(k-j)} k^j$ as $k \to \infty$. Thus as $k \to \infty$,

$$\begin{aligned} \frac{i^{(k-j)}\binom{k}{j} j!}{I^{(k-J)}\binom{k}{J} J!} \approx \frac{k^j i^{(k-j)}}{k^J I^{(k-J)}} &= \frac{k^{(j-J)} i^{(k-j)}}{I^{(k-J)}} \\ &= \exp\{(k-j)\log i + (j-J)\log k - (k-J)\log I\} \\ &\approx \exp\{k \log \frac{i}{I} + O(\log k)\} \end{aligned}$$

which tends to 0 if $i < I$.

Also, if $i = I, j < J$, the above gives

$$\frac{i^{(k-j)}\binom{k}{j} j!}{I^{(k-J)}\binom{k}{J} J!} \approx \exp\{(J-j)\log i + (j-J)\log k\} \approx \exp\{(j-J)\log k + O(1)\} \to 0.$$

Thus in (4.13), the coefficient of a_{IJ} dominates as $k \to \infty$ and we have a contradiction. $\square$

Corollary 4.14. For any $Q_j \in \mathbb{C}[\Xi]$, $j = 0, \ldots, R$, $R > 0$, if

$$F = \sum_{j=0}^{R} Q_j (\log \Xi)^j,$$

where $Q_R \neq 0$, then F is not algebraic over $\mathbb{C}[\Xi]$.

Corollary 4.15. Any Dirichlet series of the form

$$\sum_{\nu=0}^{N} \frac{\phi_\nu}{\nu!} (\log \Xi)^\nu, \quad N > 0$$

is not algebraic over $\mathbb{C}[\Xi]$.

Corollary 4.16. For any nonzero rational number c, the value $\phi_\nu = \nu c^\nu$ in (4.2) gives a Dirichlet series which is not algebraic over $\mathbb{C}[\Xi]$.

Proof.

$$\begin{aligned}\sum_{\nu=0}^{\infty} \frac{\nu c^\nu}{\nu!}(\log \Xi)^\nu &= \sum_{\nu=1}^{\infty} \frac{c^\nu}{(\nu-1)!}(\log \Xi)^\nu \\ &= c \log \Xi \sum_{\nu=0}^{\infty} \frac{c^\nu}{\nu!}(\log \Xi)^\nu \\ &= (c \log \Xi)(\exp(c \log \Xi)) = (c \log \Xi)\Xi^c.\end{aligned}$$

Since $\log \Xi$ is not algebraic over $\mathbb{C}[\Xi]$, the series is not algebraic over $\mathbb{C}[\Xi]$. □

Theorem 4.17. $\log \Xi$ is not algebraic over $\mathbb{C}[\Xi, \Xi^c]$ for any $c \in \mathbb{C}$.

Proof. If c is rational this is precisely Theorem 4.13. Thus we may assume that c is purely complex or irrational. Suppose that $\log \Xi$ is algebraic over $\mathbb{C}[\Xi, \Xi^c]$. Then

$$\sum_{j,k,l} a_{jkl}\ \Xi^{j+ck}(\log \Xi)^l = 0,$$

where $a_{jkl} \in \mathbb{C}$, not all zero. Thus

$$\begin{aligned}0 &= \sum_{j,k,l} a_{jkl} \left(\sum_{\nu=0}^{\infty} \frac{(j+ck)^\nu}{\nu!}(\log \Xi)^\nu \right) (\log \Xi)^l \\ &= \sum_{j,k,l} a_{jkl} \sum_{\nu=0}^{\infty} \frac{(j+ck)^\nu}{\nu!}(\log \Xi)^{\nu+l} \\ &= \sum_{j,k,l} a_{jkl} \sum_{\nu=0}^{\infty} \frac{(j+ck)^\nu}{\nu!} \left(\sum_{n=2}^{\infty} \frac{\xi(n)\Lambda(n)}{n^s \log n} \right)^{\nu+l} \\ &= \sum_{j,k,l} a_{jkl} \sum_{\nu=0}^{\infty} \frac{(j+ck)^\nu}{\nu!} \sum_{n=2}^{\infty} \frac{\xi(n)}{n^s} \sum_{\substack{n_1 \cdots n_{\nu+l} = n \\ n_i > 1}} \frac{\Lambda(n_1)\cdots\Lambda(n_{\nu+l})}{(\log n_1)\cdots(\log n_{\nu+l})}. \qquad (4.14)\end{aligned}$$

For m sufficiently large and $p_1, \ldots, p_m$ primes, the coefficient of $(p_1 \cdots p_m)^{-s}$ in

(4.14) is

$$0=\sum_{j,k,l}\xi(p_1\cdots p_m)a_{jkl}\frac{(j+ck)^{m-l}}{(m-l)!}\sum_{\substack{q_1\cdots q_m=p_1\cdots p_m\\ q_i\in\{p_1,\ldots,p_m\}}}\frac{\Lambda(q_1)\cdots\Lambda(q_m)}{(\log q_1)\cdots(\log q_m)}$$
$$=\xi(p_1\cdots p_m)\sum_{j,k,l}a_{jkl}\frac{(j+ck)^{m-l}}{(m-l)!}m!.$$

Since $\xi(p_1\cdots p_l)\neq 0$, then

$$0=\sum_{jkl}a_{jkl}\frac{(j+ck)^{m-l}}{(m-l)!}m!.$$

Since the term $j=k=l=0$ does not appear (for k sufficiently large) and c is purely complex or irrational, then the $j+ck$ are all distinct and nonzero. Setting

$$a_{jkl}(j+ck)^{-l}=\alpha_{jkl}$$
$$m(m-1)\cdots(m-l+1)=\frac{m!}{(m-l)!}=P_l(m),\quad P_0(m)=1$$
$$j+ck=\lambda_{jk},$$

we see that all sufficiently large integers m satisfy

$$0=\sum_{j,k,l}\alpha_{jkl}\lambda_{jk}^m P_l(m)=\sum_{jk}\lambda_{jk}^m\sum_l\alpha_{jkl}P_l(m).$$

Since the λ_{jk} are distinct and not equal to 0, it follows that , for all j,k,

$$\sum_{l=0}^{L}\alpha_{jkl}P_l(m)=0$$

for all integers m.

If $m=0$, then $P_0(0)=1, P_l(0)=0$ for all $l\geq 1$, and so

$$0=\sum_{l=0}^{L}\alpha_{jkl}P_l(0)=\alpha_{jk0}.$$

If $m=1$, then $P_0(1)=1=P_1(1), P_l(1)=0$ for all $l>1$, and so

$$0=\sum_{l=0}^{L}\alpha_{jkl}P_l(1)=\alpha_{jk0}+\alpha_{jk1}=\alpha_{jk1}.$$

For $1 \le r \le L$, assume that $\alpha_{jkt} = 0$ for $t < r$. Then $P_r(r) = r!$ and $P_l(r) = 0$, and so

$$0 = \sum_{l=0}^{L} \alpha_{jkl} P_l(r) = P_r(r)\alpha_{jkr} = r!\alpha_{jkr}, \quad \text{i.e. } \alpha_{jkr} = 0.$$

Thus $\alpha_{jkl} = 0$ for all $l = 1, \ldots, L$. Since $j + ck \neq 0$ for $(j,k) \neq (0,0)$, then $a_{jkl} = 0$ for all $(j,k) \neq (0,0)$. Therefore $0 = \sum_{l=0}^{L} a_{00l}(\log \Xi)^l$, so $\log \Xi$ is algebraic over $\mathbb{C}[\Xi]$, which is a contradiction. □

Remarks 4.18. 1. The above arguments can be applied to prove that $\log \Xi$ is not algebraic over $\mathbb{C}[\Xi^{c_1}, \ldots, \Xi^{c_r}]$ for complex $c_1, \ldots, c_r$.

2. As a consequence of Theorem 4.17, Corollary 4.16 is also valid for c irrational.

Definition 4.19. $f \in \mathcal{A}$ is *locally ν-multiplicative* if for any ν distinct primes $p_1, \ldots, p_\nu$, we have

$$f(p_1 \cdots p_\nu) = f(p_1) \cdots f(p_\nu).$$

Theorem 4.20. If $f \in \mathcal{A}$ is algebraic over $\mathbb{C}[\xi]$, and locally ν-multiplicative for all sufficiently large ν, then $f = \xi^c - b$, c rational, $b = 1 - f(1)$.

Proof. Since $f \in \overline{\mathbb{C}[\xi]} \subset \mathcal{S}$, by Theorem 4.8,

$$\sum_{\mathrm{n}=1}^{\infty} \frac{f(n)}{n^s} = \sum_{\nu=0}^{\infty} \frac{\phi_\nu}{\nu!}(\log \Xi)^\nu,$$

where $\phi_\nu = \dfrac{f(p_1 \cdots p_\nu)}{\xi(p_1 \cdots p_\nu)}$ is independent of the choice of the ν distinct primes $p_1, \ldots, p_\nu$. Then $\dfrac{f(p)}{\xi(p)} = \phi_1 = c$, a constant, for all primes p. Since f is ν-multiplicative for all sufficiently large ν, there exists an $N \in \mathbb{N}$ such that for all $n > N$,

$$\phi_n = \frac{f(p_1 \cdots p_n)}{\xi(p_1 \cdots p_n)} = \frac{f(p_1) \cdots f(p_n)}{\xi(p_1) \cdots \xi(p_n)} = c^n.$$

Thus

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{f(n)}{n^s} &= \sum_{\nu=0}^{\infty}\frac{\phi_\nu}{\nu!}(\log\Xi)^\nu \\ &= \phi_0 + \sum_{\nu=1}^{N}\frac{\phi_\nu}{\nu!}(\log\Xi)^\nu + \sum_{\nu=N+1}^{\infty}\frac{\phi_\nu}{\nu!}(\log\Xi)^\nu \\ &= f(1) + \sum_{\nu=1}^{N}\frac{\phi_\nu}{\nu!}(\log\Xi)^\nu + \sum_{\nu=N+1}^{\infty}\frac{c^\nu}{\nu!}(\log\Xi)^\nu.\end{aligned}$$

Since

$$\Xi^c = \exp(c\log\Xi) = \sum_{\nu=0}^{\infty}\frac{(c\log\Xi)^\nu}{\nu!} = 1 + \sum_{\nu=1}^{\infty}\frac{c^\nu(\log\Xi)^\nu}{\nu!},$$

and let $b = 1 - f(1)$, then

$$\begin{aligned}b + \sum_{n=1}^{\infty}\frac{f(n)}{n^s} - \Xi^c &= 1 - f(1) + f(1) + \sum_{\nu=1}^{N}\frac{\phi_\nu}{\nu!}(\log\Xi)^\nu + \sum_{\nu=N+1}^{\infty}\frac{c^\nu}{\nu!}(\log\Xi)^\nu \\ &\quad - 1 - \sum_{\nu=1}^{\infty}\frac{c^\nu(\log\Xi)^\nu}{\nu!} \\ &= \sum_{\nu=1}^{N}\frac{(\phi_\nu - c^\nu)}{\nu!}(\log\Xi)^\nu =: A.\end{aligned}$$

Since $F \in \overline{\mathbb{C}[\Xi]}$, then $A \in \mathbb{C}[\Xi, \Xi^c]$, and so

$$0 = \sum_{i=0}^{I} a_i A^i = \sum_{i=0}^{I} a_i \left(\sum_{\nu=1}^{N}\frac{(\phi_\nu - c^\nu)}{\nu!}(\log\Xi)^\nu\right)^i,$$

where $a_i \in \mathbb{C}[\Xi, \Xi^c]$, not all zero.

If $N \geq 1$, then $\log\Xi$ is algebraic over $\mathbb{C}[\Xi, \Xi^c]$, which is a contradiction.

Thus

$$b + \sum_{n=1}^{\infty}\frac{f(n)}{n^s} - \Xi^c = 0,$$

so

$$\sum_{n=1}^{\infty}\frac{f(n)}{n^s} = \Xi^c - b.$$

Since $\quad \Xi^c = \sum_{n=1}^{\infty}\frac{f(n)}{n^s} + b \in \overline{\mathbb{C}[\Xi]}$, then

$$0 = \sum_{j,k} a_{jk}\ \Xi^j\Xi^{ck} = \sum_{j,k} a_{jk}\ \Xi^{j+ck} = \sum_{j,k} a_{jk}\sum_{\nu=0}^{\infty}\frac{(j+ck)^\nu}{\nu!}(\log\Xi)^\nu,$$

where $a_{jk} \in \mathbb{C}$, not all zero.

If c is purely complex or irrational, by the proof of Theorem 4.17, $a_{jk} = 0$ for all j, k, which is a contradiction. Hence $f = \xi^c - b$, where c is rational. □

4.4 Log-series expansion

Let $\mathcal{A}_1$ be the subset of $\mathcal{A}$ consisting of $f \in \mathcal{A}$ with $f(1) = 1$.

Definition 4.21. Let p be a prime. We say that z is *multiplicative at* p (also are refered to as *locally multiplicative*), written $z \in \mathcal{M}_p$, if

$$z(mp^\alpha) = z(m)z(p^\alpha),$$

for each $\alpha, m \in \mathbb{N}$, $g.c.d.(m, p) = 1$.

Note that multiplicative functions are multiplicative at p, for each prime p.

Definition 4.22. For $f \in \mathcal{A}$, define the *support* of f to be $supp(f) = \{n \in \mathbb{N} : f(n) \neq 0\}$ and define $[supp(f)]$ to be the smallest set of primes which generates a subsemigroup of the positive integers containing $supp(f)$.

The proof of the next lemma is taken from Lemma 7.1 in [10], while the condition is weakened.

Lemma 4.23. Let $z \in \mathcal{A}_1$ be such that $[supp(z)]$ contains at least two primes and Z being its corresponding Dirichlet sereis. Let $p, q \in [supp(z)]$, $p \neq q$. If $z \in \mathcal{M}_p \cap \mathcal{M}_q$, then there does not exist an integer $l > 1$ such that

$$Z = 1 + H^l,$$

for any Dirichlet series H.

Proof. Suppose that $Z = 1 + H^l$ for some $l > 1$ and some Dirichlet series H with corresponding $h \in \mathcal{A}$. Then

$$d_p Z = lH^{l-1} d_p H.$$

Since $z \in \mathcal{M}_p$, we have

$$\begin{aligned} d_p Z &= \sum_{a=0}^{\infty} \frac{(a+1)z(p^{a+1})}{(p^a)^s} \sum_{\substack{m=1 \\ (p,m)=1}}^{\infty} \frac{z(m)}{m^s} \\ &= Z\Big(\sum_{a=0}^{\infty} \frac{(a+1)z(p^{a+1})}{(p^a)^s}\Big) \Big/ \Big(\sum_{a=0}^{\infty} \frac{z(p^a)}{(p^a)^s}\Big). \end{aligned}$$

Now H and $1 + H^l$ being relatively prime implies H divides $\displaystyle\sum_{a=0}^{\infty} \frac{(a+1)z(p^{a+1})}{(p^a)^s}$, i.e.

$$\sum_{a=0}^{\infty} \frac{(a+1)z(p^{a+1})}{(p^a)^s} = HG_p, \tag{4.15}$$

for some Dirichlet series G_p whose arithmetic counterpart is g_p.
If $\displaystyle\sum_{a=0}^{\infty} \frac{(a+1)z(p^{a+1})}{(p^a)^s} = 0$, then $z(p^a) = 0$ for all $a \geq 1$, and so $z(p^a m) = 0$ for all $a, m \in \mathbb{N}$. This yields $p \notin [supp(z)]$, which is a contradiction. Thus $\displaystyle\sum_{a=0}^{\infty} \frac{(a+1)z(p^{a+1})}{(p^a)^s} \neq 0$. Since $h(1) = 0$, then let n, m both > 1 be the smallest integers such that both $h(n)$ and $g_p(m)$ are nonzero (if m exists).
The coefficient of $(nm)^{-s}$ on the right side of (4.15) being nonzero gives $nm = p^c$ for some $c > 0$. Thus $n = p^a$ for some $a > 0$ (if m does not exist, then $g_p = I$, so $n = p^c$). Since n depends only on H, if this also holds for q, then $p^a = n = q^b$ for some $a, b > 0$, yielding a contradiction. Consequently, this can only hold for p, and so $z(q^b) = 0$ for all $b \geq 1$. By local multiplicativity $z(q^b m) = 0$ for all $b, m \in \mathbb{N}$, implying $q \notin [supp(z)]$, a contradiction. □

Theorem 4.24. Let $z \in \mathcal{A}_1$ with Z being its corresponding Dirichlet series. Assume that $Z - 1$ is not an l-th powers of a Dirichlet series for any $l > 1$. If $f \in \mathcal{A}$ is $\mathbb{C}$-algebraically dependent on z, then its corresponding Dirichlet series can uniquely be written under the form

$$F = \sum_{\nu=0}^{\infty} \frac{\phi_\nu}{\nu!} (\log Z)^\nu,$$

where $\phi_\nu \in \mathbb{C}$.

Proof. See [10], Theorem 7.1 □

Lemma 4.23 and Theorem 4.24 together give the following theorem.

Theorem 4.25. Let $z \in \mathcal{A}_1$ be such that $[supp(z)]$ contains at least two primes and Z being its corresponding Dirichlet series. Assume that $z \in \mathcal{M}_p \cap \mathcal{M}_q$ for some $p, q \in [supp(z)]$, $p \neq q$. If $f \in \mathcal{A}_1$ is $\mathbb{C}$-algebraically dependent over z, then we have uniquely the representation

$$F = \sum_{\nu=0}^{\infty} \frac{\phi_\nu}{\nu!} (\log Z)^\nu,$$

where $\phi_\nu \in \mathbb{C}$.

Theorem 4.25 slightly improves Theorem 7.2 of [10] by weakening the "multiplicative" condition to that of "local multiplicative at two primes".

4.5 Rational powers

Lemma 4.26. Let $z \in \mathcal{A}\backslash\{0\}$. For $f \in \mathcal{A}$, if f is properly $\mathbb{C}$-algebraically dependent over z, then $[supp(f)] = [supp(z)]$.

Proof. Since $f \in \overline{\mathbb{C}[z]}^*$, then $z \in \overline{\mathbb{C}[f]}^*$. First we prove that $[supp(z)] \subseteq [supp(f)]$. Suppose not. There is a $p \in [supp(z)]\backslash[supp(f)]$, and so $z(pm) \neq 0$ for some $m \in \mathbb{N}$. From $d_p z(m) = z(pm)\nu_p(pm) \neq 0$, we get $d_p z \neq 0$. Since $p \notin [supp(f)]$, then $f(np) = 0$ for all $n \in \mathbb{N}$, implying $d_p f(n) = f(np)\upsilon_p(np) = 0$ for all $n \in \mathbb{N}$, and so $d_p f = 0$. Therefore $d_p^k f = 0$ for all $k \in \mathbb{N}$, which induces $d_p g = 0$ for all $g \in \mathbb{C}[f]$. By Lemma 2.12, $z \notin \overline{\mathbb{C}[f]}$, which is a contradiction. The other inclusion $[supp(f)] \subseteq [supp(z)]$ is proved similarly. □

Note that if $f \in \mathcal{A}\backslash\{0\}$ is (properly) $\mathbb{C}$-algebraic over $z \in \mathcal{A}\backslash\{0\}$ and $[supp(z)]$ is infinite then $[supp(f)]$ is also infinite.

The next theorem strengthens Theorem 7.3 of [10] by lessening the "multiplicative" condition to that of "local multiplicative" and the proof given here corrects certain gaps in the original proof of [10].

Theorem 4.27. Let $z \in \mathcal{A}_1$ be such that $[supp(z)]$ is infinite. Assume that there is an infinite subset $S \subseteq [supp(z)]$ such that $z \in \bigcap_{p \in S} \mathcal{M}_p$. Let $f \in \mathcal{A}_1$ be $\mathbb{C}$-algebraically dependent over z. If $f \in \bigcap_{p \in S} \mathcal{M}_p$, then $f = z^c$, where c is rational.

Proof. Let $p \in S$. Then $z \in \mathcal{M}_p$ and $z(p^a m) \neq 0$ for some $a, m \in \mathbb{N}$, and $(p, m) = 1$. Thus $0 \neq z(p^a m) = z(p^a) z(m)$, i.e. $z(p^a) \neq 0$. Let a_p be the smallest such positive value of a. Let F be the corresponding Dirichlet series of f. Since $f \in \overline{\mathbb{C}[z]}$, by Theorem 4.25,

$$F = \sum_{\nu=0}^{\infty} \frac{\phi_\nu}{\nu!} (\log Z)^\nu,$$

where $\phi_\nu \in \mathbb{C}$. Then

$$\begin{aligned} \sum_{\mathrm{n}=1}^{\infty} \frac{f(n)}{n^s} &= \sum_{\nu=0}^{\infty} \frac{\phi_\nu}{\nu!} (\log Z)^\nu = \sum_{\nu=0}^{\infty} \phi_\nu \Big(\sum_{j=\nu}^{\infty} s(j,\nu) \frac{(Z-1)^j}{j!} \Big) \\ &= \sum_{j=0}^{\infty} (Z-1)^j \sum_{\nu \leq j} \phi_\nu \frac{s(j,\nu)}{j!}, \end{aligned}$$

where $s(j,\nu)$ are the Stirling numbers of the first kind ([4],p.282).

Since S is infinite, for any $p_1, \ldots, p_k \in S$, we have

$$\begin{aligned} f(p_1^{a_{p_1}} \cdots p_k^{a_{p_k}}) &= \sum_{j=1}^{\infty} \sum_{n_1 \cdots n_j = p_1^{a_{p_1}} \cdots p_k^{a_{p_k}}} z(n_1) \cdots z(n_j) \sum_{\nu \leq j} \phi_\nu \frac{s(j,\nu)}{j!} \\ &= z(p_1^{a_{p_1}} \cdots p_k^{a_{p_k}}) \sum_{j=1}^{k} \sum_{\substack{n_1 \cdots n_j = T_1 \cdots T_k \\ T_i = p_i^{a_{p_i}}}} 1 \sum_{\nu \leq j} \phi_\nu \frac{s(j,\nu)}{j!}. \end{aligned}$$

$$\frac{f(p_1^{a_{p_1}}\cdots p_k^{a_{p_k}})}{z(p_1^{a_{p_1}}\cdots p_k^{a_{p_k}})} = \sum_{\nu=1}^{k}\phi_\nu \sum_{j=\nu}^{k} s(j,\nu)S(k,j)$$
$$= \sum_{\nu=1}^{k}\phi_\nu\delta_{k\nu} = \phi_k,$$

where $S(k,j)$ are the Stirling numbers of the second kind ([4],p.150,p.281). Thus for all primes $p \in S$, $\dfrac{f(p^{a_p})}{z(p^{a_p})} = \phi_1 = c$, a constant. Since $f, z \in \mathcal{M}_{p_1} \cap \ldots \cap \mathcal{M}_{p_k}$, then

$$\phi_k = \frac{f(p_1^{a_{p_1}}\cdots p_k^{a_{p_k}})}{z(p_1^{a_{p_1}}\cdots p_k^{a_{p_k}})} = \frac{f(p_1^{a_{p_1}})\cdots f(p_k^{a_{p_k}})}{z(p_1^{a_{p_1}})\cdots z(p_k^{a_{p_k}})} = \phi_1^k = c^k,$$

and so

$$F = \sum_{\nu=0}^{\infty}\frac{\phi_\nu}{\nu!}(\log Z)^\nu = \sum_{\nu=0}^{\infty}\frac{c^\nu(\log Z)^\nu}{\nu!} = \exp(c\log Z) = Z^c.$$

Since $F \in \overline{\mathbb{C}[Z]}$, there are $a_{rj} \in \mathbb{C}$, not all zero, such that

$$0 = \sum_{r,j} a_{rj}Z^rF^j = \sum_{r,j} a_{rj}Z^{r+cj}$$
$$= \sum_{r,j} a_{rj}\sum_{\nu=0}^{\infty}\frac{(r+cj)^\nu}{\nu!}(\log Z)^\nu.$$

Equating the coefficients of $(p_1^{a_{p_1}}\cdots p_k^{a_{p_k}})^{-s}$, where $p_i \in S$, using the same reasoning as before we obtain

$$\sum_{r,j} a_{rj}(r+cj)^k = 0.$$

If c is purely complex or irrational , then $r + cj$ are all distinct not equal to zero, and this implies $a_{rj} = 0$ for all r, j, a contradiction. Hence c is rational and $f = z^c$. □

Using Theorem 4.27, an improvement of Theorem 7.4 in [10] is as follows :

Theorem 4.28. Let $z \in \mathcal{A}_1$ be such that $[supp(z)]$ is an infinite set. Assume that $f_1, f_2 \in \mathcal{A}_1$ are properly $\mathbb{C}$-algebraically dependent over z. If there exists an infinite subset $S \subseteq [supp(z)]$ such that $f_1, f_2 \in \bigcap_{p\in S}\mathcal{M}_p$, then f_2 is a rational power of f_1.

Proof. Since $f_1, f_2 \in \overline{\mathbb{C}[z]}^*$, by Lemma 4.26, $[supp(f_1)] = [supp(z)] = [supp(f_2)]$. Since $f_2 \in \overline{\mathbb{C}[z]}^*$ and $z \in \overline{\mathbb{C}[f_1]}^*$, then $f_2 \in \overline{\mathbb{C}[f_1]}^*$. By Theorem 4.27, $f_2 = f_1^c$ for some rational c. □

4.6 Dependence of Non-Units

Note that for a fixed prime p, and $F = \sum_{\text{n}=1}^{\infty} \frac{f(n)}{n^s}$,

$$d_pF = \sum_{\text{n}=1}^{\infty} \frac{d_pf(n)}{n^s} = \sum_{\text{n}=1}^{\infty} \frac{f(np)\upsilon_p(np)}{n^s}$$

$$= \sum_{\text{n}=1}^{\infty} \frac{f(np)\upsilon_p(np)}{(np/p)^s} + \sum_{\substack{n=1 \\ (n,p)=1}}^{\infty} \frac{f(n)\upsilon_p(n)}{(n/p)^s}$$
$$= \sum_{\text{n}=1}^{\infty} \frac{f(n)\upsilon_p(n)}{(n/p)^s}.$$

Lemma 4.29. If $f_1, \ldots, f_r \in \mathcal{A}$ are such that for all sets of r distinct primes $p_1, \ldots, p_r$, we have

$$J(f_1, \ldots, f_r/p_1, \ldots, p_r) = 0,$$

then $\quad \det(\upsilon_{p_i}(Nf_j)) = 0.$

Proof. This is a speacial case of Lemma 8.8 in [10]. □

The next theorem gives an interesting information about dependence of non-units and norms of elements in $\mathcal{A}$.

Theorem 4.30. The set of nonzero non-unit arithmetic functions whose norms are pairwise relatively prime is algebraically independent over $\mathbb{C}$.

Proof. Let $r \in \mathbb{N}$ and $f_1, \ldots, f_r$ be nonzero non-unit arithmetic functons whose norms are pairwise relatively prime. Then $Nf_i > 1$ for all $i = 1, \ldots, r$. Note that for each prime p , d_p annihilates all of $\mathbb{C}$. Suppose that $f_1, \ldots, f_r$ are algebraically dependent over $\mathbb{C}$. By Theorem 2.14, for all sets of primes $p_1, \ldots, p_r$, we have

$$J(f_1, \ldots, f_r / p_1, \ldots, p_r) = 0.$$

By Lemma 4.29, $\det(\upsilon_{p_i}(Nf_j)) = 0$. Thus there exist integers $\alpha_1, \ldots, \alpha_r$, not all zero, such that for all primes p,

$$\sum_{j=1}^{r} \alpha_j \upsilon_p(Nf_j) = 0,$$

and so

$$0 = \sum_{j=1}^{r} \alpha_j \upsilon_p(Nf_j) = \upsilon_p((Nf_1)^{\alpha_1} \cdots (Nf_r)^{\alpha_r}).$$

Then $(Nf_1)^{\alpha_1} \cdots (Nf_r)^{\alpha_r} = 1$. Since $Nf_1, \ldots, Nf_r$ are pairwise relatively prime, this is impossible. $\square$

For an $f \in \mathcal{A}$, let n' be the smallest integer greater than Nf such that $f(n') \neq 0$. Define $N_1 f = n'$. If n' does not exsit, define $N_1 f = Nf$.

Theorem 4.31. Let $f, g \in \mathcal{A}$ be nonzero such that $(Nf)(N_1 g) \neq (N_1 f)(Ng)$. If f and g are algebraically dependent over $\mathbb{C}$, then

(i) there exist integers x_1, x_2, not both zero, such that $(Nf)^{x_1}(Ng)^{x_2} = 1$;

(ii) there exist integers y_1, y_2, not both zero, such that $(Nf)^{y_1}(N_1 g)^{y_2} = 1$; and

(iii) there exist integers z_1, z_2, not both zero, such that $(N_1 f)^{z_1}(Ng)^{z_2} = 1$.

Proof. For ease of writing, let $Nf = n^*, N_1 f = n', Ng = m^*, N_1 g = m'$. If $n' = n^*$, then (iii) is equivalent to (i). If $m' = m^*$, then (ii) is equivalent to (i). We may assume that $n' \neq n^*, m' \neq m^*$, so $f(n^*), f(n'), g(m^*), g(m')$ all $\neq 0$. Assume that f and g are algebraically dependent over $\mathbb{C}$. Let p, q be distinct primes and F, G be the corresponding Dirichlet series of f, g, respectively. By Theorem 2.14,

$J(f, g/p, q) = 0$, and so

$$0 = J(F, G/p, q) = \begin{vmatrix} d_pF & d_pG \\ d_qF & d_qG \end{vmatrix}$$

$$= \begin{vmatrix} (\frac{f(n^*)\upsilon_p(n^*)}{(n^*/p)^s} + \frac{f(n')\upsilon_p(n')}{(n'/p)^s} + \ldots) & (\frac{g(m^*)\upsilon_p(m^*)}{(m^*/p)^s} + \frac{f(m')\upsilon_p(m')}{(m'/p)^s} + \ldots) \\ (\frac{f(n^*)\upsilon_q(n^*)}{(n^*/q)^s} + \frac{f(n')\upsilon_q(n')}{(n'/q)^s} + \ldots) & (\frac{g(m^*)\upsilon_q(m^*)}{(m^*/q)^s} + \frac{f(m')\upsilon_q(N_1g)}{(m'/q)^s} + \ldots) \end{vmatrix}$$

$$= \begin{vmatrix} \frac{f(n^*)\upsilon_p(n^*)}{(n^*/p)^s} & \frac{g(m^*)\upsilon_p(m^*)}{(m^*/p)^s} \\ \frac{f(n^*)\upsilon_q(n^*)}{(n^*/q)^s} & \frac{g(m^*)\upsilon_q(m^*)}{(m^*/q)^s} \end{vmatrix} + \begin{vmatrix} \frac{f(n^*)\upsilon_p(n^*)}{(n^*/p)^s} & \frac{g(m')\upsilon_p(m')}{(m'/p)^s} \\ \frac{f(n^*)\upsilon_q(n^*)}{(n^*/q)^s} & \frac{g(m')\upsilon_q(m')}{(m'/q)^s} \end{vmatrix} + \begin{vmatrix} \frac{f(n')\upsilon_p(n')}{(n'/p)^s} & \frac{g(m^*)\upsilon_p(m^*)}{(m^*/p)^s} \\ \frac{f(n')\upsilon_q(n')}{(n'/q)^s} & \frac{g(m^*)\upsilon_q(m^*)}{((m^*/q)^s} \end{vmatrix} + R$$

$$= \frac{f(n^*)g(m^*)(pq)^s}{(n^*m^*)^s} \begin{vmatrix} \upsilon_p(n^*) & \upsilon_p(m^*) \\ \upsilon_q(n^*) & \upsilon_q(m^*) \end{vmatrix} + \frac{f(n^*)g(m')(pq)^s}{(n^*m')^s} \begin{vmatrix} \upsilon_p(n^*) & \upsilon_p(m') \\ \upsilon_q(n^*) & \upsilon_q(m') \end{vmatrix}$$

$$+ \frac{f(n')g(n^*)(pq)^s}{(n'm^*)^s} \begin{vmatrix} \upsilon_p(n') & \upsilon_p(m^*) \\ \upsilon_q(n') & \upsilon_q(m^*) \end{vmatrix} + R,$$

where R is the sum of remaining terms all of whose denominators are greater than $(\frac{n'm^*}{pq})^s$ and $(\frac{n^*m'}{pq})^s$. Since $f(n^*), f(n'), g(m^*), g(m')$ are all $\neq 0$ and $n^*m' \neq n'm^*$, then

$$\begin{vmatrix} \upsilon_p(n^*) & \upsilon_p(m^*) \\ \upsilon_q(n^*) & \upsilon_q(m^*) \end{vmatrix} = \begin{vmatrix} \upsilon_p(n^*) & \upsilon_p(m') \\ \upsilon_q(n^*) & \upsilon_q(m') \end{vmatrix} = \begin{vmatrix} \upsilon_p(n') & \upsilon_p(m^*) \\ \upsilon_q(n') & \upsilon_q(m^*) \end{vmatrix} = 0.$$

From $\begin{vmatrix} \upsilon_p(n^*) & \upsilon_p(m^*) \\ \upsilon_q(n^*) & \upsilon_q(m^*) \end{vmatrix} = 0$, we deduce that there exist $x_1, x_2 \in \mathbb{Z}$, not all zero, such that for all primes r, $x_1\upsilon_r(n^*) + x_2\upsilon_r(m^*) = 0$, i.e. $\upsilon_r((n^*)^{x_1}(m^*)^{x_2}) = 0$, which renders $(Nf)^{x_1}(Ng)^{x_2} = (n^*)^{x_1}(m^*)^{x_2} = 1$.

The remaining assertions follow analogously by using the other two determinantal values. □

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