PENSION STRATEGY UNDER VOLATILITY CLUSTERING

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งานวิจัยนี้ได้ศึกษาผลกระทบการกระจุกของความผันผวนของการจัดสรรสินทรัพย์ที่ดีที่สุดในโครงการผลประโยชน์ที่กำหนดไว้ โดยตัวแบบทั้งสามของการกระจุกของความผันผวนคือตัวแบบการ์ซ, ตัวแบบจีเจอาร์และตัวแบบอีการ์ซถูกใช้ในการวิจัยนี้ พารามิเตอร์ของตัวแบบถูกประมาณค่าโดยใช้อนุกรมเวลาของผลตอบแทนของเอซแอนด์พี 500 และกลยุทธ์ที่ดีที่สุดถูกหามาโดยวิธีการเขียนโปรแกรมแบบพลวัต พารามิเตอร์ของการกระจุกของความผันผวนของตัวแบบการ์ซมีผลกระทบในการตัดสินใจอย่างมากในการเลือกน้ำหนักที่ดีที่สุด พารามิเตอร์ตัวแรกใช้ปรับความผันผวนในช่วงเวลาก่อนหน้ามีผลอย่างไรในความผันผวนปัจจุบันและพารามิเตอร์ตัวที่สองจับขนาดของความไม่แน่นอน ในขณะที่ตัวแบบของความผันผวนแบบไม่สมมาตรในตัวแบบจีเจอาร์และตัวแบบอีการ์ซถูกจับโดยพารามิเตอร์อีกตัวหนึ่งให้ความสำคัญกับความไม่แน่นอนที่สูงในผลตอบแทนที่ติดลบมากขึ้น เพื่อที่จะศึกษาพฤติกรรมของกลยุทธ์ที่ดีที่สุดภายใต้ตัวแบบที่แตกต่างกัน การจำลองสถานการณ์ด้วยวิธีมอนติคาร์โลถูกนำมาใช้เพื่อหาการกระจายของน้ำหนักที่ดีที่สุดในแต่ละตัวแบบ จากการสังเกตการแจกแจงของโมเดลที่แตกต่างกันทำให้เห็นว่าตัวแบบอีการ์ซให้การจัดสรรที่สมเหตุสมผลต่างกับตัวแบบการ์ซและตัวแบบจีเจอาร์ที่ให้น้ำหนักที่ค่อนข้างสูง กลยุทธ์ที่ดีที่สุดถูกทดสอบย้อนหลังเพื่อที่จะดูว่าตัวแบบไหนให้ผลลัพธ์ดีที่สุด ตัวแบบทั้งสามให้ผลตอบแทนที่คล้ายกันในการทดสอบย้อนหลังของเอซแอนด์พี 500 แต่กลยุทธ์ที่ดีที่สุดโดยสมมติตัวแบบอีการ์ซมีกลยุทธ์การลงทุนที่มีการป้องกันที่ดีที่สุดซึ่งจะเห็นผลได้ดีในช่วงวิกฤติการเงิน

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This research investigates the effect of volatility clustering on optimal asset allocation in a defined benefit pension scheme. Three models of volatility clustering, GARCH, GJR and EGARCH models, are examined. Model parameters are estimated using a time series of S&P500 returns while the optimal strategy is obtained using the numerical method to solve dynamic programming. The volatility-clustering parameters of GARCH models significantly influence the decision rules of choosing optimal weight. The first parameter adjusts how much volatility in the past affects present volatility, and the second parameter captures amplitude of the uncertainty. Meanwhile, asymmetric volatility in GJR and EGARCH models, captured by an additional parameter, put a focus on high volatility in negative returns. To study the behavior of optimal strategies under different models, Monte Carlo simulation is used to obtain the distribution of optimal weights and fund values in each model. Observing the distribution from different models show that the EGARCH model yields the most reasonable strategy and, unlike GARCH and GJR models, rarely gives extreme weight on risky assets. Subsequently, the optimal strategies are backtested to see which model gives the best outcome. The three models give the similar rate of return in backtesting using S&P500 returns data, but the optimal strategy assuming the EGARCH model has most conservative strategy which will be beneficial in the presence of a financial crisis.

Department:	Banking and Finance	Student's Signature	....
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CONTENTS

1 INTRODUCTION

1.1 Background and Problem Review

It has been observed in empirical data that large changes in prices of risky assets tend to come in clusters, and so do small changes. We call this phenomenon *volatility clustering*. This phenomenon is observed when there are extended periods of high market volatility, or the relative rate at which the price of financial asset changes, followed by a period of low volatility. Investment during high volatility clustering is riskier and risky assets can be susceptible to large price movement. Volatility clustering of risky assets, therefore, is an important factor to consider in long-term investment.

Volatility clustering also plays a crucial role in choosing investment strategies for a pension scheme. Pension schemes are also considered a type of long term investment and usually have investment time horizon of forty years. There are two popular types of pension schemes: defined contribution plan and defined benefit plan. In a defined contribution plan, the final fund value depends on the contribution to the fund and the fund's performance. On the other hand, in a defined benefit plan, the final fund target is fixed and depends on the proportion of final salaries of plan participants. The final salary can generally be predicted. For example, salaries of government officials increase in a largely predictable way. The defined benefit plan shares similar characteristics with other types of funds where the fund's managers have a fund target, for instance, a fund targeting five percent gain per year. However, the time horizon of other types of funds may be for a shorter period of time, such as five years. The frequency of rebalancing the portfolio in other types of funds could be weekly or monthly, while the frequency of rebalancing pension fund is usually every one or three months.

Protecting the wealth of the pension funds especially at the near terminal period is critical to make sure that it is on track to reach the fund target. In this research, we will focus on risk management of pension schemes under volatility clustering in risky asset returns and its implication on investment strategy in a defined benefit pension fund. Moreover, model extensions that capture specific features of risky asset returns, such as *asymmetric volatility* (a phenomenon that volatility is higher in down markets than in up markets), are explored. The results of this paper will provide additional insights in investment decision managing a defined benefit pension fund.

ARCH (Engle, 1982) and GARCH (Bollerslev, 1986) models are among the first models that aim to describe the volatility clustering phenomenon. Other models of volatility clustering include stochastic volatility models, such as event-risk framework (Duffie, Pan, and Singleton, 2000). The main idea behind these models is that the amplitude of volatility depends on its past realizations of the risky asset process. The

GARCH models are selected to model volatility clustering in this research because of its simplicity. Moreover, the autocorrelation term in the GARCH models conveniently captures the volatility clustering phenomenon and is easy to understand. Extension models of the GARCH model are also examined. The GJR and EGARCH models provide the property to capture the asymmetric volatility.

1.2 Research Questions and Objectives

1. How does the volatility clustering affect strategies in the pension fund management?
2. How do different volatility clustering models affect the optimal strategies? Under a defined benefit pension scheme setting, this research examines the behavior of optimal asset allocation given various volatility clustering models in the GARCH family models.
3. Which volatility clustering models give the best outcome, highest returns, and low standard deviations, in backtesting?

1.3 Contributions

Vigna and Haberman (2001) have studied the optimal investment strategy of risky asset in a pension strategy using simple returns. The optimal investment strategy only depends on the fund value. In this research, the volatility clustering is considered. In order to capture the volatility clustering of the risky asset, GARCH family models (GARCH, GJR and EGARCH models) are examined. We will study how the volatility clustering affects a defined benefit pension strategy.

The remaining sections are organized as followed. Section 2 explores literature review of GARCH family models, Bellman equations, and other relevant concepts. Section 3 looks further into the problem formulation and the overview of dynamic asset allocations, Monte Carlo simulation and backtesting strategies. Section 4 provides the numerical method for solving Bellman equation to find the optimal allocations. In Section 5, there are three parts. The first part is to analyze the optimal allocations, to investigate and compare the optimal asset allocations across models. Moreover, the effects of parameters of volatility clustering on optimal asset allocations are examined. The second part is Monte Carlo simulation, to simulate investment returns and variance under the decision rules of each model solved in the previous section. We compare the distribution of optimal weights and its fund value across GARCH family models. The last part is to backtest the strategies, to evaluate the performance of GARCH family models. Section 6 provides a summary of the results of this research.

2 LITERATURE REVIEW

2.1 Concept and Theoretical Background

The dynamic portfolio choice was first introduced by Merton (1969). It has been extended to many papers in finding optimal asset allocations. The investment strategies of pension schemes have been studied using various models. For instance, Vigna and Haberman (2001) have proposed the optimal investment strategy with discrete time model. Ngwira and Gerrard (2007) utilized the jump-diffusion process in the investment strategy of a pension scheme. In both important works of Vigna and Haberman (2001) and Ngwira and Gerrard (2007), volatility is assumed to be constant through the time horizon. However, there has been evidence of volatility clustering, where periods of high volatilities tend to cluster together and likewise for low volatilities periods. The volatility clustering in financial markets has been studied extensively. Lux and Marchesi (2000) have explained volatility clustering with a statistical analysis of simulated risky assets. Heteroscedasticity and leptokurtosis of returns are found within a multi-agent framework. Cont (2007) have studied volatility clustering in terms of the behavior of market participants with an agent-based model.

Volatility clustering in daily returns has long been observed. However, Jacobsen and Dannenburg (2003) also found evidence of volatility clustering in long-term stock returns in the Eurozone using the GARCH model. In this paper, we search for the optimal asset allocation in pension schemes given volatility clustering in 3-month returns of the risky asset by using GARCH family models.

This research focuses on managing a pension fund under volatility clustering. Volatility clustering presents a unique situation for the risk management of pension scheme. In this paper, we restrict volatility clustering model to the GARCH families. There are other researchers who have worked on related problems. Event-risk, for example, has impacts on financial markets and make a huge loss on security prices which cause individuals and funds to incur a big loss, including Pension fund. Liu, Longstaff, and Pan (2003) have studied the effect of jump size in optimal asset allocation.

2.2 GARCH Family models

In discrete time, the GARCH model was first introduced by Bollerslev (1986). It was adapted from the ARCH model proposed by Engle (1982). GARCH is often used for modeling stochastic volatility and it has been developed into many extensions. Later, Ben-Hameur, Breton, and Martinez (2006) provided a dynamic programming approach in GARCH model setting.

All models considered in this paper are in the GARCH family. Stock return dynamic is the same for every model as followed,

$$Y_t = \mu + \sigma_t \varepsilon_t, \quad (1)$$

$$X_t = Y_t - \mu = \sigma_t \varepsilon_t,\ \varepsilon_t \sim i.i.d. N(0,1),$$

where Y_t is a stock return at time t, μ is the mean of return, σ_t is the volatility at time t, ε_t is the white noise process assumed i.i.d. standard normal distribution, and X_t is a normally distributed random variable with volatility σ_t.

GARCH (1,1)

Generalized Autoregressive Conditional Heteroskedasticity (GARCH) model was proposed by Bollerslev (1986). The GARCH model is used to capture volatility clustering by the term $\alpha_1 \sigma_{t-1}^2$. We will study how this model affects optimal investment strategy. The GARCH model can be written as followed,

$$\sigma_t^2 = \alpha_0 + \alpha_1 \sigma_{t-1}^2 + \beta_1 X_{t-1}^2, \quad (2)$$

where $\alpha_0, \alpha_1, \beta_1$ are coefficients with the constraints of $\alpha_0 > 0, \alpha_1 > 0,\ \beta_1 > 0,\ \alpha_1 + \beta_1 < 1,\ \sigma_t^2$ is the conditional variance, $\varepsilon_t \sim i.i.d. N(0,1)$ and $X_{t-1} = \sigma_{t-1}\varepsilon_{t-1}$, is normally distributed with volatility σ_{t-1}. The long term variance equal to $\frac{\alpha_0}{1-\alpha_1-\beta_1}$.

GJR (1,1)

The GJR model was proposed by Glosten, Jagannathan, and Runkle (1993). The GJR model is an extension of the GARCH model that can capture the asymmetric volatility, also called *leverage effect* (changes in stock prices tend to be negatively correlated with changes in volatility, i.e., volatility is higher after negative returns than after positive returns of the same amplitude). The GJR model can be written as followed,

$$\sigma_t^2 = \alpha_0 + \alpha_1 \sigma_{t-1}^2 + \beta_1 X_{t-1}^2 + \omega_1 I_{t-1} X_{t-1}^2, \quad (3)$$

$$I_{t-1} = 1, if\ X_{t-1} < 0,$$

$$I_{t-1} = 0, if\ X_{t-1} \geq 0,$$

where $\alpha_0, \alpha_1, \beta_1$ and ω_1 are coefficients with the constraints of $\alpha_0 > 0\ , \alpha_1 > 0,\ \beta_1 > 0,\ \alpha_1 + \beta_1 + 0.5\omega_1 < 1, \sigma_t^2$ is the conditional variance, $\varepsilon_t \sim i.i.d. N(0,1)$, and $X_{t-1} = \sigma_{t-1}\varepsilon_{t-1}$ is normally distributed with volatility σ_{t-1}. The asymmetric volatility, captured by the dummy variable I_t, will also have an effect on the investment decision because the volatility depends on whether previous asset return is positive or negative. The long-term variance is equal to $\frac{\alpha_0}{1-\alpha_1-\beta_1-0.5\omega_1}$.

EGARCH (1,1)

The exponential GARCH model, proposed by Nelson (1991), is a logarithm function that is extended from the GARCH model by adding another feature to capture the leverage effect by the term $\omega_1\varepsilon_{t-1}$. The EGARCH model can be written as followed,

$$\log\sigma_t^2 = \alpha_0 + \alpha_1\log\sigma_{t-1}^2 + \beta_1\big[|\varepsilon_{t-1}| - E[|\varepsilon_{t-1}|]\big] + \omega_1\varepsilon_{t-1}, \tag{4}$$

where $\alpha_0, \alpha_1, \beta_1$ and ω_1 are coefficients, σ_t^2 is conditional variance, $\varepsilon_t \sim i.i.d.N(0,1)$, and $E[|\varepsilon_{t-1}|] = \sqrt{\frac{2}{\pi}}$. Even though the constraint of $\alpha_1, \beta_1, and\ \omega_1$ are relaxed because of its property of the logarithm function, the parameter α_1 and β_1 are expected to be positive and ω_1 to be negative. The long-term variance is equal to $e^{\frac{\alpha_0}{1-\alpha_1}}$.

2.3 Bellman Equation

The dynamic programming is used to solve a dynamic optimization problem with stochastic processes. In the dynamic programming, there are decisions to make in every period as the information changes over time. Let $V(T, W_T, y_1, \dots, y_{T-1})$ be a utility function or pay-off function that we try to maximize, which depends on the information at time T. In this context, V is a decreasing function of the difference between the final target fund and the terminal fund value. Here, W_t is wealth at time t, and y_t is weight invested in the risky asset at time t. Given r is the discount rate, Δt is the interval time invested and $\gamma = e^{-r\Delta t}$ is the discount factor, the dynamic decision problem can be written as the following equation,

$$J(0, W_0, \sigma_0^2) = \max_{y_i, i=0,..,T-1} \gamma^{\frac{T}{\Delta t}} V(T, W_T, y_1, \dots, y_{T-1}),$$

where $J(0, W_0, \sigma_0^2)$ is called the value function at time 0 and depends on initial wealth W_0 and initial variance σ_0^2 because these two values influence the decision making of choosing weight in the risky asset. In order to solve this problem, Bellman equation is a necessary condition to solve a dynamic programming problem. Bellman's optimality principle (Bellman, 1956): "An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision." In this research, the GARCH (1,1) model is used, so at time T the information that needs to use is up to time $T-1$. Given σ_t^2 is variance of the risky asset at time t ,solving the Bellman equation, we get

$$J(T-1, W_{T-1}, \sigma_{T-1}^2) = \max_{y_{T-1}}[\gamma E[V(T, W_T)|W_{T-1}, \sigma_{T-1}^2]],$$

From the equation above, we get the backward recursive equation that needs to be solved in optimal asset allocation,

$$J(t-1, W_{t-1}, \sigma_{t-1}^2) = \max_{y_{t-1}}[\gamma E[J(t, W_t, \sigma_t^2)|W_{t-1}, \sigma_{t-1}^2]].$$

The problem is then solved for the optimal allocation $y_t, t = 0, \dots, T-1$ from time $T-1$ to time 0 recursively.

3 PROBLEM FORMULATION AND OVERVIEW OF METHODOLOGY

In order to solve the optimal allocation of the risky asset, the dynamic programming problem is then set up. Firstly, we construct the wealth dynamic that consists of one risky asset and one risk-free asset,

$$W_t = W_{t-1}(1 + y_{t-1}Y_t + (1 - y_{t-1})r_f \Delta t) + c \qquad (5),$$

where W_t is fund value at time t,

y_t is the weight on the risky asset,

Y_t is the risky asset return following GARCH family models,

c is the dollar contribution per period,

r_f is the risk-free rate per year assumed constant,

Δt is the interval of investment between periods.

Let the retirement date T be the time horizon. In a defined benefit pension scheme, we care about the wealth at the terminal period. Therefore, we use the following quadratic objective function setup from Vigna and Haberman (2001),

$$H(T, W_T) = (W_T - F_T)^2,$$

where F_T is the fixed fund target at time T. By minimizing this objective function, we make the terminal fund value as close to the fund target as possible. From the objective function above, the Bellman equation is used to calculate the value function at time $T - 1$,

$$J(T-1, W_{T-1}, \sigma_{T-1}^2) = \min_{y_{T-1}} \left[\gamma E[H(T, W_T) | W_{T-1}, \sigma_{T-1}^2]\right].$$

Note that the value function depends on the state variables W_{T-1} and σ_{T-1}^2 because W_T depends on W_{T-1} and σ_{T-1}^2.

The Bellman equation is then applied recursively to compute $J(t, W_t, \sigma_t^2)$ for all $t < T$,

$$\boxed{J(t, W_t, \sigma_t^2) = \begin{cases} \min\limits_{0 \le y_t \le 1} E[\gamma(W_{t+1} - f)^2 | W_t, \sigma_t^2] & \text{, for } t = T-1 \\ \min\limits_{0 \le y_t \le 1} E[\gamma J(t+1, W_{t+1}, \sigma_{t+1}^2) | W_t, \sigma_t^2] & \text{, for } t = 0, \dots, T-2 \end{cases}}$$

Parameters $(\alpha_1, \beta_1, \omega_1)$ of the GARCH, GJR, and EGARCH models are estimated from the time series of S&P500 returns. The same time series is used across three models, so the results can be compared to each other. We set up the dynamic programming and solve it at period $T-1$, where T is the time horizon. Subsequently, solve the optimization problem from period $T-2$ to period 0 by optimizing the expectation of the value function $J(t, W_t, \sigma_t^2)$ of the next period. This numerical method will be explained in Section 4 thoroughly.

There are three parts of result in this paper: 1) analyzing the effects of volatility clustering $(\alpha_1, \beta_1, \omega_1)$ on the optimal allocations y_t, 2) simulation to compare the distribution of optimal allocations y_t and fund values W_t across the models and 3) backtesting the time series S&P500 to evaluate the performance of each model.

3.1 Analyzing the Effects of Volatility Clustering on the Optimal Allocation

In this part, the objective is to characterize the optimal allocation y_t surface of each model and compare it across GARCH models and its benchmark: constant volatility model. Moreover, the effects of volatility clustering parameters $(\alpha_1, \beta_1, \omega_1)$ on optimal allocations are examined by varying the parameters of GARCH family models.

3.2 Monte Carlo Simulation

The objective of Monte Carlo simulation is to study the probabilistic behavior of optimal allocations y_t and its fund value W_t under each model. Return Y_t and variance σ_t^2 of the risky asset are simultaneously simulated under the GARCH, GJR and EGARCH models for 100,000 possible paths. The simulated allocations follow the optimal allocation y_t from the solution of the dynamic programming. So, the optimal weight y_t varies with the state variables: variance σ_t^2 and fund value W_t.

3.3 Backtesting the Strategy

Backtesting the strategy evaluates the performance of GARCH family models. The optimal investment strategy for each GARCH-type model is characterized, the performance of each strategy under the realized historical data is compared to each other. The historical returns of S&P500 from the year 1961 to 2015 are used. The first 35 years of time series is used to estimate the parameters, while the latest 20 years of the time series is used for backtesting. We re-estimating parameters and solve the dynamic programming problem for the optimal allocation every 5 years.

4 NUMERICAL METHOD FOR SOLVING BELLMAN EQUATION

This section provides the numerical method for solving the dynamic programming problem as mentioned in Section 3 Analyzing the effects of volatility clustering on the optimal allocations to find the optimal allocations of the GARCH, GJR, and EGARCH models.

We numerically solve the dynamic programming problem. Grid points of the state variables are set up in order to solve the optimal allocation. There are two state variables in this research: variance σ_t^2 and fund value W_t.

The first state variable is the variance. Grid points of the variance are the same for all time t. The boundaries of the grid point are set to $\sigma_{min,t}^2 = 0.001$, and $\sigma_{max,t}^2 = 0.03$ because the simulated variance are mostly in this range. We use n equally-spaced grid points for the variance. Given $n = 20$, we get,

$$\left(\sigma_{1,t}^2, \sigma_{2,t}^2, \ldots, \sigma_{19,t}^2 \sigma_{20,t}^2\right) = (0.001, 0.0025 \ldots 0.0285, 0.03)\ , t = 0, \ldots, T-1.$$

The second state variable is fund value. The grid points of fund value are changed from period to period. At time t, the boundaries of the fund value ($W_{min,t}$ and $W_{max,t}$) depend on the fund target at that time. The equation of the fund target $F_t, t = 0, \ldots, T$ and the boundary of fund value are shown as followed.

For $t = 0, \ldots, T-1$, $$F_t = \left(1 + r_{req}\right)^{\frac{1}{n}} F_{t-1} + c,$$

where $F_0 = c$, c is the dollar contribution per period, r_{req} is the required return of reaching fund target per year, $n = 12\Delta t$, n is the amount of period invested in one year. At each time t, the boundaries of fund value are set to $0.3F_t$ and $1.5F_t$ because the simulated fund values are mostly in this range, regardless of the proportion of the risky asset. We also use n equally-spaced grid points for the fund value.

The grid points of state variables, $\left(W_{i,t}, \sigma_{j,t}^2\right)$, $i, j = 1 \ldots n, t = 0, \ldots, T-1$, are then found. For each period, the optimization problem that needs to be solved is equal to the total pairs of the state variables (n^2). From the Bellman equation, we solve the equation, $J(t, W_t, \sigma_t^2)$, at time T-1,

$$\min_{y_{T-1}} E\left[\gamma J\left(T, W_{i,T}, \sigma_{j,T}^2\right) | W_{T-1}, \sigma_{T-1}^2\right] = \min_{y_{T-1}} E\left[\gamma\left(W_{i,T} - f\right)^2 | W_{T-1}, \sigma_{T-1}^2\right].$$

Substituting equation 5 in $W_{i,T}$ will yield

$$\min_{y_{T-1}} E[\gamma\big(W_{i,T-1}\big(1 + y_{T-1}Y_T + (1 - y_{T-1})r_f\Delta t\big) + c - f\big)^2 | W_{T-1}, \sigma_{T-1}^2].$$

Substituting equation 1 in Y_T will yield

$$\min_{y_{T-1}} E[\gamma\big(W_{i,T-1}\big(1 + y_{T-1}(\mu + \sigma_T\varepsilon_{1,T}) + (1 - y_{T-1})r_f\Delta t\big) + c - f\big)^2 | W_{T-1}, \sigma_{T-1}^2] \quad (6)$$

The term σ_T is substituted by GARCH family models. We start at GARCH (1,1) model by substitute equation 2 in σ_T,

$$\min_{y_{T-1}} E[\gamma\left(W_{i,T-1}\left(1 + y_{T-1}\big(\mu + (\alpha_0 + \alpha_1\sigma_{j,T-1}^2\varepsilon_{2,T-1}^2 + \beta_1\sigma_{j,T-1}^2\big)^{\frac{1}{2}}\varepsilon_{1,T}) + (1 - y_{T-1})r_f\Delta t\right) + c - f\right)^2 | W_{T-1}, \sigma_{T-1}^2].$$

We use the numerical method to find the expectation. The variable $\varepsilon_{i,t}' \sim i.i.d.\, N(0,1)$, i = 1,2, are estimated numerically by varying it from -3 to 3 with the increment of 0.05. Let the probability $P(\varepsilon_t') = cdf(\varepsilon_t' + 0.025) - cdf(\varepsilon_t' - 0.025)$, where $cdf(\cdot)$ is the cumulative distribution function of the standard normal distribution. The sum of the probability, $\sum_{\varepsilon_t'=-3}^{\varepsilon_t'=3} P(\varepsilon_t') = 0.9975$, is adjusted to 1 by the normalization, $P(\varepsilon_t) = P(\varepsilon_t') / \sum_{\varepsilon_t'=-3}^{\varepsilon_t'=3} P(\varepsilon_t') = P(\varepsilon_t')/0.9975$. We can compute the expectation,

$$\min_{y_{T-1}} \sum_{\varepsilon_{2,T-1}=-3}^{3} \sum_{\varepsilon_1,T=-3}^{3} P\big(\varepsilon_{1,T}\big)P(\varepsilon_{2,T-1})(\gamma\left(W_{i,T-1}\left(1 + y_{T-1}(\mu + \big(\alpha_0 + \alpha_1\sigma_{j,T-1}^2\varepsilon_{2,T-1}^2 + \beta_1\sigma_{j,T-1}^2\big)^{\frac{1}{2}}\varepsilon_{1,T}) + (1 - y_{T-1})r_f\Delta t\right) + c - f\right)^2)$$

For the GJR (1,1) model setting, we substitute equation 3 in equation 6 and follow the same procedure as GARCH model. We get

$$\min_{y_{T-1}} \sum_{\varepsilon_{2,T-1}=-3}^{3} \sum_{\varepsilon_{1,T}=-3}^{3} P(\varepsilon_{1,T})P(\varepsilon_{2,T-1})(\gamma\Bigg(W_{i,T-1}\Bigg(1 + y_{T-1}(\mu + \left(\alpha_0 + \alpha_1\sigma_{j,T-1}^2\varepsilon_{2,T-1}^2 + \beta_1\sigma_{j,T-1}^2 + \omega_1 I_{T-1}\sigma_{j,T-1}^2\varepsilon_{2,T-1}^2\right)^{\frac{1}{2}}\varepsilon_{1,T}) + (1 - y_{T-1})r_f\Delta t\Bigg) + c - f\Bigg)^2)$$

In EGARCH (1,1) model setting, from equation 4, we get

$$\log\sigma_t^2 = \alpha_0 + \alpha_1\log\sigma_{t-1}^2 + \beta_1\big[|\varepsilon_{t-1}| - E[|\varepsilon_{t-1}|]\big] + \omega_1\varepsilon_{t-1},$$

$$\sigma_t = \left(e^{\alpha_0 + \alpha_1\log\sigma_{t-1}^2 + \beta_1[|\varepsilon_{t-1}| - E[|\varepsilon_{t-1}|]] + \omega_1\varepsilon_{t-1}}\right)^{\frac{1}{2}}.$$

Substituting the equation above σ_t in equation 6 using the same technique as the GARCH model will yield

$$\min_{y_{T-1}} \sum_{\varepsilon_{2,T-1}=-3}^{3} \sum_{\varepsilon_{1,T}=-3}^{3} P(\varepsilon_{1,T})P(\varepsilon_{2,T-1})\Bigg(\gamma\Bigg(W_{i,T-1}\Bigg(1 + y_{T-1}\Bigg(\mu + \left(e^{\alpha_0 + \alpha_1 \log\sigma_{t-1}^2 + \beta_1[|\varepsilon_{t-1}| - E[|\varepsilon_{t-1}|]] + \omega_1\varepsilon_{t-1}}\right)^{\frac{1}{2}}\varepsilon_{1,T}\Bigg) + (1 - y_{T-1})r_f\Delta t\Bigg) + c - f\Bigg)^2\Bigg)$$

For all GARCH models, at time $T-1$, the optimization problem is solved for the optimal strategy (y_{T-1}) and the value function ($J(T-1, W_{i,T-1}, \sigma_{j,T-1}^2)$) for every pair of state variables. The optimal strategy is then solved by backward recursion method from $t = T-2$ to $t = 0$. Given a pair of state variables,$(W_{i,t}, \sigma_{j,t}^2)$, $i, j = 1\ldots n$,

$$J_t(t, W_{i,t}, \sigma_{i,t}^2) = \min_{y_t} E[\gamma J_{t+1}(t, W_{i,t+1}, \sigma_{j,t+1}^2)|W_t, \sigma_t^2].$$

Substituting equation 2 in $W_{i,t+1}$ yield

$$\min_{y_t} E[\gamma J_{t+1}(t, W_{i,t}(1 + y_t(\mu + \sigma_t\varepsilon_{1,t}) + (1 - y_t)r_f\Delta t) + c, \sigma_{j,t+1}^2)].$$

Taking the expectation, where $\varepsilon_{2,t}$ is the term ε_t in $\sigma_{j,t+1}^2$'s equation, yield

$$\min_{y_t} \sum_{\varepsilon_{2,t-1}=-3}^{3} \sum_{\varepsilon_1,t=-3}^{3} P(\varepsilon_{1,t})P(\varepsilon_{2,t-1})(\gamma J_{t+1}(t, W_{i,t}(1 + y_t(\mu + \sigma_t \varepsilon_{1,t}) + (1 - y_t)r_f \Delta t) + c, \sigma_{j,t+1}^2))$$

We substitute equation 2, 3 and 4 in $\sigma_{j,t+1}^2$ for the GARCH, GJR and EGARCH models, respectively. The updated state variables are then interpolated to the grid of state variables at the next period to obtain the value function ($J_{t+1}(t, W_{i,t+1}, \sigma_{j,t+1}^2)$). We compute it for all scenarios of $(\varepsilon_{1,t}, \varepsilon_{2,t-1})$. For $t = 0, \ldots, T-2$, the optimization problem is solved recursively to get the solution of optimal weight y_t, and value function $J_t(t, W_{i,t}, \sigma_{j,t}^2)$.

5 RESULTS

The results are presented in three sections. The first section characterizes optimal allocations that are solved by the dynamic programming approach. The effects of volatility clustering on optimal allocations are also investigated. The second section provides the result of Monte Carlo simulation, studying the distribution of optimal weight and its fund value across GARCH family models. The last section, backtesting the strategies, evaluates the performance of GARCH family models in comparison to benchmark strategies: the constant volatility model and the buy-and-hold strategy.

The model parameters of the GARCH, GJR, and EGARCH models are estimated using the same time series data of S&P500 3-month returns from the year 1961 to 1995, a total of 35 years or 140 data points. Estimated parameters of the GARCH, GJR, and EGARCH models are shown in Table 4.1 in Appendix. Model parameters that are necessary to solve the dynamic programming, are set to be compatible with the pension scheme setting as followed. The interval between the portfolio adjustment is set to 3 months because the portfolio adjustment in the long term investment is not as frequent as in the short term investment. From equation 5, Δt is equal to 0.25. The investment window is 20 years, making the total investment of 80 periods. For simplicity, no short selling and leverage are allowed ($0 \leq y_t \leq 1, t = 0, \ldots, T-1$). The risk-free rate r_f and discount rate r are assumed to be constant at 1 percent per annum, contribution c in each period (3 months) is set to \$300, the required return of fund target r_{req} is set to 5%, fund target F_T is \$41,212.

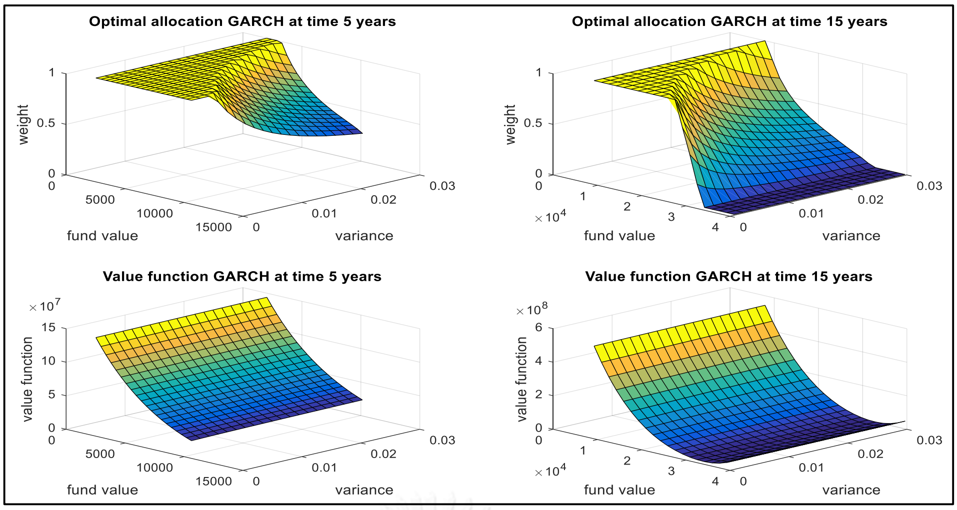
5.1 Analyzing Effects of Volatility Clustering on Optimal Allocations

In this section, we have split the results into two sub-sections. The first sub-section is to characterize the optimal allocation surface of each model and compare the optimal allocation across models. The second sub-section is to investigate the effect of volatility clustering by varying the parameters of GARCH family models.

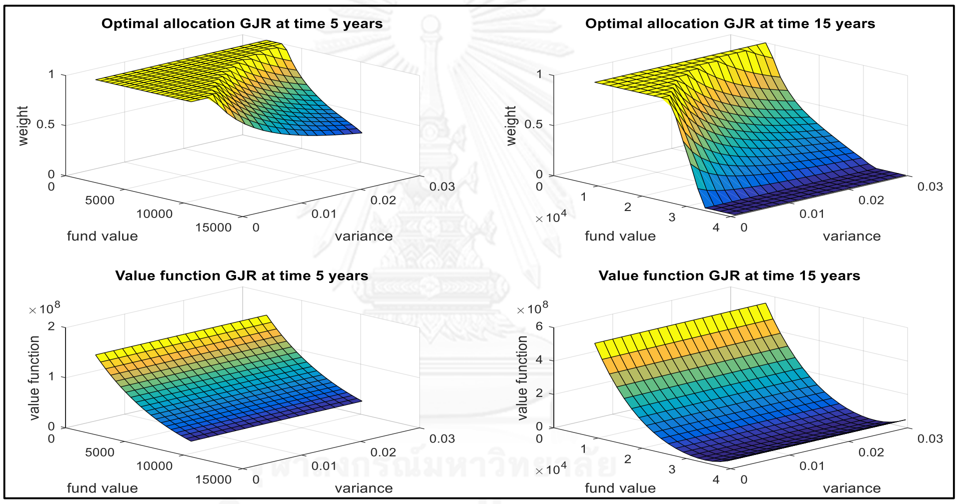
Comparing optimal allocations across volatility clustering models

The optimal strategies of the GARCH, GJR, and EGARCH models are solved using Bellman equation. The numerical method for solving the Bellman equation is provided in Section 4. After the optimal strategy is solved for each model, the optimal strategy and the value function will be plotted at 2 times: at year 5 (representing early periods) and at year 15 (representing later periods).

GARCH model



GJR model



EGRACH model

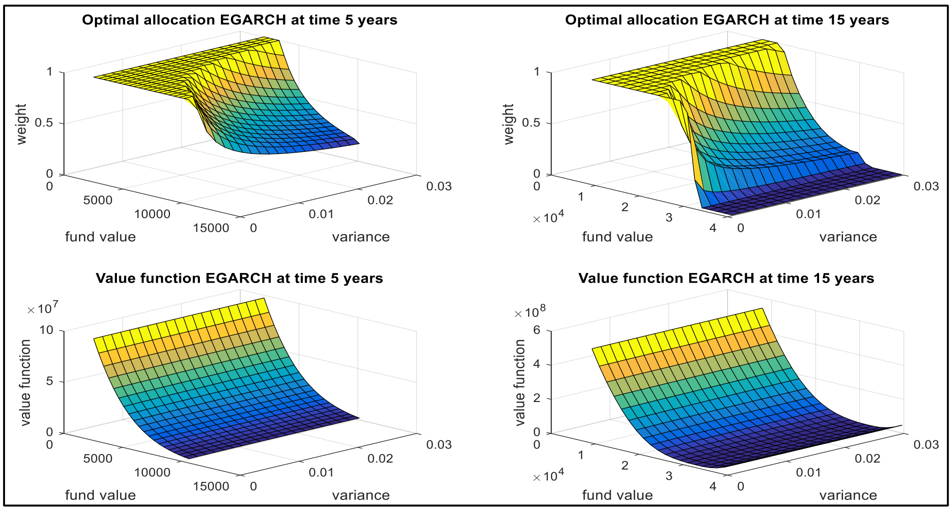


Figure 1. The surface of optimal allocation and value function of GARCH, GJR and EGARCH models at years 5 and 15.

The value function has the lowest value equal to 0 when the fund value equal to the fund target. The value function is always positive whether the fund value goes above or below the fund target. From Figure 1, if fund the value exceeds a threshold, the optimal allocation will put zero weight on the risky asset, which means that from that time on risk-free investment is enough to reach the target. This is because when the fund value is equal to the fund target, investment in the risky asset makes the expected fund value diverge more from the fund target. This threshold can be characterized by the final fund target, risk-free rate, and contribution rate as follows,

$$F_{dc,t-1} = \frac{F_{dc,t}-c}{(1+r_f)^{\frac{1}{n}}}, \qquad (7)$$

where $F_{dc,t}$ is the fund value threshold at time t, and $F_{dc,T} = F_T$ (fund target of T years). For example, the fund value threshold at year 15, calculated from equation 7, is \$33,365. If the fund value goes beyond this threshold, the optimal strategy is to invest all weight in the risk-free asset for the remaining periods. While the fund value threshold at year 5 is \$18,797, the sum of the contribution invested at that time is just \$6,000. The threshold is very high compared to the contribution so the optimal strategy is to invest high weight in the risky asset because it yields higher expected return than the risk-free asset.

The plots of the optimal allocation surface do not facilitate comparison among models. To aid comparison, we cross-section the surfaces, varying fund value and fixing the time and variance. The cross-section of optimal allocation weight of the GARCH, GJR, and EGARCH models are provided in Figure 2 below.

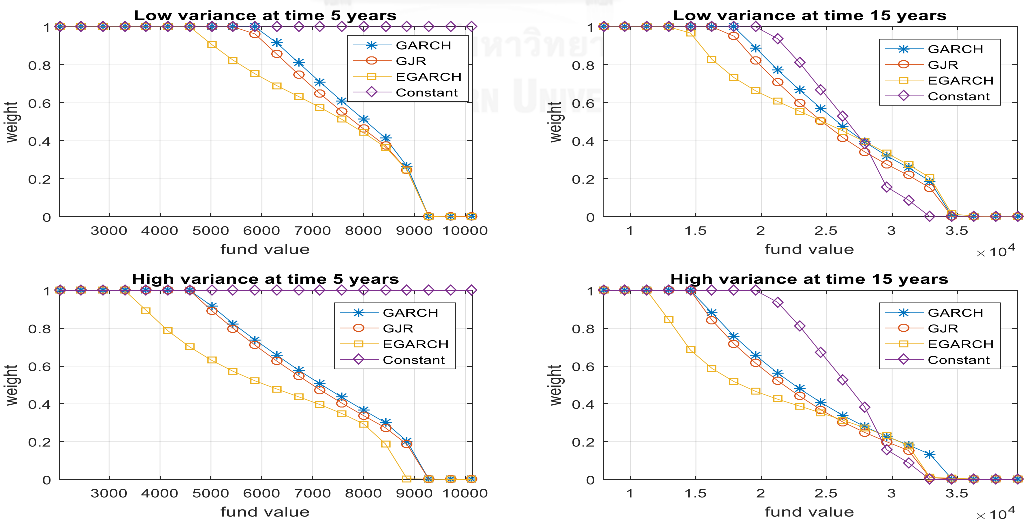


Figure 2. The cross-section of optimal allocation weight of the GARCH, GJR, EGARCH and constant volatility models at years 5 and 15, the variance of the constant volatility model = 0.063. In GARCH family models, low variance = 0.0057 and high variance = 0.0093. The parameters are estimated from the same time series, S&P500, for all models.

From Figure 2, Findings of the optimal allocations in GARCH family models are as followed, while the constant volatility model will be mentioned later.

- In all three models, the optimal weight to put in the risky asset is higher when the *variance* of risky asset and/or *fund value* is low. In the early period, low variance and low fund value result in the optimal weight closer to one.
- If the fund value already reaches the target, the optimal strategy is to put all the weight in the risk-free asset.
- In early periods, the optimal strategy will be likely to put more weight in the risky asset than in later periods. For example, at year 5, the optimal strategy will put more weight in the risky asset than at year 15.

However, as seen from Figure 2, optimal allocations differ among models. To better understand what make the characteristic of the optimal allocations difference, we look at the variance of the risky asset return of GARCH family models (equation 2,3 and 4 for the GARCH, GJR and EGARCH models, respectively). Then we examine how the variance influences the optimal allocation in each model. The variance distributions of GARCH family models are then investigated by varying ε_{t-1} on the normal distribution in the range from -3 to 3.

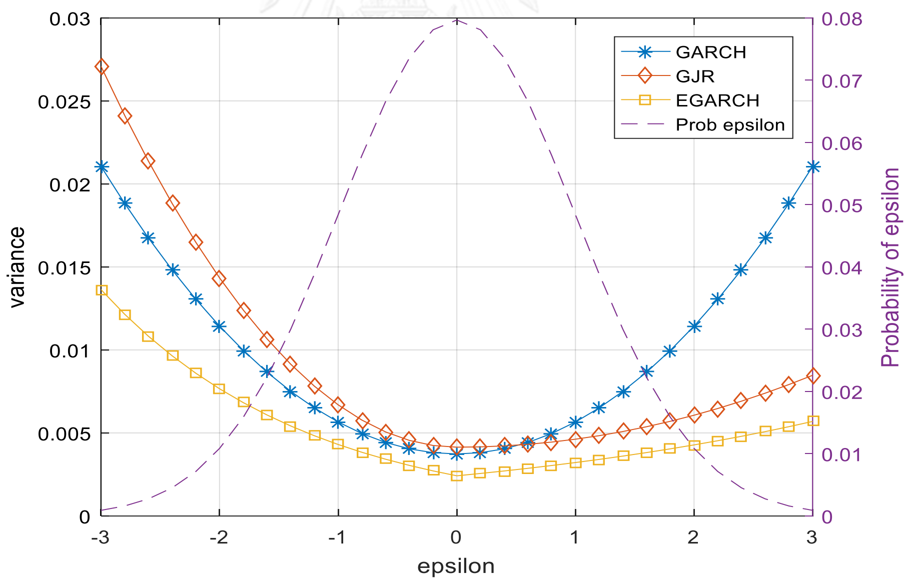


Figure 3. The distribution of the variance of the GARCH, GJR and EGARCH models with varying ε_{t-1} from -3 to 3, given $\sigma_{t-1}^2 = 0.005$.

Consider equation 6, the optimization problem at one period before the terminal period, the only difference between the three models is the variance distribution that will lead to different optimal allocations. From Figure 3, the variance of the GARCH model has the symmetric volatility. Meanwhile, the variance of the GJR and EGARCH models represent the asymmetry volatility: the effect of the term ω_1 causes the different variance in the next period, the positive ε_{t-1} yield the lower variance than the negative

ε_{t-1} of the same size. The different variance distribution provides the different optimal allocation. However, the variance of the GARCH and GJR models are very similar with the highest density of ε_{t-1} around 0. So, at one period before the terminal period, the differences of optimal investment strategies are as followed.

- The GARCH and GJR models yield very similar optimal strategies.
- The EGARCH model results in a more aggressive strategy in the later periods since the variance for all possible $-3 < \varepsilon_{t-1} < 3$ in the EGARCH model is lower than in the GARCH and GJR models.

Next, volatility clustering models are compared to the non-volatility clustering model. The constant volatility model is used as a benchmark. It has only one state variable, the fund value. The optimal allocation of this model is calculated similarly to GARCH family models.

The optimal allocation of the constant volatility model is also shown in Figure 3. The optimal allocation of the constant volatility only depends on the fund value, while volatility clustering models depend on both the fund value and the variance. At both low and high variances, in earlier periods, the result shows that the optimal strategies in GARCH family models are more conservative than in the constant volatility model. While in later periods, GARCH family models result in lower optimal weights than the constant volatility model when the fund value is low, but result in higher optimal weights when the fund value is high. As discussed, this can imply that the volatility clustering influences optimal strategies. The volatility clustering model results in more conservative investment strategy than the non-volatility clustering model in early periods.

Effect of volatility clustering on optimal strategy

This section examines the effect of volatility clustering in GARCH family models on the optimal strategy. Two important parameters of GARCH family models that can represent the volatility clustering effect are α_1 and β_1. The first parameter, α_1, represents how much volatility in the past affects present volatility. The second parameter, β_1, represents the amplitude of uncertainty in the past period. Moreover, the extension models, the GJR and EGARCH models, have the additional parameter: ω_1, allows for the asymmetric volatility that can capture both the sign and amplitude of the uncertainty.

To compare the effect of each parameter (α_1, β_1, and ω_1) on the optimal strategy, the benchmark parameters of the GARCH family models are also estimated from the S&P500 returns as same as the previous section. Each parameter is varied separately while holding the rest of parameters constant. The same dynamic programming method as in Section 4 is applied to obtain the optimal strategy. By fixing the time and the state variable: fund value, the optimal strategies result with varying variance are presented. The figures below will show the relationship between the variance and the optimal weight compared along the varying parameters in the GARCH, GJR, and EGARCH models. The parameters of the volatility clustering effect are investigated separately in each model.

Effect of volatility clustering on optimal strategy in the GARCH model

Starting from the GARCH model, we consider its constraint, $\alpha_1 + \beta_1 < 1$. The higher the value of $\alpha_1 + \beta_1$, the higher the effect of volatility clustering is. To study the effect of volatility clustering on the optimal allocation, the values of α_1 and β_1 are varied and categorized into 4 groups, given benchmark values of the GARCH model are $\alpha_0 = 0.002208$ $\alpha_1 = 0.5248, \beta_1 = 0.1272$, and $\mu = 0.01461$.

Group 1: varying α_1, fixed β_1

$\alpha_1 = 0.3$, $\beta_1 = 0.1$, $\alpha_1 + \beta_1 = 0.4$

$\alpha_1 = 0.7$, $\beta_1 = 0.1$, $\alpha_1 + \beta_1 = 0.8$

Group 2: varying β_1, fixed α_1

$\alpha_1 = 0.1$, $\beta_1 = 0.3$, $\alpha_1 + \beta_1 = 0.4$

$\alpha_1 = 0.1$, $\beta_1 = 0.7$, $\alpha_1 + \beta_1 = 0.8$

Group 3: varying $\alpha_1 + \beta_1$, equally weight of α_1 and β_1

$\alpha_1 = 0.2$, $\beta_1 = 0.2$, $\alpha_1 + \beta_1 = 0.4$

$\alpha_1 = 0.4$, $\beta_1 = 0.4$, $\alpha_1 + \beta_1 = 0.8$

Group 4: fixed $\alpha_1 + \beta_1$, varying α_1 and β_1

$\alpha_1 = 0.1$, $\beta_1 = 0.7$, $\alpha_1 + \beta_1 = 0.8$

$\alpha_1 = 0.4$, $\beta_1 = 0.4$, $\alpha_1 + \beta_1 = 0.8$

$\alpha_1 = 0.7$, $\beta_1 = 0.1$, $\alpha_1 + \beta_1 = 0.8$

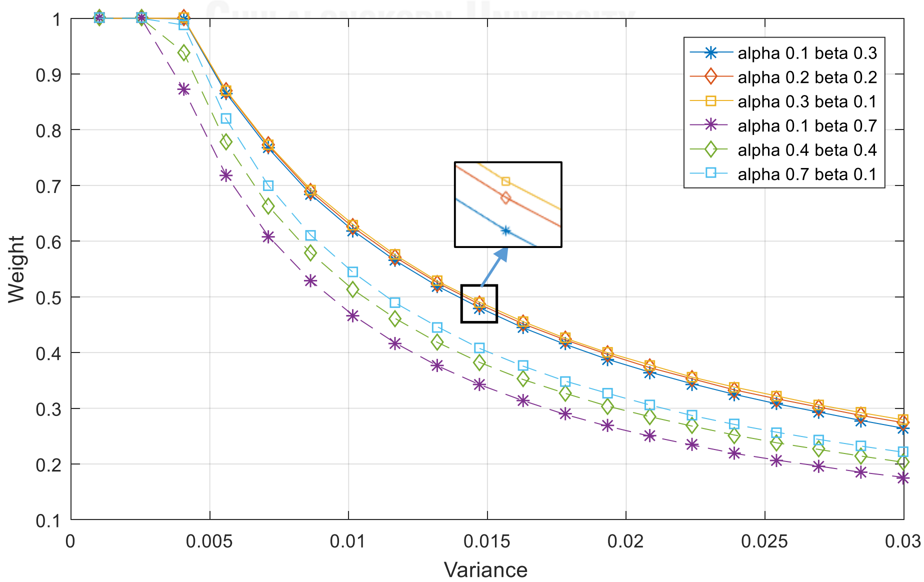


Figure 4. The cross-section of optimal allocation weights of the GARCH model with varying α_1 and β_1 by fixing fund value at year 15.

Comparing models in group 1, varying α_1 and fixed β_1, the larger the α_1, the higher the volatility clustering. In the presence of higher α_1, the optimal strategy puts less weight on the risky asset. The level of volatility clustering, α_1, has a significant impact on the optimal weight.

Comparing models in group 2, varying β_1 and fixed α_1, lower β_1 results in higher optimal weight on the risky asset than higher β_1 one. This is because higher β_1 gives more uncertainty to the model, assets become riskier, so the optimal allocation puts less weight in the risky asset. As a result, the higher β_1 leads to more conservative investment strategy with less weight in the risky asset.

Comparing models in group 3 with varying $\alpha_1 + \beta_1$ and equally weight of α_1 and β_1, at higher sum value, the optimal allocation will put less weight in the risky asset than at lower sum value. So the optimal strategy is more conservative at lower $\alpha_1 + \beta_1$.

Lastly, comparing models in group 4, given $\alpha_1 + \beta_1 = 0.8$ and varying α_1 and β_1 values, at higher β_1 model, the optimal allocation puts less weight in the risky asset than at higher α_1 model. The result implies that β_1, compared at the same level of α_1, makes the asset become riskier causing the optimal strategy to put less weight in the risky asset. Meanwhile, at $\alpha_1 + \beta_1 = 0.4$, β_1 still has a larger impact on the risky asset than α_1 although the difference is small as $\alpha_1 + \beta_1$ is small.

To verify the result, the long-run variance of the GARCH model is $\frac{\alpha_0}{1-\alpha_1-\beta_1}$. Similar to the results above, the higher the α_1, β_1 or $\alpha_1 + \beta_1$, the higher the long run variance. Moreover, from Equation 2, we get $\sigma_t^2 = \alpha_0 + \sigma_{t-1}^2(\alpha_1 + \beta_1\varepsilon_{t-1}^2)$. The fact that higher α_1 or β_1 yield higher variance also verifies the results. The term $\alpha_1 + \beta_1\varepsilon_{t-1}^2$ is the decay factor of volatility. α_1 is a fixed component while β_1 is a random component with the variation of ε_{t-1}^2. Consider variance of the GARCH model with varying ε_{t-1} in Figure 1, at $\varepsilon_{t-1} = 0$, there is no random component, causing variance to go lower than the previous period. Adding the random component, $|\varepsilon_{t-1}| > 0$ gives higher variance than at ε_{t-1}. So α_1 and β_1 are both important factors on the decision rule in choosing optimal weight.

Effect of volatility clustering on optimal strategy in the GJR model

In the GJR model, there are three volatility clustering parameters: α_1, β_1, and ω_1. Each volatility clustering parameter is varied with different values to not violate the constraint $\alpha_1 + \beta_1 + 0.5\omega_1 < 1$, given the benchmark values, $\alpha_0 = 0.002987, \alpha_1 = 0.3367, \beta_1 = 0.01317, \omega_1 = 0.4024$, and $\mu = 0.01385$.

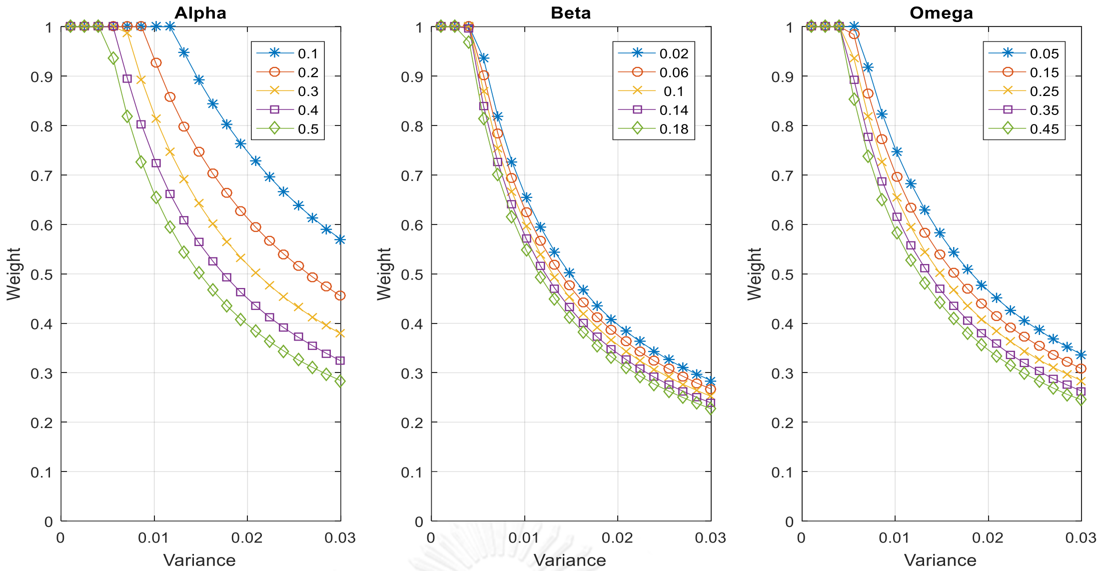


Figure 5.1. The cross-section of optimal allocation weights of GJR model varying α_1, β_1, and ω_1 by fixing fund value at year 15.

The GJR model is an extension of the GARCH model with the addition of ω_1 parameter while the rest of the model is the same as the GARCH model. Thus the effects of α_1 and β_1 are the same as in the GARCH model, as seen in Figure 5.1.

The term $\omega_1 I_{t-1}\varepsilon_{t-1}^2$ is an additional term that extends from the GARCH model characterizing asymmetric volatility. When ε_{t-1}, is more than 0, $\omega_1 I_{t-1}\varepsilon_{t-1}^2$ is equal to 0, and otherwise, positive. Thus negative ε_{t-1}, compared to positive ε_{t-1} at the same level, yields higher variance. The higher the ω_1, the lower the optimal allocation on the risky asset.

To study the relative importance of volatility clustering parameters $(\alpha_1, \beta_1, \omega_1)$, we compare the effect of volatility clustering between parameters. The constraint is set to $\alpha_1 + \beta_1 + 0.5\omega_1 = 0.8$ and the benchmark values, $\alpha_0 = 0.002987, \alpha_1 = 0.3367, \beta_1 = 0.01317, \omega_1 = 0.4024$, and $\mu = 0.01385$.

Benchmark model

$\alpha_1 = 0.3,\ \beta_1 = 0.3,\ \omega_1 = 0.4$

Comparing α_1, β_1 : lower α_1 and higher β_1

$\boldsymbol{\alpha_1 = 0.1}, \boldsymbol{\beta_1 = 0.5}, \omega_1 = 0.4$

Comparing α_1, ω_1 : lower α_1 and higher ω_1

$\boldsymbol{\alpha_1 = 0.1}, \beta_1 = 0.3, \boldsymbol{\omega_1 = 0.8}$

Comparing β_1, ω_1 : lower β_1 and higher ω_1

$\alpha_1 = 0.3, \boldsymbol{\beta_1 = 0.1}, \boldsymbol{\omega_1 = 0.8}$

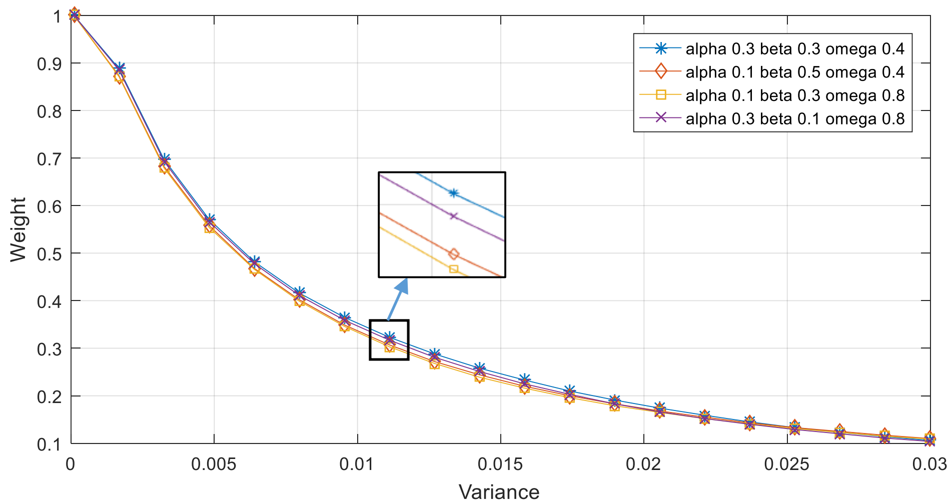


Figure 5.2. The cross-section of optimal allocation weights of GJR model for comparing pairs of two parameters $((\alpha_1, \beta_1), (\alpha_1, \omega_1), (\beta_1, \omega_1))$ by fixing fund value at year 15.

From Figure 5.2, the benchmark model $(\alpha_1 = 0.3, \beta_1 = 0.3, \omega_1 = 0.4)$ is used to compare with other models. To compare relative importance between parameters, first, we start with a pair of (α_1, β_1). The benchmark model is compared by adjusting lower α_1 and higher β_1 $(\alpha_1 = 0.1, \beta_1 = 0.5, \omega_1 = 0.4)$. The result is the higher β_1 has more sensitivity than the benchmark model just as in GARCH model. Next, we compare the benchmark model with pairs of (α_1, ω_1) and (β_1, ω_1) by lower α_1 and higher ω_1 $(\alpha_1 = 0.1, \beta_1 = 0.3, \omega_1 = 0.8)$ and lower β_1 and higher ω_1 $(\alpha_1 = 0.3, \beta_1 = 0.1, \omega_1 = 0.8)$, respectively. The result is higher ω_1 has more sensitivity than the benchmark model in both scenarios. As a summary, the term ω_1 has the most sensitivity on the optimal allocation followed by β_1 and α_1, respectively. This implies that the strategy is more concern about the leverage effect ω_1 than the amplitude of the uncertainty β_1 and the autocorrelation of volatility α_1. The uncertainty terms (β_1, ω_1) that depend on the return of risky asset ε_{t-1}, then, have more impact on optimal allocation than the predictable term (α_1) that depends on the variance of risky asset σ_t^2.

The long-run variance of the GJR model is $\frac{\alpha_0}{1-\alpha_1-\beta_1-0.5\omega_1}$. It is similar to the GARCH model with an addition of the term ω_1. Similar to the result in the GARCH model, the higher the $\alpha_1, \beta_1, \omega_1$ or $\alpha_1 + \beta_1 + 0.5\omega_1$, the higher the long run variance. Moreover, according to equation 3, $\sigma_t^2 = \alpha_0 + \sigma_{t-1}^2(\alpha_1 + (\beta_1 + \omega_1 I_{t-1})\varepsilon_{t-1}^2)$, higher α_1, β_1 or ω_1 yields higher variance. $\alpha_1 + (\beta_1 + \omega_1 I_{t-1})\varepsilon_{t-1}^2$ is the decay factor of volatility, α_1 is a fixed component while $\beta_1 + \omega_1 I_{t-1}$ is a random component with the variation of ε_{t-1}^2. α_1 and β_1 have the same effects as in the GARCH model. While the extension term, ω_1, can also capture the sign of ε_{t-1} by the indicator $I_t, I_t = 0$ when $\varepsilon_{t-1} > 0$ and $I_t = 1$ when ε_{t-1}. The term ω_1 should give us the more realistic distribution of volatility, asymmetric volatility, and provide the better strategy than in the GARCH model.

Effect of volatility clustering on optimal strategy in the EGARCH model

There are also three volatility clustering parameters (α_1, β_1, and ω_1) in the EGARCH model. As there is no constraint in the EGARCH model, each volatility clustering parameter is varied with higher different values than the GARCH and GJR models, given benchmark values of $\alpha_0 = -1.4468, \alpha_1 = 0.7163, \beta_1 = 0.3104, \omega_1 = -0.1293$, and $\mu = 0.01291$.

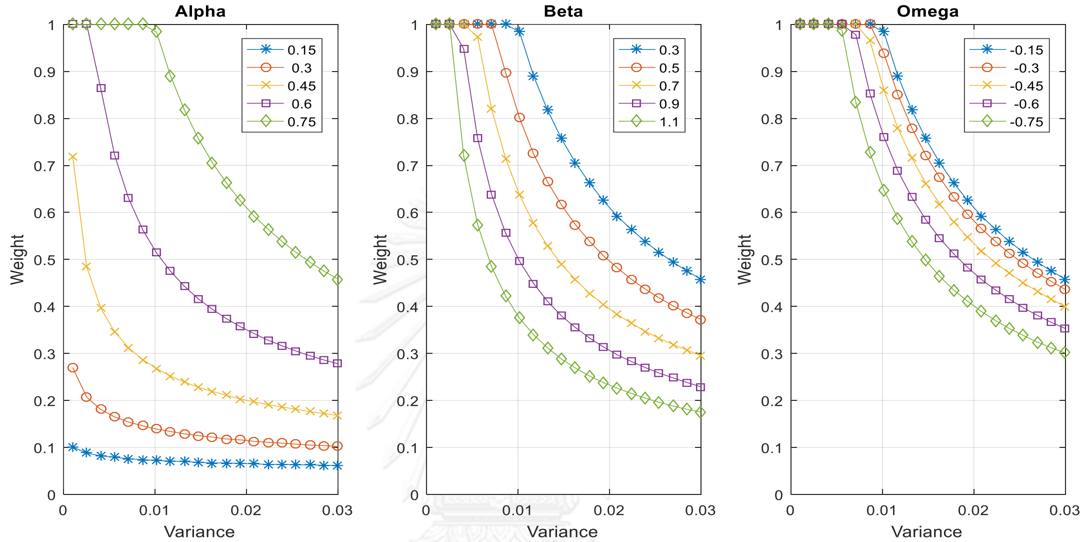


Figure 6. The cross-section of optimal allocation weights of the EGARCH model varying α_1, β_1, and ω_1 by fixing fund value at year 15.

The effect of α_1 in the EGARCH model is reverse from the GARCH model. Due to the characteristic of log function, $log(n)$ is less than 0 when $n < 1$, and the fact that $0 < \sigma < 1$, the term $\alpha_1 \log \sigma_{t-1}^2$ is more negative when α_1 is higher, resulting in overall lower variance in the current period (σ_t^2). Given low α_1 resulting in high variance, the optimal strategy will put a lower weight on the risky asset than high α_1.

The effect of β_1 in the EGARCH model is the same as in the GARCH and GJR models. The higher the β_1 is, the higher the weight of optimal allocation is.

The term ω_1 is a direct variation of ε_{t-1}. ω_1 is used to capture the asymmetric effect, i.e. high negative returns causing high volatility and lower volatility for positive returns. So high ω_1 results in the model highlighting the volatility of the risky asset, resulting in lower optimal weight in the risky asset.

In the EGARCH model, there is no constraint between parameters (α_1, β_1, and ω_1) so the parameters cannot be compared in the same way as the GARCH and GJR models ($0 < \alpha_1 + \beta_1 + \omega_1 < 1$). From Figure 6, consider when $0.15 < \alpha_1 < 0.75, 0.3 < \beta_1 < 0.9$, and $-0.75 < \omega_1 < -0.15$, the range of each parameter are fixed to 0.6. The sensitivity of parameter that has the most impact is the autocorrelation of volatility α_1 followed by the amplitude of the uncertainty β_1 and the leverage effect ω_1, respectively. The α_1 is the power of the variance σ_{t-1}^2. Even though the EGARCH model has no constraint, a very low value on α_1 could results in a lot higher variance than it should be. This is why the optimal strategy when $\alpha_1 = 0.15$ rarely put weight in the risky asset. This also applies to the higher β_1 and $|\omega_1|$ resulting in extremely high variance.

The long-run variance of the EGARCH model is $e^{\frac{\alpha_0}{1-\alpha_1}}$ when $\alpha_0 < 0$ and $0 < \alpha_1 < 1$. The higher the α_1, the lower the long-run variance, while β_1 and ω_1 have no effect on long-run variance. Moreover, according to equation 4, we get $\sigma_t^2 = e^{c_0}(\sigma_{t-1}^2)^{\alpha_1} e^{\beta_1|\varepsilon_{t-1}|+\omega_1(\varepsilon_{t-1})}$, where $c_0 = \alpha_0 - \beta_1 \mathrm{E}[|\varepsilon_{t-1}|]$ and $0 < \sigma_{t-1}^2 < 1$. The higher β_1, higher ω_1 or lower α_1 yields higher variance. e^{c_0} is the constant multiplier to σ_{t-1}^2. α_1 is the constant power of σ_{t-1}^2. $e^{\beta_1|\varepsilon_{t-1}|+\omega_1(\varepsilon_{t-1})}$ is the random component, bases on ε_{t-1}. β_1 captures the amplitude of ε_{t-1}, while ω_1 captures the sign and also amplitude of ε_{t-1}. The effect of ω_1, like in GJR model, allows the model capture more property of ε_{t-1} and provide the better strategy than in the GARCH model.

The EGARCH and GJR models have an additional feature from the GARCH model, ω term, that can capture the sign effects of return: negative residuals induce larger increases in the variance than positive residuals, asymmetric volatility. Consider when a market crashes, stock prices drop dramatically, causing a significant increase in market volatility. This phenomenon will make investment strategies of asymmetric GARCH models be more cautious about the market crash's situation than the GARCH model. By intuition, due to higher volatility in asymmetric volatility models, it will lower the weight in the risky asset compare to GARCH model in order to reduce the risk. As discussed, we think that the EGARCH and GJR models will have better investment decisions than GARCH model in the backtest period.

Table 1. Summary of the effect of α_1, β_1 and ω_1 on optimal allocation in the GARCH, GJR, and EGARCH models.

Parameter(s)	Conservative strategy / riskier model	Aggressive strategy /less risky model
α_1 (GARCH, GJR)	Higher α_1	Lower α_1
α_1 (EGARCH)	Lower α_1	Higher α_1
β_1	Higher β_1	Lower β_1
ω_1 (GJR, EGARCH)	Higher ω_1	Lower ω_1
$\alpha_1 + \beta_1$ (GARCH, GJR)	Higher $\alpha_1 + \beta_1$	Lower $\alpha_1 + \beta_1$
Fix $\alpha_1 + \beta_1$ (GARCH)	Higher β_1/ Lower α_1	Higher α_1/ Lower β_1
Fix $\alpha_1 + \beta_1 + 0.5\omega_1$ (GJR)	1) Higher β_1/ Lower α_1 2) Higher ω_1/ Lower α_1, β_1	1) Higher β_1/ Lower α_1 2) Higher α_1, β_1/ Lower ω_1

5.2 Monte Carlo Simulation

In this section, we will examine how the different volatility clustering models affect the optimal strategy. Monte Carlo simulation is used to investigate the distribution of the optimal weight and the fund value in each model. In particular, the average of the simulated allocation and the simulated fund value are compared in each model.

First, return and variance of the GARCH, GJR, and EGARCH models are simulated using the estimated parameters provided in Table 4.1 of Appendix. In the simulation, the simulated weight follows the optimal allocation in the previous section. The simulated weight invested in each period depends on the state variables: fund value and variance. Henceforth, the distribution of simulated weight and simulated fund value are found from 100,000 paths of each model. The average weight and the average fund value are calculated from the simulated weight and the simulated fund value. The distribution of simulated weight and the average weight of the GARCH, GJR, EGARCH models are shown in Figure 7.1 and Figure 7.2 respectively.

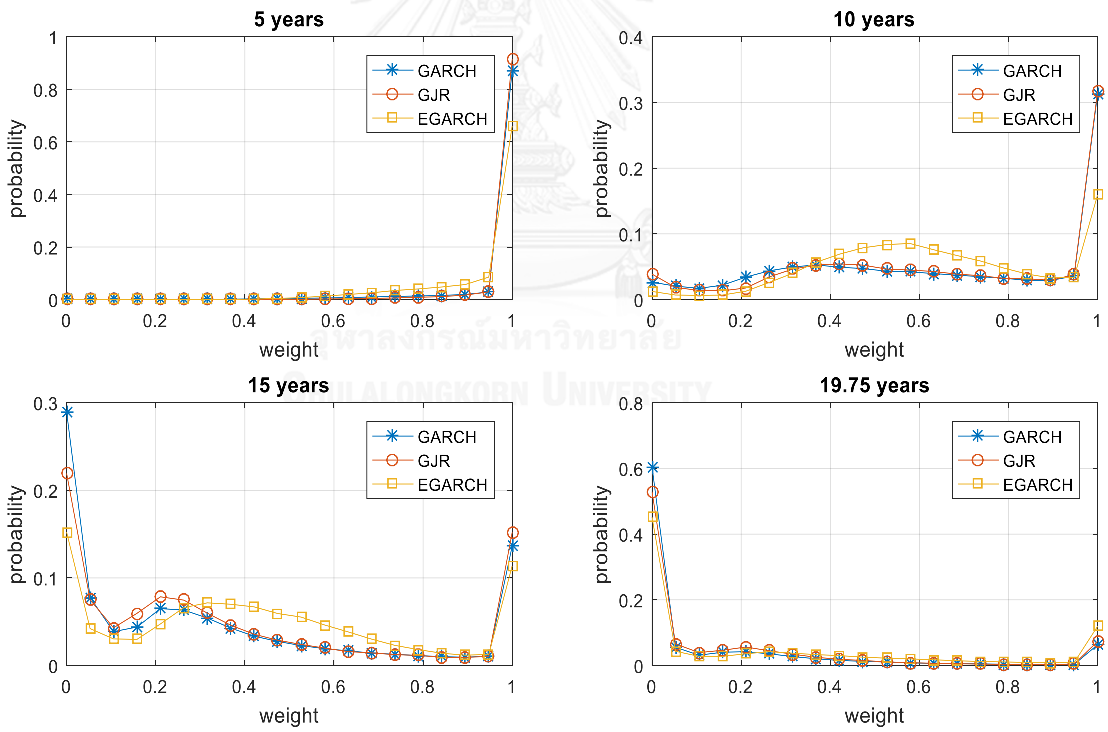


Figure 7.1. The distribution of simulated weight of the GARCH, GJR and EGARCH models at several periods.

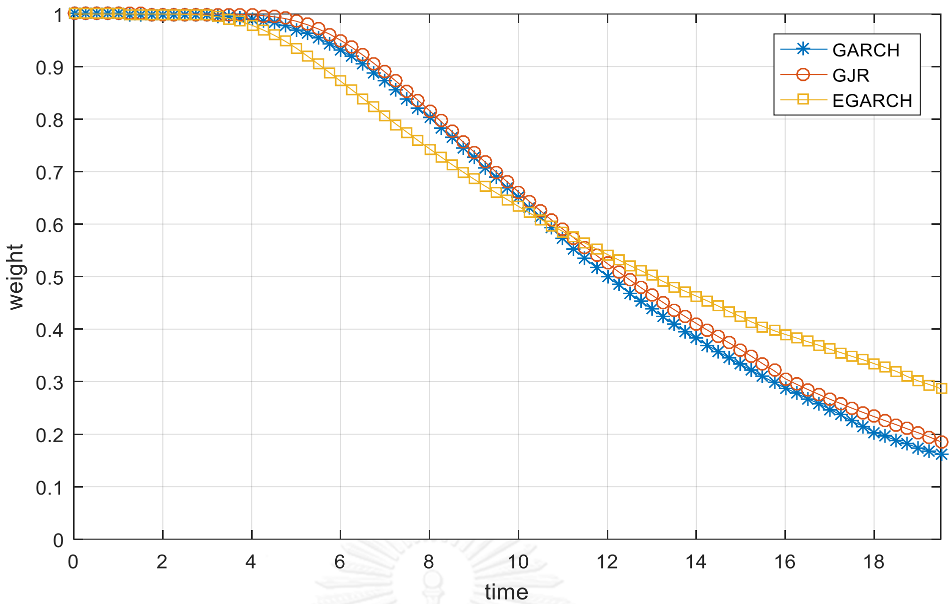


Figure 7.2. The average simulated weight in the risky asset of the GARCH, GJR, and EGARCH models.

We start simulating from year 0 to year 20 so the last period invested is one period before the time horizon, year 19.75. From Figure 7.1, the weight in risky asset is concentrated at near 0 and 1. When the weight is close to 1 for the risky asset, the fund value is lower than fund target. For example, at year 5 and 10, most of the simulated weights are invested closer to 1. Conversely, when the weight is close to 0, the fund value is close to the fund target. The closer the weight is to 0, the smaller the gap between the fund value and the fund target. This means that if the probability is high at a weight close to 0 at the terminal period, the model likely will perform well.

At early periods, every model puts all weight in the risky asset. After that, the optimal weight is decreasing to 0.8-1.0 in the risky asset at approximately year 3 for the EGARCH model, and year 4 for the GARCH and GJR models. The average of simulated weight continues to lower as time passes by. At year 5, simulated weight invests all weight in the risky asset with probabilities of 0.9, 0.85 and 0.65 for the GARCH, GJR and EGARCH models, respectively. This implies that the EGARCH model implies more conservative investment strategy at earlier periods than other models. As seen in Figure 7.2, the average weight of the EGARCH model decays slower than other models, causing the EGARCH model to put more aggressive strategy than the GARCH and GJR models in later periods. Moreover, the GARCH and GJR models have very similar optimal allocation weight, on average.

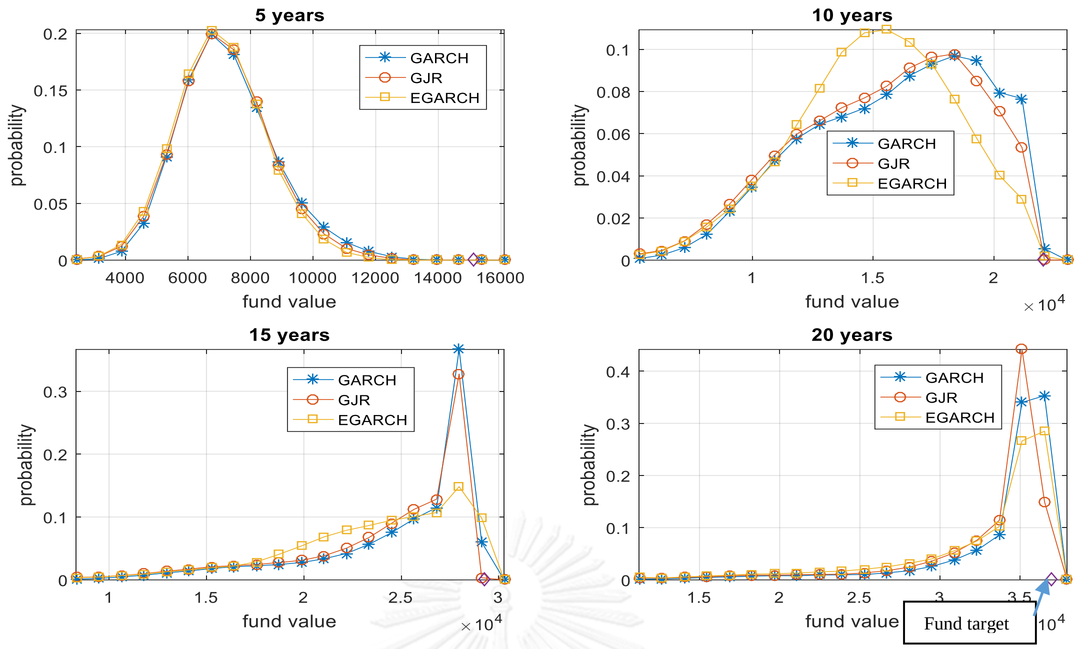


Figure 8.1. The distribution of simulated fund value of the GARCH, GJR and EGARCH models at several periods.

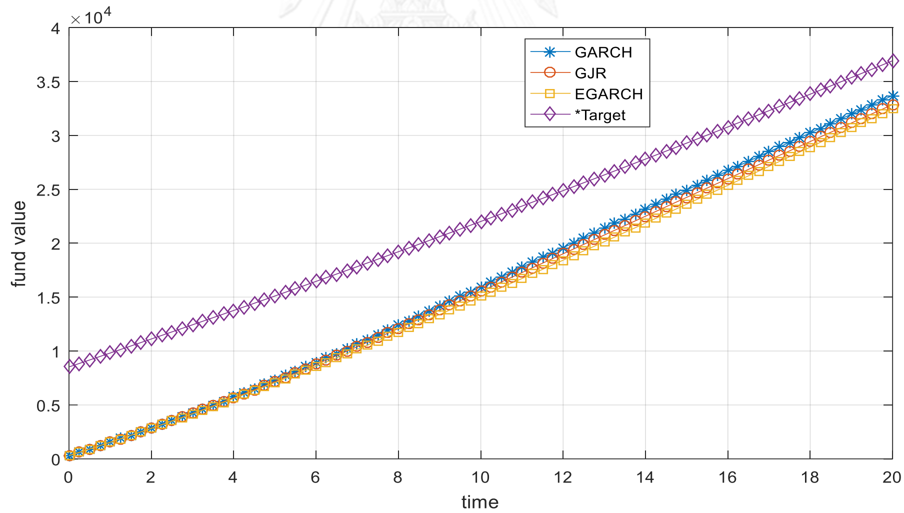


Figure 8.2. The average fund value in the risky asset of the GARCH, GJR, and EGARCH models. Target terminal wealth, discounted with risk-free and contribution rate, is calculated from equation 7.

The distribution of simulated fund value and the average fund value of the GARCH, GJR, EGARCH models are shown in Figure 8.1 and Figure 8.2 respectively. In Figure 8.2, the GARCH model resulted in the highest mean of simulated fund value, followed by the GJR and EGARCH models. Because the expected return of the risky asset is positive, on average, investing with more weight on risky asset leads to higher average return and consequently higher average fund value.

Finally, we look at the average return of simulated terminal fund value. The average return (r_{avg}) is solved from the equation, $c\sum_{t=0}^{T}\left(1+r_{avg}\right)^{t}=F_T$, where c is the contribution, and F_T is terminal fund value. Thus the fund value will reach the target if the average return is more than the fund target (4%).

Table 2. Probabilistic distribution of average return per year of simulated terminal fund value, a total of 100,000 paths from each model, given fund target equal to 4% per year. The number in each cell shows the probability that the average return of simulated terminal fund value falls within a given range.

Model	<2.5%	2.5-3%	3-3.5%	3.5-4%	>4%
GARCH	0.1494	0.0633	0.1192	0.6620	0.0061
GJR	0.1987	0.0858	0.1598	0.5556	0.0001
EGARCH	0.2505	0.0859	0.1407	0.4861	0.0368

From Table 2, the highest density of average return of simulated fund value is at range 3.5%-4%. The EGARCH model has the highest probability of reaching the target (when average return >4%). Most of the simulated fund values do not reach the fund target because the optimal strategy tries to protect wealth at the later periods. For example, at year 15, some paths of fund value nearly reach the target and hence lower weight in the risky asset, as seen in the distribution of simulated weight.

5.3 Backtesting the Strategy

In this section, the optimal strategies are compared to each other under the realized historical data. Time series of 55 years of historical return is used. The historical return of S&P500 from the year 1961 to 2015 is collected as a sample. The first 35 years of time series is used to estimate the parameters of GARCH family models. The optimal strategy is then calculated. Afterward, the latest 20 years of time series is used for backtesting. Fund portfolios are constructed assuming the GARCH, GJR, and EGARCH models. Subsequently, the portfolios are adjusted in each period using the calculated optimal strategies. The parameters of GARCH family models are re-estimated every 5 years by using a rolling window of historical returns of 35 years and the optimal allocation is also re-calculated. The estimated parameters are provided in Table 4.1 - Table 4.4 in Appendix, each table for every 5-year re-estimation in the given backtesting period. Then, the performance of funds assuming different models are compared.

There are two benchmarks to compare with volatility clustering models. The first one is constant volatility model. The optimal allocation of this model is calculated similarly to volatility clustering models with only one state variable, the fund value. The re-estimation of constant volatility is also done at every 5 years. The second benchmark model is the buy-and-hold strategy, which represents the market index. This strategy will put all the weight in the risky asset in the first period and keep all weight on the risky asset when the contribution is added in each period until reaching the terminal period. In the backtesting period, fund target is set to 4% and the risk-free rate is set to 1%.

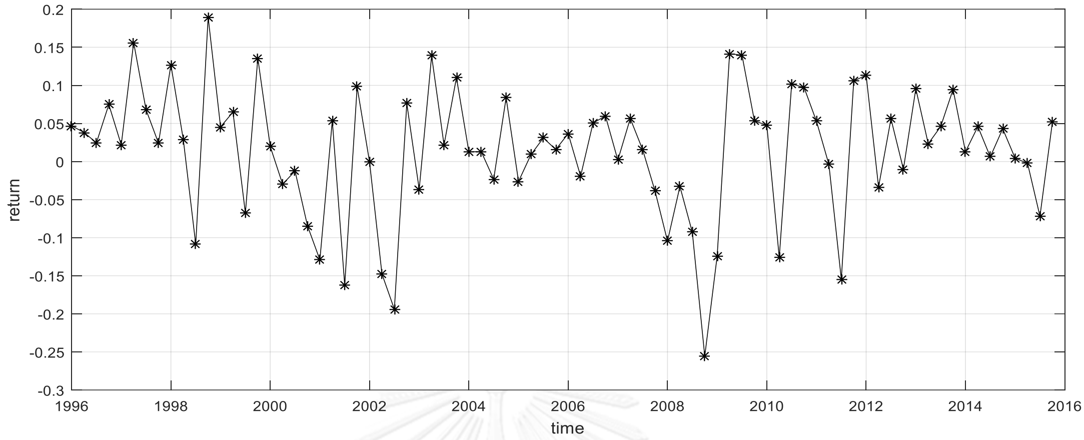


Figure 9.1. 3-month return of S&P500 between years 1996-2015.

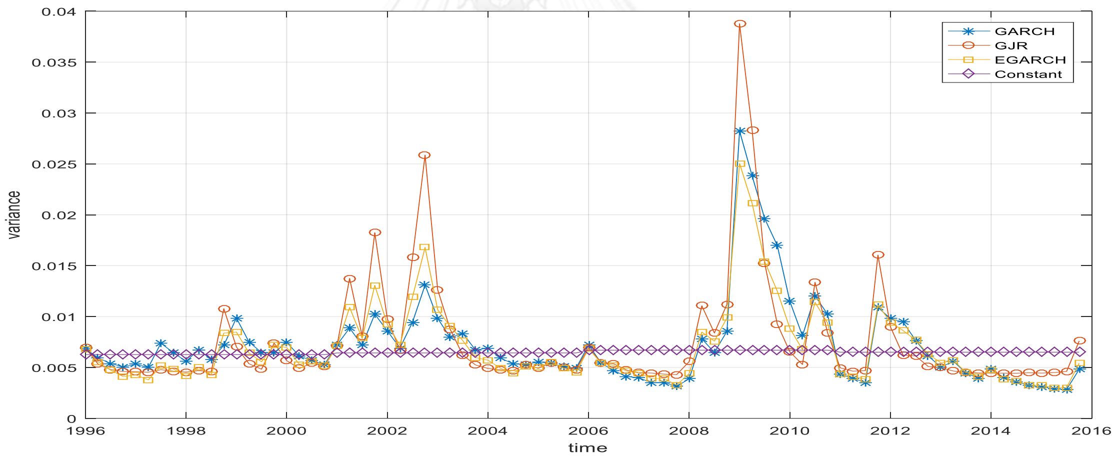


Figure 9.2. The variance of the GARCH, GJR, EGARCH and constant volatility models of S&P500 between years 1996-2015.

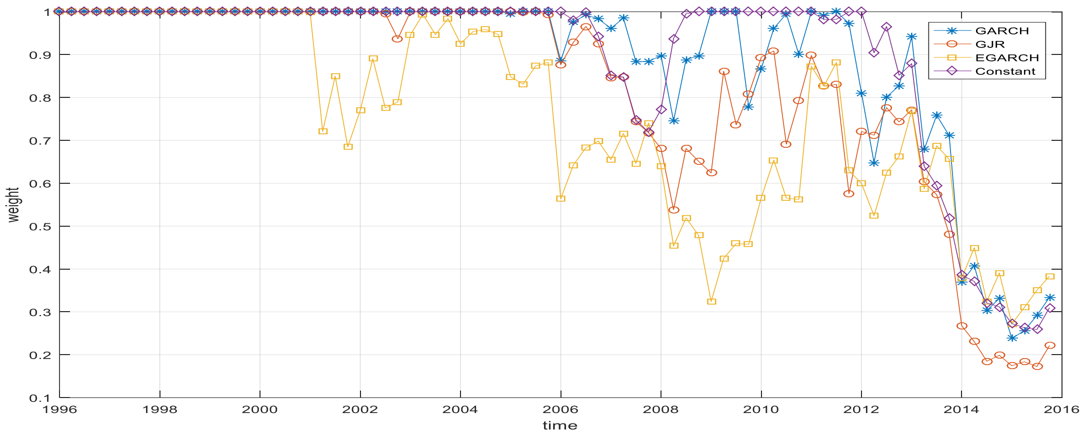


Figure 9.3. Backtesting weight invested in S&P500 of the GARCH, GJR, EGARCH and benchmark models between years 1996-2015.

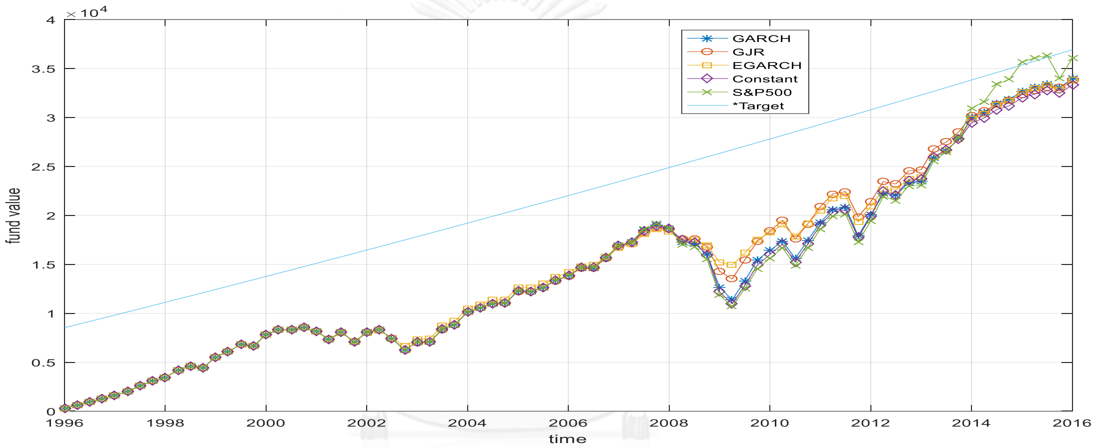


Figure 9.4. Backtesting fund value of the GARCH, GJR, EGARCH, benchmark models and fund target between years 1996-2015.

Note. *Target fund value threshold $F_{dc,t}$, discounted with the risk-free and contribution rates, is calculated from equation 7.

The backtest result is shown in Figure 9.1-9.4. In the first 5 years of backtesting, 1996-2000, strategies of the GARCH, GJR, and EGARCH models put the optimal weight in the risky asset close to 1, very much like the constant volatility model. This shows that all the volatility clustering models invest by ignoring what the level of variance is in the first 5 years. In the next 5 years, 2001-2005, the optimal strategy of the GARCH and GJR models still put weight close to 1 in the risky asset, while the EGARCH model uses a more conservative strategy than other models on the risky asset. In years 2006-2010, when financial crisis occurred in 2008, all volatility clustering models reduce the weight on the risky asset dramatically. The EGARCH model is the model with the least weight on the risky asset, followed by the GJR and GARCH models. Meanwhile, the constant volatility model still puts weight close to 1 during the financial crisis. In years 2011-2015, the optimal strategies for all models including constant volatility model reduce weights on risky asset dramatically to protect the wealth in the later periods. However, the return of S&P500 became more positive causing the buy-and-hold strategy to outperform all the optimal strategies in other models.

Table 3. Average return per year of backtest fund.

Value	Model			Benchmark	
	GARCH	GJR	EGARCH	Constant	S&P500
Return	3.31%	3.26%	3.27%	3.14%	3.86%
SD	12.83%	12.36%	11.88%	13.03%	13.13%
Sharpe ratio	0.1800	0.1828	0.1911	0.1642	0.2178

The average returns of backtested funds are shown in Table 3. The strategy with the highest average return is the buy-and-hold strategy. The average returns from strategies implied by volatility clustering models are lower than the buy-and-hold strategy, but still outperform the strategy from the constant volatility model. Overall, all the strategies of GARCH family models result in similar returns with each other. However, the EGARCH model yields returns with the lowest standard deviation. This will be beneficial in the presence of financial crises since the optimal strategy implied by the EGARCH model is more conservative. The standard deviation implied by the EGARCH model is the lowest, followed by the GJR, GARCH, constant volatility models and the buy-and-hold strategy. The buy-and-hold strategy also gives the best Sharpe ratio, the return to risk ratio. However, all strategies in volatility clustering models still yield better Sharpe ratios than the constant volatility model. Comparing volatility clustering models with the constant volatility model, the optimal strategies implied by volatility clustering models yields higher average returns with lower standard deviation and higher Sharpe ratio in backtesting. As shown above, volatility clustering models outperform non-volatility clustering models. Hence, for pension fund management, this paper highly recommends taking volatility clustering in consideration.

6 CONCLUSION

This paper asks how the volatility clustering affects strategies in the pension fund management. Moreover, how the different volatility clustering models affect the optimal strategies. Lastly, we test which volatility clustering model give the best outcome in backtesting.

This research has investigated three different volatility clustering models, the GARCH, GJR and EGARCH models, for use in the management of defined benefit pensions. First, model parameters are estimated using a time series of S&P500 3-month returns. The surface of the optimal allocation and the value function are obtained by solving the Bellman equation. The solutions of the optimal allocation depend on the variance dynamic of each model. Findings are that at lower fund value and lower variance of returns, optimal allocations have a higher weight in the risky asset. Moreover, in comparison to a model with no volatility clustering effect, GARCH family models have more conservative strategies. In volatility clustering models, volatility-clustering parameters, α_1, adjusting volatility in the past period and β_1 capturing the amplitude of uncertainties, have the impact on the decision rule in choosing optimal allocations. Meanwhile, the term ω_1 in the GJR and EGARCH models allows asymmetric volatility and highlights high uncertainties given negative returns, resulting in a more conservative strategy.

After the optimal strategies are found, Monte Carlo simulation is performed to find the distribution of optimal weights and simulated fund performance implied by optimal strategies from different models. The result shows that the EGARCH model yields the most reasonable strategy and, unlike the GARCH and GJR models, rarely gives extreme weight on the risky asset.

Lastly, in backtesting strategy, the strategies implied by three models are tested on historical returns data. All the strategies provide similar returns of S&P500 between years 1996-2015. However, the EGARCH model yields the most conservative strategy, which is beneficial if a financial crisis occurs. In comparison to the constant volatility model, the volatility clustering models result in fund management strategies that outperform the non-volatility clustering model, with higher fund value and lower standard deviation in the terminal period.

In the financial market, portfolio managers should take into account the effect of volatility clustering, using a model that can best capture the clustering effects in the actual market. Our research supports that the EGARCH model has the best performance as the evident from backtesting. Even though the strategy of the EGARCH model is more conservative than other models, the fund performance implied by the EGARCH model is similar to other volatility clustering models and would outperform other models during a financial crisis.

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APPENDIX Estimated Parameters

Table 4. Estimated parameters of the GARCH, GJR, and EGARCH models at specific periods.

Table 4.1. Year 1996-2000

Parameters		GARCH	GJR	EGARCH
α_0	value	0.002208	0.002987	-1.4468
	SE	0.001830	0.001464	1.373
α_1	value	0.5248	0.3367	0.7163
	SE	0.3442	0.3221	0.2683
β_1	value	0.1272	0.01317	0.3104
	SE	0.1009	0.1677	0.2335
ω	value	-	0.4024	-0.1293
	SE	-	0.1967	0.06781
μ	value	0.01461	0.01385	0.01291
	SE	0.007917	0.007765	0.007719

Table 4.2. Year 2001-2005

Parameters		GARCH	GJR	EGARCH
α_0	value	0.002042	0.002918	-1.603
	SE	0.001843	0.001646	1.4800
α_1	value	0.5613	0.3755	0.6829
	SE	0.3332	0.3366	0.2910
β_1	value	0.1275	-	0.2463
	SE	0.0987	-	0.2129
ω	value	-	0.3843	-0.1389
	SE	-	0.2039	0.07679
μ	value	0.01852	0.01619	0.01527
	SE	0.007482	0.007320	0.007456

Table 4.3. Year 2006-2010

Parameters		GARCH	GJR	EGARCH
α_0	value	0.001040	0.002319	-1.098
	SE	0.0007265	0.001232	0.8376
α_1	value	0.5947	0.4550	0.7811
	SE	0.1675	0.2502	0.1653
β_1	value	0.2930	-	0.3813
	SE	0.1243	-	0.2247
ω	value	-	0.4220	-0.1061
	SE	-	0.2066	0.07322
μ	value	0.01893	0.01658	0.01381
	SE	0.006937	0.007197	0.007375

Table 4.4. Year 2011-2015

Parameters		GARCH	GJR	EGARCH
α_0	value	0.0008618	0.002699	-1.033
	SE	0.0006043	0.001351	0.7636
α_1	value	0.6430	0.3904	0.7951
	SE	0.1530	0.2542	0.1483
β_1	value	0.2576	-	0.3662
	SE	0.1128	-	0.2121
ω	value	-	0.3881	-0.09203
	SE	-	0.2261	0.07556
μ	value	0.01892	0.01743	0.01442
	SE	0.006997	0.007121	0.007210

VITA

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