CHAPTER V

COMPUTATION OF ALL 2D FRICTIONAL FORCE CLOSURE GRASPS

5.1 Introduction

This chapter proposes an algorithm for computing all force closure grasps of three frictional contacts. Given a set of n frictional contact points, the proposed algorithm reports all force closure grasps in $O(n^2 \lg^2 n + K)$ where K is the number of force closure grasps. The key idea of the method is analogous to the frictionless case given in Chapter 4. The method enumerates every contact point. For each contact point, the wrench fan of which is regarded as an anchor fan and all pairs of other fans that form force closure grasp is reported. The following pseudocode gives the overview of our algorithm.

Algorithm 2 Algorithm for Computing All Frictional Closure Grasp

Require: $F_1, \ldots, F_n$: n wrenches fan,
Ensure: R: all triples of three wrenches fans that positively span the space
1: $R \Leftarrow \emptyset$
2: $W \Leftarrow F_1, \ldots, F_n$
3: **for** $i = 1$ to n **do**
4: $\quad R \Leftarrow R \cup \text{FINDFG}(F_i, W)$
5: **end for**

The routine FINDFG is a frictional variation of FINDINTS. A noticeable difference between between this algorithm and Algorithm 1 is that FINDFG always takes as input all wrench fans while FINDINTS takes only the fans that have not been enumerated. This is because the underlying condition in Algorithm 1, which is entirely implemented in FINDINTS, is a necessary and sufficient condition that is guaranteed to report all force closure grasp with respect to the given anchor fan. On the contrary, the underlying condition used in Algorithm 2 composes of several sub-conditions. Each of them is a sufficient condition but the disjunction of all sub-condition is a necessity of force closure. These sub-condition are described with respect to an anchor fan. However, there are some sub-conditions that, when they are satisfied with respect to one anchor fan, they imply other sub-conditions with respect to another anchor fan. By acknowledging this fact, FINDFG implements only some of the sub-conditions that are essentially needed, i.e., the conditions that are implied by other "reducible sub-conditions". The algorithm relies on the fact that all wrench fans must be included in FINDFG, so that grasps that are to be detected by unimplemented sub-conditions have the chance to be detected in the later iteration of FINDFG.

In this chapter, we first introduce basic condition for frictional three finger force closure grasp in Section 5.2. The condition used in the algorithm is presented in Section Section 5.3. Section 5.4 discusses the implementation of FINDFG. Numerical Example and empirical comparison are given in Section 5.5. Finally, Section 5.6 summarizes this chapter.

5.2 Frictional Force Closure Grasp

This section provides basic conditions of force closure grasp. First, let us consider the property of a frictional contact wrench set. The next lemma indicates that, it is not possible for any two fans to lie on the same plane.

Lemma 5.1 *Let F and F' be contact fans of two distinct contact points. Fans F and F' do not lie on the same plane.*

Proof: Let F be an arbitrary contact fan defined by boundary wrenches $\boldsymbol{w}_l = (x_l, y_l, z_l)$ and $\boldsymbol{w}_r = (x_r, y_r, z_r)$. Let $\boldsymbol{p} = (p_x, p_y)$ be the position of the contact point associated with the fan F. It is obvious from how a torque is calculated that $z_l = p_x y_l - p_y x_l$ and $z_r = p_x y_r - p_y x_r$. This allows us to solve for the coordinates of the contact point, $\boldsymbol{p}$, as follows:

$$p_x = \frac{x_r z_l - x_l z_r}{x_r y_l - x_l y_r} \quad (5.1)$$

$$p_y = \frac{y_r z_l - y_l z_r}{x_r y_l - x_l y_r} \quad (5.2)$$

We can rewrite (5.1) and (5.2) as $p_x = n_y/n_z$ and $p_y = n_x/n_z$ where $\boldsymbol{n} = (n_x, n_y, n_z) = \boldsymbol{w}_r \times \boldsymbol{w}_l$ is the cross product of the two boundary wrenches of F. For another fan F' to lie on the same plane as F, the cross product of the boundary wrenches of F' must be parallel to $\boldsymbol{n}$. Given how p_x and p_y are rewritten in terms of n_x, n_y and n_z, this means that the contact point of F' must be the same as that of F, which contradicts our assumption. ■

Next, we consider the key condition for frictional three finger force closure grasp.

Lemma 5.2 *Let F_1, F_2 and F_3 be three contact fans of different contact points. These fans achieve force closure if and only if the negative of one of these fans intersects with the interior of the positive span of the other two fans.*

Proof: To prove that the condition is sufficient, let $\boldsymbol{w}$ be a member of the intersection between the negative of one of the three fans and the interior of the positive span of the other two. Since $\boldsymbol{w}$ intersects with the interior of the positive span, we can always find three non-coplanar wrenches that are contained in the positive span such that $\boldsymbol{w}$ lies strictly inside the cone formed by these three wrenches. By Proposition 2.9, these three fans achieve force closure.

Let $\{i, j, k\}$ be a permutation of $\{1, 2, 3\}$. To prove that the condition is necessary, it suffices to show that force closure cannot be achieved when $-F_i$ does not intersect the interior of $\mathcal{W}$ where $\mathcal{W} = \text{SPAN}^+(F_j \cup F_k)$. Geometrically, the positive span of two fans may form (a) a pyramid, (b) a plane, (c) a half space, or (d) the entire space. From Lemma 5.1, F_j and F_k cannot be on the same plane. Case (b) is therefore not possible. In case (c), when $-F_i$ does not intersect $\mathcal{W}$ (which is a half space), F_i must be contained in $\mathcal{W}$. This means that the three fans lie in the same half space, which inhibits force closure. In case (d), since it is not possible for $-F_i$ to not intersect with $\mathcal{W}$, this case can be safely neglected. The remaining case is when $\mathcal{W}$ forms a pyramid. A pyramid can be expressed by the intersection of several bounding half spaces $\mathcal{H}_1, \ldots, \mathcal{H}_n$. We assume that there is no wrench in $-F_i$ that intersects with the interior of $\mathcal{W}$. Therefore, for each $\boldsymbol{w} \in F_i$, there exists some $\mathcal{H}_m$ that does not contain $-\boldsymbol{w}$. This means that wrench $\boldsymbol{w}$ and fans F_j and F_k must lie in the same half space $\mathcal{H}_m$ which, in turn, inhibits force closure. ∎

This lemma is a frictional counterpart of Proposition 2.9. It considers the intersection between a negative of one fan and a positive span of the other two. We can derive immediately the following corollary, which is analogous to Corollary 2.10 by alternately considering the intersection between a fan and the negative span of two other fans.

Corollary 5.3 *Let F_1, F_2 and F_3 be three contact fans of different contact points. These fans achieve force closure if and only if any fan intersects with the interior of the negative span of the other two fans.*

Lemma 5.1 also limits the shape of $\text{SPAN}^+(F_j \cup F_k)$. Since no two fans can lie on the same plane, it is not possible for a positive span of two fans to take the form of a plane, a half plan nor a fan (see Figure 2.3). Moreover, since friction is present, the case of a ray and a line are also impossible.

5.3 Identifying All Force Closure Grasp

In this section, we describe the detail of Algorithm 2 and the condition used therein. The essential idea is the same as Algorithm 1. The algorithm enumerates all fan as an anchor fan. For each anchor fan, FINDFG routine reports every pair of fans that satisfies the condition. The conditions used in FINDFG are designed such that wrench fans satisfying the conditions can be efficiently identified in force dual representation using FINDINTS or its variation.

Let us first introduce condition for a three finger force closure grasp and its decomposition into several sub-conditions. Let us assume that we are considering three wrench fans $F_1, \ldots, F_3$. Let F_1 be regarded an anchor fan. Lemma 5.2 indicates that these fans achieve force closure when $-F_1$ intersects $\text{INT}(\mathcal{W})$ where $\mathcal{W} = \text{SPAN}^+(F_2 \cup F_3)$. The decomposition of the condition is done according to the geometric property of the polyhedral convex cone $\mathcal{W}$.

Let us consider the three sets: $\text{INT}(\mathcal{W})$, $\mathcal{W}$ and $\mathbb{R}^3$. Note that $\text{INT}(\mathcal{W}) \subset \mathcal{W} \subset \mathbb{R}^3$. When the $-F_1$ intersects $\text{INT}(\mathcal{W})$, we can identify the smallest set that is a superset of $-F_1$. By this classification, there are three possibilities: (a) $-F_1$ lies entirely inside $\text{INT}(\mathcal{W})$, (b) $-F_1$ lies entirely inside $\mathcal{W}$, i.e., the some wrench being the boundary of $-F_1$ lies in the boundary of $\mathcal{W}$, and (c) Some part of $-F_1$ lies outside $\mathcal{W}$, i.e., $-F_1 \backslash \mathcal{W} \neq \emptyset$. These cases are illustrated in Figure 5.1. The following lemma formalizes the condition without proof.

Lemma 5.4 *Let F_1, F_2 and F_3 be three contact fans of different contact points. Let $\mathcal{W} = \text{SPAN}^+(F_2 \cup F_3)$ These fans achieve force closure if and only if at least one of the following conditions is satisfied.*

(a) $-F_1 \subseteq \text{INT}(\mathcal{W})$

(b) $-F_1 \not\subseteq \text{INT}(\mathcal{W})$ *and* $-F_1 \subset \mathcal{W}$

(c) $-F_1 \not\subseteq \mathcal{W}$ *and* $-F_1 \cap \mathcal{W} \neq \emptyset$

Each case in Lemma 5.4 is a sufficient condition for force closure. For each case, we derive a new condition which is a necessity of the original condition of that case. The new condition is

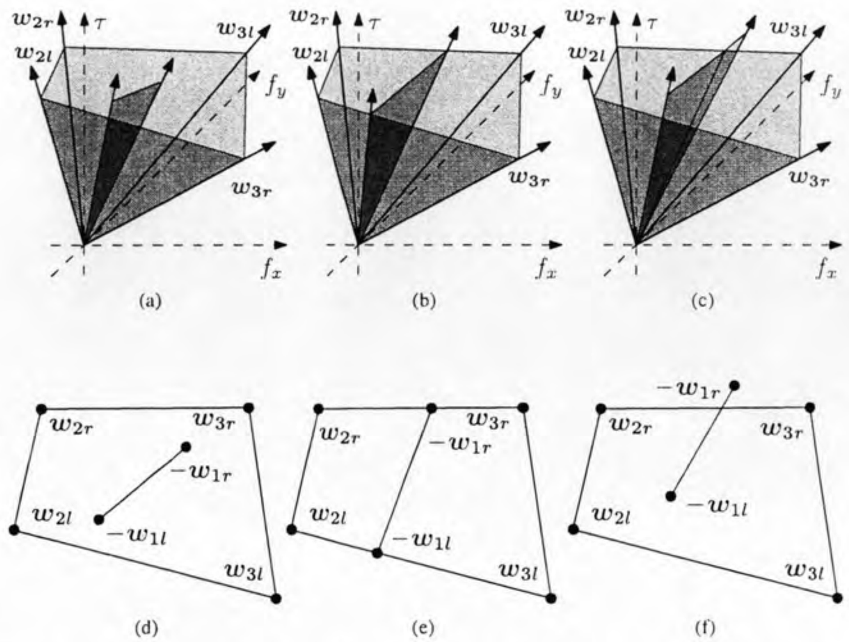

Figure 5.1: The intersection cases. (a)–(c) Case (a), Case (b) and Case (c) as observed in the wrench space. (d)–(e) Case (a), Case (b) and Case (c) as observed in the force dual representation.

also sufficient for force closure. Hence, all force closure grasps can be equivalently identified from these new conditions. These new conditions are derived to be easily verifiable in the force dual representation. The implementation of FINDFG uses these new conditions to efficiently report all force closure grasp.

Before considering each case of Lemma 5.4, we assume the following definitions. Let F_1 be regarded as an anchor fan. Let the boundary wrenches of fan F_i be $\boldsymbol{w}_{il}$ and $\boldsymbol{w}_{ir}$ where $i \in \{1, \ldots, 3\}$. Let B denotes the set of boundary wrenches of F_2 and F_3, i.e., $B = \{\boldsymbol{w}_{2l}, \boldsymbol{w}_{2r}, \boldsymbol{w}_{3l}, \boldsymbol{w}_{3r}\}$. Hence $\mathcal{W} = \text{SPAN}^+(B)$. The facets of $\mathcal{W}$ are fans, each of which is bounded by two wrenches that are the members of B. Hence, the fans that are the boundary of $\mathcal{W}$ are therefore a member of the set G of all fans that are bounded by two different wrenches chosen from B. Obviously, G contains $C_{4,2} = 6$ fans. Let $G = \{G_1, G_2, ..., G_6\}$ and let us refer to each fan in G as a *candidate fan*. Some candidate fans are actually facets of $\mathcal{W}$ while some are not. For example, the fan $\text{SPAN}^+(\boldsymbol{w}_{2r}, \boldsymbol{w}_{3l})$ and $\text{SPAN}^+(\boldsymbol{w}_{2l}, \boldsymbol{w}_{3r})$ in Figure 5.2 are candidate fans that are not facets of $\mathcal{W}$. A candidate fan that is the actual facet of $\mathcal{W}$ is referred as *a boundary candidate fan*.

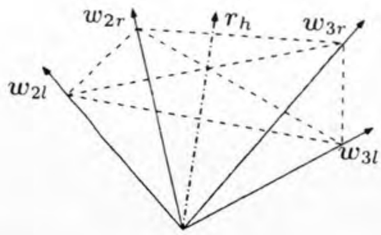

Figure 5.2: Case (c) and an exceptional ray. The candidate fans represented by dashed lines. The ray r_h is not contained in any positive span of three wrenches from B

5.3.1 Case (a) $-F_1 \subseteq \text{INT}(\mathcal{W})$

When $-F_1$ lies inside $\text{INT}(\mathcal{W})$, both of its boundary wrenches also lie inside $\text{INT}(\mathcal{W})$. Hence, it is equivalent to check whether $-\boldsymbol{w}_{1l}$ (or $-\boldsymbol{w}_{1r}$) lies inside $\text{INT}(\text{SPAN}^+(B)$. However, we consider instead $\text{SPAN}^+(\{\boldsymbol{w}_p, \boldsymbol{w}_q, \boldsymbol{w}_r\})$ where $\{\boldsymbol{w}_p, \boldsymbol{w}_q, \boldsymbol{w}_r\}$ is a combination of members of B. For each combination, we tested whether $-\boldsymbol{w}_{1l}$ (or $-\boldsymbol{w}_{1r}$) intersects $\text{INT}(\text{SPAN}^+(\{\boldsymbol{w}_p, \boldsymbol{w}_q, \boldsymbol{w}_r\}))$. According to Proposition 2.9, it is equivalent to test whether $\{\boldsymbol{w}_{1l}, \boldsymbol{w}_p, \boldsymbol{w}_q, \boldsymbol{w}_r\}$ (or $\{\boldsymbol{w}_{1r}, \boldsymbol{w}_p, \boldsymbol{w}_q, \boldsymbol{w}_r\}$) positively span $\mathbb{R}^3$.

Interestingly, the union of all $\text{INT}(\text{SPAN}^+(\{\boldsymbol{w}_p, \boldsymbol{w}_q, \boldsymbol{w}_r\}))$ is not equal to $\text{INT}(\mathcal{W})$. For example, let us consider a common arrangement of the boundary wrenches as shown in Figure 5.2. The intersection between $\text{SPAN}^+(\boldsymbol{w}_{2r}, \boldsymbol{w}_{3l})$ and $\text{SPAN}^+(\boldsymbol{w}_{2l}, \boldsymbol{w}_{3r})$ lies not in the interior of any combination. Nonetheless, this particular case causes no problem. This is because the intersection can only be a ray which is the intersection of non-boundary candidate fans. Let us refer to this ray as *the exceptional ray* $\boldsymbol{r}_h$. It is not possible for both $-\boldsymbol{w}_{1l}$ and $-\boldsymbol{w}_{1r}$ to be on the same ray. Therefore, at least one of them must be inside one of the combinations.

The other exception cases are when $\mathcal{W}$ is a half space or a wedge. In the half space case, $\text{SPAN}^+(\{\boldsymbol{w}_p, \boldsymbol{w}_q, \boldsymbol{w}_r\})$ does not include any candidate fan that lies inside the half space. In the wedge case, the candidate fan bounded by the members of B that are not on the same line is not included in any $\text{SPAN}^+(\{\boldsymbol{w}_p, \boldsymbol{w}_q, \boldsymbol{w}_r\})$. These cases will be considered afterward.

With the exception of the wedge case and the half space case, the new condition for Case (a) is formalized in Lemma 5.5. The proof is obvious from the text.

Lemma 5.5 *A necessary condition for $F_1, \ldots, F_3$ to satisfy Case (a) in Lemma 5.4, when $\mathcal{W}$ is not a wedge and is not a half space, is that $\{\boldsymbol{w}_{1l}, \boldsymbol{w}_p, \boldsymbol{w}_q, \boldsymbol{w}_r\}$ or $\{\boldsymbol{w}_{1r}, \boldsymbol{w}_p, \boldsymbol{w}_q, \boldsymbol{w}_r\}$ positively span $\mathbb{R}^3$ where $\{\boldsymbol{w}_p, \boldsymbol{w}_q, \boldsymbol{w}_r\}$ is a subset of $\{\boldsymbol{w}_{2l}, \boldsymbol{w}_{2r}, \boldsymbol{w}_{3l}, \boldsymbol{w}_{3r}\}$.*

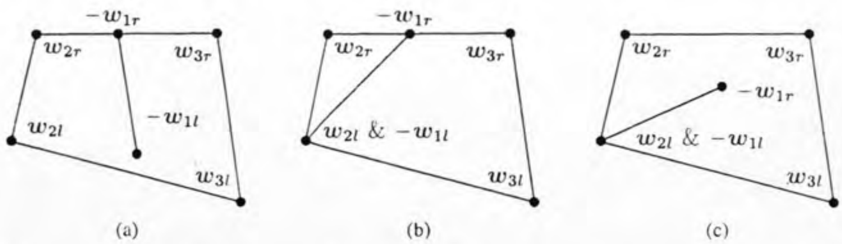

Figure 5.3: Subcases of case (b) represented in force dual representation. (a) Subcase (b1). (b) Subcase (b2). (c) Wrenches that can be categorized as both subcases.

5.3.2 Case (b) $-F_1 \not\subseteq \text{INT}(\mathcal{W})$ and $-F_1 \subset \mathcal{W}$

The wrench fans are categorized as Case (b) when at least one boundary wrench of $-F_1$ lies on the boundary of $\mathcal{W}$. According to the notion of candidate fans, we further categorize Case (b) into two subcases: Subcase (b1) and Subcase (b2). In Subcase (b1), a boundary wrench of $-F_1$ lies on $\text{RI}(G_i)$ where G_i is a boundary candidate fan. In Subcase (b2), a boundary wrench of $-F_1$ coincides with the member of B. Examples of these cases is represented in Figure 5.3. Notice that some arrangement can be categorized as both cases.

For both subcases, let us assume without lost of generality that only $-w_{1l}$ lies on a boundary candidate fan G_1. For brevity, let $w_a \in B$ and $w_b \in B$ be the boundary wrenches of G_1 and let the other two wrenches in B be w_c and w_d. Let v be the vector being perpendicular to the plane containing G_1 and pointing to the direction that makes $v \cdot w_{1r} > 0$, i.e, v lying on the same side as w_{1r}, with respect to the plane P_v containing G_1.

In Subcase (b1), $-w_{1l}$ also lies inside $\text{RI}(G_1)$. This implies that G_1 and $-w_{1l}$ positively span P_v and the span $\text{SPAN}^+(F_1 \cup G_1)$ is exactly the half space $\mathcal{H}_v$. For any wrench lying outside $\mathcal{H}_v$, the negative of that wrench intersects $\mathcal{H}_v$ and they achieve force closure. Hence, the condition for Subcase (b1) is to identify the half space $\mathcal{H}_v$ which is formed by four wrenches and then test whether w_c or w_d lies outside $\mathcal{H}_v$. Let us formalize the condition for this case as follows.

Lemma 5.6 *A necessary condition for $F_1, \ldots, F_3$ to satisfy Case (b1) is that w_c or w_d lies outside the half space $\mathcal{H}_v$.*

In Subcase (b2), we can make further assumption that G_1 is an actual contact fan F_2,

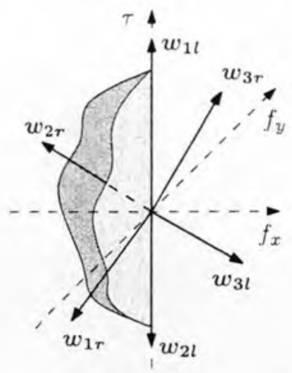

Figure 5.4: The case when both boundary wrenches of $-F_3$ coincide the boundary wrenches of F_1 and F_2. Notice that w_{3l} (resp. w_{3r}) lies on the same line as w_{2r} (resp. w_{1r}).

because w_a must be a common boundary of two boundary candidate fans and one of them must be an actual contact fan. Hence, we could relabel such that $G_1 = F_2$, $w_a = w_{2l}$ and $w_b = w_{2r}$. Let us assume that $-w_{1l}$ coincides with w_{2l}. Obviously, SPAN$^+(\{w_{1l}, w_{2l}\})$ is a line and SPAN$^+(F_1 \cup F_2)$ is a wedge (see Figure 5.6). This wedge is bounded by two half spaces; one is bounded by the plane through F_1 and the other one is bounded the plane through G_a, i.e., the half space $\mathcal{H}_{\boldsymbol{v}}$. Let the other half space be denoted by $\mathcal{H}_{\boldsymbol{u}}$. $\mathcal{H}_{\boldsymbol{u}}$ is bounded by the plane containing F_1. By Lemma 5.2, these wrenches achieve force closure when $-F_3$ intersects the interior of the wedge. This means that, when Algorithm 2 iterates until F_3 is regarded as an anchor fan, the force closure grasp of $F_1, \ldots, F_3$ will also be detected. The only exception is when $-F_3$ itself also intersects SPAN$^+(F_1 \cup F_2)$ only as Case (b2). This case can only happen when both boundary wrenches of F_3 coincide the boundary wrenches of F_1 and F_2 (see Figure 5.4). In this case, the boundary wrenches of $-F_3$ is exactly w_{2r}. The next lemma formalized the condition of this case.

Lemma 5.7 *A necessary condition for $F_1, \ldots, F_3$ to satisfy Lemma 5.4 as Subcase (b2) is one of the followings conditions.*

(i) With F_3 takes the role of F_1 in Lemma 5.4, $-F_3$ satisfies Case (a) or Subcase (b1) or Subcase (c)

(ii) With F_3 takes the role of F_1 in Lemma 5.4, the wrenches w_b and w_{1r} coincide with the boundary wrenches of $-F_3$ and w_{1l} and w_a lies antipodally.

Now, it is an appropriate time to consider the remaining exceptional cases of Case (a). The first case is when $-F_1$ lies in the interior of the half space formed by F_2 and F_3. Since F_2 and F_3 forms a half space, it is either that one boundary wrench of $-F_2$ lies in F_3, or vice versa. This implies that, when F_2 takes the roles of an anchor fan, it will be categorized as Case (b1) and the force closure will be reported in this case instead. In other words, *the exceptional half space situation of Case (a) is handled, as Subcase (b1), when the other fan takes the role of an anchor fan.* Similarly, when $-F_1$ lies in the interior of the wedge formed by F_2 and F_3, the fan F_2 (or F_3, when considered as an anchor fan, will be categorized as Case (b2), and force closure will be detected in this case instead.

However, the wedge case is recursively dependent. This is because Lemma 5.7 indicates that Case (b2) also depends on the other cases. If it is the exceptional case of Case (a) that is relied upon, force closure will never be detected. To solve this particular issue, we modify Lemma 5.7 as follows.

Lemma 5.8 *A necessary condition for $F_1, \ldots, F_3$ to satisfy Case (b) as subcase (b2) in Lemma 5.4 is one of the followings conditions.*

(i) With F_3 takes the role of F_1 in Lemma 5.4, $-F_3$ satisfies Lemma 5.5, Case (b1) or Case (c)

(ii) The wrench w_c or w_d lies in the interior of the wedge $\text{SPAN}^+(F_1 \cup \{w_a, w_b\})$.

(iii) With F_3 takes the role of F_1 in Lemma 5.4, the wrenches w_b and w_{1r} coincide with the boundary wrenches of $-F_3$ and w_{1l} and w_a lies antipodally.

5.3.3 Case (c) $-F_1 \not\subseteq \mathcal{W}$ and $-F_1 \cap \mathcal{W} \neq \emptyset$

In Case (c), it is obvious that there exists an intersection, as a ray, between $-F_1$ and the boundary of $\mathcal{W}$. This implies that there also exist an intersection, also as a ray, between $\text{RI}(G_i)$ and $\text{RI}(-F_1)$. The only exception is when the intersection of $-F_1$ and $\mathcal{W}$ is exactly on the boundary wrench of the candidate fans, i.e., on the member of B. This case will be rectified shortly afterward.

It is also possible that $\text{RI}(-F_1)$ also intersects another candidate fan $\text{RI}(G_j)$ where G_j is not a boundary candidate fan. In this case, it is obvious that $\text{RI}(G_j)$ is inside $\text{RI}(\mathcal{W})$ and the intersection of which also implies force closure. Therefore, the intersection between $\text{RI}(G_i)$ and $\text{RI}(-F_1)$ is a sufficient condition of force closure. It should be noted that the condition is equivalent to saying that F_1 and G_i positively span the wrench space. From Proposition 2.9,

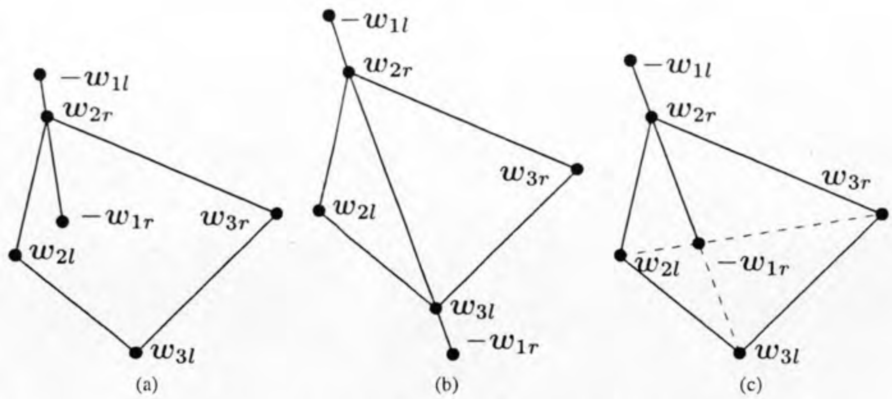

Figure 5.5: The special case of case (c) in force dual represented. $-F_1$ intersects the boundary of $\mathcal{W}$ at the boundary wrench. (a) One wrench lies in $\Omega(\mathcal{W})$. (b) No wrench lies in $\Omega(\mathcal{W})$. (c) One wrench lies in exactly in r_h.

we can also implies that a negative of one boundary wrench of G_i lies inside the interior of the positive span of F_1 and the other boundary vectors of G_i. Let us assume that the negative wrench is the boundary wrench of F_2. Hence, when F_2 takes the role of an anchor fan, it is guaranteed that the wrenches also satisfy Case (a). This means that, with the aforementioned exception, Case (c) implies case (a). This condition is formalized in Lemma 5.9.

Lemma 5.9 *A necessary condition for $F_1, \ldots, F_3$ to satisfy Case (c) in Lemma 5.4 where F_1 does not intersect any boundary wrench of F_2 and F_3 is that when $F_1, \ldots, F_3$ satisfy Case (a) in Lemma 5.4 when F_2 or F_3 take the role of an anchor fan.*

Finally, let us consider the remaining special case of Case (c), where $\text{RI}(-F_1)$ intersects the boundary of $\mathcal{W}$ at its boundary wrench (see Figure 5.5a). In this case, we could find the boundary fan of $\mathcal{W}$ that is an actual fan. Let us assume that that fan is F_2 and it is w_{2r} that intersects $\text{RI}(-F_1)$. It is clear that $-w_{2r}$ intersects $\text{RI}(F_1)$. This implies that when F_2 takes the role of an anchor fan, force closure will be detected by Case (b1).

Lemma 5.10 *A necessary condition for $F_1, \ldots, F_3$ to satisfy Case (c) in Lemma 5.4 where $\text{RI}(-F_1)$ intersects a boundary wrench of F_2 or F_3 is that when the fan whose boundary wrench intersects $-ri(F_1)$ takes the role of an anchor fan, the fans satisfy Case (b1) of Lemma 5.4.*

5.3.4 Summary of Force Closure Conditions

In summary, we see that several proposed sub-conditions can be reduced to other sub-conditions. For example, Case (c) can be entirely omitted since it is covered by Case (a) and Case (b), with respect to the other fan. Hence, the actual sub-conditions that are need to be implemented are only Lemma 5.5 for the majority of Case (a), Lemma 5.6 and Lemma 5.8 for Case (b1) and Case (b2), respectively.

5.4 Implementation

This section concentrates on the implementation of Algorithm 2. Method for identifying fans satisfying conditions presented in Section 5.3 are introduced here. In majority of the cases, the underlying implementation is exactly FINDINTS while most exceptional cases requires subtle modification to FINDINTS. In particular, we describes what FINDFG does in each iteration. Let us denote the anchor fan by $F_i = \text{SPAN}^+(\{\boldsymbol{w}_{il}, \boldsymbol{w}_{ir}\})$.

5.4.1 Implementation of Lemma 5.5

In case (a), it has to be checked whether one negative boundary wrench of the anchor fan, says $-\boldsymbol{w}_{il}$, lies inside the interior of three wrenches, says $\{\boldsymbol{w}_p, \boldsymbol{w}_q, \boldsymbol{w}_r\}$, being boundary wrenches of two other contact fans. Obviously, two wrenches of $\{\boldsymbol{w}_p, \boldsymbol{w}_q, \boldsymbol{w}_r\}$ must be the boundary wrenches of a particular contact fan, says $F_j = \text{SPAN}^+(\{\boldsymbol{w}_p, \boldsymbol{w}_q\})$. Hence, instead of identifying all wrenches that satisfy this condition, we postpone the identification until F_j is considered as an anchor fan. When the algorithm iterates until F_j is an anchor fan, we must identify every pair of wrenches that, together with F_j, positively span $\mathbb{R}^3$.

This also implies that, at the current step where we are considering F_i, we must also identify every pair of wrenches that together with F_j, positively span $\mathbb{R}^3$. Notice that the FINDINTS routine, which is described in Chapter 4, perfectly suits the task.

5.4.2 Implementation of Case (b)

The majority of Case (a) and Case (b) are actually reduced to the positively span of four wrenches. This allows them to be implemented by FINDINTS. Unfortunately, this reduction is not applicable to Case (b) because its conditions involves more than four wrenches.

However, it should be noted that the necessity of Case (b) is rather de minimis. Case (b) scrutinizes the case when a boundary wrench of an anchor fan lies exactly on a boundary fan. This case itself hardly happens in practice. Moreover, when the case happens, it is usually covered by the other case already. For example, let us consider case (b) in Figure 5.1e. It is obvious that the candidate fan SPAN$^+(\{\boldsymbol{w}_{2r}, \boldsymbol{w}_{3l}\})$ intersects the anchor fan and the grasp is detected by Case (c). This is the direct result from the first Subcase of Lemma 5.8.

The only mandatory implementation of Case (b) are Lemma 5.6 and Subcase (ii) and Subcase (iii) of Lemma 5.8. For the flow of the article, the perplexed implementation these cases is put in Appendix A. The implementation given therein has time complexity of $O(n \lg^s N + K)$.

5.4.3 Implementation Summary

In conclusion, the implementation to compute all force closure grasp in frictional case is very similar to the frictionless case. In majority, the condition is exactly the positive spanning in $\mathbb{R}^3$ of four wrenches which can be easily solved by FINDINTS. The different is that both the conditions and their implementation rely on the reduction of one condition to another condition with respect to other anchor fan. Hence, for each anchor fan, the algorithm has to take into account every wrench. However, this does not affect the time complexity of the algorithm.

This approach also introduces inevitable drawback. From the conditions given in Section 5.3, it is possible that a grasp satisfies several conditions with respect to several anchor fan. It is possible that such grasp is reported several times in the implementation. A mechanism to detect and filter out this repetitiveness is required. However, this still does not change the time complexity of the algorithm since the number of repetition is limited to the number of condition which is independent to the number of the input.

Assume that a hash table is used to guarantee uniqueness of the solutions. Since each iteration requires $O(n \lg^2 n + K)$, the overall run time complexity is $O(n^2 \lg^n + K)$.

Table 5.1: Running Time Comparison.

Test Objs.	Time of NSC-AF(s.)	Running Time Ratio (%)					
		NSC	BM	LLC	GJK	Liu	ZW
(a)	1.72	46.89	84.06	131.65	442.12	1,369.87	3,022.28
(b)	1.41	58.37	94.47	106.38	443.12	1,740.99	2,340.35
(c)	1.71	53.69	90.81	118.27	472.42	1,486.42	3,087.47
(d)	1.40	64.91	90.44	125.18	495.44	1,754.49	2,394.29
(e)	1.39	66.50	87.03	135.45	486.82	1,777.16	2,383.21
(f)	1.38	67.01	88.61	130.75	477.37	1,843.15	2,352.57
(g)	1.41	64.20	92.97	126.28	494.53	1,747.73	2,448.15
(h)	1.53	55.83	94.95	130.41	514.42	1,611.99	3,033.36
(i)	1.50	57.12	82.09	125.37	480.76	1,632.76	2,853.06
avg	1.49	59.39	89.49	125.52	478.55	1,662.73	2,657.20

5.5 Numerical Example and Comparison

Similar to the case of frictionless contact, the method presented in this chapter are shown to be theoretically efficient in term of time complexity. Nevertheless, empirical comparison are still necessary. The proposed algorithm are compared with the existing single query tests discussed in Section 3.3.1. Additionally, our newly presented method in Chapter 3 is also included. The comparison framework and test platform are the same as in Section 4.4.

5.5.1 Comparison Result

The comparison is conducted on the same objects used in Section 3.3.2 (see Figure 3.4). For each object, there are $C_{200,3}$ = 1,313,400 queries of frictional grasps to be tested for the single query approach. The frictional coefficient is assumed to be $\arctan(1120/6351) \approx 10.001$ degrees. For our algorithm, the time used to compute all solutions, including the time used to calculate the boundary wrench and to identify repetitive solutions are measured.

Similar to the example of frictionless grasps, Table 5.1 shows the time used by each algorithm. The presented algorithm is labeled as NSC-AF. The result indicates that our algorithm uses *more* time than NSC algorithm and BM algorithm. This is not unexpected because our algorithm has some inevitable overhead. The better time complexity will pay off at the larger number of input.

Table 5.2: Comparison with the modified version.

Test Object	Running Time(s.)		Ratio (%)	Case (b) Exclusive
	Original	Modified		
(a)	1.719	1.339	77.89	-
(b)	1.410	1.041	73.83	-
(c)	1.708	1.349	78.98	-
(d)	1.402	1.057	75.39	-
(e)	1.388	1.046	75.36	-
(f)	1.379	1.041	75.49	-
(g)	1.408	1.062	75.43	-
(h)	1.526	1.128	73.92	-
(i)	1.502	1.123	74.77	-
avg	1.494	1.132	75.67	-

The NSC-AF algorithm relies upon several sub-conditions in order to detect force closure grasps. For completeness, all required sub-conditions have to be implemented. However, some of the sub-conditions are of less important than the others. For example, the sub-conditions for Case (b) cover the case when wrenches lie exactly on the boundary of candidate fans, which is rather unexpected in the real situation. Though their significant is minuet, the time used by their implementation are not. In other words, it is not economical to put a significant amount of time to compute Case (b) which covers only supposedly small portion of solutions. To illustrate this points, we conduct another set of comparison where the implementation of Case (b) is disregarded. Table 5.2 represents the time used by the modified version compared to the original version. The result shows that the modified version takes less time BM algorithm but still more than NSC algorithm.

In the last column of Table 5.2, we also count the number of grasps that satisfy actually need the implementation of Case (b). Interestingly, of all test objects, there is no grasp that have to be detected by Case (b) only. This means that, we can even omit the computation of case (b) while having a high confident that all grasps are reported.

Additionally, we compare NSC-AF algorithm with NSC algorithm on different values of n. This comparison is done to show the number of n that our algorithm begins to outperform NSC algorithm. To make the difference more noticeable, the modified version of the NSC-AF algorithm is used. Figure 5.6 plots the computation time with respect to the number of contact points (n) generated from Object (d) in Figure 3.4. The number of contact points ranges from 50 to 1,500 and varies at the interval of 50. From the graph, it is clear that when n is approximately larger than 220, our modified algorithm uses less time than NSC algorithm.

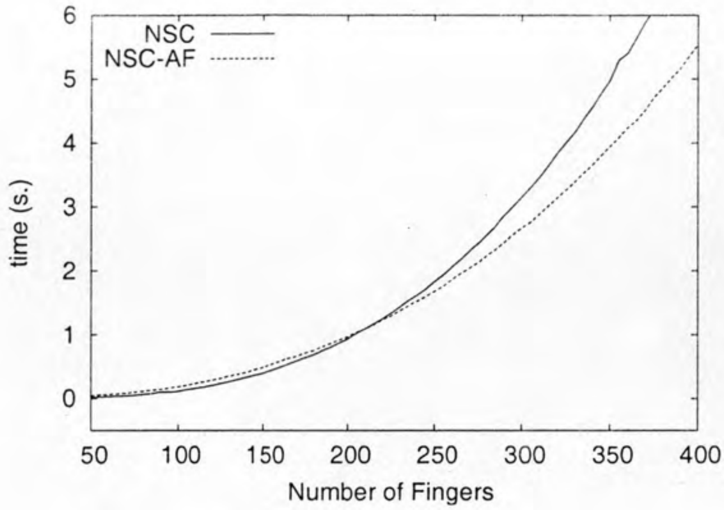

Figure 5.6: Plot between number of fingers and time.

Table 5.3: Number of reported Solutions and unique solutions

Test Object	Number of Solution		Ratio (%)
	Total	Unique	
(a)	2,001,341	538,395	26.90
(b)	523,700	137,598	26.27
(c)	1,725,166	463,537	26.87
(d)	509,383	146,353	28.73
(e)	499,764	140,013	28.02
(f)	446,864	122,081	27.32
(g)	568,683	163,627	28.77
(h)	891,927	369,807	41.46
(i)	744,453	315,531	42.38

our method also introduce another issue in its computation. It is possible that our method reports repetitive solution. These repetitive solution is removed from the final solution by the use of a hash table. The time presented in 5.1 already includes the hashing time. Nevertheless, we present the number of solutions, including repetitive ones, and the number of unique solution report by our method in Table 5.3.

5.6 Summary

In this chapter, an algorithm for identifying all frictional force closure grasp of a planar object is proposed. The algorithm relies on decomposition of a force closure grasp into several conditions for each of which efficient implementation using force dual representation are also proposed. The implementation relies on FINDINTS algorithm which is already introduced in Chapter 4. By using FINDINTS and some of its variation, the computational complexity of the algorithm is $O(n^2 \lg^2 n + K)$. An empirical experiment is also presented and it is shown that the algorithm is faster than any other method.