CHAPTER II

GRASPING PRELIMINARIES

In this chapter we describe necessary definitions and proposition which are essential to the discussion on the forthcoming chapters.

2.1 Nomenclatures

Following the definitions in (Boyd and Vandenberghe, 2004), we denote by $\text{INT}(\cdot)$, $\text{RI}(\cdot)$ and $\text{CO}(\cdot)$ the interior, the relative interior[1] and the convex hull of a set. For an arbitrary vector $\boldsymbol{v}$, let us denote by P_v the plane containing the origin and orthogonal to $\boldsymbol{v}$, i.e., $P_v = \{\boldsymbol{x} | \boldsymbol{x} \cdot \boldsymbol{v} = 0, x \in \mathbb{R}^3\}$. A point at $\boldsymbol{x}$ is said to lie in the positive side of, negative side of, or exactly on P_v when $\boldsymbol{x} \cdot \boldsymbol{v} > 0, \boldsymbol{x} \cdot \boldsymbol{v} < 0$ or $\boldsymbol{x} \cdot \boldsymbol{v} = 0$, respectively. A closed half space $\mathcal{H}(\boldsymbol{v})$ is the set of all points that lie exactly on P_v or in the positive side of P_v. An open half space $\mathcal{H}^+(\boldsymbol{v})$ is simply $\mathcal{H}(\boldsymbol{v}) - P_v$. We define $\mathcal{H}^{z+}$ to be $\mathcal{H}^+((0,0,1))$ and $\mathcal{H}^{z-}$ to be $\mathcal{H}^+((0,0,-1))$.

2.2 Contact Models

To consider a grasp analytically, one has to choose how to model a contact between a hand (fingers) and an object. Two main aspects have to be taken into account: whether a finger is hard or soft and whether friction exists. A hard finger implies that a contact is modeled as a point while a soft finger allows a face contact. A hard contact is also assumed to be unilateral, i.e., a contact can exert only inward force. Friction governs how force can be exerted by a contact. With friction, a hard contact can exert not only a force along its normal but also forces inside a cone defined by the frictional coefficient. A soft finger with friction can exert pure torque in addition to forces exertable by a hard finger. Figure 2.1 shows various contact models. In this work, we assume that all contacts are hard contacts.

Friction is represented by the Coulomb's friction model (Stewart, 2000). Coulomb's law of friction states that for a contact point exerting a normal force $\boldsymbol{f}_n$, the magnitude of friction force $\boldsymbol{f}_t$, lying tangential to the contact, is less than or equal to $\mu \|\boldsymbol{f}_n\|$ where μ is the static frictional coefficient.

[1] A relative interior of a set is the interior relative to the affine hull of the set. Intuitively speaking, a relative interior are all points not on the relative edge of the set, e.g., A relative interior of a line segment is the segment minus its endpoints, regardless of the dimension where the line is situated.

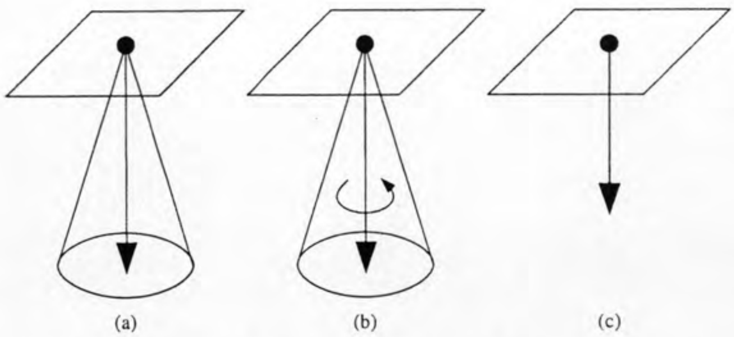

Figure 2.1: Contact Models (a) Hard finger with friction. Force can be applied inside a cone defined by the frictional coefficient. (b) Soft finger with friction, additional pure torque can be exerted by the contact. (c) Hard finger without friction, force can be applied only along the normal direction

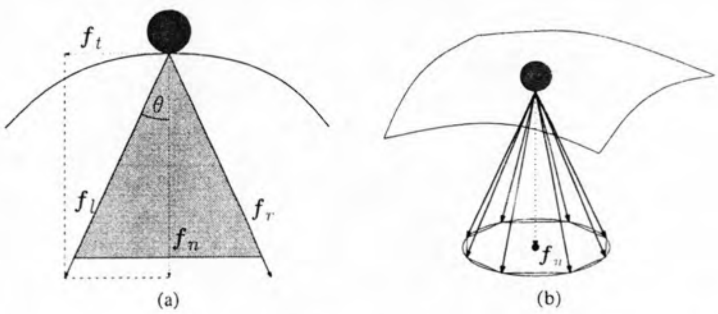

Figure 2.2: Friction cones (a) 2D contact. (b) 3D friction cone simplified as a 8-sided pyramid.

$$\|\boldsymbol{f}_t\| \leq \mu \|\boldsymbol{f}_n\| \tag{2.1}$$

This equation indicates that when the contact does not slip, the contact can exert any force in a cone whose half angle is equal to $\theta = \tan^{-1}(\mu)$. The apex of the cone locates at the contact point and the axis coincides with the contact normal. This cone is referred to as a *friction cone*. A friction cone F at the contact point with a unit normal direction $\hat{\boldsymbol{n}}$ can be written as follows.

$$F = \{\boldsymbol{f} | (\boldsymbol{f} \cdot \hat{\boldsymbol{n}}) / \|\boldsymbol{f}\| \leq \cos\theta\} \tag{2.2}$$

Figure 2.2a illustrates a friction cone of a 2D contact. A friction cone in 2D can be represented by a positive combination of two vectors: f_l and f_r in the Figure 2.2a. However, in 3D case, a cone cannot be described by any linear function. Cone introduces complexity of nonlinearity to the problem. Many works choose to simplify this problem by replacing a cone with a multi-faceted pyramid. A pyramid has planar facets which abolish nonlinearity from the problem but at a price of lesser precision. Unless the number of facets is large enough, the accuracy of the problem is reduced significantly. Figure 2.2b

2.3 Grasp and Wrenches

Force closure is a property of a grasp which is defined by a set of contacts. Each contact can be defined by its position and inward normal direction. In this work, it is assumed that every contact of the same object is represented by the same contact model.

Definition 2.1 (Grasp) *A grasp G is defined by a set of ordered pairs* $\{(\boldsymbol{p}_1, \boldsymbol{n}_1), \ldots, (\boldsymbol{p}_n, \boldsymbol{n}_n)\}$ *where* $\boldsymbol{p}_i$ *and* $\boldsymbol{n}_i$ *are the position vector and the inward normal vector of* i^{th} *contact.*

A grasp achieves force closure when the grasp is able to counterbalance any external disturbance to the object being grasped. The external disturbance and the effect of contact points are represented as a force $\boldsymbol{f}$ and a torque $\boldsymbol{\tau}$. In 2D, it is conventional to combine a force $\boldsymbol{f} = (f_x, f_y)$ and a torque $\boldsymbol{\tau}$ into an entity called a *wrench* $\boldsymbol{w} = (f_x, f_y, \boldsymbol{\tau})$. A wrench is a vector of force concatenated with a vector of torque. In 2D space, force can be described by a 2D vector while torque is described by a 1D vector, hence, a wrench in 2D space is a 3D vector. Likewise, a wrench in 3D space is 6D vector formed by a 3D force vector concatenated with a 3D torque vector. Formally, a wrench $\boldsymbol{w}$ is denoted by $(\boldsymbol{f}, \boldsymbol{t})$ where $\boldsymbol{f}$ is a force vector and $\boldsymbol{t}$ is a torque vector.

Combining force and torque into wrench makes it simpler to consider the force closure property. An effect of a contact point or external disturbance can be easily described as a wrench. For example, let us consider an equilibrium in terms of wrenches. An object is said to be under equilibrium when the summation of all force and torque acting on the object is zero. Using wrench notation, an object achieves equilibrium when the summation of acting wrenches is the zero vectors.

Analysis on force closure concerns wrenches that can be exerted by a grasp. A contact is associated with a set of wrenches that it can exert. The set of wrenches that can be exert by a contact and by a grasp are referred as *a contact wrench set* and *a grasp wrench set*, respectively. In force closure analysis, a contact wrench is allowed to take arbitrarily large magnitude[2]. Since wrenches can be added up linearly, the set of wrenches exertable by the grasp is the positive combination of wrenches of its contacts. Let us refer to a positive combination of a set of vectors as a *linear positive span*, or positive span for short. Exertable wrenches of a grasp is a positive

[2] In practice, a magnitude of a wrench is limited by the realization of the contact, e.g., the actuator of finger, the size of motor, etc. This detail is unrelated to the contact position and hence is neglected.

span of a contact wrench set of each contact.

Definition 2.2 (Positive Span) *Let W be a set of vectors. A positive span of W, denoted by* $\text{SPAN}^+(W)$, *is a set* $\{\alpha_i \boldsymbol{w}_i | \alpha_i \geq 0, \boldsymbol{w}_i \in W\}$.

2.3.1 Primitive Contact Wrenches

A contact wrench set can also be conveniently represented using positive span notation. A frictionless contact can only exert force in one direction and its contact wrench set is a ray in its respective wrench space. The ray can be represented as a positive span of a single wrench with arbitrary length lying in the same direction. For a frictional contact, a friction cone of which can be represented by positive span of its boundary force vectors. These vectors corresponds to boundary wrenches and the whole contact wrench set can be represented by a positive span of these boundary wrenches, using one single arbitrary length for each direction.

We refer to unit length boundary wrenches as *primitive contact wrenches*. A contact wrench set is a positive span of primitive contact wrenches. Similarly, a grasp wrench set is a positive span of its contact wrench sets which is also equal the to positive span all of primitive contact wrenches (from all contact points). Let $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n$ be primitive contact wrenches of a grasp. The grasp wrench set of a grasp whose primitive contact wrenches are $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n$ can be represented as follows.

$$\{\Sigma_{i=1}^{n} \alpha_i \boldsymbol{w}_i | \alpha_i \geq 0\} \tag{2.3}$$

2.3.2 Grasp Wrench Set

Primitive contact wrenches and positive span represent a grasp wrench set in a compact form. It is necessary to understand the properties of a grasp wrench set when it is represented as a positive span. A key feature of a positive span is its convexity. Convexity of a grasp wrench set is an important property exploited by most grasping works.

Other than convexity, a grasp wrench set also has other interesting properties especially in the 2D case. In 2D grasping, a friction cone is bounded by exactly two vectors hence a contact wrench set is a positive span of exactly two primitive contact wrenches which are 3D vectors. A positive span of wrenches from 2D contacts can take several forms which is are collectively known as *polyhedral convex cones* (Goldman and Tucker, 1956). A polyhedral convex cone is bounded by linear surfaces allowing several tools in linear algebra to be applicable for analysis. Figure 2.3 illustrates some examples of polyhedral convex cones of few vectors. Of special interest is the case of two vectors that form a fan. A fan is a planar cone which arises naturally in grasp analysis. For example, a friction cone is a fan in 2D and a contact wrench set of a 2D frictional contact is a fan in 3D. For the case when a polyhedral convex cone does not occupy the entire space, its facets are fans.

Positive span is an edge-based representation of a polyhedral convex cone, i.e., the cone is defined in terms of its edges. Another way to represent a convex cone is face-based representation. A cone is defined by the intersection of several half spaces each of which is described by an inward normal vector.

Polyhedral convex cone is a special case of a grasp wrench set in which the number of primitive contact wrenches is finite, e.g., when the contacts are frictionless or the grasp is 2D. In the 3D frictional contact case, a friction cone is bounded by a quadratic surface, not a finite number of wrenches. A prominent difference is that a 3D friction cone, though it still maintains convexity, is no longer a linear structure. This implies that the corresponding grasp wrench set itself is nonlinear as well.

A friction cone is a right circular cone which introduces nonlinearity to the system. In many works, a circular friction cone is simplified by an m-sided pyramid. Each boundary force vector of the pyramid yields one primitive contact wrench. Since m is finite, the number of primitive contact wrenches is also finite and thus the grasp wrench set is a polyhedral convex cone. This simplification throws away nonlinearity at the cost of accuracy.

2.4 Force Closure

A grasp achieves force closure when its grasp wrench set covers the entire wrench space. A property called *positively spanning* is defined to describe that the positive span of a vector set covers the entire space.

Definition 2.3 (Positively Span) *We say that a set V of n-dimensional vector* positively spans $\mathbb{R}^n$ *when* $\text{SPAN}^+(V) = \mathbb{R}^n$

The force closure property can be formally defined using the notion of positively spanning, namely, a grasp achieves force closure when its associated wrenches, i.e., the polyhedral convex

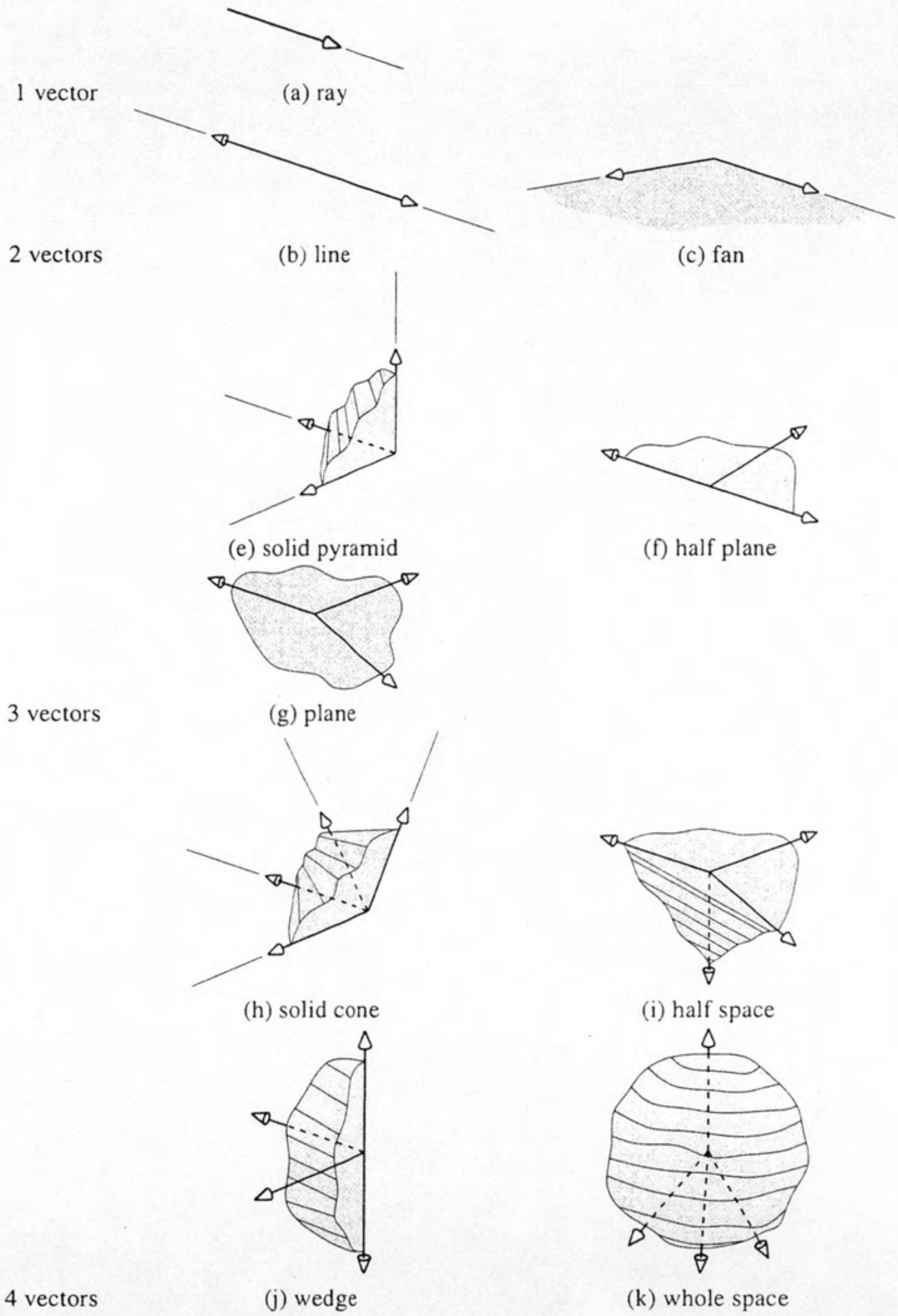

Figure 2.3: Example of polyhedral convex cones.

cone generated from the primitive contact wrenches, positively span their respective wrench space (3D wrench space in case of planar grasp and 6D wrench space in case of 3D grasp).

Definition 2.4 (Force Closure) *A grasp, whose primitive contact wrenches form the set W in $\mathbb{R}^n$, is said to achieve force closure when* $\text{SPAN}^+(W)$ *positively span* $\mathbb{R}^n$.

Since the force closure property is defined over a set of vector (wrenches) associated with a grasp, it is more convenient to say that a set of vector achieves force closure, even though a set of vector cannot literally achieve force closure. Hereafter, saying that a set of wrenches achieves force closure is a short hand of saying that a grasp whose associated set of wrenches positively span $\mathbb{R}^n$.

2.5 Condition of Force Closure

The force closure property is defined using the notion of positively spanning. However, it is still indefinite to assert whether a set of vectors positively span a space. In this section we recite some of the well known conditions that assert on positively spanning of a set of vectors.

Mishra et al. related positively spanning of a set of vectors with a convex hull of the vectors. It is shown in (Mishra et al., 1987) that a set of vectors W positively span a space when the origin of the space lies strictly inside the convex hull of W.

Proposition 2.5 *A set of wrenches W in $\mathbb{R}^n$ achieve force closure when the origin lies in the interior of the convex hull of* $\text{INT}(\text{CO}(W))$..

Proposition 2.5 transforms the force closure testing problem into a well defined computational geometry problem. A straightforward approach to solve the problem is to compute the convex hull of the primitive contact wrenches and directly whether the origin lies inside the interior. From this approach, it comes directly that if we can identify a half space through the origin that contains all primitive contact wrenches, the primitive contact wrenches cannot positively span the space.

Proposition 2.6 *A set of wrenches W do not positively span $\mathbb{R}^3$ if there exists a vector v such that the closed half space $\mathcal{H}(v)$ contains every wrench in W.*

A closely related property of force closure is equilibrium. Equilibrium indicates that the net resultant wrench of the system is a zero vector. A grasp is said to achieve equilibrium when it is possible for some contacts of the grasp to exert wrenches such that the net resultant wrench is zero vector. Formally, a grasp is an equilibrium grasp when Equation (2.4) has a non-trivial solution.

$$\Sigma_{i=1}^{n}\alpha_i \boldsymbol{w}_i = \boldsymbol{0} \tag{2.4}$$

Apparently, a grasp that achieves force closure also is an equilibrium grasp. However, the inverse is not necessary true. In the case of frictional contact, there exists a special class of equilibrium grasp called *non-marginal equilibrium.* A grasp achieves non-marginal equilibrium when the wrenches achieving equilibrium are not the wrenches associated with the boundary of a force cone. This is equivalent to substitute a less-than sign ($<$) for a less-than-or-equal sign ($\leq$) in Equation (2.2). In practice, it means that any equilibrium grasp is also a force closure grasp under any arbitrarily greater frictional coefficient.

Nguyen (1988) shows that a 2D two finger non-marginal equilibrium grasp is also a force closure grasp. Ponce and Faverjon (1995) show the same implication in the case of 2D three finger grasp and also in the case of 3D four finger grasp (Ponce et al., 1997). Care should be taken not to take this implication into general. Though it might seems that non-marginal equilibrium implies force closure, this is not always true for any number of fingers. For example a 3D two finger non-marginal equilibrium grasp *does not* achieve force closure.

Proposition 2.5 provides a general method for force closure assertion. The method is applicable in any dimension for any number of wrenches. In the case of few contacts in small dimensions, there exists conditions that need no explicit calculation of the convex hull allowing more efficient implementation. The conditions are listed using positively spanning notation as follows.

Proposition 2.7 *Three 2D vectors $\boldsymbol{w}_1, \boldsymbol{w}_2$ and $\boldsymbol{w}_3$ positively span the plane when the negative of any of these vectors lies in the interior of the polyhedral convex cone formed by the other two vectors.*

Proof: Sufficient condition: Assume that $-\boldsymbol{w}_1$ lies strictly inside the cone formed by $\boldsymbol{w}_2$ and $\boldsymbol{w}_3$. Obviously, $\boldsymbol{w}_2$ and $\boldsymbol{w}_3$ lies on different sides of a line through $-\boldsymbol{w}_1$. This means that $\boldsymbol{w}_1$ also forms a cone with $\boldsymbol{w}_2$ and also forms a cone with $\boldsymbol{w}_3$. These two cones are not overlapping and also they do not intersect the cone formed by $\boldsymbol{w}_2$ and $\boldsymbol{w}_3$. Thus, any point in the plane must be covered by one of these cones.

Necessary condition: Assume that the three vectors positively span the plane but the negative of one of them, says w_e, does not lie inside the cone formed by the others. This means that the other two vectors must lie on the same side of the line through w_e. Obviously, any point lying on the other side of the line cannot be represented by any positive combination of $w_1, \ldots, w_3$. Thus, the three vectors do not positively span the plane. ■

From Proposition 2.7, we can derive directly the following corollary.

Corollary 2.8 *Three 2D vectors w_1, w_2 and w_3 positively span the plane when any of these vectors lies in the interior of the polyhedral convex cone formed by the the negative of the other two vectors.*

Figure 2.4 shows the example of Proposition 2.7. This proposition can be easily extended to cover 3D cases as follows.

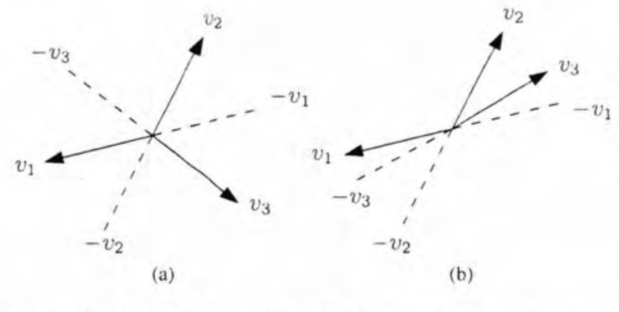

Figure 2.4: (a) Three vectors satisfying Proposition 2.7. The dashed lines represent the negative of vector. (b) Three vectors does not satisfy Proposition 2.7

Proposition 2.9 *Four 3D vectors w_1, w_2, w_3 and w_4 positively spans $\mathbb{R}^3$ when the negative of any of these vectors lies in the interior of the polyhedral convex cone formed by the other three vectors.*

Proof: Sufficient condition: Again, let us assume that $-w_1$ lies strictly inside the cone formed by $w_2, \ldots, w_4$. We will show that any point x can be represented by a positive combination of $w_1, \ldots, w_4$. If x lies inside the cone, it can be represented by a positive combination of

$w_2, \ldots, w_4$. If it does not, then there exists a plane p through the origin such that p contains x and w_1. Since $-w_1$ lies inside the cone, p must also intersect with the cone. Let w_l and w_r be the edges of the cone that intersect with p. Vector w_l and w_r are positive combination of $w_2, \ldots, w_4$. Obviously, w_1, w_l and w_r positively span on plane p. Thus, x can be represented by a positive combination of w_1, w_l and w_r. Which means that x is a positive combination of $w_1, \ldots, w_4$.

Necessary condition: Assume that the $w_1, \ldots, w_4$ positively span the space but none of them has its negative lies inside the cone formed by the others. This means that there exists a plane p contain two vectors from $\{w_1, \ldots, w_4\}$. Assume without loss of generality that p contains w_2 and w_3. From the assumption, $-w_1$ does not lie inside the cone formed by w_2, w_3 and w_4. This means that w_4 and w_1 is on the same halfspace that is bounded by plane p. Obviously, any point lying on the other half space cannot be represented by any positive combination of $w_1, \ldots, w_4$. Thus, they can not positively span the space. ■

Similarly, we can derive directly the following corollary from Proposition 2.9

Corollary 2.10 *Four 3D vectors w_1, w_2, w_3 and w_4 positively spans $\mathbb{R}^3$ when any of these vectors lies in the relative interior of the polyhedral convex cone formed by the negative of the other three vectors.*

In the case of four wrenches, there exists another condition that is equivalent to Proposition 2.9. The equivalent condition requires that the relative interior of the fan formed by two wrenches intersects with the relative interior of the fan formed by the negative of the other two wrenches. Figure 2.5 illustrates four vectors that satisfy Proposition 2.9 and Proposition 2.11.

Proposition 2.11 *Four 3D wrenches w_1, w_2, w_3 and w_4 positively span $\mathbb{R}^3$ when $\text{RI}(\text{SPAN}^+(\{w_i, w_j\}))$ intersects $\text{RI}(\text{SPAN}^+(\{-w_k, -w_l\}))$ as a ray where $\{i, j, k, l\}$ is a permutation of $\{1, \ldots, 4\}$ and $i \neq j \neq k \neq l$.*

Proof: Let us assume without loss of generality that the fans are $\text{SPAN}^+(\{w_1, w_2\})$ and $\text{SPAN}^+(\{-w_3, -w_4\})$.

Sufficient Condition: Since the intersection of the relative interior is a ray, the two fans cannot lie on the same plane and no set of three wrenches from $w_1, \ldots, w_4$ is coplanar. Hence, we can assume without lost of generality that $-w_k$ and $-w_l$ lie on the positive and the negative side of $P_{w_i \times w_j}$, respectively. Similarly, we can assume that w_i and w_j lie on the positive and the negative side of $P_{-w_l \times -w_k}$.

Consider three wrenches w_i, w_j and $-w_k$, it can be inferred directly from the assumption that w_i lies on the positive side of $P_{w_j \times -w_k} = P_{-w_k \times -w_j}$. Next, consider w_i, w_j and $-w_l$, the assumption also implies that $-w_l$ lies on the negative side of $P_{w_i \times w_j}$ which also implies that w_i lies on the positive side of $P_{-w_l \times w_j} = P_{-w_j \times -w_l}$. We can conclude that w_i lies in the interior of the cone formed by $-w_k, -w_j, -w_l$. Hence, the negative of w_l lies strictly inside the cone formed by w_k, w_j, w_l which implies force closure and conclude the sufficiency of the condition.

Necessary Condition: Assuming that the wrenches positively span the wrench space, by Proposition 2.9, we assume without lost of generality that $-w_l$ intersects with $\text{INT}(\text{SPAN}^+(\{w_i, w_j, w_k\}))$ and the no three of them is coplanar. Let P be the plane containing $-w_l$ and w_k. It is obvious that P intersects $\text{RI}(\text{SPAN}^+(\{w_i, w_j\}))$. Let a ray r be the intersection. According to Proposition 2.9, the vector $w_l, -w_k$ and r positively span the plane P. By Corollary 2.10, r also lies inside the cone formed by $-w_l$ and $-w_k$. Hence, the ray r is the intersection between $\text{RI}(\text{SPAN}^+(\{w_i, w_j\}))$ and $\text{RI}(\text{SPAN}^+(\{-w_k, -w_l\}) = \text{RI}(\text{SPAN}^-(\{w_k, w_l\})$. ■

Ding et al. (2001b) provided a generalized version of Proposition 2.9. In (Ding et al., 2001b), it is stated that, n vectors in any dimension positively span the space when a negative of one vector lying inside a positive span of the other vectors.

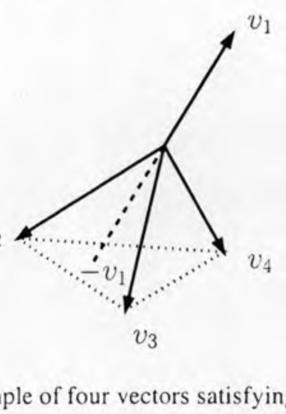

Figure 2.5: An example of four vectors satisfying Proposition 2.9

A wrench consists of a force vector and a torque vector. When the reference frame rotates, the force vector also rotates accordingly. Likewise, translation of the reference frame change the position of the force and hence affects the torque vectors. In other words, the rigid transformation of the reference frame affects wrenches. However, the intended meaning of force closure property suggests that force closure should not be affected by such transformation. The next proposition show that force closure is invariant to the choice of origin.

Proposition 2.12 *Let T be a rigid transformation. A set of wrenches W achieves force closure if and only if TW achieve force closure.*

Proof: Let T be a rigid transformation constructed from a rotation R and a translation $\boldsymbol{t}$. Let $W = \{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n\}$ and let $TW = \{\boldsymbol{w}'_1, \ldots, \boldsymbol{w}'_n\}$. It will be shown that if W achieve force closure, any arbitrary wrench $\boldsymbol{w} = (\boldsymbol{f}_v, \boldsymbol{m}_v)$ can be described by a positive combination of $\boldsymbol{w}'_1, \ldots, \boldsymbol{w}'_n$.

Let $\boldsymbol{w}^* = R^{-1}\boldsymbol{w} = (R^{-1}\boldsymbol{f}_v, R^{-1}\boldsymbol{m}_v)$. Since $\{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n\}$ achieves force closure, there exist positive values $a_1, \ldots, a_n$ such that $a_1\boldsymbol{w}_1, \ldots, a_n\boldsymbol{w}_n = \boldsymbol{w}^*$. By multiplying a_i to $\boldsymbol{w}'_i$, we obtain $a_1\boldsymbol{w}'_1, \ldots, a_n\boldsymbol{w}'_n = (R(R^{-1}\boldsymbol{f}_v), \boldsymbol{m}) = (\boldsymbol{f}_v, \boldsymbol{m})$. The value of $\boldsymbol{m}$ is equal to

$$\begin{aligned}
\Sigma\left(a_i(R\boldsymbol{p}_i + \boldsymbol{t}) \times R\boldsymbol{f}_i\right) &= \Sigma(a_iR\boldsymbol{p}_i \times R\boldsymbol{f}_i) + \Sigma(a_it \times R\boldsymbol{f}_i) \\
&= R\Sigma(a_i\boldsymbol{p}_i \times \boldsymbol{f}_i) + \Sigma(a_it \times R\boldsymbol{f}_i) \\
&= R(R^{-1}\boldsymbol{m}_v) + \Sigma\left(a_it \times R\boldsymbol{f}_i\right)
\end{aligned}$$

This means that $a_1\boldsymbol{w}'_1, \ldots, a_n\boldsymbol{w}'_n = (\boldsymbol{f}_v, \boldsymbol{m}_v + \Sigma(a_it \times R\boldsymbol{f}_i))$. Obviously, there exists a solution $b_1, \ldots, b_n$ such that $b_1\boldsymbol{w}_1, \ldots, b_n\boldsymbol{w}_n = (0, -R^{-1}\Sigma(a_it \times R\boldsymbol{f}_i))$. Since the summation of force part of $\Sigma(b_i\boldsymbol{w}_i)$ is zero, the force part of $\Sigma(b_i\boldsymbol{w}'_i)$ is also zero. Thus, $\Sigma b_i\boldsymbol{w}_i$ is equal to $(0, -\Sigma(a_it \times R\boldsymbol{f}_i))$. This means that $\boldsymbol{w}$ can be described by $(a_1 + b_1)\boldsymbol{w}'_1, \ldots, (a_n + b_n)\boldsymbol{w}'_n$.

Now, we can conclude that when W achieves force closure, TW also achieve force closure. Hence, if TW achieve force closure, $T^{-1}TW$ also achieve force closure and thus the proof is complete. ■

Since force closure is defined using positively span notation, it follows directly that any positive scaling transformation preserves force closure. Moreover, individual positive scaling of each wrenches does not affect force closure property. We offer the following proposition without proof.

Proposition 2.13 *A set of wrench $W = \{w_1, \ldots, w_n\}$ achieve force closure if and only if a set $W' = \{\alpha_1 w_1, \ldots, \alpha_n w_n\}$ achieves force closure, where α_i is a positive scaling factor.*

2.6 Force Dual Representation

Proposition 2.13 presents a very important property of force closure: force closure property considers only the direction of wrenches, but not their sizes. This means that any implementation of the aforementioned conditions that treats the primitive contact wrenches as 3D vectors might be overly complex.[3] For example, let us consider the case of 3D wrenches. A vector representation of a wrench consists of three scalars which contains the information about the size of wrenches. Instead, if a wrench is written in a spherical coordinate frame, only two scalars are required to represent the direction of the wrench.

This fact has been exploited by many researchers. Representing a wrench by a 2D entity is not a new idea in grasping. The origin of this form of representation can be traced back to (Brost and Mason, 1989; Mason, 1991) where different wrenches at a contact that lie along the same direction are represented by the same point in a plane. The technique essentially reduces wrenches with different positive scaling into one 2D representation. The idea also reappeared with its possible application in Mason's book (Mason, 2001). This representation is called *force dual representation.* Instead of representing a wrench with a point on the unit sphere centered at the origin as in spherical coordinates, Brost and Mason (1989) represent a wrench using its intersection (called *dual point*) with one of the following structures: 1) plane $\tau = 1$, 2) plane $\tau = -1$, or 3) the unit circle on the plane $\tau = 0$ with its center at the origin. Note that the dual point on the plane $\tau = 1$ of the wrench (f_x, f_y, τ) with positive τ is simply $(f_x/\tau, f_y/\tau)$, requiring two divisions. The benefit of this representation is that it allows positive combinations of wrenches to be conveniently represented. As described in (Brost and Mason, 1989), it is obvious from the geometry that all positive combinations of wrenches with positive (resp. negative) torques can

[3]This is not always the case since augmenting an entity with some information might transform the problem into a domain where the problem is easier to be solved.

be represented as the convex hull of the corresponding dual points on the plane $\tau = 1$ (resp. $\tau = -1$). See Figure 2.6a for examples.

Let us consider all primitive contact wrenches of a 2D grasp and assume that their dual points lie on the planes $\tau = 1$ or $\tau = -1$ only (this assumption can be easily fulfilled by ensuring that the origin of the primary space does not lie on a line that supports any friction cone's boundary). On each of these two planes, let us construct the convex hull of all the dual points that lie on the plane. If all the dual points lie only on one plane, it is obviously impossible for the grasp to achieve force closure. To determine force closure, Proposition 2.5 can be consulted. From geometry, it can be easily shown that the proposition is satisfied when there exists a line through the origin of the wrench space that intersects each of the two convex hulls in its relative interior (see Figure 2.6a). Instead of directly checking for the existence of such line, the following equivalent method can be used. First, label all dual points in the plane $\tau = 1$ as positive dual points. Second, for each primitive wrench w with a negative torque, draw the dual point of its inverse $-w$ on the plane $\tau = 1$ and label it a negative dual point. Third, construct the convex hull of all positive dual points and the convex hull of all negative dual points. It is clear that Proposition 2.5 is satisfied when the two convex hulls intersect and some intersection point is contained in the relative interior of both convex hulls.

Several works adopt this representation, either explicitly or implicitly. Pollard (1996) also proposes a different 2D representation and its application in finding an independent contact region. Later, Suarez, Vazquez and Ramirez (2003) use the same technique to verify form closure grasps in frictionless setting. Independently, Sudsang and Phoka (2005) propose a condition of force closure of three finger frictional grasps that utilizes the dual representation of a force cone. Many works of van der Stappen et al. (Cheong and van der Stappen, 2005; Cheong et al., 2006; Cheong and van der Stappen, 2007a,b) represent a wrench by its intersection on two perpendicular slabs.

From the previous example, it can be seen that a wrench with negative torque can be equivalently represented as a dual point of its inverse, whose torque component is positive. This means that the plane $\tau = -1$ is no longer necessary. However, a unit circle is still required to represent a wrench with zero torque. It is often complicate to derive a force closure condition which consider both the $\tau = 1$ plane and the unit circle. Many works deliberately avoid this problem by ensuring that no wrench with zero torque exists in the problem. For example, the work of Phoka and Sudsang (2005) relocate the origin such that no friction cone contain the origin. The original work of Brost and Mason (1989) does not even consider the wrench with zero torque at all.

In our work, we propose a unified representation of dual points that combines zero torque

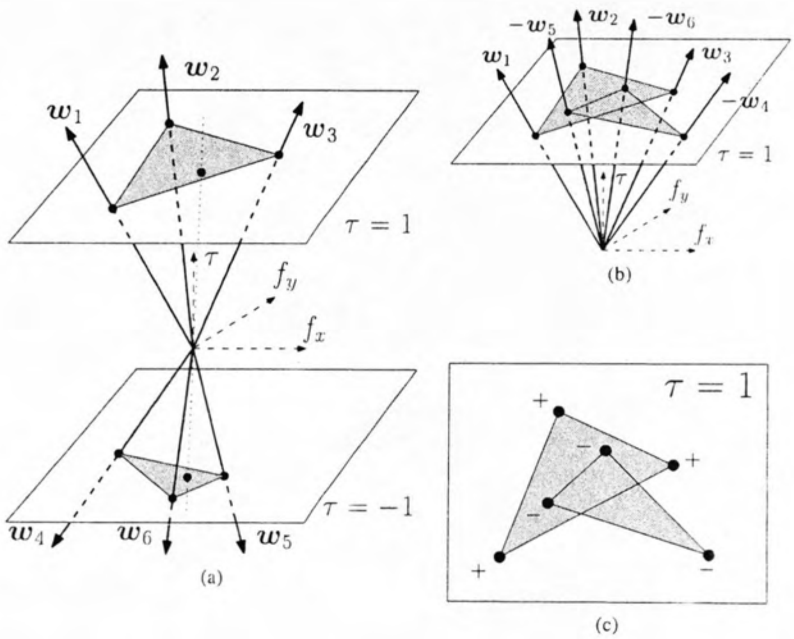

Figure 2.6: Wrenches as represented on the dual space. (a) w_1, w_2, w_3 having positive torques are represented as points on $\tau = 1$ and w_4, w_5, w_6 having negative torques are represented as points on $\tau = -1$. The shaded regions are convex hulls representing the positive combination of the wrenches on their respective planes. The dotted line represents a line through the origin that intersects the relative interiors of both convex hulls. This indicates force closure. (b) Wrenches with negative torques are represented by the intersections of their negative with $\tau = 1$ plane. (c) Simplified illustration showing only the dual points. The symbols $+$ and $-$ indicates positive and negative dual points. The intersection between the relative interiors of the two convex hulls indicates force closure.

Table 2.1: Relation between wrenches and their force dual representation

wrench	point on the extended plane	flag	remark
positive torque	$(w_x/w_\tau, w_y/w_\tau)$	+	a normal point
negative torque	$(w_x/w_\tau, w_y/w_\tau)$	−	a normal point
zero torque	(w_x, w_y)	0	a point at infinity represented by its direction

wrenches with nonzero torque wrenches. We employ a concept from Projective Geometry (Bix, 1994) to dual point representation. A wrench with zero torque is parallel to the $\tau = 1$ plane. In Euclidian geometry, the zero torque wrench does not intersect the plane and no point on the plane can be used to represent the intersection. On the contrary, Projective Geometry consider the wrench to intersects the plane at a *point at infinity*. A point at infinity does not exists on a Euclidian plane and nor can it be represented in $\mathbb{R}^2$. Informally speaking, a point at infinity is considered to be an intersection between a plane and a ray lying parallel to the plane. The point is the farthest point along a ray. Unlike an ordinary point which is defined by its position, a point at infinity is defined by its direction; rays pointing in the same direction meet at the same point at infinity. In Projective Geometry, an *extended plane* is a plane which consists of a Euclidian plane and all points at infinity. All points at infinity form a line called a line at infinity. Our work uses an extended plane to describe a wrench. By using an extended plane, any wrench can be mapped into a point in the plane. A point in an extended plane is tagged with a flag indicating whether it is a normal point or a point at infinity. The flag is simply the sign of the torque component of the wrenches. A plus (resp. minus) flag indicates that a point is from a wrench that intersects the plane $\tau = 1$ (resp. $\tau = -1$) while a zero flag indicates that the point is a point at infinity and the coordinate of the point should be regarded as a direction. Table 2.1 shows a mapping between an arbitrary wrench $w = (w_x, w_y, w_\tau)$ and a point on the extended plane.

This representation is extensively used in this work, especially in Chapter 3, 4 and 5. For convenience, let us denote by Π the extended plane containing plane $\tau = 1$ and a line at infinity. The intersection point of wrench w on Π is denoted by $\Omega(w)$. We also extend this notation to denote by $\Omega(X)$ the intersection between entity X and the plane Π. We refer to $\Omega(X)$ as the *slice* of X. It is important to note that a wrench is treated as a ray from the origin with the same direction as the wrench since arbitrary positive scaling of the wrench can be applied without affecting the force closure property. We define a 2D coordinate frame on Π. The frame has the origin that is translated such that the new origin is located on $\Omega(\tau)$. The axes of the new frame are labeled f'_x and f'_y, respectively (see Figure 2.7).

Since a fan is used extensively in this work, we need to consider a slice of a fan. Let F be a fan with boundary wrenches w_l and w_r. Let Γ be the plane that contains F. If $\Omega(F)$, the

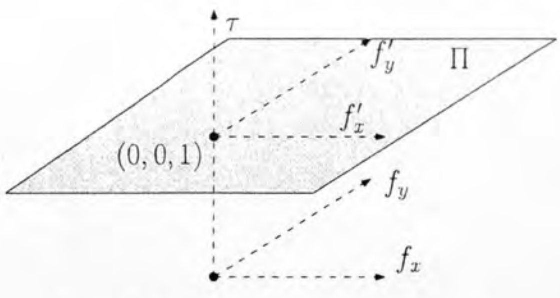

Figure 2.7: New coordination frame on Π.

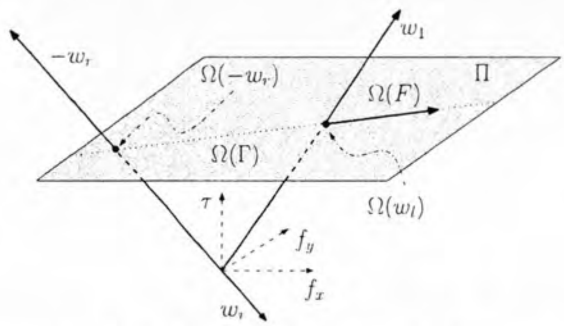

Figure 2.8: A fan partly intersects Π. The dotted line is the line $\Omega(\Gamma)$. The slice of F is a ray which begins at $\Omega \boldsymbol{w}_l$.

slice of F, exists, it must be part of the line $\Omega(\Gamma)$, i.e., $\Pi \cap \Gamma$. Since F is convex, the slice $\Omega(F)$ is a single connected set. When both $\boldsymbol{w}_l$ and $\boldsymbol{w}_r$ have positive torques, $\Omega(F)$ is the line segment joining $\Omega(\boldsymbol{w}_l)$ and $\Omega(\boldsymbol{w}_r)$. However, the slice $\Omega(F)$ also exists when one of the boundary wrench has a negative or zero torque while the other has a positive torque. Let us suppose that $\boldsymbol{w}_r$ has a negative torque. As shown in Figure 2.8, it is trivial in this case to verify that $\Omega(F)$ forms a ray that begins at $\Omega(\boldsymbol{w}_l)$ and points in the direction of the vector from $\Omega(-\boldsymbol{w}_r)$ to $\Omega(\boldsymbol{w}_l)$. For the last case, suppose that $\boldsymbol{w}_r$ has a zero torque. In this case, it is also simple to verify that $\Omega(F)$ forms a ray that begins at $\Omega(\boldsymbol{w}_l)$ and has the direction parallel to $\boldsymbol{w}_r$. It should be noted that the slice of the negative of F is a ray that begins at $\Omega(-\boldsymbol{w}_r)$ and points in the direction of the vector from $\Omega(\boldsymbol{w}_l)$ to $\Omega(-\boldsymbol{w}_r)$, i.e., the opposite vector of its positive fan counterpart.

Observe that the concept of points at infinity allows a ray to be viewed as a line segment with two endpoints: one ordinary and one at infinity. This matches perfectly with the slice of a fan. We can directly compute a slice of a fan from slices of its boundary wrenches, represented as a point or a point at infinity.