CHAPTER II

A NEW METHOD TO OBTAIN SUPER VERTEX-MAGIC TOTAL LABELINGS OF GRAPHS

The studies described in this chapter are partly rewritten from the published literature[5] because it is considered to be one of the most useful selected papers regarding to super vertex-magic total labeling. Our purpose is to construct a new super vertex-magic graph from an existing one. From a super vertex-magic graph G with some characteristics, we construct the disjoint union of k copies of G, kG which is a super vertex-magic graph.

Theorem 2.1. ([5]) *Let G be an r-regular super vertex-magic graph with the magic constant h. If the graph kG is super vertex-magic then the magic constant is* $h' = kh - \frac{(k-1)(r+1)}{2}$.

Proof.

Assume that kG is a super vertex-magic graph with the magic constant h'.

By Theorem 1.2.1,

$$\begin{aligned}
h' &= \frac{(vk+ek)(vk+ek+1)}{vk} - \frac{vk+1}{2} \\
&= \frac{(vk+ek)[(vk+ek+k)-(k-1)]}{vk} - \frac{(vk+k)-(k-1)}{2} \\
&= \frac{(vk+ek)(vk+ek+k)}{vk} - \frac{(vk+ek)(k-1)}{vk} - \frac{vk+k}{2} + \frac{k-1}{2} \\
&= \frac{k(v+e)(v+e+1)}{v} - \frac{(v+e)(k-1)}{v} - \frac{k(v+1)}{2} + \frac{k-1}{2} \\
&= \left[\frac{k(v+e)(v+e+1)}{v} - \frac{k(v+1)}{2}\right] - \left[\frac{(v+e)(k-1)}{v} - \frac{k-1}{2}\right] \\
&= k\left[\frac{(v+e)(v+e+1)}{v} - \frac{(v+1)}{2}\right] - \left[\frac{2(v+e)(k-1)-v(k-1)}{2v}\right] \\
&= k\left[\frac{(v+e)(v+e+1)}{v} - \frac{(v+1)}{2}\right] - \left[\frac{(v+2e)(k-1)}{2v}\right]
\end{aligned}$$

Since G is r-regular, $2e = rv$, and G is a super vertex-magic graph with

$h = \frac{(v+e)(v+e+1)}{v} - \frac{v+1}{2}$, then

$$= k\left[\frac{(v+e)(v+e+1)}{v} - \frac{(v+1)}{2}\right] - \left[\frac{(v+rv)(k-1)}{2v}\right]$$
$$= k\left[\frac{(v+e)(v+e+1)}{v} - \frac{(v+1)}{2}\right] - \left[\frac{(r+1)(k-1)}{2}\right].$$

Hence $h' = kh - \frac{(k-1)(r+1)}{2}$. □

From now on, we will assume that the order of the generic graph G is greater than one. Let k be a positive integer and

$$M(k) = \left\{-\frac{k-1}{2}, -\frac{k-1}{2}+1, \ldots, -\frac{k-1}{2}+(k-2), -\frac{k-1}{2}+(k-1)\right\}$$
$$= \left\{-\frac{k}{2}+\frac{1}{2}, -\frac{k}{2}+\frac{3}{2}, \ldots, \frac{k}{2}-\frac{3}{2}, \frac{k}{2}-\frac{1}{2}\right\}.$$

Definition 2.2. A *neutral labeling* of a graph G with the elements of $M(k)$ is a map β satisfying

$$\beta : V(G) \cup E(G) \to M(k)$$

and for each $v_i \in V(G)$, $w_\lambda(v_i) = 0$.

Example 2.3. Two neutral labelings of the graph $C_3 + C_6$ with the elements of $M(3) = \{-1,0,1\}$ and $M(5) = \{-2,-1,0,1,2\}$ are shown in Figure 2.1 and Figure 2.2, respectively.

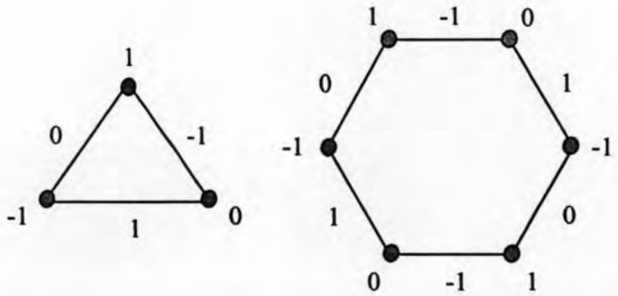

Figure 2.1 : A neutral labeling of the graph $C_3 + C_6$ with the elements of $M(3) = \{-1,0,1\}$

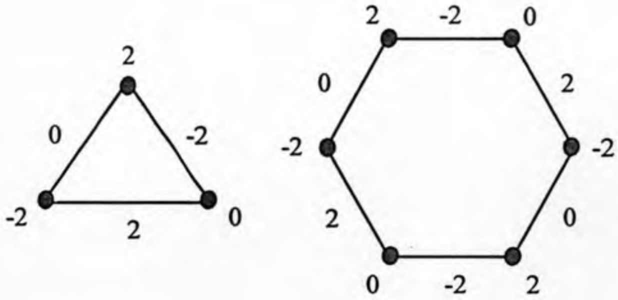

Figure 2.2 : A neutral labeling of the graph $C_3 + C_6$
with the elements of $M(5) = \{-2, -1, 0, 1, 2\}$

Theorem 2.4. ([5]) *Let r be even and G be an r-regular graph. There is no neutral labeling of G with the elements of M(k) for even k.*

Proof. Assume that k is even and $M(k) = \left\{-\frac{k}{2} + \frac{1}{2}, -\frac{k}{2} + \frac{3}{2}, \ldots, \frac{k}{2} - \frac{3}{2}, \frac{k}{2} - \frac{1}{2}\right\}$.
Therefore, the sum of any odd elements of $M(k)$ is different from 0.
Since $r + 1$ is odd, there is no neutral labeling of G with the elements of $M(k)$. □

Definition 2.5. Two neutral labelings of a graph G, β_1 and β_2, are *compatible* iff $\beta_1(v) \neq \beta_2(w)$ for each $v, w \in V(G)$ and $\beta_1(vw) \neq \beta_2(vw)$ for each $vw \in E(G)$.
A set of $q\,(\leq k)$ neutral labelings of G with the elements of $M(k)$ are *compatible* iff they are pairwise compatible.

Note that the maximum number of compatible neutral labelings of a graph G with the elements of $M(k)$ is k.

Example 2.6. Three compatible neutral labelings of the graph $C_3 + C_6$ with the elements of $M(3) = \{-1, 0, 1\}$ is shown in Figure 2.3.

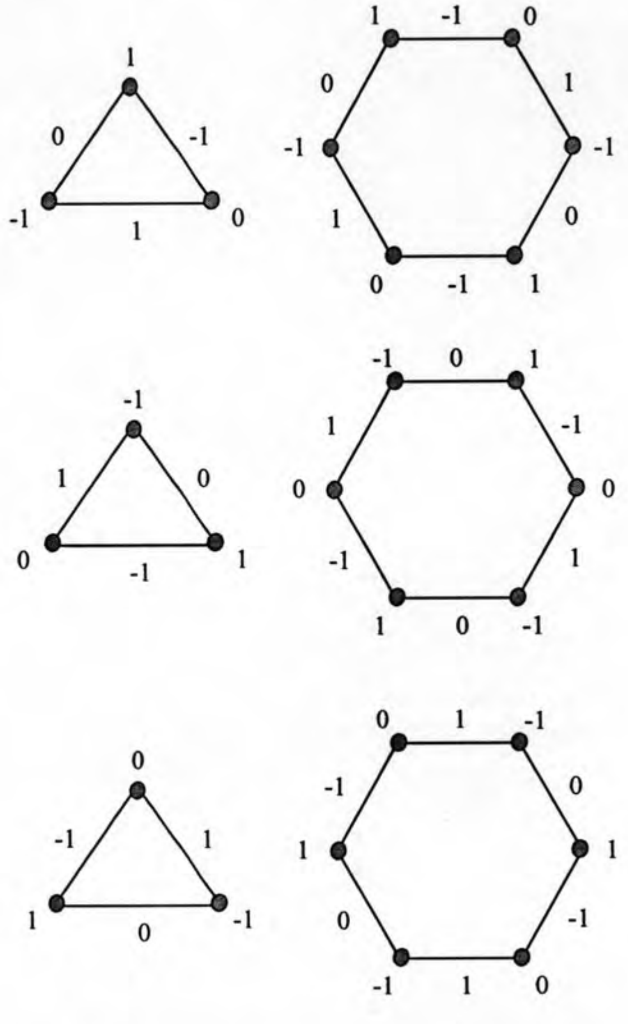

Figure 2.3 : Three compatible neutral labelings of the graph $C_3 + C_6$ with the elements of $M(3) = \{-1, 0, 1\}$

Definition 2.7. A graph is *edge-colorable* if there is a labeling from the edge set of the graph to a finite set such that incident edges have different labels. The labels are called colors, the edges of one color form a *color class*.

Theorem 2.8. ([8]) (Vizing's Theorem) *If G is a graph, then G can be edge-colored in* $\Delta(G) + 1$ *where* $\Delta(G)$ *is the maximum degree of G.*

Remark 2.9. For any graph G, by Vizing's Theorem, there are color classes $S_1, S_2, ..., S_{\Delta(G)+1}$ of $E(G)$ such that each incident edge of any vertex is in different color classes.

If G is an r-regular graph, there are color classes $S_1, S_2, ..., S_{r+1}$ of the edge set of G such that each incident edge of any vertex is in different color classes. We can also use these color classes to label the vertex set of G such that for each vertex v in G, there is $m \in \{1, 2, ..., r+1\}$ and $v \in S_m$ and incident edges with v belong to color classes $S_1, S_2, ..., S_{r+1}$ but not S_m.

To see this, Since G is r-regular, r incident edges with any vertex v belong to r different color classes from $S_1, S_2, ..., S_{r+1}$, Thus there is exactly one color class, say S_m, so we label v with S_m.

Example 2.10. The graph $C_3 + C_6$ has the maximum degree 2. There are 3 color classes S_1, S_2, S_3 of $E(C_3 + C_6)$ as shown in Figure 2.4 and by Remark 2.9, since 2 incident edges with a belong to color classes S_1 and S_2, $a \in S_3$. Similarly to $b, c, ..., i$. The labeling of $V(C_3 + C_6) \cup E(C_3 + C_6)$ with S_1, S_2, S_3 as shown in Figure 2.5.

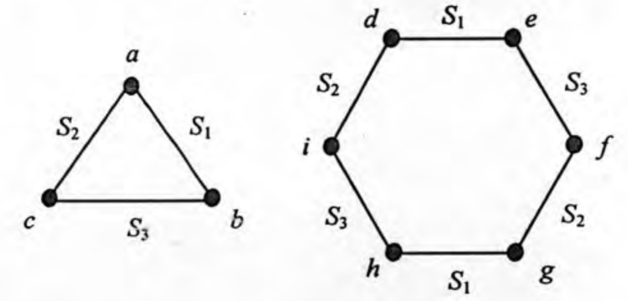

Figure 2.4 : Color classes S_1, S_2, S_3 of $E(C_3 + C_6)$

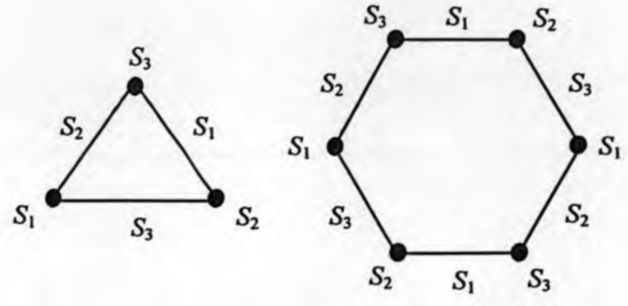

Figure 2.5 : Color classes S_1, S_2, S_3 of $V(C_3 + C_6) \cup E(C_3 + C_6)$

Theorem 2.11. ([5]) *Let k be a positive integer and G be an r-regular graph. If* $\frac{(k-1)(r+1)}{2}$ *is an integer, then the graph G has k compatible neutral labelings with the elements of M(k).*

Proof. Assume that $\frac{(k-1)(r+1)}{2}$ is an integer. By Vizing's Theorem and Remark 2.9, there are color classes $S_1, S_2, ..., S_{r+1}$ of $V(G) \cup E(G)$, for every vertex $v \in V(G)$, there is $m \in \{1, 2, ..., r+1\}$, $v \in S_m$, the incident edges with v belong to color classes $S_1, S_2, ..., S_{r+1}$ but not S_m. Let $M(k) = \left\{-\frac{k}{2}+\frac{1}{2}, -\frac{k}{2}+\frac{3}{2}, ..., \frac{k}{2}-\frac{3}{2}, \frac{k}{2}-\frac{1}{2}\right\}$.

We will construct a compatible neutral labeling of the graph kG with the elements of $M(k)$ by distinguishing into 2 cases.

Case 1 : $r+1$ is even.

For each $l = 1, 2, ..., k$, let $\beta_l : V(G) \cup E(G) \rightarrow M(k)$ be a labeling defined by

$$\beta_l(x) = \begin{cases} l - \frac{k+1}{2} & \text{if } x \in S_m,\ m \text{ is odd,} \\ \frac{k+1}{2} - l & \text{if } x \in S_m,\ m \text{ is even.} \end{cases}$$

When m is odd, we have

$$\begin{aligned} \{\beta_l(x) \mid l = 1, 2, ..., k\} &= \left\{1 - \frac{k+1}{2}, 2 - \frac{k+1}{2}, ..., k - \frac{k+1}{2}\right\} \\ &= \left\{\frac{-k+1}{2}, \frac{-k+3}{2}, ..., \frac{k-1}{2}\right\} \\ &= \left\{-\frac{k-1}{2}, -\frac{k-3}{2}, ..., \frac{k-1}{2}\right\} = M(k) \end{aligned}$$

When m is even, we have

$$\begin{aligned} \{\beta_l(x) \mid l = 1, 2, ..., k\} &= \left\{\frac{k+1}{2} - 1, \frac{k+1}{2} - 2, ..., \frac{k+1}{2} - k\right\} \\ &= \left\{\frac{k-1}{2}, \frac{k-3}{2}, ..., -\frac{k-1}{2}\right\} = M(k) \end{aligned}$$

Thus $\{\beta_l(x) \mid l = 1, 2, ..., k\} = M(k)$ for all $m = 1, 2, ..., r+1$.

Claim that $w_{\beta_l}(v)=0$ for all vertices v in G.

Case 1.1 : $v \in S_m$, m is odd. Thus v is labeled with $l-\dfrac{k+1}{2}$.

The r incident edges with v belong to color classes $S_1, S_2, \ldots, S_{r+1}$ but not S_m.

There are $\dfrac{r+1}{2}-1$ incident edges with v belong to color classes $S_1, S_3, \ldots, S_r$ and are labeled with $l-\dfrac{k+1}{2}$. Another $\dfrac{r+1}{2}$ incident edges incident with v belong to color classes $S_2, S_4, \ldots, S_{r+1}$ and are labeled with $\dfrac{k+1}{2}-l$.

Thus
$$w_{\beta_l}(v)=\beta_l(v)+\sum_{w\in N(v)}\beta_l(vw)$$
$$=\left(l-\frac{k+1}{2}\right)+\left(\frac{r+1}{2}-1\right)\left(l-\frac{k+1}{2}\right)+\left(\frac{r+1}{2}\right)\left(\frac{k+1}{2}-l\right)=0.$$

Case 1.2 : $v \in S_m$, m is even. Thus v is labeled with $\dfrac{k+1}{2}-l$.

The r incident edges with v belong to color classes $S_1, S_2, \ldots, S_{r+1}$ but not S_m.

There are $\dfrac{r+1}{2}-1$ incident edges with v belong to color classes $S_2, S_4, \ldots, S_{r+1}$ and are labeled with $\dfrac{k+1}{2}-l$. Another $\dfrac{r+1}{2}$ incident edges with v belong to color classes $S_1, S_3, \ldots, S_r$ and are labeled with $l-\dfrac{k+1}{2}$.

Thus
$$w_{\beta_l}(v)=\beta_l(v)+\sum_{w\in N(v)}\beta_l(vw)$$
$$=\left(\frac{k+1}{2}-l\right)+\left(\frac{r+1}{2}-1\right)\left(\frac{k+1}{2}-l\right)+\left(\frac{r+1}{2}\right)\left(l-\frac{k+1}{2}\right)=0.$$

Hence G has k compatible neutral labelings with the elements of $M(k)$.

Case 2 : $r+1$ is odd.

By Theorem 2.4, there is no neutral labeling of G with the elements of $M(k)$ for even k. Thus k is odd.

For each $l=1,2,\ldots,k$, let $\beta_l : V(G)\cup E(G)\to M(k)$ be the labeling defined by

$$\beta_l(x)=\begin{cases} l-\frac{k+1}{2} & \text{if } x\in S_m,\ m \text{ is odd},\ m<r,\\ \frac{k+1}{2}-l & \text{if } x\in S_m,\ m \text{ is even},\ m<r,\\ l-1 & \text{if } x\in S_r,\ l\le\frac{k+1}{2},\\ l-k-1 & \text{if } x\in S_r,\ l>\frac{k+1}{2},\\ \frac{k+3}{2}-2l & \text{if } x\in S_{r+1},\ l\le\frac{k+1}{2},\\ \frac{3k+3}{2}-2l & \text{if } x\in S_{r+1},\ l>\frac{k+1}{2}.\end{cases}$$

For $m<r$, similarly in case 1, we have $\{\beta_l(x)\mid l=1,2,\ldots,k\}=M(k)$.

For $m=r$, we have $\{\beta_l(x)\mid l\le\frac{k+1}{2}\}=\{0,1,\ldots,\frac{k-1}{2}\}$ and

$\{\beta_l(x)\mid l>\frac{k+1}{2}\}=\{-\frac{(k-1)}{2},-\frac{(k-3)}{2},\ldots,-1\}$.

Thus $\{\beta_l(x)\mid l=1,2,\ldots,k\}=M(k)$.

For $m=r+1$, we have $\{\beta_l(x)\mid l\le\frac{k+1}{2}\}=\{\frac{k-1}{2},\frac{k-5}{2},\ldots,-\frac{(k-1)}{2}\}$ and

$\{\beta_l(x)\mid l>\frac{k+1}{2}\}=\{\frac{k-3}{2},\frac{k-7}{2},\ldots,-\frac{(k-3)}{2}\}$.

Thus $\{\beta_l(x)\mid l=1,2,\ldots,k\}=M(k)$.

Hence $\{\beta_l(x)\mid l=1,2,\ldots,k\}=M(k)$ for all $m=1,2,\ldots,r+1$.

Claim that $w_{\beta_l}(v)=0$ for all vertex v in G.

Case 2.1 : $v\in S_m$, $m<r$, m is odd, and $l\le\frac{k+1}{2}$.

Thus v is labeled with $l-\frac{k+1}{2}$. The r incident edges with v belong to color classes $S_1,S_2,\ldots,S_{r+1}$ but not S_m. There are 2 incident edges with v belong to color classes S_r and S_{r+1} and they are labeled with $l-1$ and $\frac{k+3}{2}-2l$, respectively.

The $\frac{r-2}{2}$ incident edges with v belong to color classes $S_1,S_3,\ldots,S_{r-1}$ and are labeled with $l-\frac{k+1}{2}$. The last one, $\frac{r-2}{2}$ incident edges with v belong to color classes $S_2,S_4,\ldots,S_{r-2}$ and are labeled with $\frac{k+1}{2}-l$.

Thus $w_{\beta_l}(v) = \beta_l(v) + \sum_{w \in N(v)} \beta_l(vw)$

$$= \left(l - \frac{k+1}{2}\right) + (l-1) + \left(\frac{k+3}{2} - 2l\right) + \left(\frac{r-2}{2}\right)\left(l - \frac{k+1}{2}\right)$$
$$+ \left(\frac{r-2}{2}\right)\left(\frac{k+1}{2} - l\right) = 0$$

Case 2.2 : $v_i \in S_m$, $m < r$, m is even, and $l \le \frac{k+1}{2}$.

Thus v is labeled with $\frac{k+1}{2} - l$. The r incident edges with v belong to color classes $S_1, S_2, ..., S_{r+1}$ but not S_m. There are 2 incident edges with v belong to color classes S_r and S_{r+1} and they are labeled with $l-1$ and $\frac{k+3}{2} - 2l$, respectively.

The $\frac{r}{2}$ incident edges with v belong to color classes $S_1, S_3, ..., S_{r-1}$ and are labeled with $l - \frac{k+1}{2}$. The last one, $\frac{r-2}{2} - 1$ incident edges with v belong to color classes $S_2, S_4, ..., S_{r-2}$ and are labeled with $\frac{k+1}{2} - l$.

Thus $w_{\beta_l}(v) = \beta_l(v) + \sum_{w \in N(v)} \beta_l(vw)$

$$= \left(\frac{k+1}{2} - l\right) + (l-1) + \left(\frac{k+3}{2} - 2l\right) + \left(\frac{r}{2}\right)\left(l - \frac{k+1}{2}\right)$$
$$+ \left(\frac{r-2}{2} - 1\right)\left(\frac{k+1}{2} - l\right) = 0$$

Case 2.3 : $v \in S_m$, $m < r$, m is odd, and $l > \frac{k+1}{2}$.

Thus v is labeled with $l - \frac{k+1}{2}$. The r incident edges with v belong to color classes $S_1, S_2, ..., S_{r+1}$ but not S_m. There are 2 incident edges with v belong to color classes S_r and S_{r+1} and they are labeled with $l-k-1$ and $\frac{3k+3}{2} - 2l$, respectively.

The $\frac{r-2}{2}$ incident edges with v belong to color classes $S_1, S_3, ..., S_{r-1}$ and are labeled with $l - \frac{k+1}{2}$. The last one, $\frac{r-2}{2}$ incident edges with v belong to color classes $S_2, S_4, ..., S_{r-2}$ and are labeled with $\frac{k+1}{2} - l$.

Thus $w_{\beta_l}(v) = \beta_l(v) + \sum_{w \in N(v)} \beta_l(vw)$

$$= \left(l - \frac{k+1}{2}\right) + (l-k-1) + \left(\frac{3k+3}{2} - 2l\right) + \left(\frac{r-2}{2}\right)\left(l - \frac{k+1}{2}\right)$$
$$+ \left(\frac{r-2}{2}\right)\left(\frac{k+1}{2} - l\right) = 0$$

Case 2.4 : $v \in S_m$, $m < r$, m is even, and $l > \frac{k+1}{2}$.

Thus v is labeled with $\frac{k+1}{2} - l$. The r incident edges with v belong to color classes $S_1, S_2, ..., S_{r+1}$ but not S_m. There are 2 incident edges with v belong to color classes S_r and S_{r+1} and they are labeled with $l-k-1$ and $\frac{3k+3}{2} - 2l$, respectively.

The $\frac{r}{2}$ incident edges with v belong to color classes $S_1, S_3, ..., S_{r-1}$ and are labeled with $l - \frac{k+1}{2}$. The last one, $\frac{r-2}{2} - 1$ incident edges with v belong to color classes $S_2, S_4, ..., S_{r-2}$ and are labeled with $\frac{k+1}{2} - l$.

Thus $\quad w_{\beta_l}(v) = \beta_l(v) + \sum_{w \in N(v)} \beta_l(vw)$

$$= \left(\frac{k+1}{2} - l\right) + (l-k-1) + \left(\frac{3k+3}{2} - 2l\right) + \left(\frac{r}{2}\right)\left(l - \frac{k+1}{2}\right)$$
$$+ \left(\frac{r-2}{2} - 1\right)\left(\frac{k+1}{2} - l\right) = 0$$

Case 2.5 : $v \in S_r$ and $l \le \frac{k+1}{2}$. Thus v is labeled with $l-1$.

The r incident edges with v belong to color classes $S_1, S_2, ..., S_{r+1}$ but not S_r. There is 1 incident edge with v belong to color class S_{r+1} labeled with $\frac{k+3}{2} - 2l$.

The $\frac{r}{2}$ incident edges with v belong to color classes $S_1, S_3, ..., S_{r-1}$ and are labeled with $l - \frac{k+1}{2}$. The last one, $\frac{r-2}{2}$ incident edges with v belong to color classes $S_2, S_4, ..., S_{r-2}$ and are labeled with $\frac{k+1}{2} - l$.

Thus $\quad w_{\beta_l}(v) = \beta_l(v) + \sum_{w \in N(v)} \beta_l(vw)$

$$= (l-1) + \left(\frac{k+3}{2} - 2l\right) + \left(\frac{r}{2}\right)\left(l - \frac{k+1}{2}\right) + \left(\frac{r-2}{2}\right)\left(\frac{k+1}{2} - l\right) = 0$$

Case 2.6 : $v_i \in S_r$ and $l > \frac{k+1}{2}$. Thus v is labeled with $l-k-1$.

The r incident edges with v belong to color classes $S_1, S_2, ..., S_{r+1}$ but not S_r.

There is 1 incident edge with v belong to color class S_{r+1} labeled with $\frac{3k+3}{2} - 2l$.

The $\frac{r}{2}$ incident edges with v belong to color classes $S_1, S_3, ..., S_{r-1}$ and are labeled with $l - \frac{k+1}{2}$. The last one, $\frac{r-2}{2}$ incident edges with v belong to color classes $S_2, S_4, ..., S_{r-2}$ and are labeled with $\frac{k+1}{2} - l$.

Thus $$w_{\beta_l}(v) = \beta_l(v) + \sum_{w \in N(v)} \beta_l(vw)$$
$$= (l-k-1) + \left(\frac{3k+3}{2} - 2l\right) + \left(\frac{r}{2}\right)\left(l - \frac{k+1}{2}\right) + \left(\frac{r-2}{2}\right)\left(\frac{k+1}{2} - l\right) = 0$$

Case 2.7 : $v \in S_{r+1}$ and $l \le \frac{k+1}{2}$. Thus v labeled with $\frac{k+3}{2} - 2l$.

The r incident edges with v belong to color classes $S_1, S_2, ..., S_r$.

There is 1 incident edge with v belong to color class S_r labeled with $l-1$.

The $\frac{r}{2}$ incident edges with v belong to color classes $S_1, S_3, ..., S_{r-1}$ and are labeled with $l - \frac{k+1}{2}$. The last one, $\frac{r-2}{2}$ incident edges with v belong to color classes $S_2, S_4, ..., S_{r-2}$ and are labeled with $\frac{k+1}{2} - l$.

Thus $$w_{\beta_l}(v) = \beta_l(v) + \sum_{w \in N(v)} \beta_l(vw)$$
$$= \left(\frac{k+3}{2} - 2l\right) + (l-1) + \left(\frac{r}{2}\right)\left(i - \frac{k+1}{2}\right) + \left(\frac{r-2}{2}\right)\left(\frac{k+1}{2} - l\right) = 0$$

Case 2.8 : $v \in S_{r+1}(m_1 = r+1)$ and $l > \frac{k+1}{2}$. Thus v labeled with $\frac{3k+3}{2} - 2l$.

The r incident edges with v belong to color classes $S_1, S_2, ..., S_r$.

There is 1 incident edge with v belong to color class S_r labeled with $l-k-1$.

The $\frac{r}{2}$ incident edges with v belong to color classes $S_1, S_3, ..., S_{r-1}$ and are labeled with $l - \frac{k+1}{2}$. The last one, $\frac{r-2}{2}$ incident edges with v belong to color classes $S_2, S_4, ..., S_{r-2}$ and are labeled with $\frac{k+1}{2} - l$.

Thus $w_{\beta_l}(v) = \beta_l(v) + \sum_{w \in N(v)} \beta_l(vw)$

$$= \left(\frac{3k+3}{2} - 2l\right) + (l-k-1) + \left(\frac{r}{2}\right)\left(l - \frac{k+1}{2}\right) + \left(\frac{r-2}{2}\right)\left(\frac{k+1}{2} - l\right) = 0$$

Therefore for each $v_i \in V(G)$ and for each $l = 1, 2, ..., k$; $w_\lambda(v_i) = 0$.

Hence G has the k compatible neutral labelings with the element of $M(k)$. □

In fact, three compatible neutral labelings of the graph $C_3 + C_6$ in Example 2.6 are constructed by the method in Theorem 2.11.

Theorem 2.12. ([5]) *Let k be a positive integer. If G is an r-regular super vertex-magic graph and* $\frac{(k-1)(r+1)}{2}$ *is an integer, then the graph kG is super vertex-magic.*

Proof. Let $M(k) = \left\{-\frac{k}{2}+\frac{1}{2}, -\frac{k}{2}+\frac{3}{2}, ..., \frac{k}{2}-\frac{3}{2}, \frac{k}{2}-\frac{1}{2}\right\}$,

G be an r-regular super vertex-magic graph with a super vertex-magic total labeling λ and the magic constant $h = \frac{(v+e)(v+e+1)}{v} - \frac{v+1}{2}$.

Assume that $\frac{(k-1)(r+1)}{2}$ is an integer, by Theorem 2.9, G has k compatible neutral labelings $\beta_1, \beta_2, ..., \beta_k$ with the elements of $M(k)$.

Let $\alpha : V(G) \cup E(G) \to A$ where $A = \left\{\frac{a}{2} \mid a \in \mathbb{Z}^+\right\}$ be a labeling defined by

$$\alpha(x) = k\lambda(x) - \frac{k-1}{2}.$$

Let the graph kG consist of $G_1, G_2, ..., G_k$.

We will construct a super vertex-magic total labeling λ' of the graph kG as follows.

Let $\lambda' : V(kG) \cup E(kG) \to \{1, 2, ..., kv, kv+1, ..., kv+ke\}$ be a labeling defined by

$$\lambda'(v_c) = \alpha(v) + \beta_c(v),$$

$$\lambda'(v_c w_c) = \alpha(vw) + \beta_c(vw).$$

where v_c is a vertex in G_c corresponding to v in G and $v_c w_c$ is an edge in G_c corresponding to vw in G, $c = 1, 2, ..., k$.

From the labeling λ', every $x \in V(G) \cup E(G)$, $k\lambda(x) - \frac{k-1}{2}$ $(\lambda(x) = 1, 2, ..., v+e)$ must be summed individually with each element of $M(k)$.

For $\lambda(x) = 1$: $k\lambda(x) - \frac{k-1}{2} = k - \frac{k-1}{2}$.

Sum of $k - \frac{k-1}{2}$ and each of k elements of $M(k)$ is $\{1, 2, ..., k\}$.

For $\lambda(x) = 2$: $k\lambda(x) - \frac{k-1}{2} = 2k - \frac{k-1}{2}$.

Sum of $2k - \frac{k-1}{2}$ and each of k elements of $M(k)$ is $\{k+1, k+2, ..., 2k\}$.

$\vdots$ $\vdots$

For $\lambda(x) = v$: $k\lambda(x) - \frac{k-1}{2} = kv - \frac{k-1}{2}$.

Sum of $kv - \frac{k-1}{2}$ and each of k elements of $M(k)$ is $\{kv-k+1, kv-k+2, ..., kv\}$.

For $\lambda(x) = v+1$: $k\lambda(x) - \frac{k-1}{2} = kv + k - \frac{k-1}{2}$.

Sum of $vk + k - \frac{k-1}{2}$ and each of k elements of $M(k)$ is $\{kv+1, kv+2, ..., kv+k\}$.

$\vdots$ $\vdots$

For $\lambda(x) = v+e$: $k\lambda(x) - \frac{k-1}{2} = kv + ke - \frac{k-1}{2}$.

Sum of $kv + ke - \frac{k-1}{2}$ and each of k elements of $M(k)$ is

$\{kv+ke-k+1, kv+ke-k+2, ..., kv+ke\}$.

Therefore $\lambda' : V(kG) \cup E(kG) \to \{1, 2, ..., kv, kv+1, ..., kv+ke\}$ is an bijective map.

Claim that $w_{\lambda'}(v_c) = kh - \frac{(k-1)(r+1)}{2}$ for each vertex v_c in G_c.

Let $w_{\lambda'}(v_c)$ be the weight of the vertex $v_i \in G_c$.

We have

$$
\begin{aligned}
w_{\lambda'}(v_c) &= \lambda'(v_c) + \sum_{(w_c) \in N(v_c)} \lambda'(v_c w_c) \\
&= \alpha(v) + \beta_c(v) + \sum_{(w_c) \in N(v_c)} [\alpha(vw) + \beta_c(vw)] \\
&= k\lambda(v) - \frac{k-1}{2} + \beta_c(v) + \sum_{(w_c) \in N(v_c)} [k\lambda(vw) - \frac{k-1}{2} + \beta_c(vw)]
\end{aligned}
$$

$$= k\lambda(v) - \frac{k-1}{2} + \beta_c(v) + \sum_{(w_c)\in N(v_c)} k\lambda(vw) - \sum_{(w_c)\in N(v_c)} \left(\frac{k-1}{2}\right) + \sum_{(w_c)\in N(v_c)} \beta_c(vw)$$

$$= \left[k\lambda(v) + \sum_{(w_c)\in N(v_c)} k\lambda(vw)\right] - \left[\frac{k-1}{2} + \sum_{(w_c)\in N(v_c)} \left(\frac{k-1}{2}\right)\right]$$

$$+ \left[\beta_c(v) + \sum_{(w_c)\in N(v_c)} \beta_c(vw)\right]$$

$$= k\left[\lambda(v) + \sum_{(w_c)\in N(v_c)} \lambda(vw)\right] - \left[\frac{k-1}{2} + \frac{r(k-1)}{2}\right] + 0$$

$$= kh - \frac{(k-1)(r+1)}{2}.$$

Hence the graph kG is a super vertex-magic graph. □

Example 2.13. Since the graph $2K_4$ is a super vertex-magic graph with super vertex-magic total labeling λ and $2K_4$ is 3-regular as shown in Figure 2.6, there are color classes S_1, S_2, S_3, S_4 of $V(C_3+C_6) \cup E(C_3+C_6)$ as shown in Figure 2.7, and two compatible neutral labelings β_1, β_2 of $2K_4$ with the elements of $M(2)=\{-0.5, 0.5\}$ are obtained as shown in Figure 2.8.

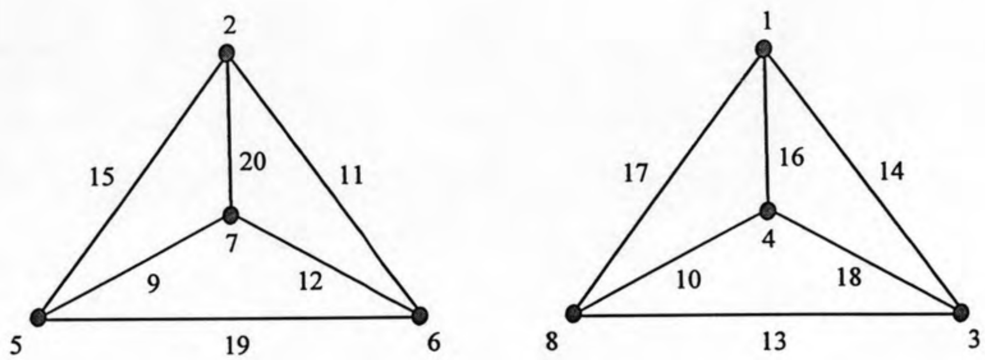

Figure 2.6 : The super vertex-magic total labeling λ of $2K_4$

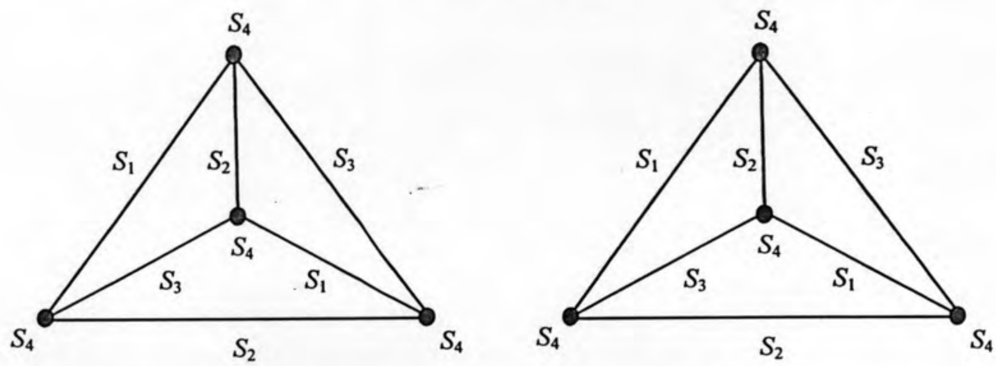

Figure 2.7 : Color classes S_1, S_2, S_3, S_4 of $V(C_3+C_6) \cup E(C_3+C_6)$

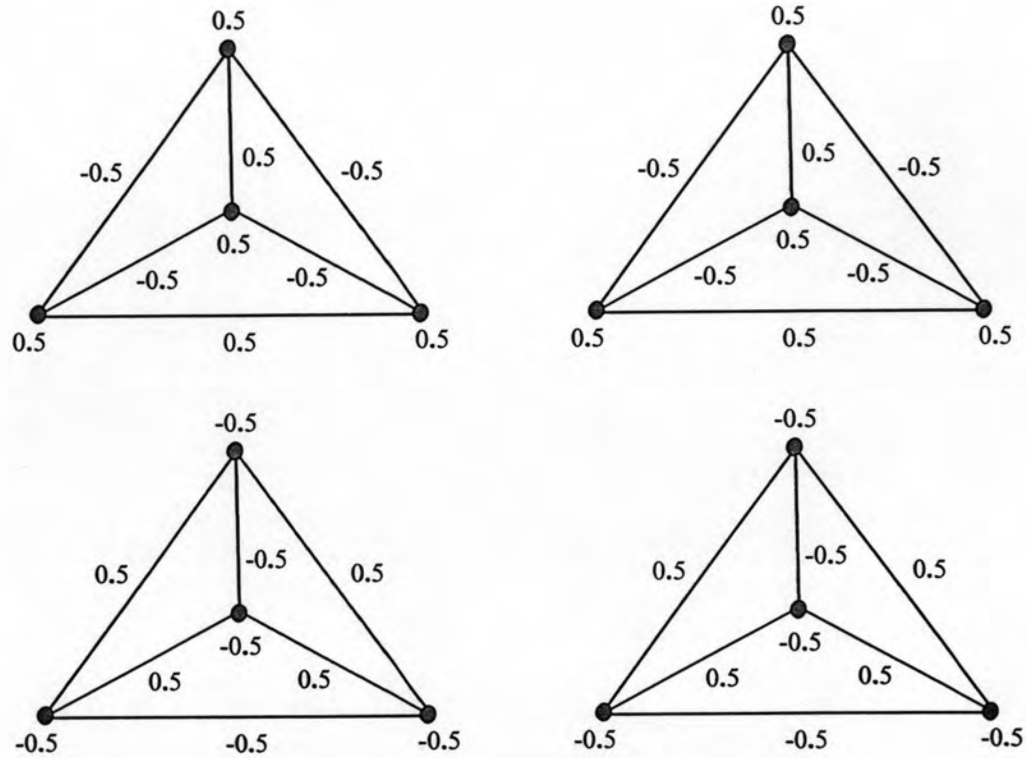

Figure 2.8 : Two compatible neutral labelings β_1, β_2 of $2K_4$ with the elements of $M(2) = \{-0.5, 0.5\}$

By Theorem 2.11, the labeling α of $2K_4$ and the super vertex-magic total labeling λ' of $4K_4$ are shown in Figure 2.9 and Figure 2.10, respectively.

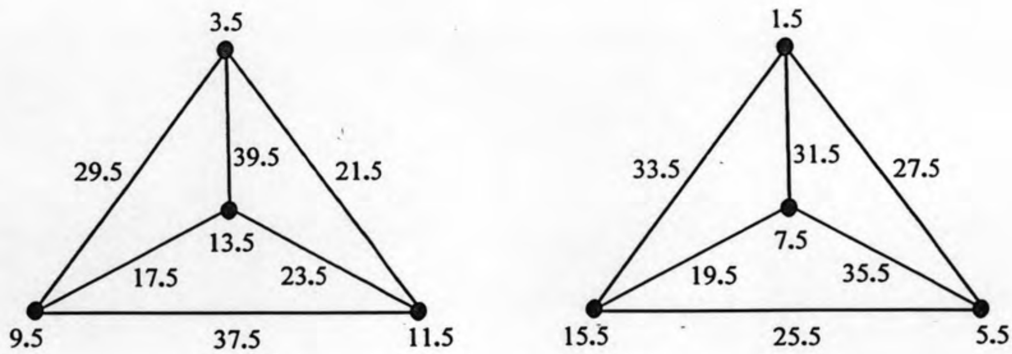

Figure 2.9 : The labeling α of $2K_4$

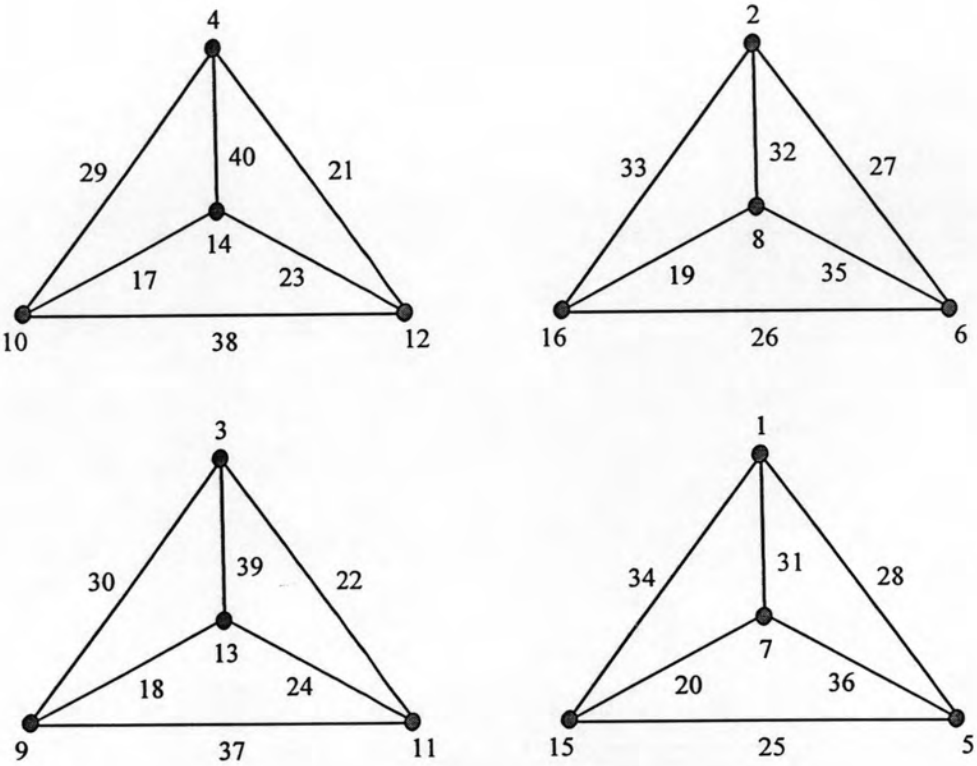

Figure 2.10 : The super vertex-magic total labeling λ' of $4K_4$

Example 2.14. The graph $C_3 + C_6$ is a super vertex-magic graph as shown in Figure 2.11 and a super vertex-magic graph $3(C_3 + C_6)$ is shown in Figure 2.12.

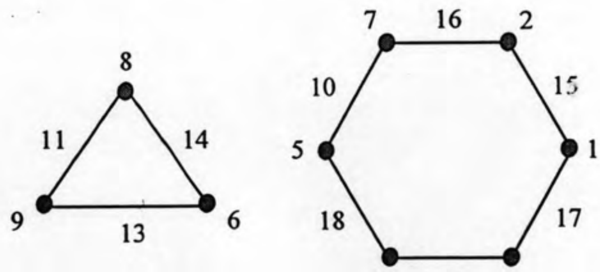

Figure 2.11 : Super vertex-magic graph $C_3 + C_6$

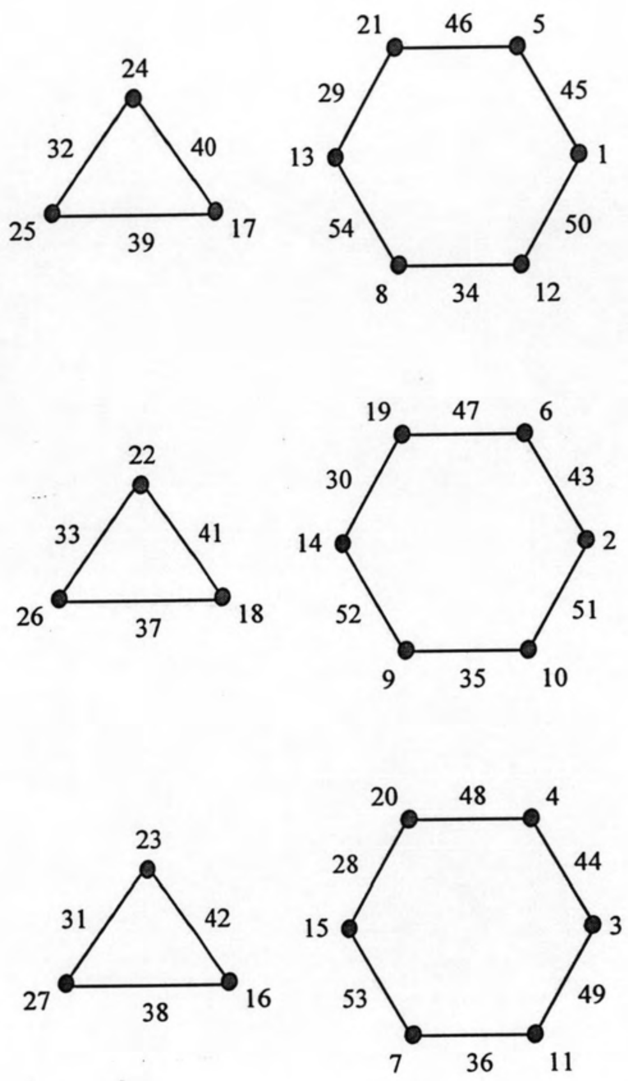

Figure 2.12 : Super vertex-magic graph $3(C_3 + C_6)$