CHAPTER III

THEORETICAL BASIS OF MASWM

3.1 Overview

To involve with MASWM, investigator should be familiar with the digital signal processing and the theory of wave propagation. Therefore, in this chapter the author intends to provide some useful materials to easily get access to MASWM, for instance, spatial sampling, temporal sampling, and Two Dimensional Fast Fourier Transform (2DFFT).

Since the MASWM relies on the Rayleigh waves, it is significant to be aware that these waves travel along the boundary of the earth and air. Indeed, the wave with short wavelength interacts only with the top layer while the long wave penetrates deeper and involves with underlying ground layer. Therefore, the waves contain the information of only the layer they are subject to.

3.2 Notation of Useful Relationship

Since we know that the incident waves of different frequencies (or different wavelengths) travel along the surface with its own unique phase velocities, we can recall the relationship in equation (2.1):

$$\lambda = \frac{V}{f} \tag{3.1}$$

Due to the wavelength and wavenumber have a connection (equation 3.1), it is possible to make a link between phase velocity and wavenumber to easily obtain the dispersion curve from f-k processing.

$$\lambda = \frac{2\pi}{k} \tag{3.2}$$

By substituting the equation (3.1) into (3.2), the phase velocity expression can be simply derived:

$$V = f \cdot \frac{2\pi}{k} \tag{3.3}$$

3.3 Spatial Sampling Rate for MASWM Test

The length of the geophone array is one of the most important field parameter that controls the maximum depth of investigation. It is configured and varied in according with the inversion algorithms, the total number of receivers, the available space in field, and the energy of the source. For example, if the number of receivers is fixed, a narrow spacing array implies a short spread with shallow prospecting depth. The wide geophone spacing might cause the loss of the uppermost layer information; nevertheless, the longer spread length allows a better resolution of wavenumber (k) in f-k domain. The maximum wavenumber not affected by spatial aliasing is associated with the receiver spacing (*d*) by Nyquist criterion:

$$k_{Nyquist} = \frac{2\pi}{2d} = \frac{\pi}{d} \tag{3.4}$$

The resolution of wavenumber k can be determined by:

$$\Delta k = \frac{2\pi}{N.d} \tag{3.5}$$

where N is the total number of geophone in the array line.

For connection with the application prospect, the maximum depth of investigation is equal to one-half of the longest wavelength (Rix et al., 1991). To deal with near and far field effects (explain in the next section), another perspective has been drawn form Tokimatsu (1995) and Stokoe et al. (1998) that the estimated wavelength should not smaller than one-third and larger than two times of inter-geophone distance. In addition, SeisImager/SW Manual (2005) also proposed that inter-geophone distance and spread length are connected to one-half of minimum and maximum wavelength respectively.

3.4 Temporal Sampling Rate for MASWM test

The temporal sampling rate is used to discretize an analog signal in equal interval in time domain. This concept duplicates the same idea to section 3.3 but in time axis. The temporal interval controls the resolution in frequency domain through the relationship:

$$\Delta f = \frac{2\pi}{M.dt} \tag{3.6}$$

Where Δf is the interval between consecutive discrete frequency

M is the number of samples

dt is the temporal sampling rate

From equation (3.6), a small sampling interval leads to a low resolution in frequency; however, it can be improve by increasing the number of samples. Due to digitization in time, it is known that the highest meaningful frequency from the measurement is limited to one-half of sampling frequency. This criterion is known as Nyquist frequency, expressed by

$$f_{Nyquist} = \frac{1}{2\Delta t} \tag{3.7}$$

During test design step, it is important to match the Nyquist frequency with the frequency range of interest. For instance, in geotechnical investigation, the useful frequency range is around 100Hz to 150Hz corresponding to the sampling frequency of 200 to 300 Hz.

3.5 Two Dimensional Fast Fourier Transform (2D FFT)

2D FFT is an interesting geophysical approach for data analysis. The idea is extended from discrete Fourier transform by treating spatial variable like temporal variable. The 2D-FFT is used to transform the signal in time-space domain to frequency-wavenumber domain which can be expressed in mathematical form as

$$P(f,k) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} p(x,t)e^{-i2\pi(ft+kx)}dxdt \tag{3.8}$$

where $p(x,t)$ present a wavefield function in time-space domain (*t*-*x*). Since the wavefield of time and space are gathered in discrete form, the real operation of 2D-FFT is done by its discrete version:

$$P(u,v) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} p(x,y)e^{-i2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)} \tag{3.9}$$

where M, N, *u*, *v* are the number of samples at the interval *dt* in time domain, the number of geophones at the interval *dx* in spatial domain, *u* and *v* are the temporal and spatial frequencies, respectively.

The advantage of this method is that it guarantees the transformation from the unreadable raw data in t-x domain to f-k domain without loss of information.

3.6 Body Wave and Surface Wave

If an infinite homogeneous medium is subjected to wave propagation, two types of waves can be deduced that are the compressional wave P or primary wave, and the distorsional wave S or secondary wave. The speed of propagation varies according to the elastic properties of the material. For compressional wave, its velocity, V_P, can be expressed by

$$V_P = \sqrt{\frac{\lambda + 2\mu}{\rho}} \tag{3.10}$$

where $\mu = G$. (3.11)

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} , \tag{3.12}$$

μ and λ are Lame's elastic constants of the medium, G is the shear modulus, E is the Young's modulus, ν is the Poisson's ratio, and ρ is the mass density. For distorsional wave, its velocity, V_S, is equal to

$$V_S = \sqrt{\frac{\mu}{\rho}} . \tag{3.13}$$

3.7 Rayleigh Waves in an Infinite Homogeneous Half-Space

The interaction between P and S waves generates Rayleigh wave which attenuates exponentially with depth and induces no stresses at the free interface. The Rayleigh wave was first introduced by John Strutt Lord of Rayleigh (1885) to solve the problem of free vibration for an elastic half-space. When the free boundary (elastic media) condition is imposed on general equation of wave propagation, the behavior of soil response is described by using elastic model.

The general equilibrium equation can be written as:

$$\sigma_{ij,j} + \rho . f_i = \rho . \ddot{u}_i \tag{3.14}$$

where σ_{ij} is the stress tensor, u_i is the displacement vector of a material point, ρ is the mass density per unit volume and f_i is the body force per unit mass. For the assumption of isotropic linear elastic medium, the stress-strain relationship can be written as:

$$\sigma_{ij} = \lambda \cdot \varepsilon_{kk} \cdot \delta_{ij} + 2\mu \cdot \varepsilon_{ij} \tag{3.15}$$

The small stain tensor is formulated by:

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{3.16}$$

Differentiating equation (3.15) and substituting with equation (3.16) into equation (3.14) we can obtain the Navier's dynamical equilibrium:

$$(\lambda + \mu) \cdot u_{j,ji} + \mu \cdot u_{i,jj} + \rho \cdot f_i = \rho \cdot \ddot{u}_i \tag{3.17}$$

For plane strain condition and discarding the infinite solutions at the infinite depth, the solution of Rayleigh wave can be obtained by characteristic equation:

$$K^6 - 8K^4 + (24 - 16\gamma^2) \cdot K^2 + 16 \cdot (\gamma^2 - 1) = 0 \tag{3.18}$$

where K and γ are the following ratios between velocities of longitudinal (P), distortional (S) and Rayleigh (R) waves:

$$K = \frac{V_R}{V_S} \tag{3.19}$$

$$\gamma = \frac{V_S}{V_P}\ . \tag{3.20}$$

The roots of the K^2cubic equation are a function of Poisson's ratio since $\gamma^2 = \frac{1-2\nu}{2(1-\nu)}$. After Viktorov (1967), only one real and acceptable solution exists for real media ($0 < \nu < 0.5$). Figure 3.1 shows the relationship between the velocities of waves and Poisson's ratio.

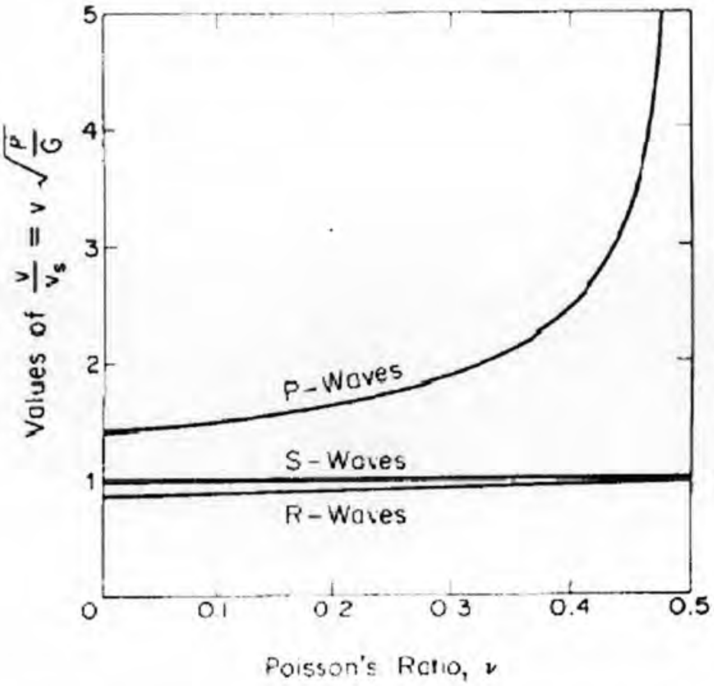

Figure 3-1 **Relation between Poisson's ratio and velocity of propagation of compression (P), shear (S) and Rayleigh (R) waves in a linear elastic homogeneous half-space (Foti, 2000)**

Therefore, the approximated solution can be drawn:

$$K = \frac{0.87 + 1.12\nu}{1 + \nu} \tag{3.21}$$

From Figure 3-1 and eq. (3.21), it can be seen that the difference between shear wave velocity and Rayleigh wave velocity is limited in a range:

$$0.87 < \frac{V_R}{V_S} < 0.96 \ . \tag{3.22}$$

From the solution, the phase velocity of Rayleigh wave does not depend on the frequency, thus the dispersion does not exist when wave propagates in homogeneous half-space.

The particle motion of Rayleigh wave is retrograde on free surface, composed of the horizontal and vertical components which are physically perpendicular. Since the vertical component have bigger amplitude than that of the horizontal one, thus during propagation it generates the elliptical shape (Figure 3-2). Below a certain depth about $1/2\pi$ times of wavelength, the ellipse becomes prograde (Figure 3-3). The two components travel along the surface with the same velocity, but their amplitudes attenuate differently along depth. The particle motion degrades quickly and becomes negligible at a certain depth (Figure 3-3). This phenomenon implies the characteristic of Rayleigh wave which disturbs only the shallow one and interacts with the mechanical properties of such layers.

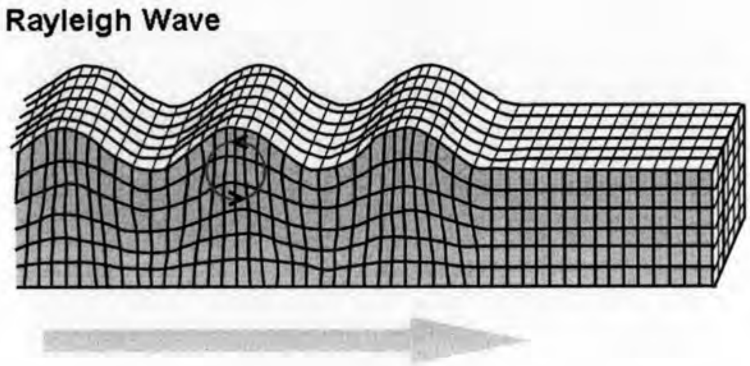

Figure 3-2 Rayleigh disturbance on the free surface of a homogeneous half-space

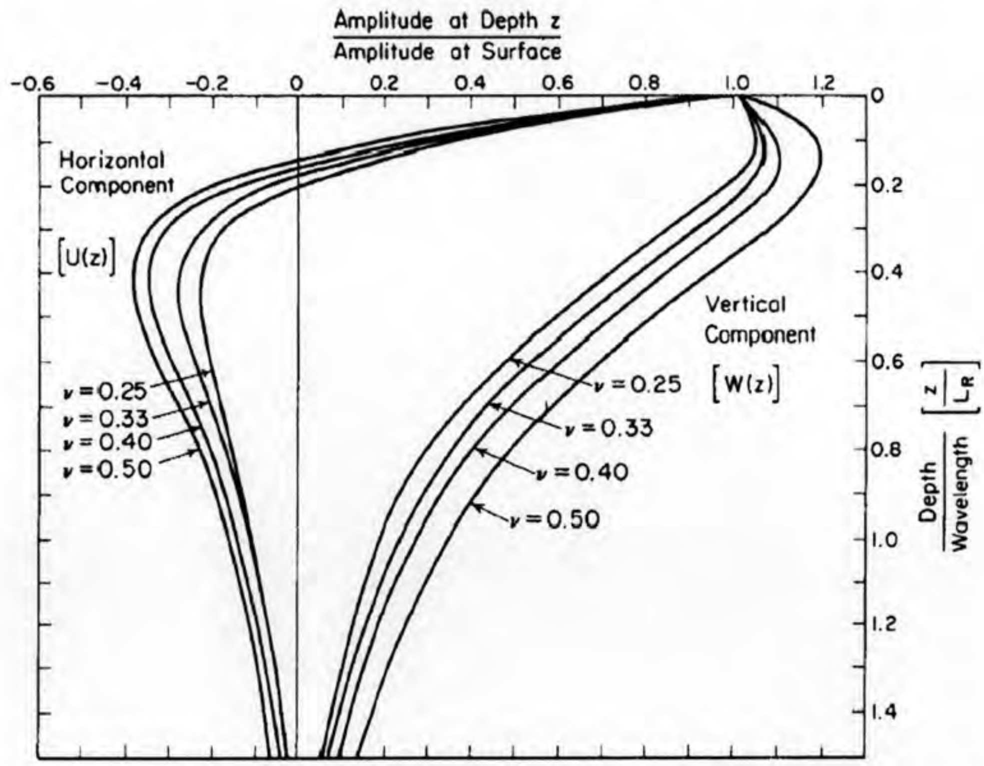

Figure 3-3 Amplitude ratio vs. dimensionless depth for Rayleigh wave in a homogenous half-space (Foti, 2000)

3.8 Rayleigh Wave in an Heterogeneous Media

The mathematical formulation for Rayleigh wave in heterogeneous and anisotropic media is more complex than the homogenous case and can be expressed as

$$(\lambda + \mu)\nabla\nabla.\boldsymbol{u} + \mu\nabla^2\boldsymbol{u} + e_z\frac{d\lambda}{dz}\nabla.\boldsymbol{u} + \frac{d\mu}{dz}\left(e_z \times \nabla \times \boldsymbol{u} + 2 \cdot \frac{\partial \boldsymbol{u}}{\partial z}\right) = \rho\ddot{\boldsymbol{u}} \qquad (3.20)$$

where e_z is the base vector for the direction perpendicular to the free surface

Generally, a numerical solution is needed to solve this problem because it is not possible to solve it analytically. Therefore, it is better to explain its physical meaning in phenomenological way. The geometrical dispersion characteristic, i.e. Rayleigh phase velocity dependence on frequency is easily described by definition in equation 3.1. Since the wavelength λ also has connection with frequency *f*, it is clear that low frequency wave will penetrate deeper into soil layer. The surface waves at different frequency will propagate and interact at different layers in a vertically heterogeneous ground and consequently the phase velocity will be related to their mechanical properties. The phenomenon is summarized in figure 3-4 in term of vertical displacements of two different frequency waves in layered medium.

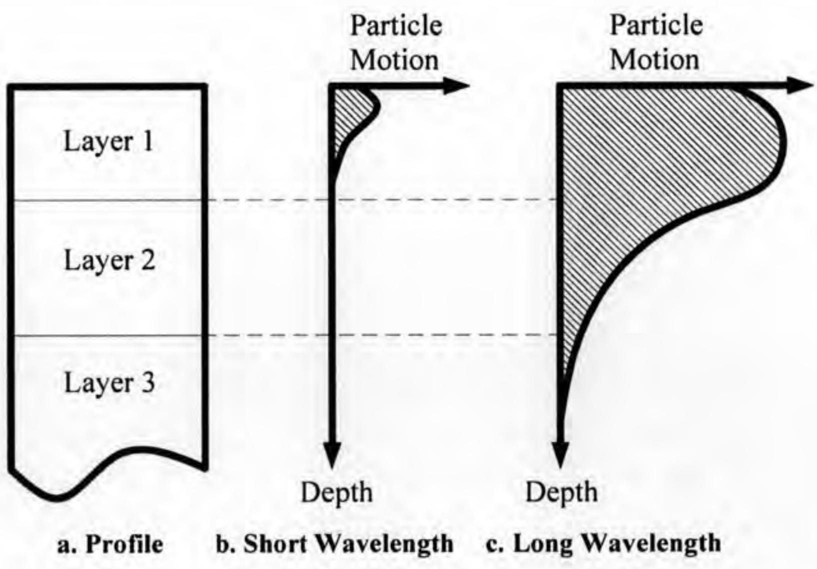

Figure 3-4 The geometrical dispersion in a layered medium (Rix, 1988)

3.9 Near-Field and Far-Field Effects

The near-field effect happens when the distance between seismic sources to the first geophone is not sufficient for planar or fundamental mode of Rayleigh wave to occur. It will result in a lack of linear coherency in phase at the lower frequencies. For the fact that the wave with high frequency components attenuate quickly with distance away from the source, its energy is not able to dominate a large receiver offset. This phenomenon is called far-field effect. It consequently shows the reduction in linear coherency and amplitude of the arrival waves due to interference between low-velocity surface waves and high-velocity body waves.

To avoid near field effect, the near-offset (the distance between the source and the first geophone) should be estimated by one-half of wavelength (λ) to let Rayleigh wave develops as horizontally travelling plane wave (Stokoe et al., 1994). Nevertheless, other investigators also reported that Rayleigh waves would appear where the offset is larger than one-half of the first layer thickness (Xu et al., 2006). However, after many site tests the nearest field should be set shorter when using small energy sources and longer with strong impact sources. For common practice, SeisImager/SW Manual (2005) suggests that the minimum shot location to the first geophone should be set about 10 to 20% of spread length.