CHAPTER II

THEORY OF COMPLEX RANDOM MATRIX

In this chapter, the density of non-central quadratic form on complex random matrices and their joint eigenvalues densities are derived for applications to information theory. These densities are represented by complex hypergeometric functions of matrix arguments, which can be expressed in terms of complex zonal polynomials and invariant polynomials.

Once these joint densities are established, in the next chapter, simplified but convenient expressions will be used to evaluate and analyze one of the most important information theoretic measures, namely, the ergodic capacities for the various MIMO configurations in Rayleigh fading correlated and uncorrelated channel conditions.

2.1 Quadratic Forms on Complex Random Matrices

The probability distributions of Complex Random Matrix (CRM) are often derived in terms of complex hypergeometric functions of matrix arguments. Complex hypergeometric function of two complex matrix arguments with Hermitian $A \in \mathbb{C}^{n \times n}$, $B \in \mathbb{C}^{m \times m}$ $(n \leq m)$, is given as

$$
\begin{aligned}
&{}_p\widetilde{F}_q^{(n)}\left(a_1,...,a_p;b_1...,b_q;A,B\right) \triangleq \\
&\sum_{k=0}^{\infty}\sum_{\kappa}\frac{[a_1]_\kappa\cdots[a_p]_\kappa}{[b_1]_\kappa\cdots[b_q]_\kappa}\,\frac{\widetilde{C}_\kappa(A)\widetilde{C}_\kappa(B)}{k!\,\widetilde{C}_\kappa(\mathbf{I}_m)},
\end{aligned}
\tag{2.1}
$$

where

- $a_1,\cdots,a_p;\ b_1,\cdots,b_q$ are complex constants,

- $\kappa = (k_1, k_2, \ldots k_n)$ is a partition of the integer k not more than n part with $k_1 \geq k_2 \geq \cdots \geq k_n \geq 0$ and $k = k_1 + k_2 + \cdots + k_n$,
- $[a]_\kappa \triangleq \prod_{i=1}^{n} (a - i + 1)_{k_i}$ is the multivariate hypergeometric coefficient with $[a]_0 = 1$.

For the case that of ${}_0\widetilde{F}_0^{(n)}(A, B)$ is the complex hypergeometric function of two Hermitian matrix arguments with Wishart distribution, it is defined by

$$
{}_0\widetilde{F}_0^{(n)}(A, B) \triangleq \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\widetilde{C}_\kappa(A)\, \widetilde{C}_\kappa(B)}{k!\, \widetilde{C}_\kappa(\mathbf{I}_m)}, \tag{2.2}
$$

where $\widetilde{C}_\kappa(A) \triangleq \chi_{[\kappa]}(1)\, \chi_{[\kappa]}(A)$ is complex zonal polynomial of a Hermitian matrix A [86], $\chi_{[\kappa]}(1)$ is the dimension of the representation $[\kappa]$ of the symmetric group and expressed as

$$
\chi_{[\kappa]}(1) = k! \frac{\prod_{i<j}^{n} (k_i - k_j - i + j)}{\prod_{i=1}^{n} (k_i + n - i)!}, \tag{2.3}
$$

and $\chi_{[\kappa]}(A)$ is the character of the representation $[\kappa]$ of the linear group, given as a symmetric function of the eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$ of A by the following equation

$$
\chi_{[\kappa]}(A) = \left. \frac{\left| \lambda_i^{k_j + n - j} \right|}{\left| \lambda_i^{n-j} \right|} \right|_{(i, j = 1, \ldots, n;\, i<j)}. \tag{2.4}
$$

Note that both the real and complex zonal polynomials are particular cases of the Jack polynomial $C_\kappa^{(\alpha)}(X)$ with general α. Refer [86] for more details. Complex and real zonal polynomial are $\alpha = 1$ and $\alpha = 2$ respectively. The superscript of Jack polynomials will be dropped, i.e., $C_\kappa(X) \triangleq C_\kappa^{(1)}(X)$.

The complex zonal polynomials own the essential properties as given in [87]:

1. The unique decomposition property:

$$\left(\mathrm{tr}(X)\right)^k = \sum_\kappa C_\kappa(X) \tag{2.5}$$

where $\mathrm{tr}(X)$ is the trace of matrix.

2. The splitting property:

$$\begin{aligned} &\int_{U(m)} C_\kappa\left(AXBX^\dagger\right)(dX) \\ &= \frac{C_\kappa(A)\, C_\kappa(B)}{C_\kappa(\mathbf{I}_n)}, \end{aligned} \tag{2.6}$$

where

- $d(X)$ is the invariant measure on the unitary group $U(m)$, normalized to make the total measure unity.

3. The reproductive property:

$$\begin{aligned} &\frac{1}{\tilde{\Gamma}_n(a)} \int \mathrm{etr}(X) \det(X)^{a-m}\, \tilde{C}_\kappa(XY)(dX) \\ &= [a]_\kappa \tilde{C}_\kappa(Y), \end{aligned} \tag{2.7}$$

where

- $\mathrm{Re}(a) > n-1$,

Moreover, the complex multivariate Gamma function is denoted as

$$\begin{aligned}\widetilde{\Gamma}_n(a) &\triangleq \pi^{n(n-1)/2} \underbrace{\prod_{i=1}^{n} \Gamma(a-i+1)}_{\triangleq \Gamma_n(a)} \\ &= \pi^{n(n-1)/2} \Gamma_n(a),\end{aligned} \tag{2.8}$$

$$\mathbb{R}(a) > n-1,$$

with $\Gamma(n) = \int_0^\infty t^{n-1} e^{-t} dt$, being the standard Gamma function [88].

and $\widetilde{C}_\kappa(\mathbf{I}_m)$ (in Eq. (2.2)) is expressed as

$$\widetilde{C}_\kappa(\mathbf{I}_m) = k! \frac{\left[\prod_{i<j}^{m} (k_i - k_j - i + j)\right]^2}{\left[\prod_{i=1}^{m} \Gamma(k_i + m - i + 1) \prod_{i=1}^{m} \Gamma(m-i+1)\right]}. \tag{2.9}$$

Next is a description a class of homogeneous polynomials $C_\phi^{\kappa,\tau}(X,Y)$ of degrees k and t in the elements of the $m \times m$ Hermitian matrices X, and Y, respectively. These polynomials are invariants under the simultaneous transformations:

$$\begin{aligned}&X \rightarrow E^H X E, \\ &Y \rightarrow E^H Y E, \\ &E \in U(m),\end{aligned} \tag{2.10}$$

where

- $U(m)$ denotes the group of unitary $m \times m$ matrices, i.e.,

$$U(m) = \left\{E \in \mathbb{C}^{m \times m};\ E^H E = \mathbf{I}_m\right\}, \tag{2.11}$$

Moreover, these polynomials satisfy the relationship

$$\int_{U(m)} C_\kappa\left(AE^H XE\right) C_\tau\left(BE^H YE\right)(dE)$$
$$= \sum_{\phi \in \kappa,\tau} \frac{C_\phi^{\kappa,\tau}(A,B) C_\phi^{\kappa,\tau}(X,Y)}{C_\phi(\mathbf{I})}, \tag{2.12}$$

where

- C_κ, C_τ, and C_ϕ are zonal polynomials indexed by the ordered partitions
- κ, τ, and ϕ of the nonnegative integers k, t, and $f = k + t$, respectively, in not more than m parts.

If we assume $Gl(m, \mathbb{C})$ denote the general linear group of $m \times m$ nonsingular complex matrices, then $\phi \in \kappa, \tau$ denotes the irreducible representation of $Gl(m, \mathbb{C})$ indexed by 2ϕ that occurs in the decomposition of the Kronecker product $2\kappa \otimes 2\tau$ of the irreducible representations indexed by 2κ and 2τ.

2.2 Complex Central Wishart Matrix

If assuming that there exists a type of correlated Rayleigh fading, the channel matrix $\mathbf{H}$ is a complex Gaussian random matrix with zero mean. The covariance matrix of $\mathbf{H}$ is assumed to be of the particular form $\mathbf{\Psi}_\mathrm{T} \otimes \mathbf{\Psi}_\mathrm{R}$, where $\otimes$ denotes the Kronecker product, $\mathbf{\Psi}_\mathrm{T} \in \mathbb{C}^{N_\mathrm{T} \times N_\mathrm{T}}$, and $\mathbf{\Psi}_\mathrm{R} \in \mathbb{C}^{N_\mathrm{R} \times N_\mathrm{R}}$ are Complex Positive Definite Hermitian (CPDH) matrices, $\mathbf{\Psi}_\mathrm{T}$ and $\mathbf{\Psi}_\mathrm{R}$ can be viewed as the covariance matrices at the transmitter side and at the receiver side, respectively. The probability density function (pdf) of $\mathbf{H}$ is well known by James to be given who has proposed in [87] with following theorem:

Theorem 1 *If we suppose that there exists a correlated Rayleigh type of fading and let the channel matrix* $\mathbf{H} \sim \widetilde{N}_{N_\mathrm{T},N_\mathrm{R}}\left(0, \mathbf{\Psi}_\mathrm{T} \otimes \mathbf{\Psi}_\mathrm{R}\right)$ *be a* $N_\mathrm{R} \times N_\mathrm{T}$ $(N_\mathrm{T} \geq N_\mathrm{R})$ *CRM with zero mean* $E[\mathbf{H}] = 0$, *distributed as Gaussian,* $\widetilde{N}_{N_\mathrm{T},N_\mathrm{R}}(\cdot,\cdot)$ *denotes the complex normal distribution, the pdf of* $\mathbf{H}$ *is given by*

$$P(\mathbf{H}) = \pi^{-N_\mathrm{T}N_\mathrm{R}} \left|\mathbf{\Psi}_\mathrm{T}\right|^{-N_\mathrm{R}} \left|\mathbf{\Psi}_\mathrm{R}\right|^{-N_\mathrm{T}}$$

$$\times \exp\left[-\mathrm{tr}\left(\mathbf{\Psi}_\mathrm{R}^{-1}\mathbf{H}\ \mathbf{\Psi}_\mathrm{T}^{-1}\mathbf{H}^H\right)\right], \tag{2.13}$$

In the next theorem, the density of quadratic forms on CRM will be presented.

Theorem 2 *From Theorem 1, the pdf of the positive definite quadratic form* $\mathbf{S}$ *in* $\mathbf{H}$ *associated with L is denoted by* $\mathbf{S} = \mathbf{H}L\mathbf{H}^H \sim \widetilde{Q}_{N_\mathrm{T},N_\mathrm{R}}\left(L, \mathbf{\Psi}_\mathrm{T} \otimes \mathbf{\Psi}_\mathrm{R}\right)$. $\widetilde{Q}_{N_\mathrm{T},N_\mathrm{R}}(\cdot,\cdot)$ *denotes the distribution of matrix* $\mathbf{S}$ *and* $L \in \mathbb{C}^{N_\mathrm{R} \times N_\mathrm{R}}$ *is a CPDH, is given by*

$$P(\mathbf{S}) = \left(\tilde{\Gamma}_{N_\mathrm{R}}(N_\mathrm{T}) \left|L\mathbf{\Psi}_\mathrm{T}\right|^{N_\mathrm{R}} \left|\mathbf{\Psi}_\mathrm{R}\right|^{N_\mathrm{T}}\right)^{-1} \left|\mathbf{S}\right|^{N_\mathrm{T}-N_\mathrm{R}}$$

$$\times \exp\left[-q^{-1}\mathrm{tr}\left(\mathbf{\Psi}_\mathrm{R}^{-1}\mathbf{S}\right)\right] {}_0\widetilde{F}_0^{(N_\mathrm{T})}\left(\mathrm{T}, q^{-1}\mathbf{\Psi}_\mathrm{R}^{-1}\mathbf{S}\right), \tag{2.14}$$

where

- q is positive scalar chosen to make $P(\mathbf{S})$ converge fast, and
- $\mathrm{T} = \mathbf{I}_{N_\mathrm{T}} - qL^{-1/2}\mathbf{\Psi}_\mathrm{T}^{-1}L^{-1/2}$.

From the generalized density, we can easily derive other well-known densities. For example, if $L = \mathbf{I}_{N_\mathrm{T}}$, $\mathbf{\Psi}_\mathrm{T} = \mathbf{I}_{N_\mathrm{T}}$ and $\mathbf{\Psi}_\mathrm{R} = \mathbf{\Psi}$, then $\mathbf{S} = \mathbf{H}\mathbf{H}^H$ is a complex Wishart distribution, denoted by $\widetilde{W}_{N_\mathrm{R}}\left(N_\mathrm{T}, \mathbf{\Psi}\right)$.

In the next theorem, the joint eigenvalue density of quadratic forms on CRMs is given. From this theorem, the joint eigenvalue densities are easily derived.

Theorem 3 *Consider the* $N_{\mathrm{R}} \times N_{\mathrm{R}}$ *CPDH matrix* $\mathbf{S} = \mathbf{H} L \mathbf{H}^{H} \sim \widetilde{Q}_{N_{\mathrm{T}}, N_{\mathrm{R}}} \left(L, \boldsymbol{\Psi}_{\mathrm{T}} \otimes \boldsymbol{\Psi}_{\mathrm{R}}\right)$, $\boldsymbol{\Lambda} = diag\left(\lambda_1, \ldots, \lambda_{N_{\mathrm{R}}}\right)$ *is a diagonal matrix composed of the ordered nonzero eigenvalues of* $\mathbf{S}$. *Then, the joint pdf of the* N_{R} *nonzero ordered positive eigenvalues* $\lambda_1 \geq \cdots \geq \lambda_{N_{\mathrm{R}}} > 0$ *of* $\mathbf{S}$ *is*

$$P(\boldsymbol{\Lambda}) = \frac{\pi^{N_{\mathrm{R}}(N_{\mathrm{R}}-1)} \prod_{i<j}^{N_{\mathrm{R}}} \left(\lambda_i - \lambda_j\right)^2 \prod_{j=1}^{N_{\mathrm{R}}} \lambda_j^{N_{\mathrm{T}}-N_{\mathrm{R}}}}{\left|\boldsymbol{\Psi}_{\mathrm{R}}\right|^{N_{\mathrm{T}}} \left|L \boldsymbol{\Psi}_{\mathrm{T}}\right|^{N_{\mathrm{R}}} \widetilde{\Gamma}_{N_{\mathrm{R}}}(N_{\mathrm{T}}) \widetilde{\Gamma}_{N_{\mathrm{R}}}(N_{\mathrm{R}})} \times {}_0\widetilde{F}_0^{(N_{\mathrm{R}}),(N_{\mathrm{T}})}\left(-L^{-1/2} \boldsymbol{\Psi}_{\mathrm{T}}^{-1} L^{-1/2}, -\boldsymbol{\Psi}_{\mathrm{R}}^{-1}, \boldsymbol{\Lambda}\right), \tag{2.15}$$

where

- ${}_0\widetilde{F}_0^{(N_{\mathrm{R}}),(N_{\mathrm{T}})}(\cdot,\cdot,\cdot)$ is the hypergeometric function of three matrix arguments, and is defined similarly as in (2.2). Therefore, $P(\boldsymbol{\Lambda})$ can be expressed as

$$P(\boldsymbol{\Lambda}) = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\widetilde{C}_{\kappa}\left(L^{-1/2} \boldsymbol{\Psi}_{\mathrm{T}}^{-1} L^{-1/2}\right) \widetilde{C}_{\kappa}\left(-\boldsymbol{\Psi}_{\mathrm{R}}^{-1}\right) \widetilde{C}_{\kappa}(\boldsymbol{\Lambda})}{k!\, \widetilde{C}_{\kappa}\left(\mathbf{I}_{N_{\mathrm{T}}}\right) \widetilde{C}_{\kappa}\left(\mathbf{I}_{N_{\mathrm{R}}}\right)} \times \frac{\pi^{N_{\mathrm{R}}(N_{\mathrm{R}}-1)} \prod_{i<j}^{N_{\mathrm{R}}} \left(\lambda_i - \lambda_j\right)^2 \prod_{j=1}^{N_{\mathrm{R}}} \lambda_j^{N_{\mathrm{T}}-N_{\mathrm{R}}}}{\left|\boldsymbol{\Psi}_{\mathrm{R}}\right|^{N_{\mathrm{T}}} \left|L \boldsymbol{\Psi}_{\mathrm{T}}\right|^{N_{\mathrm{R}}} \widetilde{\Gamma}_{N_{\mathrm{R}}}(N_{\mathrm{T}}) \widetilde{\Gamma}_{N_{\mathrm{R}}}(N_{\mathrm{R}})}. \tag{2.16}$$

Finally, these densities will be used to evaluate and analyze one of the most important information theoretic measures, namely the channel ergodic capacity of the MIMO Rayleigh correlated and uncorrelated channels, where spatial fading correlation at the receiver side is examined. This topic will be investigated in the next chapter.