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ANALYSIS OF ROTATIVE MAPPINGS ON $\mathbb{R}$

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In this work, we prove a fixed point theorem of rotative mappings on the set of real numbers and determine the fixed point sets of these mappings. Also, the rotativeness of mappings in some classical classes in fixed point theory is investigated.

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CHAPTER I

INTRODUCTION

The concept of rotative mappings introduced by Goebel and Koter [8], was first used on a map defined on a closed convex subset of a Banach space. It can be considered on a map defined on a metric space as well. In this chapter, let us recall the terminology that will be used throughout our work.

Definition 1.1. *Let (X, d) be a metric space, and $T : X \to X$ a mapping. For a positive constant k, T is called* **k-Lipschitzian** *if*

$$d(Tx, Ty) \leq kd(x, y)$$

for all $x, y \in X$. The mapping T is called **Lipschitzian** *if it is k-Lipschitzian for some $k > 0$, T is called* **nonexpansive** *if it is 1-Lipschitzian, and T is called* **contraction** *if it is k-Lipschitzian for some $k < 1$.*

Definition 1.2. *Let X be a metric space and $n \in \mathbb{N}$. A function $T : X \to X$ is said to be* **n-periodic** *if $T^n = I$. (Here $T^n = T \circ T \circ \cdots \circ T$ is the n-fold composition.)*

It is widely known that Banach contraction principle was formulated in Banach's thesis in 1922. He showed that any contraction mapping on a complete metric space has a unique fixed point. Banach's result has inspired many authors for further investigations. Several authors try to generalize his result by giving some conditions of geometrical nature or finding more classes of mappings. One of them is replacing contraction mappings by nonexpansive mappings. However, nonexpansive mappings on a complete metric space may or may not have fixed points. In order to assure the existence of fixed points for nonexpansive mappings on Banach spaces, some conditions need to be imposed on the space. The following theorems are some of the well-known results.

Recall that for a space X and for a class τ of mappings $T : X \to X$, the space X is said to have **fixed point property** with respect to τ if each mapping T in τ, T has a fixed point. A Banach space X is said to be **uniformly convex** if for any ε, $0 < \varepsilon \leq 2$ and $x, y \in X$ with $\|x\| \leq 1$, $\|y\| \leq 1$, and $\|x - y\| \geq \varepsilon$, there exists a $\delta = \delta(\varepsilon) > 0$ such that $\|(x + y)/2\| \leq 1 - \delta$. A nonempty convex subset C of a Banach space X is said to have **normal structure** if each nonempty convex bounded subset D of C there exists $x_0 \in D$ such that

$$\sup\{\|x_0 - x\| : x \in D\} < \operatorname{diam}(D)$$

and the Banach space X is said to have **normal structure** if every closed convex bounded subset C of X with $\operatorname{diam}(C) > 0$ has normal structure.

Theorem 1.3. [4, Theorem 2.5, p. 20] *If C is a compact convex subset of a Banach space X and $T : C \to C$ is nonexpansive, then T has a fixed point in C.*

Theorem 1.4. (Browder's theorem and Göhde's theorem [1, p. 229]) *Let X be a uniformly convex Banach space and C a nonempty closed convex bounded subset of X. Then every nonexpansive mapping $T : C \to C$ has a fixed point in C.*

Theorem 1.5. [16, p. 51] *Let X be a reflexive Banach space and C a nonempty closed convex bounded subset of X which has normal structure. Then every nonexpansive mapping $T : C \to C$ has a fixed point.*

Theorem 1.6. (Kirk's fixed point theorem [1, p. 230]) *Let X be a Banach space and C a nonempty weakly compact convex subset of X with normal structure. Then every nonexpansive mapping $T : C \to C$ has a fixed point in C.*

Another way to guarantee the existence of a fixed point for a nonexpansive mapping, is to put some additional properties on the mapping itself. In 1981, Goebel and Koter [8] introduced a new class of mappings called a rotative mapping. These mappings are quite natural in the class of nonexpansive mappings and there are plenty of examples.

Remark 1.7. *Let $x \in X$ and $f : X \to X$. For simplicity, we will sometimes write fx instead of $f(x)$.*

Definition 1.8. *Let (X, d) be a metric space and C a nonempty subset of X. A mapping $f : C \to C$ is said to be a* **rotative mapping** *on C if there exist $n \geq 2$, $0 \leq a < n$ such that for any $x \in C$*

$$d(x, f^n x) \leq ad(x, fx).$$

If f is rotative on C with the parameters n $(n \geq 2)$ and a $(0 \leq a < n)$, we may refer to f as an (n, a)-rotative mapping, or simply by an n-rotative mapping.

We denote by $\Phi(C, n, a, k)$ the class of all (n, a)-rotative and k-Lipschitzian mappings on a nonempty closed convex subset C of a Banach space.

The following theorem shows that the condition of rotativeness is actually quite strong. It assures the existence of fixed point(s) of nonexpansive mappings without boundedness or any special geometric structure on C required.

Theorem 1.9. [8] Let C be a nonempty closed convex subset of a Banach space. If $T \in \Phi(C, n, a, 1)$ for some $n \geq 2$ and $0 \leq a < n$, then T has a fixed point.

Moreover, Goebel and Koter [9] showed that rotativeness also assures the existence of fixed points in the case of k-Lipschitzian mappings with k slightly greater than 1 as stated in the next theorem.

Theorem 1.10. ([9], see also [14, p. 324–327]) Let C be a nonempty closed convex subset of a Banach space X. For each $n \geq 2$ and $0 \leq a < n$, there exists $\gamma > 1$ such that if $k < \gamma$, every mapping T in $\Phi(C, n, a, k)$ has a fixed point.

Clearly, γ in Theorem 1.10 depends on X, n, and a. Therefore it is natural to consider the function $\gamma(X, n, a)$ defined by

$$\begin{aligned}\gamma(X, n, a) = \inf \{\, k \in [0, \infty) \mid & \text{ there is a nonempty closed convex subset } C \text{ of} \\ & X \text{ and } T \in \Phi(C, n, a, k) \text{ with } \operatorname{Fix} T = \varnothing \,\}.\end{aligned}$$

We can write γ in another form as follow.

$$\begin{aligned}\gamma(X, n, a) = \sup\{k \in [0, \infty) \mid & \text{ for every nonempty closed convex subset } C \text{ of } X, \\ & \text{if } T \in \Phi(C, n, a, k), \text{ then } T \text{ has a fixed point}\}.\end{aligned}$$

According to Theorem 1.10, it is known that $\gamma(X, n, a) > 1$ for any Banach space X, $n \geq 2$, and $a \in [0, n)$. By now upper bounds and lower bounds of $\gamma(X, n, a)$ have been obtained for some X, n, and a. However, the precise value of $\gamma(X, n, a)$ is completely unknown for any X, n, a. Here is a list of known results on $\gamma(X, n, a)$, and $\gamma(H, n, a)$ where X is a Banach space and H is a Hilbert space.

The first estimation came from the work of Goebel and Kirk. Goebel [6] obtained that for Lipschitzian mapping $T : C \to C$ satisfying $T^2 = I$ has a fixed point, where C is any closed convex subset of a Banach space, and Kirk [15] generalized Goebel's result on the case when $T^n = I$ for some integer $n > 1$. From their results, it is obtained that

$$\gamma(X, n, 0) \geq \begin{cases} 2, & \text{if } n = 2; \\ \sqrt[n-1]{\frac{1}{n-2}\left(-1 + \sqrt{n(n-1) - \frac{1}{n-1}}\right)}, & \text{if } n > 2. \end{cases} \tag{1.1}$$

See also in [9]. Moreover, Goebel and Koter [9], and Goebel and Kirk [7, Chapter

17], obtained some information on $\gamma(X,2,a)$ for $a \in (0,2)$ that

$$\gamma(X,2,a) \geq \max\left\{\frac{1}{2}\left(2-a+\sqrt{(2-a)^2+a^2}\right),\right.$$
$$\left.\frac{1}{8}\left(a^2+4+\sqrt{(a^2+4)^2-64a+64}\right)\right\}.$$

Note that the first term gives a better estimation for $a \in [0, 2(\sqrt{2}-1)]$, while the second one for $a \in [2(\sqrt{2}-1), 2)$.

As mentioned before that $\gamma(X,n,a)$ may depend on space X, so many researchers examined on different spaces to find this precise value γ. Goebel [5], Goebel and Koter [9], obtained an estimation for $\gamma(C[0,1],2,a)$, where $a \in (1,2)$ and $C[0,1]$ is the space of real-valued continuous function on $[0,1]$,

$$\gamma(C[0,1],2,a) \leq \frac{1}{a-1}.$$

Koter [18] obtained $\gamma(H,2,0) \geq \sqrt{\pi^2-3} \approx 2.6209$, see also in [7], and Komorowski [17] obtained $\gamma(H,2,a) \geq \sqrt{\frac{5}{a^2+1}}$ for any $a \in [0,2)$, which is the best known estimation for a Hilbert space (see also in [19]). After that Koter [19] obtained $\gamma(H,n,0)$ for $n = 3,4,5,6$.

$$\gamma(H,3,0) \geq 1.3666, \qquad \gamma(H,4,0) \geq 1.1962,$$
$$\gamma(H,5,0) \geq 1.0849, \text{ and } \quad \gamma(H,6,0) \geq 1.0228.$$

All this evaluations are slightly better than those obtained in (1.1) in a general Banach space X. Indeed, it follows from (1.1) that

$$\gamma(X,3,0) \geq 1.3452, \qquad \gamma(X,4,0) \geq 1.065,$$
$$\gamma(X,5,0) \geq 1.0351, \text{ and } \quad \gamma(X,6,0) \geq 1.022.$$

But Koter's procedure cannot be applied in order to estimate the value of $\gamma(H,n,0)$ when $n > 6$.

In 2005, Górnicki and Pupka [12] gave a better estimation for $\gamma(X, n, a)$ for $n \geq 3$ by using Halpern's idea [13] of iterative procedure, that

$$\gamma(X,3,0) \geq 1.3821, \qquad \gamma(X,4,0) \geq 1.2524,$$
$$\gamma(X,5,0) \geq 1.1777, \text{ and } \quad \gamma(X,6,0) \geq 1.1329.$$

Recently, García and Nathansky [3] gave the best estimation known nowadays for Hilbert spaces. They are

$$\gamma(H,3,0) \geq 1.5549, \qquad \gamma(H,4,0) \geq 1.3267,$$
$$\gamma(H,5,0) \geq 1.2152, \text{ and } \quad \gamma(H,6,0) \geq 1.1562.$$

In addition, different spaces from above were evaluated, such as L^p or ℓ^p for $1 < p < \infty$, Banach space X with the modulus convexity δ_X, and Banach space X with uniformly convex norm [11, 18, 19].

From the list of results given above, we see that even the largest lower bound of $\gamma(X, n, a)$ is smaller than 3. So it is natural to ask the following questions:

Q1: In what space X is $\gamma(X, n, a)$ the largest?

Q2: Can we find a Banach space X, $n \geq 2$, $a \in [0, n)$, and a function $T \in \Phi(C, n, a, k)$, for $k > 3$ with $\operatorname{Fix} T \neq \varnothing$?

Other questions concerning the function γ are the following.

Q3: For a Banach space X, what is a good estimation for $\gamma(X, n, 0)$?
Is $\gamma(X, n, 0) < \infty$?

Q4: From the list given above, we know that $\gamma(C[0,1], 2, a) \leq \frac{1}{a-1}$ for $a \in (1, 2)$.
But nothing is known for $a \in [0, 1)$.
Is $\gamma(C[0,1], 2, a) < \infty$ for some $a \in [0, 1)$?
Is $\gamma(C[0,1], 2, a) = \infty$ for some $a \in [0, 1)$?

Q5: Can we find a precise value of $\gamma(X, n, a)$ for some X, n, and a?

Q6: For a Banach space X and $n \geq 2$, is $\gamma(X, n, \cdot) : [0, n) \to (1, \infty]$ continuous?

The purpose of our research is to investigate the properties of rotative mappings and give answer to the above questions, at least in case that the mappings are real valued.

This thesis is organized as follows:

In Chapter 2, we consider various examples of functions in some classical classes in fixed point theory such as contraction, contractive, n-periodic, etc. Some necessary and sufficient conditions for the rotativeness of those functions are provided.

In Chapter 3, we prove that every continuous rotative mappings on a closed real interval has a fixed point. The result leads us to obtain a precise value of $\gamma(\mathbb{R}, n, a)$ for any $n \geq 2$ and $a \in [0, n)$. Finally, a characterization of the fixed point sets of continuous rotative mappings is obtained.

CHAPTER II

ANALYSIS ON A CLASS OF ROTATIVE MAPPINGS

It is mentioned by K. Goebel and M. Koter in [8, p. 115] that "We feel that rotativeness is quite natural metrical assumption. However, we are aware of the fact that for concrete mapping T, it may be difficult to check whether it is rotative or not." This statement motivates us not only to investigate the existence of fixed points of rotative mappings but also to study some characterizations of rotative mappings.

We first investigate the relations between rotative mappings and other types of mappings such as contractions, periodic mappings, linear maps, piecewise linear maps, etc.

2.1 Contraction, Contractive, Nonexpansive, and n-Periodic Maps

Proposition 2.1. *Let (X, d) be a metric space and $f : X \to X$. If f is a contraction, then f is n-rotative for every $n \geq 2$.*

Proof. Let f be a contraction and let $r \in (0, 1)$ be such that $d(fx, fy) \leq rd(x, y)$ for all $x, y \in X$. Then

$$\begin{aligned} d(f^2x, x) &\leq d(f^2x, fx) + d(fx, x) \\ &\leq rd(fx, x) + d(fx, x) \\ &= (r + 1)d(fx, x) \quad \text{for all } x \in X. \end{aligned}$$

So f is a $(2, r+1)$-rotative mapping. In general, for every $n \geq 2$,

$$\begin{aligned} d(f^n x, x) &\leq d(f^n x, f^{n-1}x) + d(f^{n-1}x, f^{n-2}x) + \cdots + d(fx, x) \\ &\leq \left(r^{n-1} + r^{n-2} + \cdots + r + 1\right) d(fx, x) \quad \text{for all } x \in X. \end{aligned}$$

So f is an n-rotative mapping for every $n \geq 2$. This completes the proof. $\square$

Although every contraction map is n-rotative for every $n \geq 2$, a contractive map or a nonexpansive map may not be n-rotative for any n. By the following examples we will show that there exists a contractive map which is not n-rotative for any $n \geq 2$. Also there exists a 2-rotative mapping which is not n-rotative for any $n \geq 3$.

Example 2.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \ln(1 + e^x)$ for all $x \in \mathbb{R}$. Then by mean value theorem, for some c lies between x and y*

$$|f(x) - f(y)| = |f'(c)(x - y)| = \left|\frac{e^c}{1 + e^c}\right| |x - y| < |x - y|,$$

for every $x \neq y$. Therefore f is a contractive map. Next let $b, c > 0$. By L'Hôpital rule, we have

$$\lim_{x\to\infty} \frac{x - \ln(c + e^x)}{x - \ln(b + e^x)} = \frac{c}{b} \lim_{x\to\infty} \frac{b + e^x}{c + e^x} = \frac{c}{b}.$$

In particular,

$$\lim_{x\to\infty} \left|\frac{x - f^2(x)}{x - f(x)}\right| = \lim_{x\to\infty} \left|\frac{x - \ln(2 + e^x)}{x - \ln(1 + e^x)}\right| = 2.$$

So for each $a \in (0, 2)$, there exists $M > 0$ such that $|x - f^2(x)| > a|x - f(x)|$ for every $x > M$. This implies that f is not 2-rotative. In general, we have

$$\lim_{x\to\infty} \left|\frac{x - f^n(x)}{x - f(x)}\right| = \lim_{x\to\infty} \left|\frac{x - \ln(n + e^x)}{x - \ln(1 + e^x)}\right| = n.$$

So for each $a \in (0, n)$, there exists an x such that $|x - f^n(x)| > a\,|x - f(x)|$. Therefore f is not n-rotative for any $n \geq 2$.

The next example shows that some 2-rotative mapping is not n-rotative for any $n \geq 3$.

Example 2.3. *Let X be a normed linear space and $f : X \to X$ given by $f(x) = -2x$. We will show that f is 2-rotative but not n-rotative for any $n \geq 3$. First consider*

$$\left\|x - f^2 x\right\| = \|x - 4x\| = 3\|x\| = \|x - fx\| \quad \text{for all } x \in X.$$

So f is $(2,1)$-rotative. But

$$\frac{\|x - f^3 x\|}{\|x - fx\|} = \frac{9\|x\|}{3\|x\|} = 3 \quad \text{for every } x \in X - \{0\}.$$

So f is not 3-rotative. In general,

$$\frac{\|x - f^n x\|}{\|x - fx\|} \geq \frac{2^n - 1}{3} \quad \text{for every } x \in X - \{0\}.$$

It is easy to prove by induction that $\frac{2^n-1}{3} \geq n$ for every $n \geq 4$. Therefore f is not n-rotative for any $n \geq 3$.

However, if a mapping T is nonexpansive and n-rotative, then T is also m-rotative for any $m > n$. This can be seen in the next proposition.

Proposition 2.4. *Let (X, d) be a metric space and $T : X \to X$ a nonexpansive mapping. If T is n-rotative, then it is also m-rotative for all $m > n$.*

Proof. Let T be nonexpansive and n-rotative. Firstly we note that for any $x \in X$ and $k \in \mathbb{N}$ $d(f^{k+1}x, f^k x) \leq d(fx, x)$. For $m > n$, we obtain

$$\begin{aligned}
d(f^m x, x) &\leq d(f^m x, f^{m-1}x) + d(f^{m-1}x, f^{m-2}x) + \cdots + d(f^{n+1}x, f^n x) + d(f^n x, x) \\
&\leq d(fx, x) + \cdots + d(fx, x) + ad(fx, x) \\
&= (m - n + a)d(fx, x).
\end{aligned}$$

So T is $(m, m - n + a)$-rotative. $\square$

We end this section by giving a class of rotative mappings. These mappings are called n-periodic. Recall that a mapping f on a metric space into itself is n-periodic on (X, d) if $d(f^n x, x) = 0$ for every $x \in X$. So we have the following proposition.

Proposition 2.5. *Let (X, d) be a metric space. Then for $n \geq 2$, every n-periodic mapping on X is n-rotative.*

Proof. It is obvious, since $d(f^n x, x) = 0 \leq d(fx, x)$ for all $x \in X$. □

2.2 Affine Maps

It is noticed that some of the affine maps on a normed linear space are rotative. In this section, we give a characterization of affine maps which are rotative.

Theorem 2.6. *Let X be a normed linear space. Let $n \geq 2$, $x_0 \in X$, and $f : X \to X$ given by $f(x) = cx + x_0$, where c is a scalar.*

(i) If $c \neq 1$, then f is n-rotative if and only if $\left|\frac{c^n - 1}{c-1}\right| < n$.

(ii) If $c = 1$, then f is n-rotative if and only if $x_0 = 0$.

Proof. We have $fx = cx + x_0$, $f^2 x = c^2 x + cx_0 + x_0$, and in general, $f^n x = c^n x + c^{n-1} x_0 + c^{n-2} x_0 + \cdots + cx_0 + x_0$. So

$$f^n x - x = (c^n - 1)x + (c^{n-1} + c^{n-2} + \cdots + c + 1)x_0.$$

If $c = 1$, then $\|f^n x - x\| = n \|x_0\|$ and $\|fx - x\| = \|x_0\|$. From this it is easy to see that

$$f \text{ is } n\text{-rotative if and only if } x_0 = 0.$$

If $c \neq 1$, then

$$\begin{aligned} f^n x - x &= (c^n - 1)x + \frac{c^n - 1}{c - 1} x_0 \\ &= \frac{c^n - 1}{c - 1}((c-1)x + x_0) = \frac{c^n - 1}{c - 1}(fx - x). \end{aligned}$$

Therefore $\|f^n x - x\| = \left|\frac{c^n-1}{c-1}\right| \|fx - x\|$ for all $x \in X$. Hence

$$f \text{ is } n\text{-rotative if and only if } \left|\frac{c^n - 1}{c - 1}\right| < n.$$

This completes the proof. □

Corollary 2.7. *Let X be a normed space over $\mathbb{R}$, $x_0 \in X$, and $f : X \to X$ be of the form $f(x) = cx + x_0$. Then*

(i) f is 2-rotative if and only if $-3 < c < 1$.

(ii) f is 3-rotative if and only if $-2 < c < 1$.

Proof. By Theorem 2.6 and $|c + 1| < 2$ if and only if $c \in (-3, 1)$, f is 2-rotative if and only if $-3 < c < 1$. And since $|c^2 + c + 1| < 3$ if and only if $c \in (-2, 1)$, f is 3-rotative if and only if $-2 < c < 1$. □

2.3 Functions of Dirichlet Type

It is well-known that the Dirichlet function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 1$ if x is rational and $f(x) = 0$ if x is irrational, is nowhere continuous. However, many functions similarly to this one are rotative. In this section, we give some conditions to assure that any Dirichlet type is rotative.

Example 2.8. *Let $b \in \mathbb{Q}$, $c \in \mathbb{Q}^c$, and let $f : \mathbb{R} \to \mathbb{R}$ be given by*

$$f(x) = \begin{cases} b, & \text{if } x \in \mathbb{Q}; \\ c, & \text{if } x \notin \mathbb{Q}. \end{cases}$$

If $x \in \mathbb{Q}$, then $f(x) = b$ and $f^2(x) = f(b) = b$. Similarly if $x \notin \mathbb{Q}$, then $f(x) = c = f^2(x)$. Therefore $|x - f^2(x)| = |x - f(x)|$ for all $x \in \mathbb{R}$. This implies that

$$|x - f(x)| = |x - f^n(x)| \quad \textit{for every } x \in \mathbb{R} \textit{ and } n \in \mathbb{N}.$$

Hence f is n-rotative for every $n \geq 2$.

In general, we have the following theorem.

Theorem 2.9. *Let $g, h : \mathbb{R} \to \mathbb{R}$ be n-rotative and $A \subseteq \mathbb{R}$, and let $f : \mathbb{R} \to \mathbb{R}$ be given by*

$$f(x) = \begin{cases} g(x), & \textit{if } x \in A; \\ h(x), & \textit{if } x \notin A. \end{cases}$$

If $\{g(x) \mid x \in A\} \subseteq A$ and $\{h(x) \mid x \in A^c\} \subseteq A^c$, then f is n-rotative.

Proof. Let $a_1, a_2 \in (0, n)$ be such that $|x - g^n(x)| \leq a_1|x - g(x)|$, for all $x \in \mathbb{R}$ and $|x - h^n(x)| \leq a_2|x - h(x)|$, for all $x \in \mathbb{R}$.

Let $x \in A$. Since $f(x) = g(x)$ and $g(x) \in A$, we have

$$f^2(x) = f(f(x)) = f(g(x)) = g(g(x)) = g^2(x).$$

In general, $f^n(x) = g^n(x)$ for every $n \in \mathbb{N}$. Therefore

$$|x - f^n(x)| = |x - g^n(x)| \leq a_1|x - g(x)| = a_1|x - f(x)|.$$

Similarly, if $x \notin A$, then $f^n(x) = h^n(x)$ for every $n \in \mathbb{N}$, and we have

$$|x - f^n(x)| \leq a_2|x - f(x)|.$$

Let $a = \max\{a_1, a_2\}$. Then $a \in (0, n)$ and $|x - f^n(x)| \leq a|x - f(x)|$ for all $x \in \mathbb{R}$. This shows that f is n-rotative. $\square$

Corollary 2.10. *Let c_1, c_2, d_1, d_2 be rational numbers. Let $f : \mathbb{R} \to \mathbb{R}$ be given by*

$$f(x) = \begin{cases} c_1 x + d_1, & \text{if } x \in \mathbb{Q}; \\ c_2 x + d_2, & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Then the following statements hold.

(i) If c_1, $c_2 \in (-3, 1)$ and $c_2 \neq 0$, then f is 2-rotative.

(ii) If $c_1 \in (-3, 1)$ and $c_2 = 0$, then f is 2-rotative if and only if $c_1 d_2 + d_1 - d_2 = 0$.

Proof. It follows immediately from Corollary 2.7 and Theorem 2.9 that (i) holds. For (ii), let $c_2 = 0$. If $c_1 d_2 + d_1 - d_2 \neq 0$, then

$$\lim_{\substack{x \to d_2 \\ x \in \mathbb{Q}^c}} \frac{|f^2 x - x|}{|fx - x|} = \lim_{\substack{x \to d_2 \\ x \in \mathbb{Q}^c}} \left| \frac{c_1 d_2 + d_1 - x}{d_2 - x} \right| = +\infty,$$

which implies that f is not 2-rotative. Next assume that $c_1 d_2 + d_1 - d_2 = 0$. If $x \in \mathbb{Q}$, then similar to the calculation in Theorem 2.6, we obtain that

$$\left| f^2 x - x \right| \leq |1 + c_1| \, |fx - x| \, .$$

If $x \notin \mathbb{Q}$, then

$$\left| f^2 x - x \right| = |c_1 d_2 + d_1 - x| = |d_2 - x| = |fx - x| \, .$$

Let $a = \max\{1, |1 + c_1|\}$. Then $a \in [0, 2)$ and $|f^2 x - x| \leq a \, |fx - x|$ for every $x \in \mathbb{R}$. Therefore f is 2-rotative. □

Next we give an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is n-rotative for infinitely many n but is not m-rotative for infinitely many m.

Example 2.11. *Let $b \in \mathbb{Q}$, $c \in \mathbb{Q}^c$, and let $f : \mathbb{R} \to \mathbb{R}$ be given by*

$$f(x) = \begin{cases} c, & x \in \mathbb{Q}; \\ b, & x \in \mathbb{Q}^c. \end{cases}$$

It is easy to check that

$$\textit{if } x \in \mathbb{Q}, \textit{ then } f^n(x) = \begin{cases} c, & \textit{if } n \textit{ is odd;} \\ b, & \textit{if } n \textit{ is even,} \end{cases} \quad \textit{and}$$

$$\textit{if } x \in \mathbb{Q}^c, \textit{ then } f^n(x) = \begin{cases} b, & \textit{if } n \textit{ is odd;} \\ c, & \textit{if } n \textit{ is even.} \end{cases}$$

So if n is odd, then $x - f^n(x) = x - f(x)$ for all $x \in \mathbb{R}$. Therefore f is $(n,1)$-rotative for every odd integer $n \geq 3$. If $m \geq 2$ is even, we let $x = \frac{mb-c}{m-1}$, so that $x \in \mathbb{Q}^c$ and $\frac{x-f^m(x)}{x-f(x)} = \frac{x-c}{x-b} = m$. Therefore f is not m-rotative for any even integer $m \geq 2$.

2.4 Piecewise Linear Maps

Throughout this section, let c_1, c_2, b_1, $b_2 \in \mathbb{R}$ be such that $c_1 < c_2 < b_1 < b_2$.

Proposition 2.12. *Let $c = \frac{c_2-c_1}{b_2-b_1}$ and let $f : \mathbb{R} \to \mathbb{R}$ be given by*

$$f(x) = \begin{cases} c_1, & x \leq b_1; \\ c(x-b_1) + c_1, & b_1 < x < b_2; \\ c_2, & x \geq b_2. \end{cases}$$

Then

(i) f is continuous and $\operatorname{Fix} f = \{c_1\}$.

(ii) f is n-rotative if and only if $b_1 > \frac{nc_2-c_1}{n-1}$.

Proof. It is obvious that f is continuous and $\operatorname{Fix} f = \{c_1\}$. Next we will prove that f is n-rotative if and only if $b_1 > \frac{nc_2-c_1}{n-1}$. Let $x \in \mathbb{R}$ and $n \geq 2$.

If $x \leq b_1$, then $f(x) = c_1$ and $f^n(x) = c_1$, and so

$$|x - f^n(x)| = |x - c_1| = |x - f(x)| \tag{2.1}$$

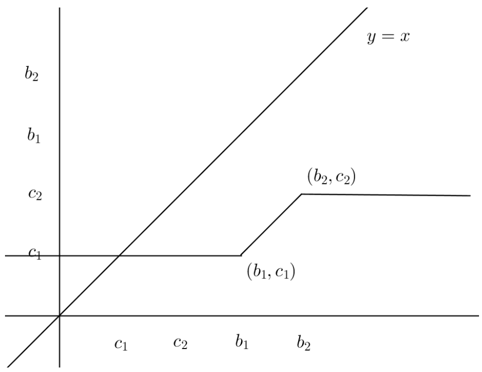

Figure 2.1: The graph of f when $c = \frac{c_2 - c_1}{b_2 - b_1}$.

If $x \geq b_2$, then $f(x) = c_2$ and $f^n(x) = c_1$, and thus

$$\frac{|x - f^n(x)|}{|x - f(x)|} = \frac{x - c_1}{x - c_2} = 1 + \frac{c_2 - c_1}{x - c_2} \tag{2.2}$$

If $b_1 < x < b_2$, then $f(x) < c_2$, $f^n(x) = c_1$, and therefore

$$\frac{|x - f^n(x)|}{|x - f(x)|} = \frac{x - c_1}{x - f(x)} < \frac{x - c_1}{x - c_2} = 1 + \frac{c_2 - c_1}{x - c_2} \tag{2.3}$$

In conclusion, we have for every $n \geq 2$

$$\frac{|x - f^n(x)|}{|x - f(x)|} \leq \begin{cases} 1, & \text{if } x \leq b_1 \text{ and } x \neq c_1; \\ 1 + \frac{c_2 - c_1}{x - c_2}, & \text{if } x > b_1. \end{cases}$$

From this, we see that

$$\sup\left\{\left|\frac{x - f^n(x)}{x - f(x)}\right| \mid x \in \mathbb{R} - \{c_1\}\right\} = \sup\left\{1 + \frac{c_2 - c_1}{x - c_2} \mid x > b_1\right\} = 1 + \frac{c_2 - c_1}{b_1 - c_2}. \tag{2.4}$$

Now assume that $b_1 > \frac{nc_2 - c_1}{n-1}$. Then we let $a = 1 + \frac{c_2 - c_1}{b_1 - c_2}$ so that $a \in (1, n)$ and by (2.4),

$$|x - f^n(x)| \leq a\,|x - f(x)| \text{ for all } x \in \mathbb{R} - \{c_1\}. \tag{2.5}$$

Since $\operatorname{Fix} f = \{c_1\}$, the inequality in (2.5) also holds for $x = c_1$. Therefore (2.5) holds for every $x \in \mathbb{R}$. This shows that f is n-rotative.

Conversely, assume that f is n-rotative. Then there exists $a \in (0, n)$ such that

$$|x - f^n(x)| \leq a\,|x - f(x)| \text{ for all } x \in \mathbb{R}.$$

Then $a \geq \frac{|x-f^n(x)|}{|x-f(x)|}$ for every $x \in \mathbb{R} - \{c_1\}$. So we obtain by (2.4) that

$$a \geq 1 + \frac{c_2 - c_1}{b_1 - c_2}.$$

Since $a < n$, $1 + \frac{c_2-c_1}{b_1-c_2} < n$. Hence $b_1 > \frac{nc_2-c_1}{n-1}$, as required. □

Example 2.13. *Let $f, g : \mathbb{R} \to \mathbb{R}$ be given by*

$$f(x) = \begin{cases} 1, & x \leq 12; \\ 5(x-12)+1, & 12 < x < 13; \\ 6, & x \geq 13, \end{cases}$$

$$g(x) = \begin{cases} 1, & x \leq \frac{199}{99}; \\ x - \frac{100}{99}, & \frac{199}{99} < x < \frac{298}{99}; \\ 2, & x \geq \frac{298}{99}. \end{cases}$$

The function f corresponds to the case $c_1 = 1$, $c_2 = 6$, $b_1 = 12$, $b_2 = 13$ in Proposition 2.12. Since $b_1 > \frac{nc_2-c_1}{n-1}$ for every $n \geq 2$, f is n-rotative for every $n \geq 2$. The function g corresponds to the case $c_1 = 1$, $c_2 = 2$, $b_1 = 2 + \frac{1}{99}$, $b_2 = 3 + \frac{1}{99}$. It is easy to check that $b_1 > \frac{nc_2-c_1}{n-1}$ if and only if $n > 100$. Therefore g is not n-rotative for $n \in [2, 100]$ and g is n-rotative for $n \geq 101$.

Next we study various functions similar to the one given in Proposition 2.12.

Proposition 2.14. *Let $c = \frac{c_2-c_1}{b_1-b_2}$ and let $f : \mathbb{R} \to \mathbb{R}$ be given by*

$$f(x) = \begin{cases} c_2, & x \leq b_1; \\ c(x-b_1)+c_2, & b_1 < x < b_2; \\ c_1, & x \geq b_2. \end{cases}$$

Then f is $(n, 1)$-rotative for every $n \geq 2$.

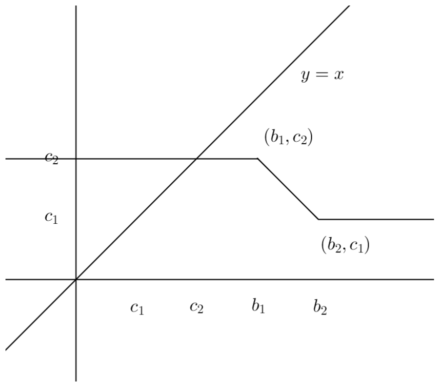

Figure 2.2: The graph of f when $c = \frac{c_2 - c_1}{b_1 - b_2}$.

Proof. Let $n \geq 2$ and $x \in \mathbb{R}$. If $x \leq b_1$, then $f(x) = c_2$ and $f^n(x) = c_2$, and therefore

$$|x - f^n(x)| = |x - c_2| = |x - f(x)|.$$

If $x \geq b_2$, then $f(x) = c_1$, $f^n(x) = c_2$, and therefore

$$\frac{|x - f^n(x)|}{|x - f(x)|} = \frac{|x - c_2|}{|x - c_1|} = \frac{x - c_2}{x - c_1} = 1 + \frac{c_1 - c_2}{x - c_1} < 1.$$

If $b_1 < x < b_2$, then $f(x) < c_2$, $f^2(x) = c_2$, and thus

$$\frac{|x - f^n(x)|}{|x - f(x)|} = \frac{x - c_2}{x - f(x)} < \frac{x - c_2}{x - c_2} = 1.$$

This shows that $|x - f^n(x)| \leq |x - f(x)|$ for all $x \in \mathbb{R}$. Therefore f is $(n, 1)$-rotative for every $n \geq 2$. $\square$

Proposition 2.15. *Let $f : \mathbb{R} \to \mathbb{R}$ be given by*

$$f(x) = \begin{cases} c_1, & x \leq c_1; \\ x, & c_1 < x < c_2; \\ c_2, & x \geq c_2. \end{cases}$$

Then f is n-rotative for every $n \geq 2$.

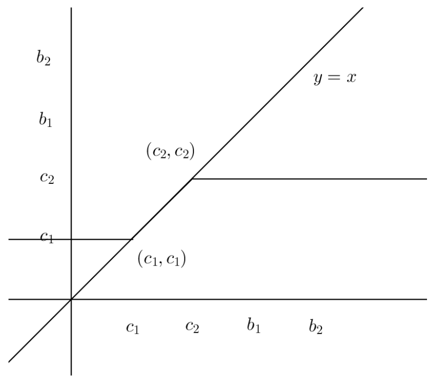

Figure 2.3: The graph of f when $f(x) = x$ on (c_1, c_2).

Proof. The proof is straightforward and similar to the proof in the previous proposition. □

Proposition 2.16. *Let* $c = \frac{b_2 - c_2}{b_1 - c_1}$ *be such that* $c < 1$ *and* $f : \mathbb{R} \to \mathbb{R}$ *be given by*

$$f(x) = \begin{cases} c_2, & x \leq c_1; \\ c(x - c_1) + c_2, & c_1 < x < b_1; \\ b_2, & x \geq b_1. \end{cases}$$

Then f *is* 2*-rotative.*

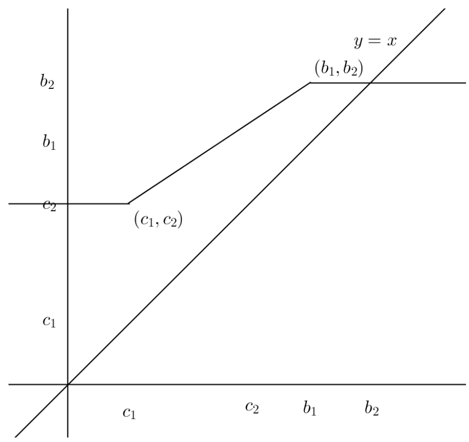

Figure 2.4: The graph of f when $c = \frac{b_2 - c_2}{b_1 - c_1}$.

Proof. If $x \le c_1$, then $f(x) = c_2$ and $f^2(x) = c(c_2 - c_1) + c_2$, and therefore

$$\frac{x - f^2(x)}{x - f(x)} = \frac{x - (c(c_2 - c_1) + c_2)}{x - c_2} = 1 - \frac{c(c_2 - c_1)}{x - c_2} = 1 + \frac{c(c_2 - c_1)}{c_2 - x}$$
$$\le 1 + \frac{c(c_2 - c_1)}{c_2 - c_1} = 1 + c < 2.$$

If $x \ge b_1$, then $f(x) = b_2$, $f^2(x) = b_2$, and therefore $|x - f^2(x)| = |x - f(x)|$.

If $c_1 < x < b_1$, then $f(x) = c(x - c_1) + c_2$.

Case 1. If $f(x) < b_1$, then $f^2(x) = c(c(x - c_1) + c_2 - c_1) + c_2$ and therefore

$$\begin{aligned} f^2(x) - x &= c(c(x - c_1) + c_2 - c_1) + c_2 - x \\ &= c^2(x - c_1) + c(c_2 - c_1) + (c_2 - c_1) - (x - c_1) \\ &= (c^2 - 1)(x - c_1) + (c + 1)(c_2 - c_1) \\ &= (c + 1)((c - 1)(x - c_1) + (c_2 - c_1)) \\ &= (c + 1)(f(x) - x). \end{aligned}$$

Case 2. If $f(x) \ge b_1$, then $f^2(x) = b_2$, and therefore

$$\begin{aligned} \left|\frac{x - f^2(x)}{x - f(x)}\right| &= \frac{f^2(x) - x}{f(x) - x} = \frac{f^2(x) - f(x)}{f(x) - x} + 1 \\ &= \frac{b_2 - f(x)}{f(x) - x} + 1. \end{aligned}$$

Since $\lim_{x \to b_1^-} \frac{b_2 - f(x)}{f(x) - x} = 0$, there exists $\delta > 0$ such that $|b_2 - f(x)| \le |f(x) - x|$ for all $x \in (b_1 - \delta, \delta)$. Since $f(x) \ge b_1$, $x \ge \frac{b_1 - c_2}{c} + c_1$. So we will consider that for $x \in \left[\frac{b_1 - c_2}{c} + c_1, b_1 - \delta\right]$,

$$\begin{aligned} 1 + \frac{b_2 - f(x)}{f(x) - x} &= \frac{b_2 - (c(x - c_1) + c_2)}{c(x - c_1) + c_2 - x} + 1 \\ &\le 1 + \frac{b_2 - b_1}{c(x - c_1) + c_2 - x} \\ &\le 1 + \frac{b_2 - b_1}{c(b_1 - c_1) - c\delta + c_2 - b_1 + \delta} \\ &= 1 + \frac{b_2 - b_1}{b_2 - b_1 + (1 - c)\delta} < 2. \end{aligned}$$

Let $a = \max\left\{1+c, 1+\frac{b_2-b_1}{b_2-b_1+(1-c)\delta}\right\}$. Then $a \in [0,2)$ and $|x-f^2(x)| \leq a|x-f(x)|$ for every $x \in \mathbb{R}$. Therefore f is 2-rotative. $\square$

Example 2.17. *Let $c = \frac{b_2-c_1}{b_2-b_1}$ and let $f : \mathbb{R} \to \mathbb{R}$ be given by*

$$f(x) = \begin{cases} c_1, & x \leq b_1; \\ c(x-b_1)+c_1, & b_1 < x < b_2; \\ b_2, & x \geq b_2. \end{cases}$$

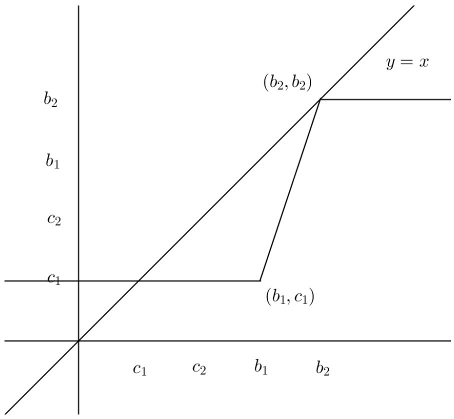

Figure 2.5: The graph of f when $c = \frac{b_2-c_1}{b_2-b_1}$.

We will show that f is not 2-rotative. We consider $x \in (b_1, b_2)$ that is closed to b_2. So let $b_1 + \frac{(b_1-c_1)(b_2-b_1)}{b_2-c_1} < x < b_2$. Then $f(x) = c(x-b_1)+c_1$. It is easy to check that $f(x) \in (b_1, b_2)$. Then $f^2(x) = c\left(c(x-b_1)+c_1-b_1\right)+c_1$. Then by L'Hôpital rule, we obtain

$$\begin{aligned} \lim_{x \to b_2^-} \frac{x-f^2(x)}{x-f(x)} &= \lim_{x \to b_2^-} \frac{x - \left(c\left(c(x-b_1)+c_1-b_1\right)+c_1\right)}{x-(c(x-b_1)+c_1)} \\ &= \frac{1-c^2}{1-c} = 1+c \\ &= 2 + \frac{b_1-c_1}{b_2-b_1} > 2. \end{aligned}$$

This implies that f is not 2-rotative.

In general, suppose that $n \geq 3$ is given. We can choose $x \in (b_1, b_2)$ very closed *to* b_2 *so that* $f^n x = c^n(x - b_1) + (c^{n-1} + c^{n-2} + \cdots + c)(c_1 - b_1) + c_1$. *Then by L'Hôpital rule, we obtain*

$$\begin{aligned}\lim_{x \to b_2^-} \frac{x - f^n(x)}{x - f(x)} &= \lim_{x \to b_2^-} \frac{x - c^n(x - b_1) - (c^{n-1} + c^{n-2} + \cdots + c)(c_1 - b_1) - c_1}{x - c(x - b_1) - c_1} \\ &= \frac{1 - c^n}{1 - c} = 1 + c + c^2 + \cdots + c^{n-1} \\ &> n.\end{aligned}$$

Therefore f *is not* n*-rotative for any* $n \geq 2$.

Example 2.18. *Let* $c = \frac{b_2 - c_1}{b_1 - c_2}$ *and let* $f : \mathbb{R} \to \mathbb{R}$ *be given by*

$$f(x) = \begin{cases} c_1, & x \leq c_2; \\ c(x - c_2) + c_1, & c_2 < x < b_1; \\ b_2, & x \geq b_1. \end{cases}$$

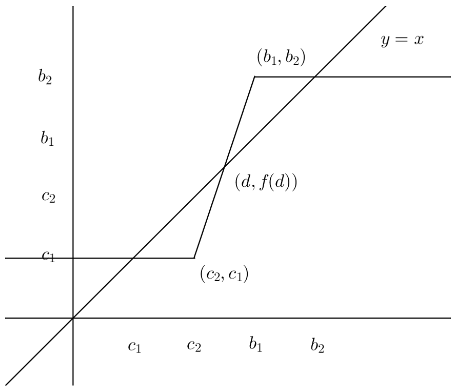

Figure 2.6: The graph of f when $c = \frac{b_2 - c_1}{b_1 - c_2}$.

We will show that f *is not* n*-rotative for any* $n \geq 2$. *Let* d *be the fixed point of* f *lying in* (c_2, b_1) *as shown in Figure 2.6. Then* $d = f(d) = c(d - c_2) + c_1$. *Similarly to the calculation in Example 2.17, we obtain that for every* $n \geq 2$,

$$\lim_{x \to d^-} \frac{x - f^n(x)}{x - f(x)} = \frac{1 - c^n}{1 - c} = 1 + c + c^2 + \cdots + c^{n-1} > n.$$

So f *is not* n*-rotative for any* $n \geq 2$.

Example 2.19. *Let* $c = \frac{b_2 - c_2}{b_1 - c_1}$ *and* $f : \mathbb{R} \to \mathbb{R}$ *be given by*

$$f(x) = \begin{cases} c_2, & x \le c_1; \\ c(x - c_1) + c_2, & c_1 < x < b_1; \\ b_2, & x \ge b_1. \end{cases}$$

We will show that f *is not* n*-rotative for any* $n \ge 2$ *if* $c \ge 1$. *The calculation is similar to that in Example 2.17. Let* $n \ge 2$.

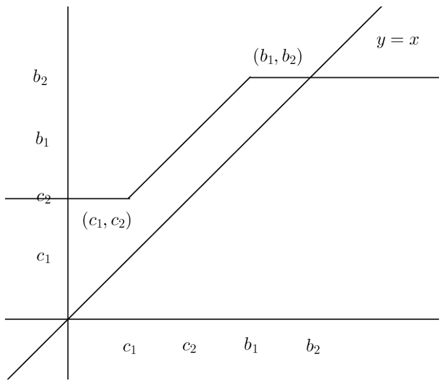

Figure 2.7: The graph of f when $c = \frac{b_2 - c_2}{b_1 - c_1} = 1$.

Case 1. Assume that $c = 1$. *Then*

$$\lim_{x \to c_1^+} \frac{x - f^n(x)}{x - f(x)} = \lim_{x \to c_1^+} \frac{x - (x - nc_1 + nc_2)}{x - (x - c_1 + c_2)} = \frac{nc_1 - nc_2}{c_1 - c_2} = n.$$

Then for each $a \in (0, n)$ *there exists a* $\delta > 0$ *such that* $|x - f^n(x)| > a\,|x - f(x)|$ *for every* $x \in (c_1 - \delta, c_1 + \delta)$. *Therefore* f *is not* n*-rotative.*

Case 2. Assume that $c > 1$. *Then*

$$\lim_{x \to c_1^+} \frac{x - f^n(x)}{x - f(x)} = 1 + c + c^2 + \cdots + c^{n-1} > n.$$

Therefore f *is not* n*-rotative.*

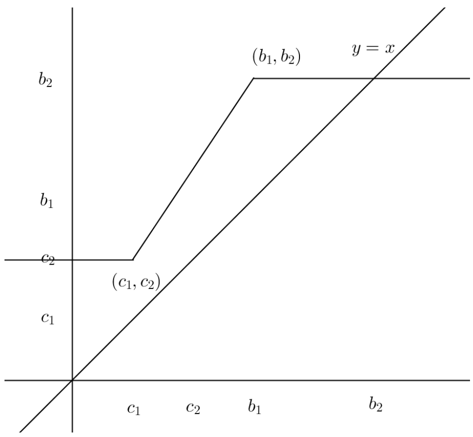

Figure 2.8: The graph of f when $c = \frac{b_2 - c_2}{b_1 - c_1} > 1$.

2.5 The Square Root Function

It is noted that rotativeness of a function may depend on its domain. In this section, we will consider the square root function on various subsets of $\mathbb{R}$. The L'Hôpital monotone rule will be used.

Lemma 2.20. (L'Hôspital monotone rule (H. Chen [2, Chapter 4])) *Let $f, g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$, differentiable on (a, b) and $g'(x) \neq 0$ for all $x \in (a, b)$. If $\frac{f'}{g'}$ is increasing (decreasing) on (a, b), then the functions given by*

$$\frac{f(x) - f(a)}{g(x) - g(a)} \quad \text{and} \quad \frac{f(x) - f(b)}{g(x) - g(b)}$$

are increasing (decreasing) on (a, b).

Proof. The proof can be found, for example, in [2] and [10]. □

Example 2.21. *Let $f : [0, \infty) \to [0, \infty)$ be given by $f(x) = \sqrt{x}$. Then*

$$\begin{aligned} \lim_{x \to 0^+} \frac{x - f^2(x)}{x - f(x)} &= \lim_{x \to 0^+} \frac{x - x^{\frac{1}{4}}}{x - x^{\frac{1}{2}}} = \lim_{x \to 0^+} \frac{1 - \frac{1}{4} x^{-\frac{3}{4}}}{1 - \frac{1}{2} x^{-\frac{1}{2}}} \\ &= \lim_{x \to 0^+} \frac{x^{\frac{3}{4}} - \frac{1}{4}}{x^{\frac{3}{4}} - \frac{1}{2} x^{\frac{1}{4}}} = +\infty. \end{aligned}$$

This implies that f is not 2-rotative. In general,

$$\lim_{x\to 0^+} \frac{x - f^n(x)}{x - f(x)} = \lim_{x\to 0^+} \frac{x - x^{\frac{1}{2^n}}}{x - x^{\frac{1}{2}}} = \lim_{x\to 0^+} \frac{1 - \frac{1}{2^n} x^{\frac{1}{2^n}-1}}{1 - \frac{1}{2}x^{-\frac{1}{2}}}$$
$$= \lim_{x\to 0^+} \frac{x^{1-\frac{1}{2^n}} - \frac{1}{2^n}}{x^{1-\frac{1}{2^n}} - \frac{1}{2}x^{\frac{1}{2}-\frac{1}{2^n}}}$$
$$= +\infty.$$

So f is not n-rotative for any $n \geq 2$.

Remark 2.22. *Since* $\lim_{x\to+\infty} \frac{x - f^2(x)}{x - f(x)} = \lim_{x\to+\infty} \frac{x - x^{\frac{1}{4}}}{x - x^{\frac{1}{2}}} = 1$, *there is an $M > 0$ such that*

$$\left|x - f^2(x)\right| \leq \frac{3}{2}\left|x - f(x)\right| \quad \textit{for all } x \geq M.$$

From this remark we may expect that $f\mid_{[N,\infty)}$ is 2-rotative for some N. However, $f\mid_{[N,\infty)}$ is not a selfmap if $N > 1$, so some careful consideration is needed. We notice that if $\left|\frac{x-f^2(x)}{x-f(x)}\right| = 2$, then by using calculator we have $x \approx 0.145$. So we will show that $f\mid_{[M,\infty)}$ is 2-rotative if $0.146 < M \leq 1$. To see this, let $g, h : [0.146, \infty) \to \mathbb{R}$ be given by $g(x) = x^{-\frac{3}{4}}$ and $h(x) = x^{-\frac{1}{2}}$. Then

$$\frac{g'(x)}{h'(x)} = \frac{-\frac{3}{4}x^{-\frac{7}{4}}}{-\frac{1}{2}x^{-\frac{3}{2}}} = \frac{3}{2}x^{-\frac{1}{4}}.$$

So $\frac{g'}{h'}$ is decreasing on $[0.146, 1]$ and $[1, \infty)$. Define $T : [0.146, \infty) \to \mathbb{R}$ by

$$T(x) = \begin{cases} \frac{x-f^2(x)}{x-f(x)}, & x \neq 1; \\ \frac{3}{2}, & x = 1. \end{cases}$$

Since $\lim_{x\to 1} \frac{x - f^2(x)}{x - f(x)} = \frac{3}{2}$, T is continuous. In addition,

$$T(x) = \frac{x - f^2(x)}{x - f(x)} = \frac{x - x^{\frac{1}{4}}}{x - x^{\frac{1}{2}}} = \frac{x^{-\frac{3}{4}} - 1}{x^{-\frac{1}{2}} - 1} = \frac{g(x) - g(1)}{h(x) - h(1)}.$$

Applying L'Hôspital monotone rule on $[0.146, 1]$ and $[1, \infty)$, we see that T is decreasing on $[0.146, \infty)$. Now assume that $0.146 < M \leq 1$. Then $T(x) \leq T(M)$

for all $x \in [M, \infty)$. Let $a = T(M)$. Then $a \in [0, 2)$ and

$$\left|x - f^2(x)\right| \leq a\,|x - f(x)| \quad \text{for all } x \in [M, \infty).$$

This shows that $f\,|_{[M,\infty)}\colon [M, \infty) \to [M, \infty)$ is 2-rotative.

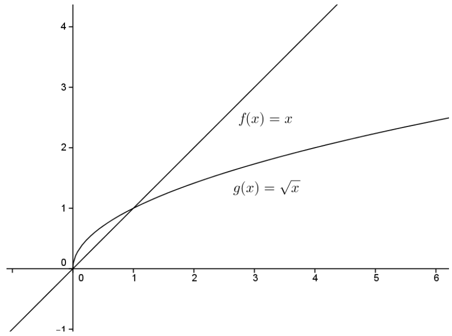

Figure 2.9: The graph of x and $\sqrt{x}$.

2.6 Constructions of Rotative Mappings on Product Spaces

For each $i \in \{1, 2, \ldots, m\}$, let (X_i, d_i) be a metric space. We can define metrics on $X_1 \times X_2 \times \cdots \times X_m$ in several ways. A natural metric on $X_1 \times X_2 \times \cdots \times X_m$ imitating the Euclidean metric on $\mathbb{R}^m$ is as follows:

$$\rho(x, y) = \sqrt{(d_1(x_1, y_1))^2 + (d_2(x_2, y_2))^2 + \cdots + (d_m(x_m, y_m))^2} \tag{2.6}$$

for every $x = (x_1, x_2, \ldots, x_m)$ and $y = (y_1, y_2, \ldots, y_m)$ in $X_1 \times X_2 \times \cdots \times X_m$. Some other natural metrics are, for example,

$$\rho_1(x, y) = d_1(x_1, y_1) + d_2(x_2, y_2) + \cdots + d_m(x_m, y_m), \quad \text{and}$$

$$\rho_\infty(x, y) = \max\left\{d_1(x_1, y_1), d_2(x_2, y_2), \ldots, d_m(x_m, y_m)\right\}.$$

The metrics $\rho, \rho_1, \rho_\infty$ induce the product topology on $X_1 \times X_2 \times \cdots \times X_m$. We will use ρ as the metric on the product space although the result also holds for the other metrics.

Theorem 2.23. *For each $i \in \{1, 2, \ldots, m\}$, let (X_i, d_i) be a metric space and let $f_i : X_i \to X_i$. Let $X = \prod_{i=1}^{m} X_i$ and ρ the metric on X defined in (2.6). Let $f : X \to X$ be given by*

$$f(x_1, x_2, \ldots, x_m) = (f_1(x_1), f_2(x_2), \ldots, f_m(x_m)) .$$

For every $n \geq 2$, if $f_1, f_2, \ldots, f_m$ are n-rotative, then f is n-rotative.

Proof. Let $n \geq 2$. Assume that $f_1, f_2, \ldots, f_m$ are n-rotative. For each $i \in \{1, 2, \ldots, m\}$, there exists $a_i \in (0, n)$ such that

$$d_i\left(x, f_i^n x\right) \leq a_i d_i\left(x, f_i x\right) \quad \text{for all } x \in X_i.$$

Let $a = \max\{a_1, a_2, \ldots, a_m\}$ and let $x = (x_1, x_2, \ldots, x_m) \in X$. Then $a \in (0, n)$ and

$$\begin{aligned}
\rho\left(x, f^n x\right) &= \sqrt{\left(d_1\left(x_1, f_1^n x_1\right)\right)^2 + \left(d_2\left(x_2, f_2^n x_2\right)\right)^2 + \cdots + \left(d_m\left(x_m, f_m^n x_m\right)\right)^2} \\
&\leq \sqrt{a_1^2\left(d_1\left(x_1, f_1 x_1\right)\right)^2 + a_2^2\left(d_2\left(x_2, f_2 x_2\right)\right)^2 + \cdots + a_m^2\left(d_m\left(x_m, f_m x_m\right)\right)^2} \\
&\leq \sqrt{a^2\left(d_1\left(x_1, f_1 x_1\right)\right)^2 + a^2\left(d_2\left(x_2, f_2 x_2\right)\right)^2 + \cdots + a^2\left(d_m\left(x_m, f_m x_m\right)\right)^2} \\
&= a\rho\left(x, f x\right).
\end{aligned}$$

This shows that f is n-rotative. ☐

Remark 2.24. *The result in Theorem 2.23 also holds if we replace the metric ρ by ρ_1 or ρ_∞.*

We restate here the above theorem when $X_i = \mathbb{R}$ for every i.

Corollary 2.25. *For each $i \in \{1, 2, \ldots, m\}$, let $f_i : \mathbb{R} \to \mathbb{R}$, and $f : \mathbb{R}^m \to \mathbb{R}^m$ be given by*

$$f(x_1, x_2, \ldots, x_m) = (f_1(x_1), f_2(x_2), \ldots, f_m(x_m)) .$$

If $f_1, f_2 \ldots, f_m$ are n-rotative, then f is n-rotative.

Corollary 2.26. *Let* $c_1, c_2, \ldots, c_m,\ d_1, d_2, \ldots, d_m \in \mathbb{R}$ *and let* $T : \mathbb{R}^m \to \mathbb{R}^m$ *be given by*

$$T(x_1, x_2, \ldots, x_m) = (c_1 x_1 + d_1, c_2 x_2 + d_2, \ldots, c_m x_m + d_m).$$

Then the following holds.

(i) If $c_1, c_2, \ldots, c_m \in (-3, 1)$*, then* T *is* 2*-rotative.*

(ii) If $c_1, c_2, \ldots, c_m \in (-2, 1)$*, then* T *is* 3*-rotative.*

Proof. This follows immediately from Corollary 2.7 and Corollary 2.25. □

Notice that the condition in Theorem 2.23 is that all $f_1, f_2, \ldots, f_m$ are n-rotative. If f_1 is n_1-rotative, f_2 is n_2-rotative, and $n_1 \neq n_2$, then the map $(x_1, x_2) \mapsto (f_1(x_1), f_2(x_2))$ may not be n-rotative for any n as shown in the next example.

Example 2.27. *Let* $f, g : \mathbb{R} \to \mathbb{R}$ *be given by*

$$f(x) = -\frac{5}{2}x \quad \textit{and}$$

$$g(x) = \begin{cases} 1, & \textit{if } x \leq 0; \\ 3, & \textit{if } 0 < x \leq 2; \\ -1, & \textit{if } x > 2. \end{cases}$$

Then f *is* 2*-rotative by Theorem 2.6. It is easy to verify that*

$$g^3(x) = \begin{cases} -1, & \textit{if } x \leq 0; \\ 1, & \textit{if } 0 < x \leq 2; \\ 3, & \textit{if } x > 2, \end{cases}$$

and that $|x - g^3(x)| \leq |x - g(x)|$ *for all* $x \in \mathbb{R}$*. So* g *is* 3*-rotative. Next define* $T : \mathbb{R}^2 \to \mathbb{R}^2$ *by* $T(x, y) = (f(x), g(y))$*. We will show that* T *is not* n*-rotative for* *any* $n \geq 2$. *Let* $z = (0, -1)$. *Then* $Tz = (0, 1)$, $T^2z = (0, 3)$, *and therefore*

$$\frac{\|z - T^2z\|_2}{\|z - Tz\|_2} = \frac{4}{2} = 2.$$

So T *is not* 2*-rotative. Next let* $n \geq 3$ *and* $z = (1, 0)$. *Then*

$$T^n z = \left(\left(-\frac{5}{2}\right)^n, a_n\right) \quad \textit{where} \quad a_n = \begin{cases} 1, & \textit{if } n \equiv 1 \pmod 3; \\ 3, & \textit{if } n \equiv 2 \pmod 3; \\ -1, & \textit{if } n \equiv 0 \pmod 3. \end{cases}$$

Then $\|z - T^n z\|_2 = \sqrt{\left(1 - \left(-\frac{5}{2}\right)^n\right)^2 + a_n^2} \geq \left(\frac{5}{2}\right)^n - 1$ *and* $\|z - Tz\|_2 = \frac{\sqrt{53}}{2}$. *So it is enough to show that*

$$\left(\frac{5}{2}\right)^n - 1 \geq n\frac{\sqrt{53}}{2} \tag{2.7}$$

If $n = 3$, *then* $\left(\frac{5}{2}\right)^3 - 1 > 14 > \frac{3\sqrt{53}}{2}$. *So (2.7) holds for* $n = 3$. *If (2.7) holds for* $n \geq 3$, *then*

$$\begin{aligned} \left(\frac{5}{2}\right)^{n+1} - 1 &\geq \left(\frac{5}{2}\right)^n + \left(\frac{5}{2}\right)^n - 1 \\ &\geq \left(\frac{5}{2}\right)^n + \frac{n\sqrt{53}}{2} \\ &> 15 + \frac{n\sqrt{53}}{2} \\ &> \frac{\sqrt{53}}{2} + \frac{n\sqrt{53}}{2} \\ &> (n+1)\frac{\sqrt{53}}{2}. \end{aligned}$$

Therefore (2.7) holds for all $n \geq 3$. *Hence* T *is not* n*-rotative for* $n \geq 2$.

2.7 Further Examples

From the study of various mappings above, we see that some of them are rotative but some are not. In this section, we will show that the cosine function is 2-rotative while the sine function is not n-rotative for any $n \geq 2$.

To show the cosine function is 2-rotative, we will use the following lemma.

Lemma 2.28. *The following statements hold.*

(i) The function $f : \mathbb{R} \to \mathbb{R}$ *given by* $f(x) = \cos x$ *has a unique fixed point* $x_0 \in \left(0, \frac{\pi}{4}\right)$.

(ii) $x < \cos\cos x$ *for every* $x < x_0$*, and* $x > \cos\cos x$ *for every* $x > x_0$.

(iii) $x < \cos x$ *for every* $x < x_0$*, and* $x > \cos x$ *for every* $x > x_0$.

We demonstrate the graph of x, $\cos x$, and $\cos\cos x$ as below.

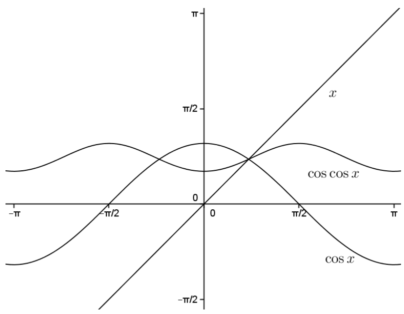

Figure 2.10: The graph of x, $\cos x$, and $\cos\cos x$.

Proof. Let $g(x) = x - \cos x$. Then $g'(x) = 1 + \sin x \geq 0$ for every $x \in \mathbb{R}$. So g is increasing on $\mathbb{R}$. Since $g(0) = -1 < 0$ and $g\left(\frac{\pi}{4}\right) = \frac{\pi}{4} - \frac{\sqrt{2}}{2} > 0$, there exists $x_0 \in \left(0, \frac{\pi}{4}\right)$ such that $g(x_0) = 0$. That is $f(x_0) = x_0$. This implies that x_0 is the unique fixed point of f. This proves (i). Next let $h(x) = x - \cos\cos x$. Then $h'(x) = 1 - (\sin\cos x)(\sin x) > 0$ for every $x \in (x_0, \infty)$. So h is strictly increasing on (x_0, ∞). Therefore for every $x > x_0$,

$$x - \cos\cos x = h(x) > h(x_0) = 0.$$

$\square$

Lemma 2.29. *Let* $g(x) = \cos x - \cos\cos x$ *and* x_0 *the fixed point of* f *given in Lemma 2.28. Then* *(i)* g *is increasing on* $[-\pi, 0]$ *and is decreasing on* $[0, \pi]$*,*

(ii) $g(x_0) = 0$ *and there exists* $x_1 \in \left(-\frac{\pi}{2}, 0\right)$ *such that* $g(x_1) = 0$*,*

(iii) $g(x) \leq 0$ *if* $x \in [-\pi, x_1] \cup [x_0, \pi]$*,*

(iv) $g(x) \geq 0$ *if* $x \in [x_1, x_0]$*.*

Proof. We have $g'(x) = -\sin x(1 + \sin\cos x)$. Since $1 + \sin\cos x \geq 0$, $g'(x) \geq 0$ if and only if $-\sin x \geq 0$, $g'(x) \leq 0$ if and only if $-\sin x \leq 0$. Therefore $g'(x) \geq 0$ on $[-\pi, 0]$ and $g'(x) \leq 0$ on $[0, \pi]$. So g is increasing on $[-\pi, 0]$ and is decreasing on $[0, \pi]$. This proves (i). Since $g\left(-\frac{\pi}{2}\right) < 0$ and $g(0) > 0$, there exists $x_1 \in \left(-\frac{\pi}{2}, 0\right)$ such that $g(x_1) = 0$. In addition, $g(x_0) = \cos x_0 - \cos\cos x_0 = x_0 - x_0 = 0$. This proves (ii). (iii) and (iv) follow from (i) and (ii). □

Note that the point x_1 in Lemma 2.29 is $-x_0$ where x_0 is the fixed point of f given in Lemma 2.28.

Theorem 2.30. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be given by* $f(x) = \cos x$ *for* $x \in \mathbb{R}$*. Then* f *is* 2*-rotative.*

Proof. Let $x \in \mathbb{R}$. We prove the assertion in 5 partitions of $\mathbb{R}$.

1) Let x be such that $|x| \geq \pi$. By Lemma 2.28, $\cos x \neq x$. Then

$$\begin{aligned}\left|\frac{x - \cos\cos x}{x - \cos x}\right| &= \left|1 + \frac{\cos x - \cos\cos x}{x - \cos x}\right| \\ &\leq 1 + \frac{|\cos x| + |\cos\cos x|}{||x| - |\cos x||} \\ &\leq 1 + \frac{2}{\pi - 1}.\end{aligned}$$

2) Let $x \in \left[-\pi, -\frac{\pi}{2}\right] \cup \left[\frac{\pi}{2}, \pi\right]$. Then $-1 \leq \cos x \leq 0$. Since cosine is increasing on $[-\pi, 0]$, we have $\cos x \leq 0 < \cos(-1) \leq \cos(\cos x) \leq \cos 0 = 1$. So $\cos x < \cos\cos x$. Then

$$x - \cos x > x - \cos\cos x. \tag{2.8}$$

By Lemma 2.28,

$$x - \cos\cos x > 0. \tag{2.9}$$

By (2.8) and (2.9), we obtain $0 < \frac{x-\cos\cos x}{x-\cos x} < 1$.

3) Let x be such that $x_0 < x < \frac{\pi}{2}$. Since cosine is decreasing on $\left[x_0, \frac{\pi}{2}\right]$ and on $[0, x_0]$, we have $x_0 = \cos x_0 > \cos x > \cos\frac{\pi}{2} = 0$, and $x_0 = \cos x_0 < \cos(\cos x) < \cos 0 = 1$. So $\cos\cos x > x_0 > \cos x$. Therefore

$$x - \cos\cos x < x - \cos x. \tag{2.10}$$

By Lemma 2.28,

$$x - \cos\cos x > 0. \tag{2.11}$$

By (2.10) and (2.11), we obtain $0 < \frac{x-\cos\cos x}{x-\cos x} < 1$.

4) Let x be such that $-\frac{\pi}{2} < x < x_1$. We obtain by Lemmas 2.28 and 2.29 that

$$\begin{aligned}\left|\frac{\cos x - \cos\cos x}{x - \cos x}\right| &\leq \frac{\left|\cos\left(-\frac{\pi}{2}\right) - \cos\cos\left(-\frac{\pi}{2}\right)\right|}{|x_1 - \cos x_1|} \\ &= \frac{1}{|x_1 - \cos x_1|} < 1.\end{aligned}$$

So $\left|\frac{x-\cos\cos x}{x-\cos x}\right| < 1 + \frac{1}{|x_1-\cos x_1|} < 2$.

5) Let x be such that $-x_0 < x < x_0$. By Lemma 2.28, $\cos\cos x < \cos x$. Then $-x + \cos\cos x < -x + \cos x$. By Lemma 2.28, $-x + \cos\cos x > 0$. So $0 < \frac{-x+\cos\cos x}{-x+\cos x} < 1$. Therefore $\left|\frac{x-\cos\cos x}{x-\cos x}\right| < 1$.

From 1) to 5), we have for every $x \in \mathbb{R}$

$$\left|\frac{x - \cos\cos x}{x - \cos x}\right| \leq \max\left\{1, 1 + \frac{2}{\pi - 1}, 1 + \frac{1}{|x_1 - \cos x_1|}\right\}.$$

Therefore f is 2-rotative. $\square$

Although the cosine function is 2-rotative, we will show that the sine function is not n-rotative for any $n \geq 2$.

Example 2.31. *Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \sin x$. We will show that f is not n-rotative for any $n \geq 2$. First we evaluate the limit by applying L'Hôspital rule as follows:*

$$\begin{aligned} \lim_{x\to 0} \frac{x - f^2(x)}{x - f(x)} &= \lim_{x\to 0} \frac{x - \sin(\sin x)}{x - \sin x} \\ &= \lim_{x\to 0} \cos(\sin x) + \frac{(\cos x)^2(\sin(\sin x))}{\sin x} \end{aligned} \tag{2.12}$$

Now

$$\lim_{x\to 0} \frac{\sin(\sin x)}{\sin x} = \lim_{x\to 0} \frac{(\cos(\sin x))(\cos x)}{\cos x} = 1 \tag{2.13}$$

From (2.12) and (2.13), we see that $\lim_{x\to 0} \dfrac{x - f^2(x)}{x - f(x)} = 2$. *Then for each $a \in (0, 2)$, there exists a $\delta > 0$ such that $|x - f^2(x)| > a|x - f(x)|$ for every $x \in (-\delta, \delta) - \{0\}$. This implies that f is not 2-rotative. In general, f is not n-rotative for any $n \geq 2$. The same idea above still works for the general case but the calculation is more complicated. To show this, let $g_0(x) = \cos x$ and $g_n(x) = \cos(\sin^n x)$ for each $n \geq 1$, where $\sin^n x$ means $(\sin \circ \sin \circ \cdots \circ \sin)(x)$, the n-fold composition of the sine function. Note that by L'Hôspital rule, we have*

$$\lim_{x\to 0} \frac{\sin^n x}{\sin x} = \lim_{x\to 0} \frac{g_{n-1}(x)g_{n-2}(x)\cdots g_1(x)g_0(x)}{\cos x} = 1 \text{ for every } n \in \mathbb{N}. \tag{2.14}$$

For every $n \in \mathbb{N}$, we also have

$$\begin{aligned} \lim_{x\to 0} \frac{x - f^n x}{x - fx} &= \lim_{x\to 0} \frac{x - \sin^n x}{x - \sin x} = \lim_{x\to 0} \frac{1 - g_{n-1}(x)g_{n-2}(x)\cdots g_1(x)g_0(x)}{1 - \cos x} \\ &= -\lim_{x\to 0} \frac{(g_{n-1}g_{n-2}\cdots g_0)(x)\left(\frac{g'_{n-1}}{g_{n-1}} + \frac{g'_{n-2}}{g_{n-2}} + \cdots + \frac{g'_0}{g_0}\right)(x)}{\sin x} \\ &= -\lim_{x\to 0} (g_{n-1}g_{n-2}\cdots g_0)(x)\left(\frac{g'_{n-1}(x)}{g_{n-1}(x)\sin x} + \frac{g'_{n-2}(x)}{g_{n-2}(x)\sin x} + \cdots + \frac{g'_0(x)}{g_0(x)\sin x}\right). \end{aligned} \tag{2.15}$$

Let $m \in \mathbb{N}$. Then

$$\begin{aligned}
\lim_{x\to 0} \frac{g'_m(x)}{g_m(x)\sin x} &= \lim_{x\to 0} \frac{-\sin(\sin^m x) g_{m-1}(x) g_{m-2}(x)\cdots g_0(x)}{g_m(x)\sin x} \\
&= -\lim_{x\to 0} \left(\frac{\sin^{m+1} x}{\sin x}\right)\left(\frac{g_{m-1}(x) g_{m-2}(x)\cdots g_0(x)}{g_m(x)}\right) \\
&= -1, \quad \text{by (2.14).} \qquad (2.16)
\end{aligned}$$

From (2.15) and (2.16), we obtain

$$\lim_{x\to 0} \frac{x - f^n x}{x - fx} = n \quad \text{for every } n \in \mathbb{N}. \qquad (2.17)$$

Therefore for each $a \in (0, n)$, there exists a $\delta > 0$ such that $|x - f^n x| > a|x - fx|$ for every $x \in (-\delta, \delta) - \{0\}$. This implies that f is not n-rotative for any $n \geq 2$.

Note that if we consider a function $f(x) = \sin x$ on $X = [-2\pi, 2\pi]$, we see that $f : X \to X$ is continuous on a compact and connected but f is not rotative. This shows that there is a continuous function on a compact convex subset of $\mathbb{R}$ which is not rotative.

CHAPTER III

FIXED POINTS OF CONTINUOUS ROTATIVE MAPPINGS

3.1 Fixed Point Theorems

Most results in fixed point theory are concerned with conditions on the map or on its domain that guarantee the existence of a fixed point of the map. Here we consider a map f which is rotative and k-Lipschitzian and is defined on a closed convex subset C of a Banach space X. A condition which guarantees the existence of a fixed point of f is that k is not too large. So it is natural to define $\gamma(X, n, a)$ to be the supremum of the values of k which assures the existence of a fixed point f, so that we obtain

If $k \leq \gamma(X, n, a)$, then every rotative k-Lipschitzian defined on a closed convex subset C of X has a fixed point.

So it is desirable to obtain the precise value or a large lower bound of $\gamma(X, n, a)$. However, all authors deal with general Banach spaces X and obtain a very small lower bound of $\gamma(X, n, a)$, which leads to somewhat unsatisfactory development of the theory. For example, the first estimation [19] of $\gamma(H, 3, 0)$ is that $\gamma(H, 3, 0) \geq 1.3666$ and the best estimation [3] known to date is that $\gamma(H, 3, 0) \geq 1.5549$, where H is a Hilbert space. Perhaps, this is because we try to think big while the supporting tools are not enough. This motivates us to think small and develop a theory from the most basic nontrivial examples. So we restrict ourselves to the space $\mathbb{R}$, which leads us to the first precise value of $\gamma(X, n, a)$ in the literature.

We hope that this will shed some light on the current state of knowledge on rotative mappings and the function γ, and will lead to a better development of the theory in the future. In the next theorem, we prove that every rotative continuous selfmap on closed (not necessary bounded) interval in $\mathbb{R}$ has a fixed point.

Theorem 3.1. *Let f be a selfmap on a closed (not necessary bounded) interval I in $\mathbb{R}$. If f is rotative and continuous, then* $\operatorname{Fix} f \neq \varnothing$.

Proof. Let $f : I \to I$ be rotative and continuous where I is a closed interval in $\mathbb{R}$. Then there exists $n \geq 2$, and $a \in [0, n)$ such that $|f^n(x) - x| \leq a\,|f(x) - x|$ for all $x \in I$. Suppose that for every $x \in I$, $f(x) \neq x$. By the intermediate value property of f on I, we have that either

$$f(x) > x \text{ for all } x \in I \text{ or } f(x) < x \text{ for all } x \in I.$$

Case I. Assume that $f(x) > x$ for all $x \in I$. Then for each $x \in I$, we have $\cdots > f^{n+1}(x) > f^n(x) > \cdots > f^2(x) > f(x) > x$. For each $x \in I$ and each $m \in \mathbb{N}$, let $a_m(x) = f^{m+1}(x) - f^m(x)$ and $a_0(x) = f(x) - x$. Then $a_m(x) > 0$ and

$$f^m(x) - x = a_{m-1}(x) + a_{m-2}(x) + \cdots + a_0(x) \quad \text{for all } m = 0, 1, 2, \ldots. \tag{3.1}$$

By (3.1) and rotativeness of f, we have

$$\frac{a_0(x) + a_1(x) + \cdots + a_{n-1}(x)}{a_0(x)} = \frac{f^n(x) - x}{f(x) - x} \leq a. \tag{3.2}$$

For each nonnegative integer k, replacing x by $f^k(x)$ in (3.2), we have

$$\frac{a_k(x) + a_{k+1}(x) + \cdots + a_{k+n-1}(x)}{a_k(x)} \leq a.$$

For $k = 0$, $\frac{a_0(x)}{a_0(x)} + \cdots + \frac{a_{n-1}(x)}{a_0(x)} \leq a$ implies that there exists $m_0 \in \{0, 1, 2, \ldots, n-1\}$ such that

$$\frac{a_{m_0}(x)}{a_0(x)} \leq \frac{a}{n}.$$

Because $\frac{a}{n} < 1$, so $m_0 \neq 0$ and we note that $a_0(x) + \cdots + a_{m_0}(x) \leq aa_0(x)$.

For $k = m_0$, we have $\frac{a_{m_0}(x)}{a_{m_0}(x)} + \cdots + \frac{a_{m_0+n-1}(x)}{a_{m_0}(x)} \leq a$, therefore there exists $m_1 \in \{m_0, m_0+1, \ldots, m_0+n-1\}$ such that

$$\frac{a_{m_1}(x)}{a_{m_0}(x)} \leq \frac{a}{n}.$$

Again $m_1 \neq m_0$ and $a_{m_0}(x) + \cdots + a_{m_1}(x) \leq aa_{m_0}(x) \leq a\left(\frac{a}{n}\right)a_0(x)$.

For $k = m_1$, since $\frac{a_{m_1}(x)}{a_{m_1}(x)} + \cdots + \frac{a_{m_1+n-1}(x)}{a_{m_1}(x)} \leq a$, there exists $m_2 \in \{m_1, m_1 + 1, \ldots, m_1+n-1\}$ such that

$$\frac{a_{m_2}(x)}{a_{m_1}(x)} \leq \frac{a}{n}.$$

So $m_2 \neq m_1$ and

$$\begin{aligned} a_{m_1}(x) + \cdots + a_{m_2}(x) \leq aa_{m_1}(x) &\leq a\left(\frac{a}{n}\right)a_{m_0}(x) \\ &\leq a\left(\frac{a}{n}\right)^2 a_0(x). \end{aligned}$$

In general, if $m_0, m_1, m_2, \ldots, m_\ell$ are chosen, then there exists $m_{\ell+1} \in \{m_\ell + 1, \ldots, m_\ell+n-1\}$ such that

$$\frac{a_{m_{\ell+1}}(x)}{a_{m_\ell}(x)} \leq \frac{a}{n},$$

and

$$\begin{aligned} a_0(x) + \cdots + a_{m_{\ell+1}(x)} &\leq \left(1 + \frac{a}{n} + \left(\frac{a}{n}\right)^2 + \cdots + \left(\frac{a}{n}\right)^{\ell+1}\right) aa_0(x) \\ &\leq \frac{na}{n-a}a_0(x). \end{aligned} \tag{3.3}$$

If $k \in \mathbb{N}$ is given, then there exists $\ell \in \mathbb{N}$ such that $k \leq m_\ell$ and by (3.3),

$$a_0(x) + \cdots + a_k(x) \leq a_0(x) + \cdots + a_{m_\ell}(x) \leq \frac{na}{n-a}a_0(x).$$

This shows that for each $x \in I$ the series $\sum_{i=0}^{\infty} a_i(x)$ has bounded partial sums and hence is convergent. Since $f^k(x) - x = a_k(x) + \cdots + a_0(x)$ and $\sum_{i=0}^{\infty} a_i(x)$ converges, $(f^k(x))_{k\in\mathbb{N}}$ converges.

Case II. Assume that $f(x) < x$ for all $x \in I$. Then for each $x \in I$, we obtain $\cdots < f^{n+1}(x) < f^n(x) < \cdots < f^2(x) < f(x) < x$. Similar to Case I, we obtain that $(f^k(x))_{k\in\mathbb{N}}$ converges.

In both cases, we have $(f^k(x))_{k\in\mathbb{N}}$ converges with limit in the closed interval I. Let $(f^k(x))$ converge to a point $x_0 \in I$. Since f is continuous at x_0, $f\left(f^k(x)\right)$ converges to $f(x_0)$. But $\left(f^{k+1}(x)\right)$ also converges to x_0. Therefore $f(x_0) = x_0$. This is a contradiction. So $\operatorname{Fix} f \neq \varnothing$. $\square$

Now recall that for a nonempty closed convex subset C of a Banach space X, a mapping $T : C \to C$ is said to belong to the class $\Phi(C, n, a, k)$ if T is (n, a)-rotative and k-Lipschitzian. It is proved by Goebel and Koter ([8], [9]) that for $n \geq 2$ and $a \in [0, n)$ if $T \in \Phi(C, n, a, 1)$, then $\operatorname{Fix} T \neq \varnothing$. Moreover, for some $\gamma > 1$ for any $k < \gamma$, if $T \in \Phi(C, n, a, k)$, then $\operatorname{Fix} T \neq \varnothing$.

Since every Lipschitzian mapping is continuous, we immediately obtain the following corollary.

Corollary 3.2. *Let I be a nonempty closed interval in $\mathbb{R}$, $k > 0$, $n \geq 2$, and $a \in [0, n)$. If $T \in \Phi(I, n, a, k)$, then T has a fixed point. In other words, every (n, a)-rotative k-Lipschitzian mapping on a closed interval has a fixed point.*

Recall that by letting $\inf \varnothing = +\infty$ and $\sup \varnothing = -\infty$, then we have the following result.

Corollary 3.3. $\gamma(\mathbb{R}, n, a) = +\infty$*, for every* $n \geq 2$, $a \in [0, n)$.

Proof. By the definition of γ and Corollary 3.2, we obtain

$$\begin{aligned}\gamma(\mathbb{R}, n, a) &= \inf\{k \in [0, \infty) \mid \text{ there is a nonempty closed interval } I \text{ of } \mathbb{R} \text{ and} \\ &\qquad\qquad T \in \Phi(I, n, a, k) \text{ such that } \operatorname{Fix} T = \varnothing\} \\ &= \inf \varnothing \\ &= +\infty\end{aligned}$$

□

We remark that Corollary 3.3 gives a partial answer to Q3, Q5, and Q6.

Corollary 3.4. *For any $k > 0$ and $n \geq 2$, there exists a function $f : \mathbb{R} \to \mathbb{R}$ such that f is k-Lipschitzian, n-rotative, and* $\operatorname{Fix} f \neq \varnothing$.

Proof. Let $c_1 = 1$, $c_2 = 2$, $b_1 = 4$, $b_2 = 4 + \frac{1}{k}$, and let $f : \mathbb{R} \to \mathbb{R}$ be defined as in Proposition 2.12. Then $|f(x) - f(y)| \leq k|x - y|$ for all $x, y \in \mathbb{R}$, and $|f(b_2) - f(b_1)| = k|b_2 - b_1|$. So f is k-Lipschitzian where $k = \frac{c_2 - c_1}{b_2 - b_1}$, and $\operatorname{Fix} f = \{1\}$. Since $b_1 > \frac{2n-1}{n-1} = \frac{nc_2 - c_1}{n-1}$, we see that f is n-rotative by Theorem 2.12. □

By Corollary 3.4, there exists a function $T \in \Phi(\mathbb{R}, n, a, k)$ with $\operatorname{Fix} T \neq \varnothing$ and k is arbitrarily large. This gives an answer to Q2.

It is noticed that there is an alternative proof of a particular case of Theorem 3.1 when $n = 2$. The proof is much simpler, as shown in the following theorem.

Theorem 3.5. *Let I be a nonempty closed interval in $\mathbb{R}$. Then every 2-rotative continuous function $f : I \to I$ has a fixed point.*

Proof. Let $f : I \to I$ be 2-rotative and continuous. Then there exists $b \in [0, 2)$ such that $|f^2(x) - x| \leq b\,|f(x) - x|$ for all $x \in I$. Suppose for a contradiction that f has no fixed point. By the intermediate value property of continuous function on an interval, we have that either

$$f(x) > x \text{ for all } x \in I \text{ or } f(x) < x \text{ for all } x \in I.$$

In both cases, we note that

$$f^2(x) > f(x) > x \quad \text{for all } x \in I, \text{ or}$$

$$f^2(x) < f(x) < x \quad \text{for all } x \in I.$$

Therefore $\frac{f^2(x)-x}{f(x)-x} > 0$ for all $x \in I$. This implies that $b \in (1, 2)$ and

$$1 + \frac{f^2(x) - f(x)}{f(x) - x} = \frac{f^2(x) - x}{f(x) - x} = \left| \frac{f^2(x) - x}{f(x) - x} \right| \leq b$$

for each $x \in I$. That is for some $a \in (0, 1)$

$$\left| \frac{f^2(x) - f(x)}{f(x) - x} \right| \leq a \quad \text{for all } x \in I. \tag{3.4}$$

Now consider for each fixed $x \in I$. We will show that the sequence $(f^n(x))$ converges. For each $n \in \mathbb{N}$, we obtain by (3.4) that

$$\left| \frac{f^{n+1}(x) - f^n(x)}{f^n(x) - f^{n-1}(x)} \right| = \left| \frac{f^2\left(f^{n-1}(x)\right) - f\left(f^{n-1}(x)\right)}{f\left(f^{n-1}(x)\right) - f^{n-1}x} \right| \leq a.$$

Therefore $|f^{n+1}(x) - f^n(x)| \leq a\,|f^n(x) - f^{n-1}(x)|$ for every $n \in \mathbb{N}$. This implies that $|f^{n+1}(x) - f^n(x)| \leq a^n\,|f(x) - x|$ for all $n \in \mathbb{N}$. Now for $m, n \in \mathbb{N}$ and $m > n$, we have

$$\begin{aligned}
\left|f^{m+1}(x) - f^n(x)\right| &\leq \left|f^{m+1}(x) - f^m(x)\right| + \left|f^m(x) - f^{m-1}(x)\right| + \cdots + \\
&\qquad \left|f^{n+1}(x) - f^n(x)\right| \\
&\leq \left(a^m + a^{m-1} + \cdots + a^n\right)|f(x) - x| \\
&\leq \frac{a^n}{1-a}\,|f(x) - x|\,.
\end{aligned} \tag{3.5}$$

Since $a \in (0, 1)$,

$$\lim_{n\to\infty} \frac{a^n}{1-a} = 0. \tag{3.6}$$

By (3.5) and (3.6), we obtain that $(f^n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence in I. Then $(f^n(x))$ converges to a point $x_0 \in I$. Since f is continuous, $f\left(f^n(x)\right)$ converges to $f(x_0)$. But $\left(f\left(f^n(x)\right)\right) = \left(f^{n+1}(x)\right)$ is a subsequence of $(f^n(x))$, it converges to x_0. Therefore $f(x_0) = x_0$, a contradiction. This completes the proof. $\square$

3.2 The Fixed Point Set

Some results concerning fixed point sets of continuous rotative mappings are given in this section.

Proposition 3.6. *Let I be a closed (not necessary bounded) interval and let $f : I \to I$ be n-rotative and continuous. Assume that $f(x) \geq x$ for all $x \in I$ or $f(x) \leq x$ for all $x \in I$. Then for every $x \in I$, the sequence $(f^k(x))_{k\geq 0}$ converges to a fixed point of f and $d(x, \operatorname{Fix} f) \leq \frac{na}{n-a}|f(x) - x|$.*

Proof. Let $x \in I$. From the proof of Theorem 3.1, we see that the sequence $(f^k(x))_{k\geq 0}$ is monotone and satisfies $|f^k(x) - x| \leq \frac{na|f(x)-x|}{n-a}$ for every $k \in \mathbb{N}$. In addition, $(f^k(x))_{k\geq 0}$ converges to a fixed point, say z. Therefore $|z-x| \leq \frac{na|f(x)-x|}{n-a}$. Thus $d(x, \operatorname{Fix} f) \leq |z - x| \leq \frac{na|f(x)-x|}{n-a}$. □

Proposition 3.7. *Let I be a closed (not necessary bounded) interval and $f : I \to I$ an n-rotative and continuous mapping.*

(i) If $f(x) \geq x$ for all $x \in I$ and z is a fixed point of f, then $\operatorname{Fix} f \supseteq [z, \infty) \cap I$.

(ii) If $f(x) \leq x$ for all $x \in I$ and z is a fixed point of f, then $\operatorname{Fix} f \supseteq (-\infty, z] \cap I$.

Proof. Assume that $f(x) \geq x$ for all $x \in I$ and z is a fixed point of f. Let $u \in (z, \infty) \cap I$. Suppose for a contradiction that $f(u) \neq u$. Let $y = \sup A$ where $A = \{w \in I \mid z \leq w < u \text{ and } w = f(w)\}$. Since A is closed, $y \in A$ and therefore $y < u$ and $f(y) = y$. Pick an $x \in (y, u)$ and consider the increasing sequence $(f^k(x))_{k\geq 0}$. By Proposition 3.6, $(f^k(x))_{k\geq 0}$ converges to a point z_0 and $|z_0 - x| \leq \frac{na}{n-a}(f(x) - x)$. Since $\lim_{x\to y^+}(f(x) - x) = 0$, $z_0 < u$. Thus $y < z_0 < u$ and $f(z_0) = z_0$. This contradicts the definition of y. The proof of (ii) is similar. □

It is obviously seen that if the hypothesis in Proposition 3.6 holds, then by

Proposition 3.7 the fixed point set of f is a closed subinterval of I. So we state here the following corollary.

Corollary 3.8. *Under the assumption of Proposition 3.6, we have that* $\operatorname{Fix} f$ *is a closed subinterval of* I*.*

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