สำนักงานวิทยทรัพยากร จุฬาลงกรณ์มหาวิทยาลัย

# CHAPTER V

# CONCLUSION AND OPEN PROBLEMS

## 5.1 Conclusion

We have studied properties of glued graphs which do not have a new clique for any original graphs. We also have investigated clique covering numbers of glued graphs at clone which is an induced subgraph of both original graphs, $K_n$ and $K_2$. The results are as follows:

**A trivial bound of clique covering numbers of glued graphs:**

$$1 \leq cc(G_1 \triangleleft\!\!\triangleright G_2) \leq cc(G_1) + cc(G_2).$$

**Clique covering numbers of glued graphs without new cliques:**

1. If $G_1 \triangleleft\!\!\triangleright G_2$ does not have a new clique with at least 3 vertices for any original graphs, then $cc(G_1 \triangleleft\!\!\triangleright G_2) \geq \max\{cc(G_1), cc(G_2)\}$.
2. For $G_1 \underset{H}{\triangleleft\!\!\triangleright} G_2$ which does not have a new clique for any original graphs, $cc(G_1) + cc(G_2) - 2cc(H) \leq cc(G_1 \underset{H}{\triangleleft\!\!\triangleright} G_2) \leq cc(G_1) + cc(G_2)$.

**Clique covering numbers of glued graphs at induced clone:**

For a glued graph at induced clone $G_1 \underset{H}{\oplus} G_2$,

$$cc(G_1) + cc(G_2) - 2cc(H) \leq cc(G_1 \underset{H}{\oplus} G_2) \leq cc(G_1) + cc(G_2).$$

**Clique covering numbers of glued graphs at clone $K_n$:**

For any graphs $G_1$ and $G_2$ containing $K_n$ as a subgraph:

1. $cc(G_1) + cc(G_2) - 2 \leq cc(G_1 \underset{K_n}{\oplus} G_2) \leq cc(G_1) + cc(G_2)$.

2. If there exists a minimum clique covering of $G_1 \underset{K_n}{\oplus} G_2$ containing a nontrivial subgraph of the clone $K_n$, then $cc(G_1 \underset{K_n}{\oplus} G_2) = cc(G_1) + cc(G_2) - 1$.

3. If $cc(G_1 \underset{K_n}{\oplus} G_2) = cc(G_1) + cc(G_2) - 1$ then there exists a minimum clique covering of $G_1$ or $G_2$ containing the clone $K_n$.

4. If $cc(G_1 \underset{K_n}{\oplus} G_2) = cc(G_1) + cc(G_2) - 2$ then there exists minimum clique coverings of $G_1$ and $G_2$, both containing the clone $K_n$.

5. $cc(G_1) + cc(G_2) - 2 \leq cc(G_1 \underset{K_n}{\oplus} G_2) \leq cc(G_1) + cc(G_2) - 1$ if and only if there exists a minimum clique covering of $G_1$ or $G_2$ containing the clone $K_n$.

6. $cc(G_1 \underset{K_n}{\oplus} G_2) = cc(G_1) + cc(G_2)$ if and only if there is no minimum clique covering of $G_1$ or $G_2$ containing the clone $K_n$.

7. $cc(G_1 \underset{K_n}{\oplus} G_2) = cc(G_1) + cc(G_2) - 2$ if and only if there exist minimum clique coverings of $G_1$ and $G_2$ where both contain the clone $K_n$ and the union of them deleting the clone $K_n$ is a clique covering of $G_1 \underset{K_n}{\oplus} G_2$.

**Clique covering numbers of glued graphs at clone $K_2$:**

For any graphs $G_1$ and $G_2$ containing $K_2$ as a subgraph:

1. $cc(G_1) + cc(G_2) - 1 \leq cc(G_1 \underset{K_2}{\Phi} G_2) \leq cc(G_1) + cc(G_2)$.

2. The following statements are equivalent:

   **(i)** $cc(G_1 \underset{K_2}{\Phi} G_2) = cc(G_1) + cc(G_2) - 1$.

   **(ii)** There exists a minimum clique covering of $G_1$ or $G_2$ containing the clone $K_2$.

   **(iii)** $cc(G_1 - e) = cc(G_1) - 1$ or $cc(G_2 - e) = cc(G_2) - 1$ where $e$ is the edge of the clone $K_2$.

3. The following statements are equivalent:

   **(i)** $cc(G_1 \underset{K_2}{\Phi} G_2) = cc(G_1) + cc(G_2)$.

   **(ii)** There is no minimum clique covering of $G_1$ and $G_2$ which contains the clone $K_2$.

   **(iii)** $cc(G_1 - e) \geq cc(G_1)$ and $cc(G_2 - e) \geq cc(G_2)$ where $e$ is the edge of the clone $K_2$.

In this thesis, we have obtained a characterization of all possible values of $cc(G_1 \underset{K_2}{\Phi} G_2)$, while we have obtained characterizations of some possible values of $cc(G_1 \underset{K_n}{\Phi} G_2)$ such as $cc(G_1 \underset{K_n}{\Phi} G_2) = cc(G_1) + cc(G_2)$ and $cc(G_1 \underset{K_n}{\Phi} G_2) = cc(G_1) + cc(G_2) - 2$. When a glued graph does not have a new clique for any original graphs or its clone is an induced subgraph of both original graphs, we obtain only bounds of clique covering numbers of such glued graph.

## 5.2 Open problems

We have some open problems for future work as follows:

1. In Section 2.2, we have introduced a new clique of glued graphs. An open problem is to find values or improve bounds of the clique covering number of glued graphs with a new clique.

2. In Section 3.2, we have obtained $cc(G_1) + cc(G_2) - 2cc(H) \leq cc(G_1 \underset{H}{\Diamond} G_2) \leq cc(G_1) + cc(G_2)$ where $H$ is an induced subgraph of both $G_1$ and $G_2$. An open problem is to investigate a characterization of each possible values of $cc(G_1 \underset{H}{\Diamond} G_2)$.

3. In Section 4.1, we show characterizations of $cc(G_1 \underset{K_n}{\Diamond} G_2)$ such that $cc(G_1 \underset{K_n}{\Diamond} G_2) = cc(G_1) + cc(G_2)$ and $cc(G_1 \underset{K_n}{\Diamond} G_2) = cc(G_1) + cc(G_2) - 2$. An open problem is to investigate a characterization of $cc(G_1 \underset{K_n}{\Diamond} G_2) = cc(G_1) + cc(G_2) - 1$.

4. The related topic of a clique covering of $G$ is a clique partition of a graph $G$. A *clique partition* of a graph $G$ is a set of cliques of $G$ which together contain each edge of $G$ exactly once. The *clique partition number* of a graph $G$, denoted by $cp(G)$, is the smallest cardinality of clique partitions of $G$. Many people have studied clique partitions of some graphs. This motivates a future work to study clique partitions of glued graphs.