CHAPTER IV

GLUED GRAPHS AT CLONE K_n

When the clone of a glued graph is K_n, the clone is always an induced subgraph of both original graphs. In this chapter, we consider clique coverings of glued graphs at clone K_n. First, we study properties of clique coverings of glued graphs at clone K_n and bounds of clique covering numbers of glued graphs at clone K_n, and a characterization of each possible value of the clique covering number of glued graphs at clone K_n, in Section 4.1. In the last section, we investigate bounds of the clique covering number of glued graphs at clone K_2 and give a characterization of all possible values of the clique covering number of glued graphs at clone K_2.

4.1 Clique coverings of glued graphs at clone K_n

For $n \geq 2$, $G_1 \underset{K_n}{\oplus} G_2$ denotes an arbitrary glued graph between graphs G_1 and G_2 containing a subgraph K_n at any clone which is isomorphic to K_n.

Remark 4.1.1. Since $G_1 \underset{K_n}{\oplus} G_2$ is a glued graph at induced clone, $G_1 \underset{K_n}{\oplus} G_2$ does not have a new clique for any original graphs by Remark 3.2.2.

We use the previous remark to investigate bounds of the clique covering number of $G_1 \underset{K_n}{\oplus} G_2$ in the following theorem.

Theorem 4.1.2. *For any graphs G_1 and G_2 containing K_n as a subgraph,* $cc(G_1) + cc(G_2) - 2 \leq cc(G_1 \underset{K_n}{\Phi} G_2) \leq cc(G_1) + cc(G_2)$.

Proof. Let $G_1 \underset{Q}{\Phi} G_2$ be the glued graph between G_1 and G_2 at clone $Q \cong K_n$. By Remark 4.1.1, Q is an induced subgraph of G_1 and G_2. From Corollary 3.2.4, we have that $cc(G_1) + cc(G_2) - 2cc(Q) \leq cc(G_1 \underset{Q}{\Phi} G_2) \leq cc(G_1) + cc(G_2)$. Because $Q \cong K_n$, we have $cc(Q) = 1$. Therefore $cc(G_1) + cc(G_2) - 2 \leq cc(G_1 \underset{Q}{\Phi} G_2) \leq cc(G_1) + cc(G_2)$. □

Example 4.1.3. *The sharpness of the lower bound in Theorem 4.1.2.* We give G_1 and G_2 containing K_n as a subgraph such that $cc(G_1 \underset{K_n}{\Phi} G_2) = cc(G_1) + cc(G_2) - 2$. Let $|E(K_n)| = k$. Let Q be a graph such that $Q \cong K_n$, $V(Q) = \{v_1, ..., v_n\}$ and $E(Q) = \{e_1, ..., e_k\}$. Let Q' be a graph such that $Q' \cong K_n$, $V(Q') = \{v_1', ..., v_n'\}$ and $E(Q') = \{e_1', ..., e_k'\}$. Thus $Q \cong Q'$ by isomorphism f such that $f(v_i) = v_i'$. We can assume that $e_i = e_i'$.

- Let G_1 be an $(n+1)$-vertex graph containing Q as a subgraph such that the remaining vertex a_1 joins with endpoints of e_1 in Q.
- Let G_2 be a graph with $V(G_2) = V(Q') \cup \{a_2..., a_k\}$ and contains Q' as a subgraph such that each vertex a_i joins with endpoints of each e_i in Q' where $i = 2, ..., k$.

G_1, G_2 and $\underset{Q \cong_f Q'}{G_1 \Phi G_2}$ are shown in Figure 4.1.1. In particular, when $n = 4$ $G_1 \underset{K_4}{\Phi} G_2$ is shown in Figure 4.1.2.

Since G_1 is not a complete graph, $cc(G_1) \geq 2$. Because a_1 is adjacent to both endpoints of e_1 in G_1, we obtain a 3-clique in G_1. We can use this 3-clique and Q to cover G_1. Thus $cc(G_1) \leq 2$. Therefore $cc(G_1) = 2$. So, the set of these cliques is a minimum clique covering of G_1.

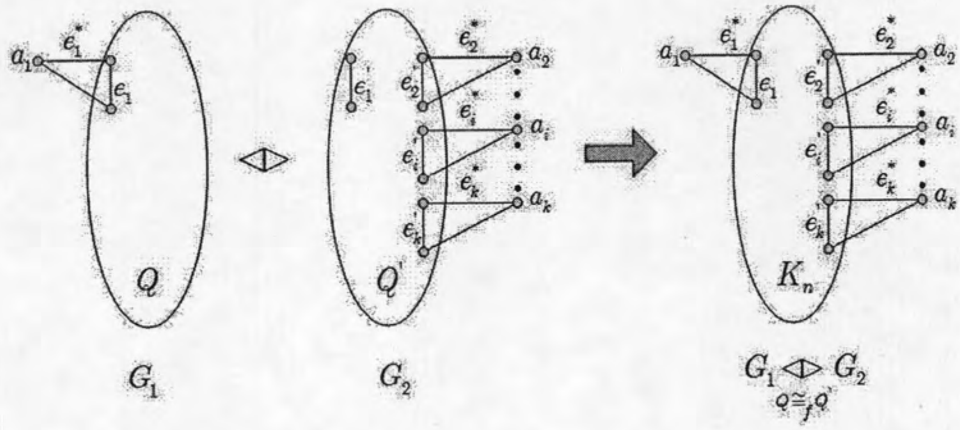

Figure 4.1.1: The sharpness of the lower bound in Theorem 4.1.2

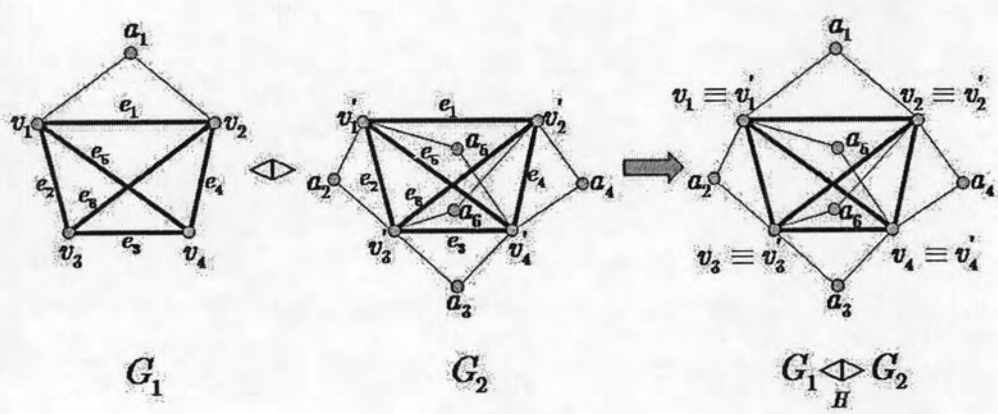

Figure 4.1.2: The sharpness of the lower bound in Theorem 4.1.2 with $n = 4$

Since each vertex a_i joins with both endpoints of each e_i where $i = 2, ..., k$, we obtain $k-1$ 3-cliques in G_2. We can use these $k-1$ 3-cliques and Q' to cover G_2. Thus $cc(G_2) \leq k$. Let e_i^* be an edge incident with a_i for $i = 1, ..., k$ and let $I = \{e_1, e_2^*, ..., e_k^*\}$. Because a_i and a_j are not adjacent in G_2 for $i \neq j$, I is a clique-independent set of G_2. Thus $cc(G_2) \geq k$. Therefore $cc(G_2) = k$. So, the set of these $k-1$ 3-cliques and Q' is a minimum clique covering of G_2.

In $G_1 \underset{Q \cong_f Q'}{\Phi} G_2$, each vertex a_i joins with both endpoints of each e_i where $i = 1, ..., k$. We obtain k 3-cliques in $G_1 \underset{Q \cong_f Q'}{\Phi} G_2$ which can be used to cover $G_1 \underset{Q \cong_f Q'}{\Phi} G_2$. Hence $cc(G_1 \underset{Q \cong_f Q'}{\Phi} G_2) \leq k$. Now, let $I' = \{e_1^*, e_2^*, ..., e_k^*\}$. Because a_i and a_j are not adjacent in $G_1 \underset{Q \cong_f Q'}{\Phi} G_2$, I' is a clique-independent set of $G_1 \underset{Q \cong_f Q'}{\Phi} G_2$. Thus $cc(G_1 \underset{Q \cong_f Q'}{\Phi} G_2) \geq k$. Hence $cc(G_1 \underset{Q \cong_f Q'}{\Phi} G_2) = k$. Therefore, $cc(G_1 \underset{Q \cong_f Q'}{\Phi} G_2) = k = cc(G_1) + cc(G_2) - 2$. □

4.1.1 Properties of $\mathscr{C}[G_1]$ and $\mathscr{C}[G_2]$ for a minimum clique covering $\mathscr{C}$ in glued graphs at clone K_n

From Remark 4.1.1, we have that $G_1 \underset{K_n}{\Phi} G_2$ satisfies all properties of a glued graph which does not have a new clique for any original graphs in Section 3.1. Next, we consider properties of $\mathscr{C}[G_1]$ and $\mathscr{C}[G_2]$ for a minimum clique covering $\mathscr{C}$ in $G_1 \underset{K_n}{\Phi} G_2$.

Remark 4.1.1.1. Let $G_1 \underset{Q}{\Phi} G_2$ be a glued graph between G_1 and G_2 at clone $Q \cong K_n$ and let $\mathscr{C}$ be a minimum clique covering of $G_1 \underset{Q}{\Phi} G_2$. Then

1. There exists at most one nontrivial subgraph of Q in $\mathscr{C}$. Suppose not, let S and S' be nontrivial subgraphs of Q in $\mathscr{C}$ such that $S' \neq S$. Thus $(\mathscr{C} \setminus \{S, S'\}) \cup \{Q\}$ is a clique covering of $G_1 \underset{Q}{\Phi} G_2$. But $|(\mathscr{C} \setminus \{S, S'\}) \cup \{Q\}| < |\mathscr{C}|$. This contradicts the fact that $\mathscr{C}$ is a minimum clique covering of $G_1 \underset{Q}{\Phi} G_2$.

2. By definition of $\mathscr{C}[G_1]$ and $\mathscr{C}[G_2]$, and (1), we have

$$\mathscr{C}[G_1] \cap \mathscr{C}[G_2] = \begin{cases} \{S\} & \text{, if there exists a nontrivial subgraph } S \text{ of } Q \text{ in } \mathscr{C}; \\ \varnothing & \text{, otherwise.} \end{cases}$$

3. Because $|\mathscr{C}| = |\mathscr{C}[G_1] \cup \mathscr{C}[G_2]|$ and by (2), we have

$$|\mathscr{C}| = \begin{cases} |\mathscr{C}[G_1]| + |\mathscr{C}[G_2]| - 1 & \text{, if there exists a nontrivial subgraph } S \\ & \text{of } Q \text{ in } \mathscr{C}; \\ |\mathscr{C}[G_1]| + |\mathscr{C}[G_2]| & \text{, otherwise.} \end{cases}$$

Next proposition, we obtain possible values of the cardinality of $\mathscr{C}[G_1]$ and $\mathscr{C}[G_2]$ for a minimum clique covering $\mathscr{C}$ of $G_1 \underset{K_n}{\oplus} G_2$.

Proposition 4.1.1.2. *For a minimum clique covering $\mathscr{C}$ of $G_1 \underset{K_n}{\oplus} G_2$, $cc(G_1) - 1 \leq |\mathscr{C}[G_1]| \leq cc(G_1)$ and $cc(G_2) - 1 \leq |\mathscr{C}[G_2]| \leq cc(G_2)$.*

Proof. Let $G_1 \underset{Q}{\oplus} G_2$ be a glued graph between G_1 and G_2 at clone $Q \cong K_n$. By Remark 4.1.1, $G_1 \underset{Q}{\oplus} G_2$ does not have a new clique for any original graphs. By Proposition 3.1.10, we have $|\mathscr{C}[G_1]| \geq cc(G_1) - cc(Q) = cc(G_1) - 1$ and $|\mathscr{C}[G_2]| \geq cc(G_2) - cc(Q) = cc(G_2) - 1$.

Suppose that $|\mathscr{C}[G_1]| > cc(G_1)$. Let $\mathscr{D}$ be a minimum clique covering of G_1. Thus $|\mathscr{D}| = cc(G_1)$ and $\mathscr{D} \cup \mathscr{C}[G_2]$ is a clique covering of $G_1 \underset{Q}{\oplus} G_2$. Consider $|\mathscr{D} \cup \mathscr{C}[G_2]| = |\mathscr{D}| + |\mathscr{C}[G_2]| - |\mathscr{D} \cap \mathscr{C}[G_2]| = cc(G_1) + |\mathscr{C}[G_2]| - |\mathscr{D} \cap \mathscr{C}[G_2]|$ and $|\mathscr{C}[G_1] \cup \mathscr{C}[G_2]| = |\mathscr{C}[G_1]| + |\mathscr{C}[G_2]| - |\mathscr{C}[G_1] \cap \mathscr{C}[G_2]| > cc(G_1) + |\mathscr{C}[G_2]| - |\mathscr{C}[G_1] \cap \mathscr{C}[G_2]|$. Because $\mathscr{D}$ is a minimum clique covering of G_1 and $\mathscr{C}[G_2]$ contains only cliques in G_2, there exists at most one nontrivial subgraph of Q in $\mathscr{D} \cap \mathscr{C}[G_2]$.

Case 1. $\mathscr{D} \cap \mathscr{C}[G_2] \neq \varnothing$. Thus there is unique nontrivial subgraph of Q, say C, in $\mathscr{D} \cap \mathscr{C}[G_2]$. We have that $|\mathscr{D} \cup \mathscr{C}[G_2]| = cc(G_1) + |\mathscr{C}[G_2]| - 1$. Since $C \in \mathscr{C}[G_2]$ and C is a clique of Q, $C \in \mathscr{C}[G_1]$. By Remark 4.1.1.1(2), we have $|\mathscr{C}[G_1] \cup \mathscr{C}[G_2]| > cc(G_1) + |\mathscr{C}[G_2]| - 1$. Thus $|\mathscr{C}[G_1] \cup \mathscr{C}[G_2]| > |\mathscr{D} \cup \mathscr{C}[G_2]|$. This contradicts the fact that $\mathscr{C}[G_1] \cup \mathscr{C}[G_2]$ is a minimum clique covering of $G_1 \underset{Q}{\oplus} G_2$.

Case 2. $\mathscr{D} \cap \mathscr{C}[G_2] = \varnothing$. Thus $|\mathscr{D} \cup \mathscr{C}[G_2]| = cc(G_1) + |\mathscr{C}[G_2]|$.

Case 2.1. $\mathscr{C}[G_2]$ does not contain a nontrivial subgraph of Q, also is $\mathscr{C}[G_1]$. We have that $|\mathscr{C}[G_1] \cup \mathscr{C}[G_2]| > cc(G_1) + |\mathscr{C}[G_2]|$. Hence $|\mathscr{C}[G_1] \cup \mathscr{C}[G_2]| > |\mathscr{D} \cup \mathscr{C}[G_2]|$. This contradicts the fact that $\mathscr{C}[G_1] \cup \mathscr{C}[G_2]$ is a minimum clique covering of $G_1 \underset{Q}{\oplus} G_2$.

Case 2.2. $\mathscr{C}[G_2]$ contains a nontrivial subgraph of Q, say C'. Since $\mathscr{D} \cap \mathscr{C}[G_2] = \varnothing$, $C' \notin \mathscr{D}$. Thus $(\mathscr{D} \cup \mathscr{C}[G_2]) \smallsetminus C'$ is a clique covering of $G_1 \underset{Q}{\oplus} G_2$. Consider $\left|(\mathscr{D} \cup \mathscr{C}[G_2]) \smallsetminus C'\right| = cc(G_1) + |\mathscr{C}[G_2]| - 1$. Since $C' \in \mathscr{C}[G_2]$ and C' is a clique of Q, $C' \in \mathscr{C}[G_1]$. By Remark 4.1.1.1(2), we have $|\mathscr{C}[G_1] \cup \mathscr{C}[G_2]| > cc(G_1) + |\mathscr{C}[G_2]| - 1$. Thus $|\mathscr{C}[G_1] \cup \mathscr{C}[G_2]| > \left|(\mathscr{D} \cup \mathscr{C}[G_2]) \smallsetminus C'\right|$. This contradicts the fact that $\mathscr{C}[G_1] \cup \mathscr{C}[G_2]$ is a minimum clique covering of $G_1 \underset{Q}{\oplus} G_2$.

By both cases, we have $|\mathscr{C}[G_1]| \leq cc(G_1)$. Similarly, $cc(G_2) - 1 \leq |\mathscr{C}[G_2]| \leq cc(G_2)$. □

We next give conditions to obtain that the cardinality of $\mathscr{C}[G_1]$(or $\mathscr{C}[G_2]$) is $cc(G_1)$(or $cc(G_2)$) for a minimum clique covering $\mathscr{C}$ of $G_1 \underset{K_n}{\oplus} G_2$. We illustrate these in Proposition 4.1.1.3 and Remark 4.1.1.7.

Proposition 4.1.1.3. *For a minimum clique covering $\mathscr{C}$ of $G_1 \underset{K_n}{\oplus} G_2$. If $\mathscr{C}[G_1]$(or $\mathscr{C}[G_2]$) is a clique covering of G_1(or G_2), then $|\mathscr{C}[G_1]| = cc(G_1)$(or $|\mathscr{C}[G_2]| = cc(G_2)$).*

Proof. Assume that $\mathscr{C}$ is a minimum clique covering of $G_1 \underset{K_n}{\Leftrightarrow} G_2$ and $\mathscr{C}[G_1]$ is a clique covering of G_1. Since $\mathscr{C}[G_1]$ is a clique covering of G_1, $|\mathscr{C}[G_1]| \geq cc(G_1)$. By Proposition 4.1.1.2, $|\mathscr{C}[G_1]| = cc(G_1)$. □

Next, we show an example of Proposition 4.1.1.3.

Example 4.1.1.4. Let G_1 and G_2 be graphs and $G_1 \underset{H}{\Leftrightarrow} G_2$ be the glued graph whose clone $H \cong K_4$ is shown as bold edges in Figure 4.1.1.1.

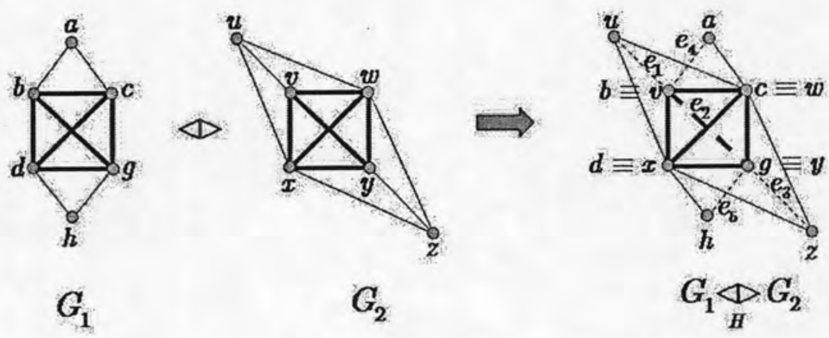

Figure 4.1.1.1: A glued graph in Example 4.1.1.4

Since $\mathscr{C}$ as shown in Figure 4.1.1.2 is a clique covering of $G_1 \underset{H}{\Leftrightarrow} G_2$ and $I = \{e_1, e_2, e_3, e_4, e_5\}$ which is the set of all dashed edges as illustrated in Figure 4.1.1.1 is a clique-independent set of $G_1 \underset{H}{\Leftrightarrow} G_2$, $cc(G_1 \underset{H}{\Leftrightarrow} G_2) = 5$. Hence $\mathscr{C}$ is a minimum clique covering of $G_1 \underset{H}{\Leftrightarrow} G_2$. We obtain $\mathscr{C}[G_1]$ and $\mathscr{C}[G_2]$ from $\mathscr{C}$ as shown in Figure 4.1.1.2. Note that $\mathscr{C}[G_1]$ and $\mathscr{C}[G_2]$ are clique coverings of G_1 and G_2, respectively. Besides, it is easy to see that $cc(G_1) = 3$ and $cc(G_2) = 3$. Hence, $|\mathscr{C}[G_1]| = cc(G_1)$ and $|\mathscr{C}[G_2]| = cc(G_2)$. □

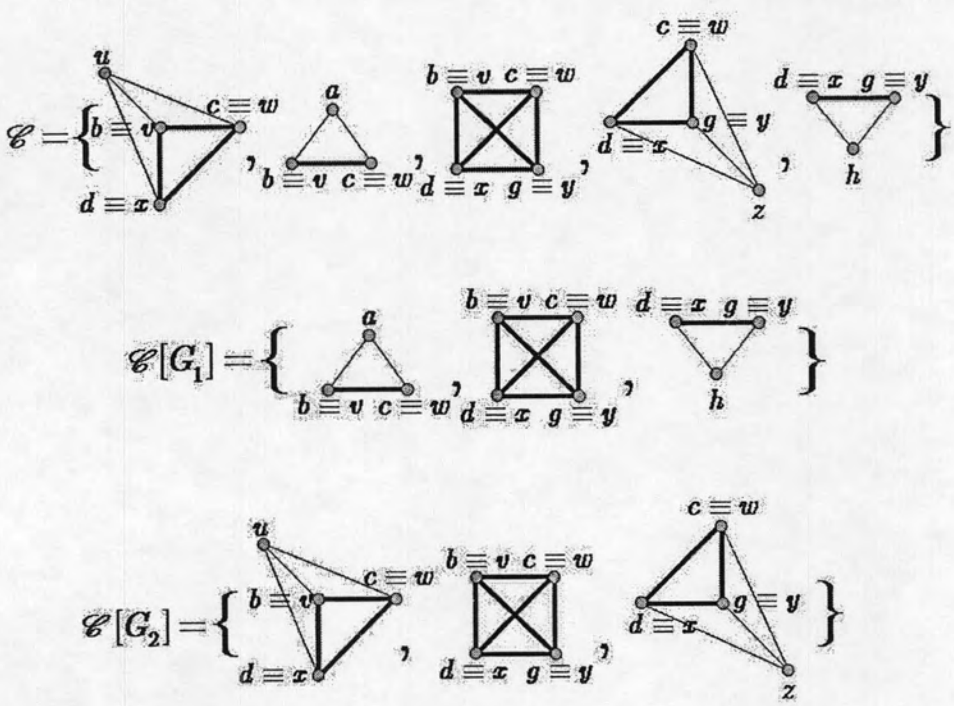

Figure 4.1.1.2: $\mathscr{C}$, $\mathscr{C}[G_1]$ and $\mathscr{C}[G_2]$ of a glued graph in Example 4.1.1.4

The converse of Proposition 4.1.1.3 does not hold as shown in Example 4.1.1.5.

Example 4.1.1.5. Let G_1 and G_2 be graphs and $G_1 \underset{H}{\Leftrightarrow} G_2$ be the glued graph whose clone $H \cong K_3$ is shown as bold edges in Figure 4.1.1.3.

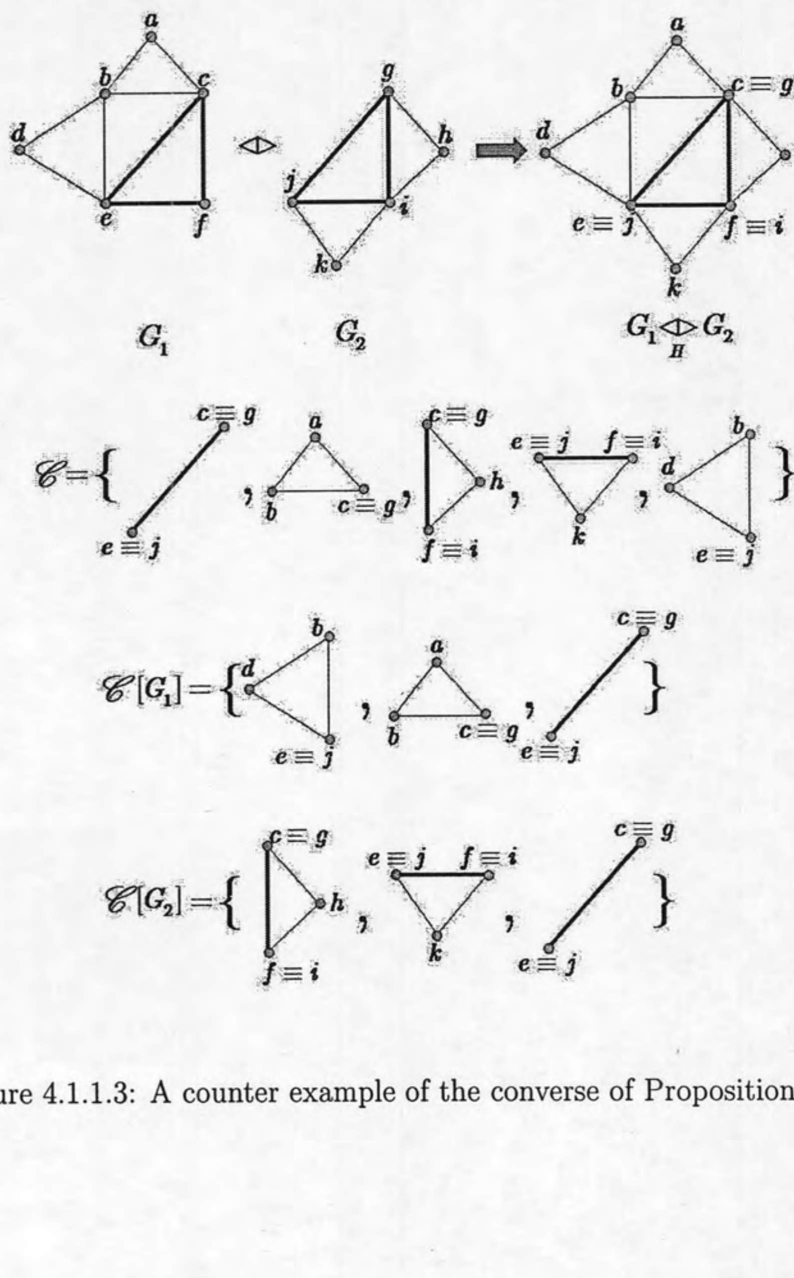

Figure 4.1.1.3: A counter example of the converse of Proposition 4.1.1.3

We can see that $\mathscr{C}$ in Figure 4.1.1.3 is a minimum clique covering of $G_1 \underset{H}{\diamond} G_2$. Therefore, we obtain $\mathscr{C}[G_1]$ and $\mathscr{C}[G_2]$ from $\mathscr{C}$ as shown in Figure 4.1.1.3. It is easy to see that $cc(G_1) = 3$ and $cc(G_2) = 3$. Hence $|\mathscr{C}[G_1]| = 3 = cc(G_1)$, while $\mathscr{C}[G_1]$ is not a clique covering of G_1. Moreover, $\mathscr{C}[G_2]$ is a clique covering of G_2 and $|\mathscr{C}[G_2]| = 3 = cc(G_2)$.

□

The following example shows that for a minimum clique covering $\mathscr{C}$ of $G_1 \underset{K_n}{\diamond} G_2$, $|\mathscr{C}[G_1]| \neq cc(G_1)$ and $|\mathscr{C}[G_2]| \neq cc(G_2)$ while $\mathscr{C}[G_1]$ and $\mathscr{C}[G_2]$ are not clique coverings of G_1 and G_2, respectively.

Example 4.1.1.6. Let G_1 and G_2 be graphs and $G_1 \underset{H}{\diamond} G_2$ be the glued graph whose clone $H \cong K_3$ is shown as bold edges in Figure 4.1.1.4.

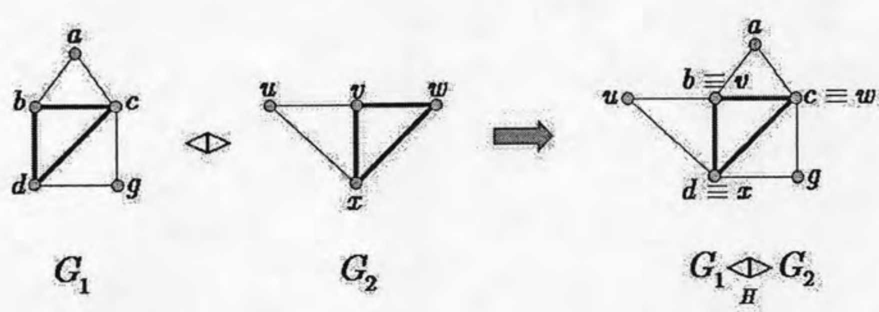

Figure 4.1.1.4: A glued graph in Example 4.1.1.6

We can easily see that $cc(G_1) = 3$, $cc(G_2) = 2$ and $\mathscr{C}$ in Figure 4.1.1.5 is a minimum clique covering of $G_1 \underset{H}{\diamond} G_2$. We obtain $\mathscr{C}[G_1]$ and $\mathscr{C}[G_2]$ from $\mathscr{C}$ as shown in Figure 4.1.1.5 such that $\mathscr{C}[G_1]$ and $\mathscr{C}[G_2]$ are not clique coverings of G_1 and G_2, respectively. Furthermore, $|\mathscr{C}[G_1]| = 2 = cc(G_1) - 1$ and $|\mathscr{C}[G_2]| = 1 = cc(G_2) - 1$.

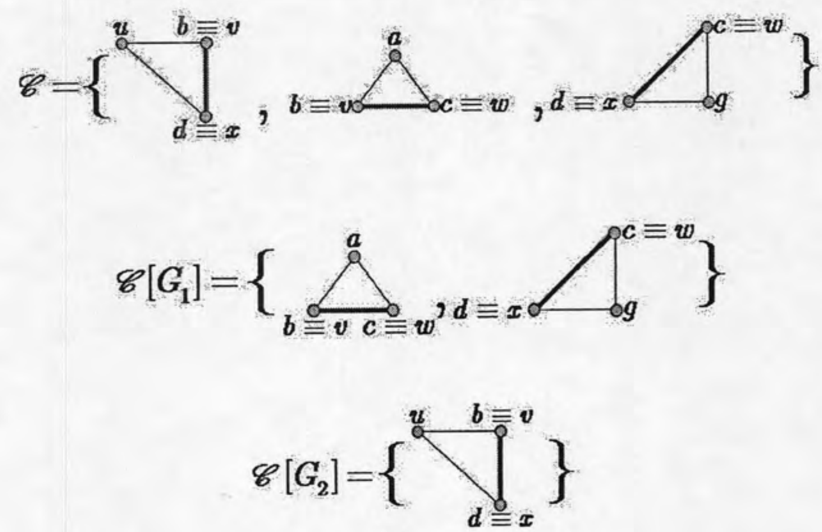

Figure 4.1.1.5: $\mathscr{C}$, $\mathscr{C}[G_1]$ and $\mathscr{C}[G_2]$ of a glued graph in Example 4.1.1.6

□

From Remark 4.1.1.1(1), there exists at most one nontrivial subgraph of Q in $\mathscr{C}$, here we consider the case that there exists exactly one nontrivial subgraph of Q in $\mathscr{C}$.

Proposition 4.1.1.7. *Let $G_1 \underset{Q}{\oplus} G_2$ be the glued graph between G_1 and G_2 at clone $Q \cong K_n$ and let $\mathscr{C}$ be a minimum clique covering of $G_1 \underset{Q}{\oplus} G_2$. If there exists a nontrivial subgraph S of Q such that $S \in \mathscr{C}$, then*

(i) $|\mathscr{C}[G_1]| = cc(G_1)$ *and* $|\mathscr{C}[G_2]| = cc(G_2)$, *and*

(ii) $(\mathscr{C}[G_1] \cup \{Q\}) \setminus \{S\}$ *and* $(\mathscr{C}[G_2] \cup \{Q\}) \setminus \{S\}$ *are minimum clique coverings of G_1 and G_2, respectively.*

Proof. Assume that there exists a nontrivial subgraph S of Q such that $S \in \mathscr{C}$. Hence $(\mathscr{C}[G_1] \cup \{Q\}) \setminus \{S\}$ and $(\mathscr{C}[G_2] \cup \{Q\}) \setminus \{S\}$ are clique coverings of G_1 and G_2, respectively. Therefore, $|(\mathscr{C}[G_1] \cup \{Q\}) \setminus \{S\}| \geq cc(G_1)$ and $|(\mathscr{C}[G_2] \cup \{Q\}) \setminus \{S\}| \geq cc(G_2)$. Consider $|(\mathscr{C}[G_1] \cup \{Q\}) \setminus \{S\}| = |\mathscr{C}[G_1]|$ and $|(\mathscr{C}[G_2] \cup \{Q\}) \setminus \{S\}| = |\mathscr{C}[G_2]|$. Hence $|\mathscr{C}[G_1]| \geq cc(G_1)$ and $|\mathscr{C}[G_2]| \geq cc(G_2)$. By Proposition 4.1.1.2, we have $|\mathscr{C}[G_1]| = cc(G_1)$ and $|\mathscr{C}[G_2]| = cc(G_2)$. Moreover, $(\mathscr{C}[G_1] \cup \{Q\}) \setminus \{S\}$ and $(\mathscr{C}[G_2] \cup \{Q\}) \setminus \{S\}$ are minimum clique coverings of G_1 and G_2, respectively. $\square$

4.1.2 Characterization of clique covering numbers of glued graphs at clone K_n

In Theorem 4.1.2, we have that $cc(G_1) + cc(G_2) - 2 \leq cc(G_1 \underset{K_n}{\Phi} G_2) \leq cc(G_1) + cc(G_2)$. This section, we obtain a characterization of glued graphs with the clique covering number of each possible value.

We first give a condition to obtain $cc(G_1 \underset{K_n}{\Phi} G_2) = cc(G_1) + cc(G_2) - 1$ as illustrated this in the next lemma.

Lemma 4.1.2.1. *If there exists a minimum clique covering of* $G_1 \underset{K_n}{\Phi} G_2$ *containing a nontrivial subgraph of the clone* K_n*, then* $cc(G_1 \underset{K_n}{\Phi} G_2) = cc(G_1) + cc(G_2) - 1$.

Proof. Let $G_1 \underset{Q}{\Phi} G_2$ be the glued graph between G_1 and G_2 at clone $Q \cong K_n$ and $\mathscr{C}$ be a minimum clique covering of $G_1 \underset{Q}{\Phi} G_2$ and S be a nontrivial subgraph of Q such that $S \in \mathscr{C}$. By Proposition 4.1.1.7(i), we have $|\mathscr{C}[G_1]| = cc(G_1)$ and $|\mathscr{C}[G_2]| = cc(G_2)$. Hence, by Remark 4.1.1.1(3), we have $cc(G_1 \underset{Q}{\Phi} G_2) = |\mathscr{C}| = |\mathscr{C}[G_1]| + |\mathscr{C}[G_2]| - 1 = cc(G_1) + cc(G_2) - 1$. $\square$

The converse of Lemma 4.1.2.1 does not hold as shown in Example 4.1.2.2.

Example 4.1.2.2. Let G_1 and G_2 be graphs and $G_1 \underset{H}{\triangleleft\!\triangleright} G_2$ be the glued graph whose clone $H \cong K_3$ is shown as bold edges in Figure 4.1.2.1.

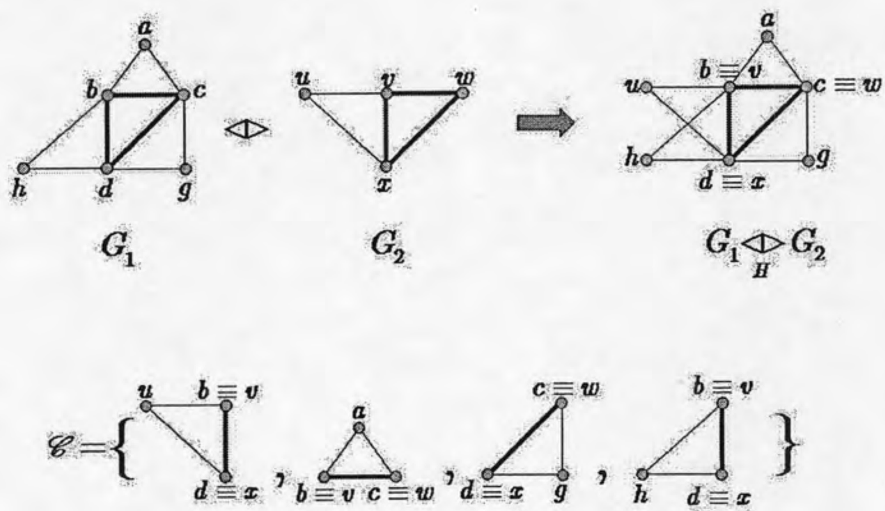

Figure 4.1.2.1: A counter example of the converse of Lemma 4.1.2.1

We can easily see that $\mathscr{C}$ as shown in Figure 4.1.2.1 is the only minimum clique covering of $G_1 \underset{H}{\triangleleft\!\triangleright} G_2$. It is evident that $cc(G_1) = 3$ and $cc(G_2) = 2$. So, there is no minimum clique covering of $G_1 \underset{H}{\triangleleft\!\triangleright} G_2$ containing a nontrivial subgraph of H while $cc(G_1 \underset{H}{\triangleleft\!\triangleright} G_2) = 3 + 2 - 1 = cc(G_1) + cc(G_2) - 1$.

□

In Theorem 4.1.2, we obtain three possible values of $cc(G_1 \underset{K_n}{\triangleleft\!\triangleright} G_2)$. We prove Lemma 4.1.2.3 and Lemma 4.1.2.5 to help us to characterize $cc(G_1 \underset{K_n}{\triangleleft\!\triangleright} G_2)$ of each possible value.

Lemma 4.1.2.3. *For any graphs G_1 and G_2 containing K_n as a subgraph, if $cc(G_1 \underset{K_n}{\triangleleft\!\triangleright} G_2) = cc(G_1) + cc(G_2) - 1$ then there exists a minimum clique covering of G_1 or G_2 containing the clone K_n.*

Proof. Let $G_1 \underset{Q}{\Phi} G_2$ be the glued graph between G_1 and G_2 at clone $Q \cong K_n$. Assume that $cc(G_1 \underset{Q}{\Phi} G_2) = cc(G_1) + cc(G_2) - 1$. Let $\mathscr{C}$ be a minimum clique covering of $G_1 \underset{Q}{\Phi} G_2$. Thus $|\mathscr{C}| = cc(G_1) + cc(G_2) - 1$.

Case 1. There exists a nontrivial subgraph S of Q such that $S \in \mathscr{C}$. By Proposition 4.1.1.7(ii), $(\mathscr{C}[G_1] \cup \{Q\}) \smallsetminus \{S\}$ and $(\mathscr{C}[G_2] \cup \{Q\}) \smallsetminus \{S\}$ are minimum clique coverings of G_1 and G_2, respectively.

Case 2. There is no any nontrivial subgraph of Q contained in $\mathscr{C}$. Thus $Q \notin \mathscr{C}$. By Remark 4.1.1.1(3), we have $|\mathscr{C}| = |\mathscr{C}[G_1]| + |\mathscr{C}[G_2]|$. Assume that all minimum clique coverings of G_1 do not contain Q.

If $\mathscr{C}[G_1]$ is a clique covering of G_1 then $|\mathscr{C}[G_1]| = cc(G_1)$ by Proposition 4.1.1.3. Otherwise, suppose that $\mathscr{C}[G_1]$ is not a clique covering of G_1. Note that $\mathscr{C}[G_1] \cup \{Q\}$ is a clique covering of G_1. By Proposition 4.1.1.2, we have $|\mathscr{C}[G_1]| \geq cc(G_1) - 1$. If $|\mathscr{C}[G_1]| = cc(G_1) - 1$, then $\mathscr{C}[G_1] \cup \{Q\}$ is a minimum clique covering of G_1 which contains Q, a contradiction. Thus $|\mathscr{C}[G_1]| > cc(G_1) - 1$. By Proposition 4.1.1.2, $|\mathscr{C}[G_1]| = cc(G_1)$. Consider $cc(G_1) + cc(G_2) - 1 = |\mathscr{C}| = |\mathscr{C}[G_1]| + |\mathscr{C}[G_2]|$. Thus $|\mathscr{C}[G_2]| = cc(G_2) - 1$. Since $\mathscr{C}[G_2] \cup \{Q\}$ is a clique covering of G_2 and $|\mathscr{C}[G_2] \cup \{Q\}| = cc(G_2)$, $\mathscr{C}[G_2] \cup \{Q\}$ is a minimum clique covering of G_2. □

The converse of Lemma 4.1.2.3 does not hold as shown in Example 4.1.2.4.

Example 4.1.2.4. Let G_1 and G_2 be graphs and $G_1 \underset{H}{\Phi} G_2$ be the glued graph whose clone $H \cong K_3$ is shown as bold edges in Figure 4.1.2.2. It is easy to see that $cc(G_1) = 3$, $cc(G_2) = 2$ and $cc(G_1 \underset{H}{\Phi} G_2) = 3$. It is evident that $\mathscr{C}_1$ and $\mathscr{C}_2$ as shown in Figure 4.1.2.2 are minimum clique coverings of G_1 and G_2, respectively. Consider $cc(G_1 \underset{H}{\Phi} G_2) = 3 = 3+2-2 = cc(G_1)+cc(G_2)-2$. Therefore $\mathscr{C}_2$ is a minimum clique covering of G_2 containing the clone H while $cc(G_1 \underset{H}{\Phi} G_2) \neq cc(G_1) + cc(G_2) - 1$. □

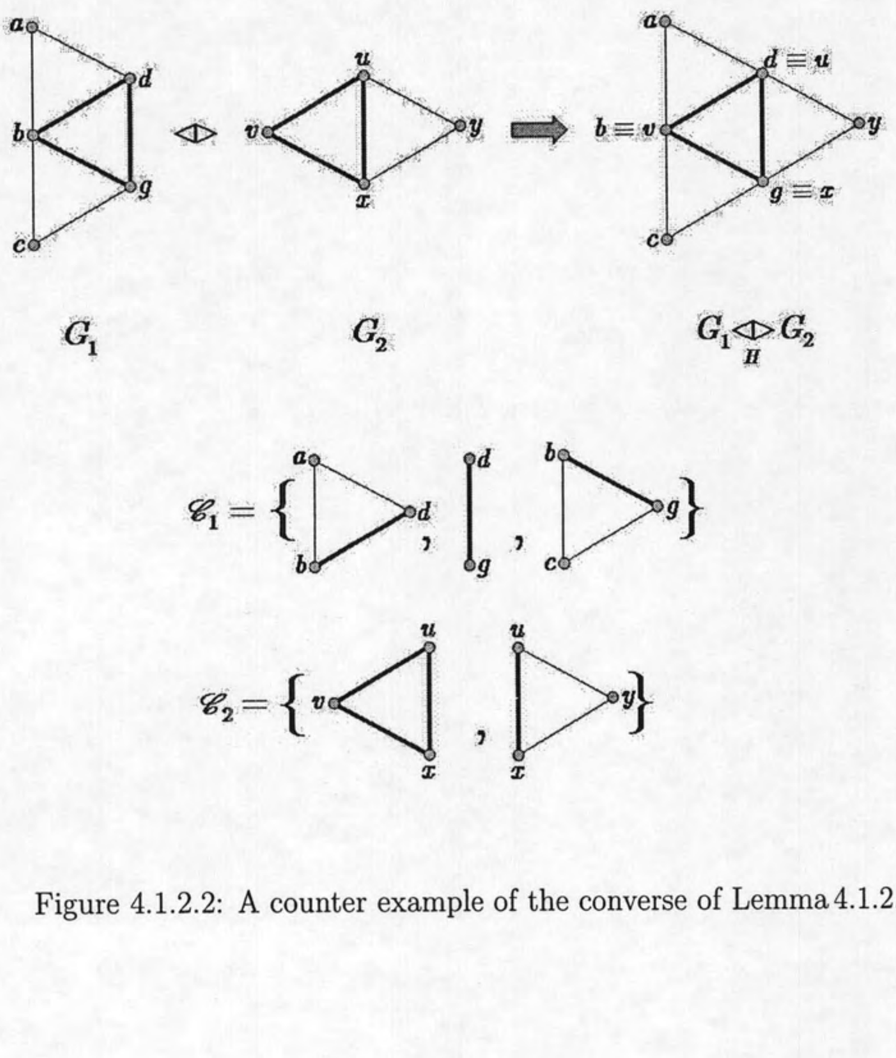

Figure 4.1.2.2: A counter example of the converse of Lemma 4.1.2.3

Lemma 4.1.2.5. *For any graphs G_1 and G_2 containing K_n as a subgraph, if $cc(G_1 \underset{K_n}{\oplus} G_2) = cc(G_1) + cc(G_2) - 2$ then there exists minimum clique coverings of G_1 and G_2, both containing the clone K_n.*

Proof. Let $G_1 \underset{Q}{\oplus} G_2$ be the glued graph between G_1 and G_2 at clone $Q \cong K_n$. Assume that $cc(G_1 \underset{Q}{\oplus} G_2) = cc(G_1) + cc(G_2) - 2$. Let $\mathscr{C}$ be a minimum clique covering of $G_1 \underset{Q}{\oplus} G_2$. Thus $|\mathscr{C}| = cc(G_1) + cc(G_2) - 2$.

By Lemma 4.1.2.1, we have $Q \notin \mathscr{C}$, otherwise, $cc(G_1 \underset{Q}{\oplus} G_2) = cc(G_1)+cc(G_2)-1$, a contradiction. Note that $\mathscr{C}[G_1] \cup \{Q\}$ and $\mathscr{C}[G_2] \cup \{Q\}$ are clique coverings of G_1 and G_2, respectively. By Proposition 4.1.1.2, we have $|\mathscr{C}[G_1]| \geq cc(G_1) - 1$ and $|\mathscr{C}[G_2]| \geq cc(G_2) - 1$. By Remark 4.1.1.1(3), we have $|\mathscr{C}| = |\mathscr{C}[G_1]| + |\mathscr{C}[G_2]|$. But $|\mathscr{C}| = cc(G_1) + cc(G_2) - 2$, $|\mathscr{C}[G_1]| = cc(G_1) - 1$ and $|\mathscr{C}[G_2]| = cc(G_2) - 1$. Hence $|\mathscr{C}[G_1] \cup \{Q\}| = |\mathscr{C}[G_1]| + 1 = cc(G_1)$ and $|\mathscr{C}[G_2] \cup \{Q\}| = |\mathscr{C}[G_2]| + 1 = cc(G_2)$. Therefore, $\mathscr{C}[G_1] \cup \{Q\}$ and $\mathscr{C}[G_2] \cup \{Q\}$ are minimum clique coverings of G_1 and G_2, respectively. □

The converse of Lemma 4.1.2.5 does not hold as shown in Example 4.1.2.4.

Example 4.1.2.6. Let G_1 and G_2 be graphs and $G_1 \underset{H}{\oplus} G_2$ be the glued graph whose clone $H \cong K_3$ is shown as bold edges in Figure 4.1.2.3. It is easy to see that $cc(G_1) = 2$, $cc(G_2) = 2$ and $cc(G_1 \underset{H}{\oplus} G_2) = 3$. It is evident that $\mathscr{C}_1$ and $\mathscr{C}_2$ as shown in Figure 4.1.2.3 are minimum clique coverings of G_1 and G_2, respectively. Consider $cc(G_1 \underset{H}{\oplus} G_2) = 3 = 2 + 2 - 1 = cc(G_1) + cc(G_2) - 1$. Therefore $\mathscr{C}_1$ and $\mathscr{C}_2$ are minimum clique covering of G_1 and G_2, respectively, both containing H while $cc(G_1 \underset{H}{\oplus} G_2) \neq cc(G_1) + cc(G_2) - 2$.

□

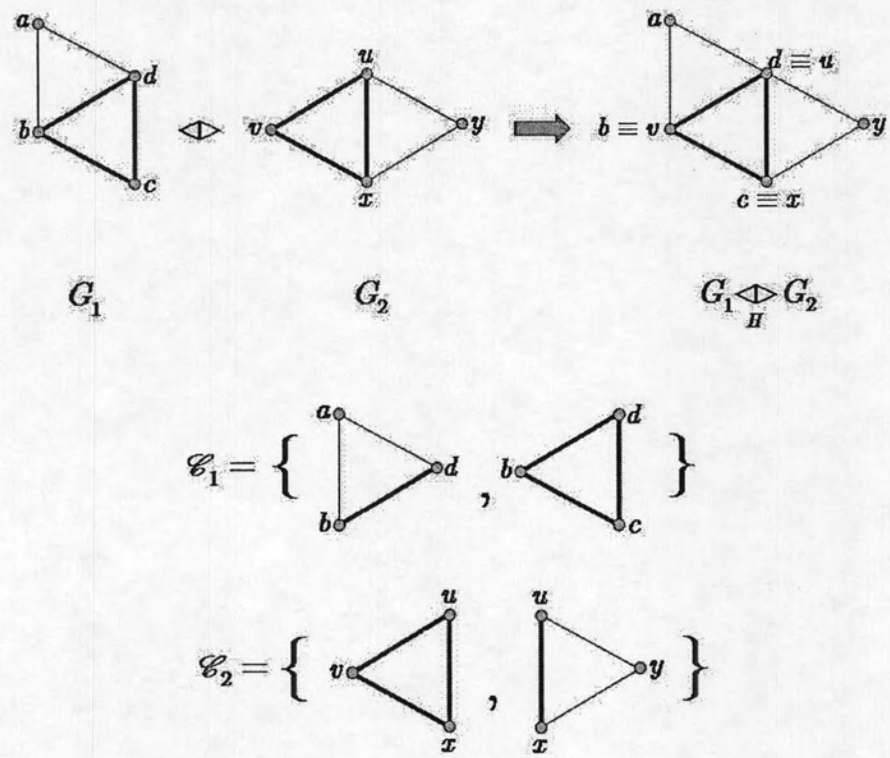

Figure 4.1.2.3: A counter example of the converse of Lemma 4.1.2.5

We next characterize $cc(G_1) + cc(G_2) - 2 \leq cc(G_1 \underset{K_n}{\oplus} G_2) \leq cc(G_1) + cc(G_2) - 1$ by using Lemma 4.1.2.3 and Lemma 4.1.2.5.

Theorem 4.1.2.7. *For any graphs G_1 and G_2 containing K_n as a subgraph, $cc(G_1) + cc(G_2) - 2 \leq cc(G_1 \underset{K_n}{\oplus} G_2) \leq cc(G_1) + cc(G_2) - 1$ if and only if there exists a minimum clique covering of G_1 or G_2 containing the clone K_n.*

Proof. Let $G_1 \underset{Q}{\oplus} G_2$ be the glued graph between G_1 and G_2 at clone $Q \cong K_n$.

For sufficiency, assume that Q is contained in at least one minimum clique covering of G_1 or G_2. Without loss of generality, we choose a minimum clique covering of G_1 containing Q, say $\mathscr{C}_1$. Let $\mathscr{C}_2$ be a minimum clique covering of G_2. Then $(\mathscr{C}_1 \setminus \{Q\}) \cup \mathscr{C}_2$ is a clique covering of $G_1 \underset{Q}{\oplus} G_2$. Thus $cc(G_1 \underset{Q}{\oplus} G_2) \leq |(\mathscr{C}_1 \setminus \{Q\}) \cup \mathscr{C}_2|$. Since $(\mathscr{C}_1 \setminus \{Q\}) \cap \mathscr{C}_2 = \varnothing$,

$$\begin{aligned} |(\mathscr{C}_1 \setminus \{Q\}) \cup \mathscr{C}_2| &= |\mathscr{C}_1 \setminus \{Q\}| + |\mathscr{C}_2| \\ &= |\mathscr{C}_1| - 1 + |\mathscr{C}_2| \\ &= cc(G_1) + cc(G_2) - 1. \end{aligned}$$

Hence $cc(G_1 \underset{Q}{\oplus} G_2) \leq cc(G_1) + cc(G_2) - 1$. By Theorem 4.1.2, $cc(G_1 \underset{Q}{\oplus} G_2) \geq cc(G_1) + cc(G_2) - 2$.

For necessity, assume that $cc(G_1)+cc(G_2)-2 \leq cc(G_1 \underset{K_n}{\oplus} G_2) \leq cc(G_1)+cc(G_2)-1$.

Case 1. $cc(G_1 \underset{K_n}{\oplus} G_2) = cc(G_1) + cc(G_2) - 1$. It follows immediately from Lemma 4.1.2.3.

Case 2. $cc(G_1 \underset{K_n}{\oplus} G_2) = cc(G_1) + cc(G_2) - 2$. It follows immediately from Lemma 4.1.2.5. □

A contrapositive of the following theorem helps us to characterize $cc(G_1 \underset{K_n}{\oplus} G_2) = cc(G_1) + cc(G_2)$.

Corollary 4.1.2.8. *For any graphs* G_1 *and* G_2 *containing* K_n *as a subgraph,* $cc(G_1 \underset{K_n}{\oplus} G_2) = cc(G_1) + cc(G_2)$ *if and only if there is no minimum clique covering of* G_1 *or* G_2 *containing the clone* K_n.

Proof. It follows immediately from Theorem 4.1.2 and Theorem 4.1.2.7. □

Now, we give a characterization of $cc(G_1 \underset{K_n}{\oplus} G_2) = cc(G_1) + cc(G_2) - 2$.

Theorem 4.1.2.9. *For any graphs* G_1 *and* G_2 *containing* K_n *as a subgraph,* $cc(G_1 \underset{K_n}{\oplus} G_2) = cc(G_1) + cc(G_2) - 2$ *if and only if there exist minimum clique coverings of* G_1 *and* G_2 *where both contain the clone* K_n *and the union of them deleting the clone* K_n *is a clique covering of* $G_1 \underset{K_n}{\oplus} G_2$.

Proof. Let $G_1 \underset{Q}{\oplus} G_2$ be the glued graph between G_1 and G_2 at clone $Q \cong K_n$.

For sufficiency, assume that there exist minimum clique coverings of G_1 and G_2 where both contain the clone K_n and the union of them deleting the clone K_n is a clique covering of $G_1 \underset{K_n}{\oplus} G_2$. We can choose a minimum clique covering of G_1 containing Q, say $\mathscr{C}_1$ and a minimum clique covering of G_2 containing Q, say $\mathscr{C}_2$ such that $(\mathscr{C}_1 \smallsetminus \{Q\}) \cup (\mathscr{C}_2 \smallsetminus \{Q\})$ is a clique covering of $G_1 \underset{Q}{\oplus} G_2$. Hence $|(\mathscr{C}_1 \smallsetminus \{Q\}) \cup (\mathscr{C}_2 \smallsetminus \{Q\})| \geq cc(G_1 \underset{Q}{\oplus} G_2)$. Suppose that $(\mathscr{C}_1 \smallsetminus \{Q\}) \cap (\mathscr{C}_2 \smallsetminus \{Q\}) \neq \varnothing$. Thus there is a clique Q' contained in $(\mathscr{C}_1 \smallsetminus \{Q\}) \cap (\mathscr{C}_2 \smallsetminus \{Q\})$. Since an edge of G_1 and G_2 in $G_1 \underset{Q}{\oplus} G_2$ must be in Q , Q' is a subgraph of Q. Hence $Q, Q' \in \mathscr{C}_1$ and $Q, Q' \in \mathscr{C}_2$. This contradicts Remark 4.1.1.1(1). Therefore $(\mathscr{C}_1 \smallsetminus \{Q\}) \cap (\mathscr{C}_2 \smallsetminus \{Q\}) = \varnothing$. Thus

$$\begin{aligned}
|(\mathscr{C}_1 \smallsetminus \{Q\}) \cup (\mathscr{C}_2 \smallsetminus \{Q\})| &= |\mathscr{C}_1 \smallsetminus \{Q\}| + |\mathscr{C}_2 \smallsetminus \{Q\}| \\
&= |\mathscr{C}_1| - 1 + |\mathscr{C}_2| - 1 \\
&= cc(G_1) - 1 + cc(G_2) - 1 \\
&= cc(G_1) + cc(G_2) - 2.
\end{aligned}$$

Hence $cc(G_1)+cc(G_2)-2 \geq cc(G_1 \underset{Q}{\Phi} G_2)$. By Theorem 4.1.2, $cc(G_1 \underset{Q}{\Phi} G_2) \geq cc(G_1)+cc(G_2) - 2$. Therefore $cc(G_1 \underset{Q}{\Phi} G_2) = cc(G_1) + cc(G_2) - 2$.

For necessity, assume that $cc(G_1 \underset{Q}{\Phi} G_2) = cc(G_1) + cc(G_2) - 2$. Let $\mathscr{C}$ be a minimum clique covering of $G_1 \underset{Q}{\Phi} G_2$. By Lemma 4.1.2.1, we have that $Q \notin \mathscr{C}$. From the proof of Lemma 4.1.2.5, we see that $\mathscr{C}[G_1] \cup \{Q\}$ and $\mathscr{C}[G_2] \cup \{Q\}$ are minimum clique coverings of G_1 and G_2, respectively. Consider $[(\mathscr{C}[G_1] \cup \{Q\}) \cup (\mathscr{C}[G_2] \cup \{Q\})] \smallsetminus \{Q\} = \mathscr{C}[G_1] \cup \mathscr{C}[G_2]$. By Remark 4.1.1, $G_1 \underset{Q}{\Phi} G_2$ does not have a new clique for any original graphs. By Proposition 3.1.7, $\mathscr{C}[G_1] \cup \mathscr{C}[G_2] = \mathscr{C}$. Thus $\mathscr{C}[G_1] \cup \mathscr{C}[G_2]$ is a minimum clique covering of $G_1 \underset{Q}{\Phi} G_2$. We obtain minimum clique coverings of G_1 and G_2 where both contain the clone K_n and the union of them deleting the clone K_n is a clique covering of $G_1 \underset{K_n}{\Phi} G_2$. □

We can use Theorem 4.1.2.9 to determine the clique covering number of the glued graph in Example 4.1.3 as shown in the following example.

Example 4.1.2.10. From Example 4.1.3, we can choose $\mathscr{C}_1$ and $\mathscr{C}_2$ is minimum clique coverings of G_1 and G_2, respectively, both containing the clone H. It is evident that $(\mathscr{C}_1 \cup \mathscr{C}_2) \smallsetminus \{H\}$ is a clique covering of $G_1 \underset{H}{\Phi} G_2$. By Theorem 4.1.2.9, $cc(G_1 \underset{H}{\Phi} G_2) = cc(G_1) + cc(G_2) - 2 = k$. □

The following example shows a glued graph at clone K_3 such that the clique covering number of a glued graph is $cc(G_1) + cc(G_2) - 2$ and there are minimum clique coverings of G_1 and G_2 both containing a proper subgraph in the clone of a glued graph.

Example 4.1.2.11. Let G_1 and G_2 be graphs and $G_1 \underset{H}{\Diamond} G_2$ be the glued graph whose clone $H \cong K_3$ is shown as bold edges in Figure 4.1.2.4.

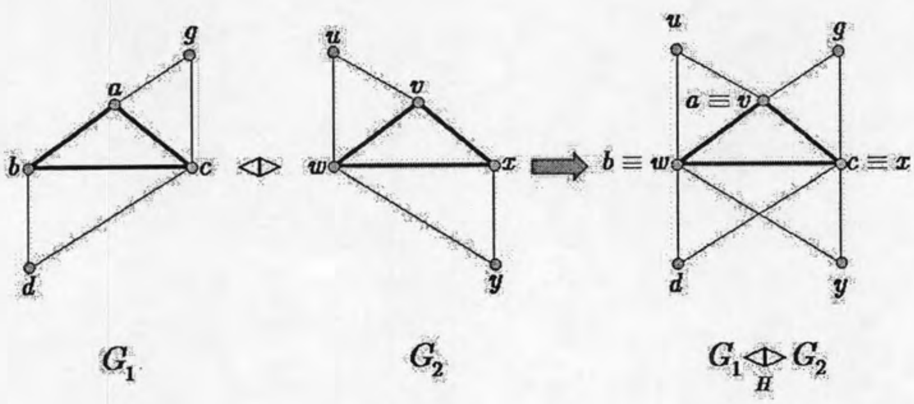

Figure 4.1.2.4: A glued graph in Example 4.1.2.11

It is easy to see that $cc(G_1) = 3$ and $cc(G_2) = 3$. Let $H \cong K_3(a, b, c)$ in G_1 and $H \cong K_3(v, w, x)$ in G_2. Consider $\mathscr{C}_1 = \{K_3(a, g, c),\ K_2(a, b), K_3(b, c, d)\}$ and $\mathscr{C}_2 = \{K_3(u, v, w), K_2(v, x), K_3(w, x, y)\}$ illustrated in Figure 4.1.2.5. Therefore, $\mathscr{C}_1$ and $\mathscr{C}_2$ are minimum clique coverings of G_1 and G_2, respectively.

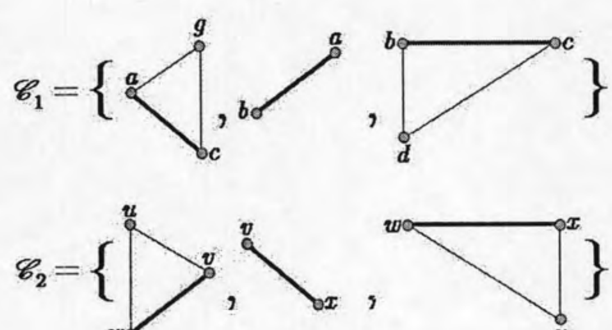

Figure 4.1.2.5: Minimum clique coverings of G_1 and G_2 do not contain the clone H

Since $\mathscr{C}_1$ contain $K_2(a, b)$ which is a subgraph of H, $\mathscr{C}_1' = (\mathscr{C}_1 \cup \{H\}) \setminus \{K_2(a, b)\}$

is a minimum clique covering of G_1. Similarly, $\mathscr{C}_2' = (\mathscr{C}_2 \cup \{H\}) \setminus \{K_2(v,x)\}$ is a minimum clique covering of G_2. Thus $\mathscr{C}_1'$ and $\mathscr{C}_2'$ are minimum clique covering of G_1 and G_2, respectively, both containing the clone H. We can use $(\mathscr{C}_1' \cup \mathscr{C}_2') \setminus \{H\}$ as shown in Figure 4.1.2.6 to cover $G_1 \underset{H}{\oplus} G_2$. By Theorem 4.1.2.9, we have that $cc(G_1 \underset{H}{\oplus} G_2) = cc(G_1) + cc(G_2) - 2 = 3 + 3 - 2 = 4$.

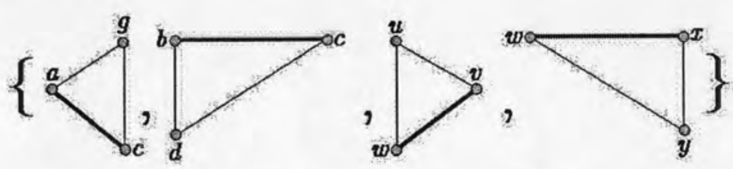

Figure 4.1.2.6: $\mathscr{C}_1' \cup \mathscr{C}_2'$ deleting the clone H

□

By Theorem 4.1.2, any glued graph at clone K_n, $G_1 \underset{K_n}{\oplus} G_2$, which does not satisfy the conditions in Corollary 4.1.2.8 and Theorem 4.1.2.9 has $cc(G_1 \underset{K_n}{\oplus} G_2) = cc(G_1) + cc(G_2) - 1$ immediately. However, the opposite of these two conditions is not simple. It is interesting to find another characterization of $G_1 \underset{K_n}{\oplus} G_2$ with $cc(G_1 \underset{K_n}{\oplus} G_2) = cc(G_1) + cc(G_2) - 1$.

4.2 Clique Coverings of Glued graphs at Clone K_2

From Section 4.1, we obtained bounds of $G_1 \underset{K_n}{\oplus} G_2$ and characterizations of $cc(G_1 \underset{K_n}{\oplus} G_2)$ for each possible values in terms of $cc(G_1)$ and $cc(G_2)$. In this section, we now focus on a glued graph at clone K_2.

Theorem 4.2.1. *For any graphs G_1 and G_2,*
$cc(G_1) + cc(G_2) - 1 \leq cc(G_1 \underset{K_2}{\oplus} G_2) \leq cc(G_1) + cc(G_2)$.

Proof. The upper bound has been already examined by Remark 2.1.1. Here we present the lower bound. Let $G_1 \underset{Q}{\Diamond} G_2$ be the glued graph between G_1 and G_2 at clone $Q \cong K_2$. Let e be the only edge in Q. Let $\mathscr{C}$ be a minimum clique covering of $G_1 \underset{Q}{\Diamond} G_2$. By Remark 4.1.1.1(2), only clique Q can possibly belong to $\mathscr{C}[G_1] \cap \mathscr{C}[G_2]$. Thus we consider two cases.

Case 1. $Q \in \mathscr{C}$. By Lemma 4.1.2.1, we have that $cc(G_1 \underset{Q}{\Diamond} G_2) = cc(G_1) + cc(G_2) - 1$.

Case 2. $Q \notin \mathscr{C}$. By Lemma 4.1.1.1(3), we have that $|\mathscr{C}| = |\mathscr{C}[G_1]| + |\mathscr{C}[G_2]|$. From Remark 4.1.1, $G_1 \underset{Q}{\Diamond} G_2$ does not have a new clique for any original graphs. Therefore $\mathscr{C} = \mathscr{C}[G_1] \cup \mathscr{C}[G_2]$ by Proposition 3.1.7. Since $\mathscr{C}[G_1] \cup \mathscr{C}[G_2]$ is a clique covering of $G_1 \underset{Q}{\Diamond} G_2$, at least one of $\mathscr{C}[G_1]$ and $\mathscr{C}[G_2]$ contains a clique that covers e, say $\mathscr{C}[G_1]$. Hence $\mathscr{C}[G_1]$ is a clique covering of G_1, so $|\mathscr{C}[G_1]| \geq cc(G_1)$. Besides $\mathscr{C}[G_2] \cup \{Q\}$ is a clique covering of G_2. Thus $|\mathscr{C}[G_2] \cup \{Q\}| \geq cc(G_2)$. Consider $|\mathscr{C}[G_2] \cup \{Q\}| = |\mathscr{C}[G_2]| + 1 - |\mathscr{C}[G_2] \cup \{Q\}| \leq |\mathscr{C}[G_2]| + 1$. So, $|\mathscr{C}[G_2]| \geq cc(G_2) - 1$. Hence $cc(G_1 \underset{Q}{\Diamond} G_2) = |\mathscr{C}| = |\mathscr{C}[G_1]| + |\mathscr{C}[G_2]| \geq cc(G_1) + cc(G_2) - 1$.

Therefore, for both cases, $cc(G_1 \underset{Q}{\Diamond} G_2) \geq cc(G_1) + cc(G_2) - 1$ as desired. □

The previous theorem reveals that when the clone is only an edge or K_2, the clique covering number of a glued graph could be either equal to the sum of clique covering numbers of original graphs or one less. Theorem 4.2.2 and Corollary 4.2.3 give a characterization of glued graphs satisfying each case.

Theorem 4.2.2. *For any graphs G_1 and G_2, the following statements are equivalent:*

(i) $cc(G_1 \underset{K_2}{\Diamond} G_2) = cc(G_1) + cc(G_2) - 1$.

(ii) There exists a minimum clique covering of G_1 or G_2 containing the clone K_2.

(iii) $cc(G_1 - e) = cc(G_1) - 1$ *or* $cc(G_2 - e) = cc(G_2) - 1$ *where* e *is the edge of the clone* K_2.

Proof. Let $G_1 \underset{Q}{\oplus} G_2$ be the glued graph between G_1 and G_2 at clone $Q \cong K_2$. Let e be the only edge in Q.

$(i) \Rightarrow (ii)$ It follows immediately from Lemma 4.1.2.3.

$(ii) \Rightarrow (iii)$ Assume that a minimum clique covering of G_1, say $\mathscr{C}_1$, contains Q as a member. We have that $\mathscr{C}_1 \setminus \{Q\}$ is a clique covering $G_1 - e$. Thus $cc(G_1 - e) \leq |\mathscr{C}_1 \setminus \{Q\}| = |\mathscr{C}_1| - 1 = cc(G_1) - 1$. By Proposition 1.2.1.23, $cc(G_1 - e) = cc(G_1) - 1$.

$(iii) \Rightarrow (i)$ Assume that $cc(G_1 - e) = cc(G_1) - 1$. Let $\mathscr{C}_1$ be a minimum clique covering of $G_1 - e$. Thus $|\mathscr{C}_1| = cc(G_1 - e) = cc(G_1) - 1$. Let $\mathscr{C}_2$ be a minimum clique covering of G_2. Since $\mathscr{C}_2$ contains a clique of G_2 containing e, $\mathscr{C}_1 \cup \mathscr{C}_2$ is a clique covering of $G_1 \underset{Q}{\oplus} G_2$. Thus $cc(G_1 \underset{Q}{\oplus} G_2) \leq |\mathscr{C}_1 \cup \mathscr{C}_2|$. Since $\mathscr{C}_1$ does not contain clique of G_1 containing e , $\mathscr{C}_1 \cap \mathscr{C}_2 = \varnothing$. So, $|\mathscr{C}_1 \cup \mathscr{C}_2| = |\mathscr{C}_1| + |\mathscr{C}_2| = cc(G_1) + cc(G_2) - 1$. Hence $cc(G_1 \underset{Q}{\oplus} G_2) \leq cc(G_1) + cc(G_2) - 1$. Therefore, by Theorem 4.2.1, $cc(G_1 \underset{Q}{\oplus} G_2) = cc(G_1) + cc(G_2) - 1$. □

The next corollary give a characterization in case $cc(G_1 \underset{K_2}{\oplus} G_2) = cc(G_1) + cc(G_2)$.

Corollary 4.2.3. *For any graphs* G_1 *and* G_2*, the following statements are equivalent:*

(i) $cc(G_1 \underset{K_2}{\oplus} G_2) = cc(G_1) + cc(G_2)$.

(ii) There is no minimum clique covering of G_1 *and* G_2 *containing the clone* K_2.

(iii) $cc(G_1 - e) \geq cc(G_1)$ *and* $cc(G_2 - e) \geq cc(G_2)$ *where* e *is the edge of the clone* K_2.

Proof. It follows immediately from Proposition 1.2.1.23 and Theorem 4.2.2. □