CHAPTER III

GLUED GRAPHS WITHOUT NEW CLIQUES

In this chapter, we study clique coverings of glued graphs which does not have a new clique for any original graphs. From Remark 2.1.1, we give the upper bound of the clique covering number of $G_1 \triangleleft\!\triangleright G_2$. Our purpose in this chapter is to study the lower bound of clique covering numbers of glued graph which does not have a new clique for any original graphs. We separate this chapter into two sections. The first section, we study bounds and many properties of a glued graph which does not have a new clique for any original graphs. In the last section, we study clique coverings of glued graphs at clone where is an induced subgraph of both original graphs.

3.1 Clique coverings of glued graphs without new cliques

First, we tighten the lower bound of the clique covering number of $G_1 \triangleleft\!\triangleright G_2$ in Remark 2.1.1 where $G_1 \triangleleft\!\triangleright G_2$ does not have a new clique for any original graphs.

Theorem 3.1.1. *If $G_1 \triangleleft\!\triangleright G_2$ does not have a new clique for any original graphs, then $cc(G_1 \triangleleft\!\triangleright G_2) \geq max\{cc(G_1), cc(G_2)\}$.*

Proof. Assume that $G_1 \triangleleft\!\triangleright G_2$ does not have a new clique for any original graphs. Thus $G_1 \triangleleft\!\triangleright G_2$ does not have a new clique for G_1. Hence at least $cc(G_1)$ cliques are needed to cover the copy of G_1 in $G_1 \triangleleft\!\triangleright G_2$. Therefore $cc(G_1) \leq cc(G_1 \triangleleft\!\triangleright G_2)$. Similarly, $cc(G_2) \leq cc(G_1 \triangleleft\!\triangleright G_2)$. Hence $cc(G_1 \triangleleft\!\triangleright G_2) \geq max\{cc(G_1), cc(G_2)\}$. □

Note that if $G_1 \diamond G_2$ has a new clique with 2 vertices for some original graph, while it does not have a new clique of a larger size, then we still have the result that $cc(G_1 \diamond G_2) \geq max\{cc(G_1), cc(G_2)\}$. Hence, we can weaken the assumption of Theorem 3.1.1 as the following corollary.

Corollary 3.1.2. *If $G_1 \diamond G_2$ does not have a new clique with at least 3 vertices for any original graphs, then $cc(G_1 \diamond G_2) \geq max\{cc(G_1), cc(G_2)\}$.*

The converse of Corollary 3.1.2 does not hold as shown in Example 3.1.3.

Example 3.1.3. Let G_1 and G_2 be graphs and $G_1 \underset{H}{\diamond} G_2$ be the glued graph whose clone H is shown as bold edges in Figure 3.1.1.

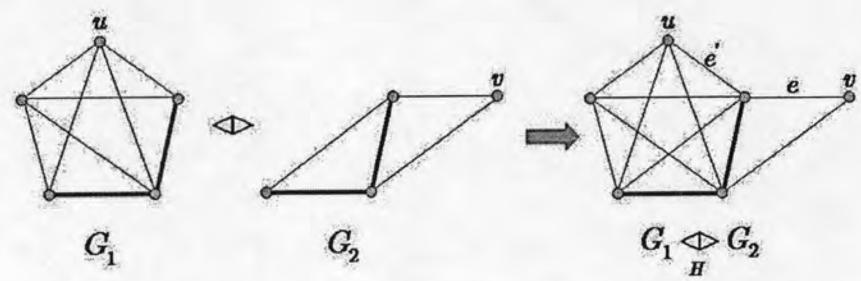

Figure 3.1.1: A counter example of the converse of Corollary 3.1.2

Note that $cc(K_n - e) = 2$ where e is an edge in K_n. Hence $cc(G_1) = 2$ and $cc(G_2) = 2$. We can use a 5-clique and a 3-clique to cover $G_1 \underset{H}{\diamond} G_2$. Thus $cc(G_1 \underset{H}{\diamond} G_2) \leq 2$. Note that e and e' in $G_1 \underset{H}{\diamond} G_2$ as shown in Figure 3.1.1 do not be contained in a common clique. Let $I = \{e, e'\}$. Thus I is a clique-independent set of $G_1 \underset{H}{\diamond} G_2$. Therefore, $cc(G_1 \underset{H}{\diamond} G_2) \geq |I| = 2$. Hence $cc(G_1 \underset{H}{\diamond} G_2) = 2$. We can see that $cc(G_1 \underset{H}{\diamond} G_2) = 2 = max\{cc(G_1), cc(G_2)\}$. But 5-clique in $G_1 \underset{H}{\diamond} G_2$ is a new clique for G_1.

□

When $\mathscr{C}$ is a minimum clique covering of a glued graph $G_1 \lhd\!\rhd G_2$, considering the set of all cliques in $\mathscr{C}$ which belong to each original graph is useful to study bounds of $cc(G_1 \lhd\!\rhd G_2)$. The next notations are defined for convenience.

Definition 3.1.4. For a glued graph $G_1 \underset{H}{\lhd\!\rhd} G_2$, let $\mathscr{C}$ be a minimum clique covering of $G_1 \underset{H}{\lhd\!\rhd} G_2$. We define $\mathscr{C}[G_1] = \{C \in \mathscr{C} \mid C \text{ is a clique of } G_1\}$, $\mathscr{C}[G_2] = \{C \in \mathscr{C} \mid C \text{ is a clique of } G_2\}$ and $\mathscr{C}[H] = \{C \in \mathscr{C} \mid C \text{ is a clique of } H\}$.

Example 3.1.5. Let G_1 and G_2 be graphs and $G_1 \underset{H}{\lhd\!\rhd} G_2$ be the glued graph whose clone H is shown as bold edges in Figure 3.1.2.

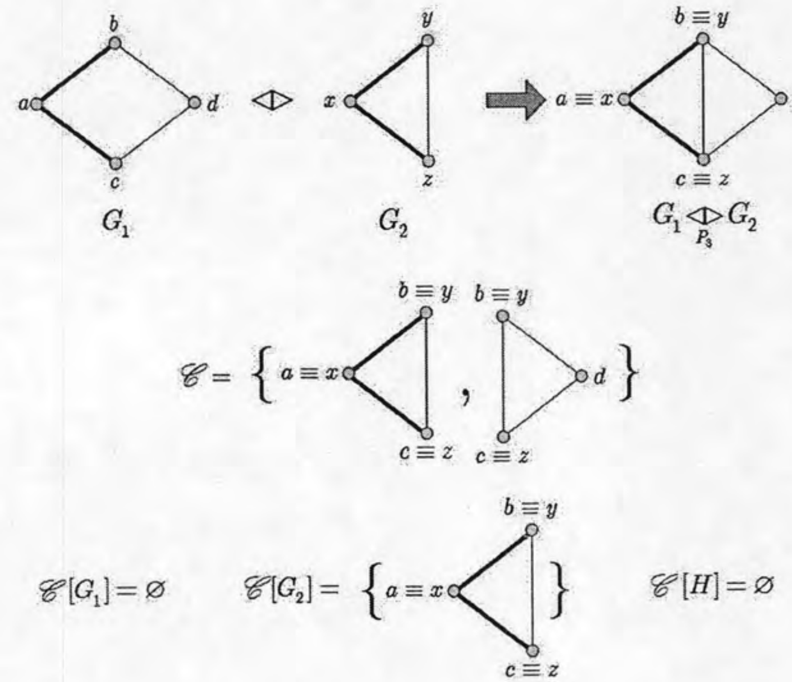

Figure 3.1.2: $\mathscr{C}[G_1]$, $\mathscr{C}[G_2]$ and $\mathscr{C}[H]$ of a glued graph

Let $\mathscr{C} = \{K_3(a \equiv x, b \equiv y, c \equiv z), K_3(b \equiv y, c \equiv z, d)\}$ as shown in Figure 3.1.2. Then $\mathscr{C}$ is a minimum clique covering of $G_1 \underset{H}{\lhd\!\rhd} G_2$. We can see that $\mathscr{C}[G_1] = \varnothing$, $\mathscr{C}[G_2] = \{K_3(a \equiv x, b \equiv y, c \equiv z)\}$ and $\mathscr{C}[H] = \varnothing$. Moreover, $\mathscr{C} \neq \mathscr{C}[G_1] \cup \mathscr{C}[G_2]$.

□

Remark 3.1.6. Note that $\mathscr{C}$ may contain a new clique which is in neither $\mathscr{C}[G_1]$ nor $\mathscr{C}[G_2]$. In general, $\mathscr{C}$ may not be $\mathscr{C}[G_1] \cup \mathscr{C}[G_2]$, as shown by Example 3.1.5.

Proposition 3.1.7. *For a minimum clique covering* $\mathscr{C}$ *of* $G_1 \diamond G_2$ *which does not have a new clique for any original graphs,* $\mathscr{C} = \mathscr{C}[G_1] \cup \mathscr{C}[G_2]$.

Proof. Let $G_1 \diamond G_2$ be a glued graph which does not have a new clique for any original graphs. Assume that $\mathscr{C}$ is a minimum clique covering of $G_1 \diamond G_2$. By Definition 3.1.4, we have $\mathscr{C}[G_1] = \{C \in \mathscr{C} \mid C \text{ is a clique of } G_1\}$ and $\mathscr{C}[G_2] = \{C \in \mathscr{C} \mid C \text{ is a clique of } G_2\}$. Then $\mathscr{C}[G_1] \cup \mathscr{C}[G_2] \subseteq \mathscr{C}$. Since $G_1 \diamond G_2$ does not have a new clique for any original graphs, every clique in the glued graph must be a copy of cliques in G_1 or G_2. Thus, $\mathscr{C} \subseteq \mathscr{C}[G_1] \cup \mathscr{C}[G_2]$. Hence $\mathscr{C} = \mathscr{C}[G_1] \cup \mathscr{C}[G_2]$. □

The following example shows that $\mathscr{C} = \mathscr{C}[G_1] \cup \mathscr{C}[G_2]$ for a minimum clique covering $\mathscr{C}$ of a glued graph without a new clique for any original graphs.

Example 3.1.8. Let G_1 and G_2 be graphs and $G_1 \underset{H}{\diamond} G_2$ be the glued graph whose clone H is shown as bold edges in Figure 3.1.3.

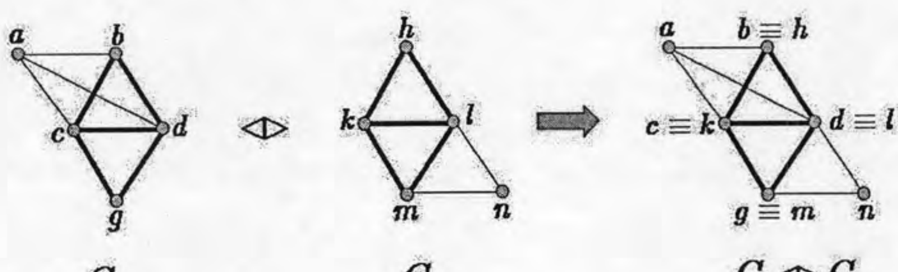

Figure 3.1.3: A glued graph without new cliques

We can see that $G_1 \underset{H}{\oplus} G_2$ does not have a new clique for any original graphs. Let $\mathscr{C} = \{K_4(a, b \equiv h, c \equiv k, d \equiv l), K_3(c \equiv k, d \equiv l, g \equiv m), K_3(d \equiv l, g \equiv m, n)\}$ as shown in Figure 3.1.4. Thus $\mathscr{C}$ is a clique covering of $G_1 \underset{H}{\oplus} G_2$. We can see that $\mathscr{C}[G_1] = \{K_4(a, b \equiv h, c \equiv k, d \equiv l), K_3(c \equiv k, d \equiv l, g \equiv m)\}$ and $\mathscr{C}[G_2] = \{K_3(c \equiv k, d \equiv l, g \equiv m), K_3(d \equiv l, g \equiv m, n)\}$ as shown in Figure 3.1.4. Moreover, $\mathscr{C} = \mathscr{C}[G_1] \cup \mathscr{C}[G_2]$.

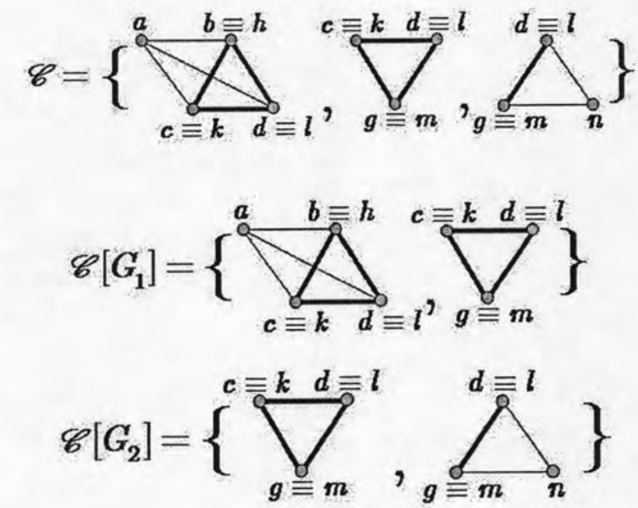

Figure 3.1.4: $\mathscr{C}$, $\mathscr{C}[G_1]$ and $\mathscr{C}[G_2]$ of a glued graph without new cliques

□

Next remark, we consider properties of $\mathscr{C}[G_1]$ and $\mathscr{C}[G_2]$ of a glued graph $G_1 \oplus G_2$ which does not have a new clique for any original graphs.

Remark 3.1.9. Let $G_1 \underset{H}{\oplus} G_2$ be a glued graph at clone H which does not have a new clique for any original graphs and $\mathscr{C}$ be a minimum clique covering of $G_1 \underset{H}{\oplus} G_2$. Then

1. If Q_1 and Q_2 are cliques of H where $Q_1 \neq Q_2$ such that $Q_1, Q_2 \in \mathscr{C}$, then Q_1 and Q_2 cannot be contained in the same clique of H in $G_1 \underset{H}{\oplus} G_2$.

Otherwise, if Q is a clique of H containing both Q_1 and Q_2 such that $Q_1, Q_2 \in \mathscr{C}$, then $(\mathscr{C} \setminus \{Q_1, Q_2\}) \cup \{Q\}$ is a clique covering of $G_1 \underset{H}{\oplus} G_2$ such that $|(\mathscr{C} \setminus \{Q_1, Q_2\}) \cup \{Q\}| < |\mathscr{C}|$.

2. Since H is a subgraph of both G_1 and G_2, $\mathscr{C}[H] \subseteq \mathscr{C}[G_1]$ and $\mathscr{C}[H] \subseteq \mathscr{C}[G_2]$. So, $\mathscr{C}[H] \subseteq \mathscr{C}[G_1] \cap \mathscr{C}[G_2]$. Because all members of $\mathscr{C}[G_1] \cap \mathscr{C}[G_2]$ are cliques of H, we have that $\mathscr{C}[G_1] \cap \mathscr{C}[G_2] \subseteq \mathscr{C}[H]$. Therefore, $\mathscr{C}[H] = \mathscr{C}[G_1] \cap \mathscr{C}[G_2]$.

3. Suppose that $|\mathscr{C}[H]| > cc(H)$. Let $\mathscr{D}$ be a minimum clique covering of the clone H. Thus $|\mathscr{D}| = cc(H)$. Note that $(\mathscr{C} \setminus \mathscr{C}[H]) \cup \mathscr{D}$ is a clique covering of $G_1 \underset{H}{\oplus} G_2$. Since $\mathscr{C} = \mathscr{C}[G_1] \cup \mathscr{C}[G_2]$, we have

$$\begin{aligned} |(\mathscr{C} \setminus \mathscr{C}[H]) \cup \mathscr{D}| &\leq |(\mathscr{C} \setminus \mathscr{C}[H])| + |\mathscr{D}| \\ &= |\mathscr{C}| - |\mathscr{C}[H]| + |\mathscr{D}| \\ &= |\mathscr{C}| - |\mathscr{C}[H]| + cc(H) \\ &< |\mathscr{C}|. \end{aligned}$$

This contradicts the fact that $\mathscr{C}$ is a minimum clique covering of $G_1 \underset{H}{\oplus} G_2$. Therefore, $|\mathscr{C}[H]| \leq cc(H)$.

Next proposition, we obtain the lower bound of the cardinality of $\mathscr{C}[G_1]$ and $\mathscr{C}[G_2]$ for a minimum clique covering $\mathscr{C}$ of a glued graph which does not have a new clique for any original graphs.

Proposition 3.1.10. *For a minimum clique covering $\mathscr{C}$ of $G_1 \underset{H}{\oplus} G_2$ which does not have a new clique for any original graphs, $|\mathscr{C}[G_1]| \geq cc(G_1) - cc(H)$ and $|\mathscr{C}[G_2]| \geq cc(G_2) - cc(H)$.*

Proof. Let $G_1 \underset{H}{\oplus} G_2$ be a glued graph which does not have a new clique for any original graphs. Let $\mathscr{C}$ and $\mathscr{D}$ be minimum clique coverings of $G_1 \underset{H}{\oplus} G_2$ and H, respectively. By Proposition 3.1.7, $\mathscr{C}[G_1] \cup \mathscr{C}[G_2]$ is a clique covering of $G_1 \underset{H}{\oplus} G_2$. Note that $\mathscr{C}[G_1] \cup \mathscr{D}$ and $\mathscr{C}[G_2] \cup \mathscr{D}$ are clique coverings of G_1 and G_2, respectively. Therefore $|\mathscr{C}[G_1] \cup \mathscr{D}| \geq cc(G_1)$. Thus $|\mathscr{C}[G_1] \cup \mathscr{D}| = |\mathscr{C}[G_1]| + |\mathscr{D}| - |\mathscr{C}[G_1] \cap \mathscr{D}| = |\mathscr{C}[G_1]| + cc(H) - |\mathscr{C}[G_1] \cap \mathscr{D}|$. So, $|\mathscr{C}[G_1]| \geq cc(G_1) - cc(H) + |\mathscr{C}[G_1] \cap \mathscr{D}| \geq cc(G_1) - cc(H)$. Similarly, $|\mathscr{C}[G_2]| \geq cc(G_2) - cc(H)$. □

The following theorem shows the new lower bound of a glued graph which does not have a new clique for any original graphs.

Theorem 3.1.11. *For $G_1 \underset{H}{\oplus} G_2$ which does not have a new clique for any original graphs, $cc(G_1) + cc(G_2) - 2cc(H) \leq cc(G_1 \underset{H}{\oplus} G_2) \leq cc(G_1) + cc(G_2)$.*

Proof. The upper bound has been already examined by Remark 2.1.1. Here we present the lower bound. Assume that $G_1 \underset{H}{\oplus} G_2$ is a glued graph which does not have a new clique for any original graphs. Let $\mathscr{C}$ be a minimum clique covering of $G_1 \underset{H}{\oplus} G_2$. By Proposition 3.1.7, $\mathscr{C} = \mathscr{C}[G_1] \cup \mathscr{C}[G_2]$. Therefore $cc(G_1 \underset{H}{\oplus} G_2) = |\mathscr{C}| = |\mathscr{C}[G_1] \cup \mathscr{C}[G_2]| = |\mathscr{C}[G_1]| + |\mathscr{C}[G_2]| - |\mathscr{C}[G_1] \cap \mathscr{C}[G_2]|$. By Remark 3.1.9(2), we have $cc(G_1 \underset{H}{\oplus} G_2) = |\mathscr{C}[G_1]| + |\mathscr{C}[G_2]| - |\mathscr{C}[H]|$.

Let $\mathscr{D}$ be a minimum clique covering of H. We can see that $\mathscr{C}[G_1] \smallsetminus \mathscr{C}[H]$ and $\mathscr{C}[G_2] \smallsetminus \mathscr{C}[H]$ do not contain a clique of H. Therefore $(\mathscr{C}[G_1] \smallsetminus \mathscr{C}[H]) \cap \mathscr{D} = \varnothing$ and $(\mathscr{C}[G_2] \smallsetminus \mathscr{C}[H]) \cap \mathscr{D} = \varnothing$. Consider

$$\begin{aligned}
|(\mathscr{C}[G_1] \smallsetminus \mathscr{C}[H]) \cup \mathscr{D}| &= |\mathscr{C}[G_1] \smallsetminus \mathscr{C}[H]| + |\mathscr{D}| - |(\mathscr{C}[G_1] \smallsetminus \mathscr{C}[H]) \cap \mathscr{D}| \\
&= |\mathscr{C}[G_1] \smallsetminus \mathscr{C}[H]| + |\mathscr{D}| \\
&= |\mathscr{C}[G_1]| - |\mathscr{C}[H]| + cc(H).
\end{aligned}$$

Similarly, $|(\mathscr{C}[G_2] \smallsetminus \mathscr{C}[H]) \cup \mathscr{D}| = |\mathscr{C}[G_2]| - |\mathscr{C}[H]| + cc(H)$. Note that $(\mathscr{C}[G_1] \smallsetminus \mathscr{C}[H]) \cup \mathscr{D}$ and $(\mathscr{C}[G_2] \smallsetminus \mathscr{C}[H]) \cup \mathscr{D}$ are clique coverings of G_1 and G_2, respectively. Hence $|(\mathscr{C}[G_1] \smallsetminus \mathscr{C}[H]) \cup \mathscr{D}| \geq cc(G_1)$ and $|(\mathscr{C}[G_2] \smallsetminus \mathscr{C}[H]) \cup \mathscr{D}| \geq cc(G_2)$.

Therefore $|\mathscr{C}[G_1]|-|\mathscr{C}[H]|+cc(H) \geq cc(G_1)$ and $|\mathscr{C}[G_2]|-|\mathscr{C}[H]|+cc(H) \geq cc(G_2)$. So, $|\mathscr{C}[G_1]| \geq cc(G_1) + |\mathscr{C}[H]| - cc(H)$ and $|\mathscr{C}[G_2]| \geq cc(G_2) + |\mathscr{C}[H]| - cc(H)$. Thus

$$\begin{aligned}
cc(G_1 \underset{H}{\diamond} G_2) &= |\mathscr{C}[G_1]| + |\mathscr{C}[G_2]| - |\mathscr{C}[H]| \\
&\geq cc(G_1) + |\mathscr{C}[H]| - cc(H) + cc(G_2) + |\mathscr{C}[H]| - cc(H) - |\mathscr{C}[H]| \\
&= cc(G_1) + cc(G_2) - 2cc(H) + |\mathscr{C}[H]| \\
&\geq cc(G_1) + cc(G_2) - 2cc(H).
\end{aligned}$$

□

3.2 Clique coverings of glued graphs at induced clone

Since a glued graph which does not have a new clique for any original graphs is a large class of graphs, we now focus a smaller subclass of it. In this section, we consider clique coverings of glued graphs at clone which is an induced subgraph of both original graphs. First, we will show that $G_1 \underset{H}{\diamond} G_2$ which H is an induced subgraph of both G_1 and G_2 does not have a new clique for any original graphs.

Definition 3.2.1. A *glued graph at induced clone* is a glued graph at clone which is an induced subgraph for both original graphs.

Proposition 3.2.2. *Any glued graph at induced clone $G_1 \underset{H}{\diamond} G_2$ does not have a new clique for any original graphs.*

Proof. Let $G_1 \underset{H}{\diamond} G_2$ be a glued graph at induced clone H. Therefore H is an induced subgraph of both G_1 and G_2. Suppose that an edge $e = ab$ in $G_1 \underset{H}{\diamond} G_2$ is a new edge for G_1. Thus a and b are not adjacent in G_1. By Remark 2.2.3(3), both endpoints of a new edge of a glued graph must be in the clone H. Hence

a and b are vertices in the clone H. Because H is an induced subgraph of G_1, a and b are not adjacent in H. However, since a and b are adjacent in $G_1 \underset{H}{\triangleleft\!\triangleright} G_2$, a and b are adjacent in G_2. Because H is an induced subgraph of G_2, a and b are adjacent in H, a contradiction. Therefore $G_1 \underset{H}{\triangleleft\!\triangleright} G_2$ does not have a new edge for any original graphs. This yields that $G_1 \underset{H}{\triangleleft\!\triangleright} G_2$ does not have a new clique for any original graphs. □

The following example shows that a glued graph at clone which is an induced subgraph of G_1 but not an induced subgraph of G_2 has a new clique for any original graphs.

Example 3.2.3. Let G_1 and G_2 be graphs and $G_1 \underset{H}{\triangleleft\!\triangleright} G_2$ be the glued graph whose clone H is shown as bold edges in Figure 3.2.1.

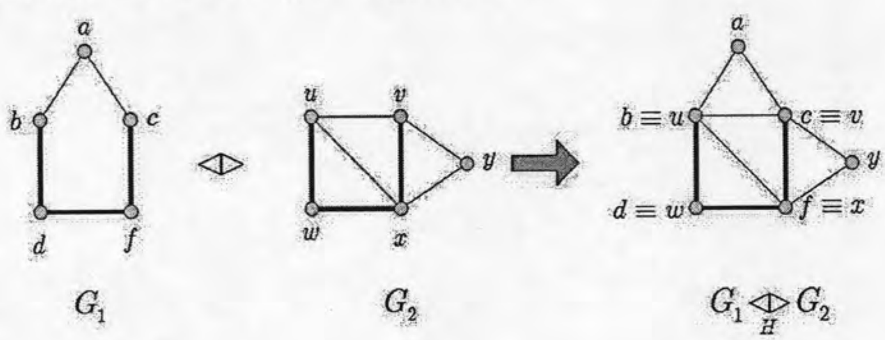

Figure 3.2.1: A glued graph at clone H which is an induced subgraph of G_1 but not an induced subgraph of G_2

From Figure 3.2.1, we can see that H is an induced subgraph of G_1 but not an induced subgraph of G_2. It is evident that $K_3(b \equiv u, d \equiv w, f \equiv x)$ is a new clique for G_1. □

Next, we use Proposition 3.2.2 to investigate the bound of clique covering numbers of glued graphs at induced clone.

Corollary 3.2.4. *For a glued graph at induced clone* $G_1 \underset{H}{\lhd\!\rhd} G_2$,
$cc(G_1) + cc(G_2) - 2cc(H) \leq cc(G_1 \underset{H}{\lhd\!\rhd} G_2) \leq cc(G_1) + cc(G_2)$.

Proof. It follows immediately from Proposition 3.2.2 and Theorem 3.1.11. □

The following example shows a glued graph at induced clone $G_1 \underset{H}{\lhd\!\rhd} G_2$ such that $cc(G_1 \underset{H}{\lhd\!\rhd} G_2) = cc(G_1) + cc(G_2) - 2cc(H)$.

Example 3.2.5. Let G_1 and G_2 be graphs and $G_1 \underset{H}{\lhd\!\rhd} G_2$ be the glued graph whose clone H is shown as bold edges in Figure 3.2.2.

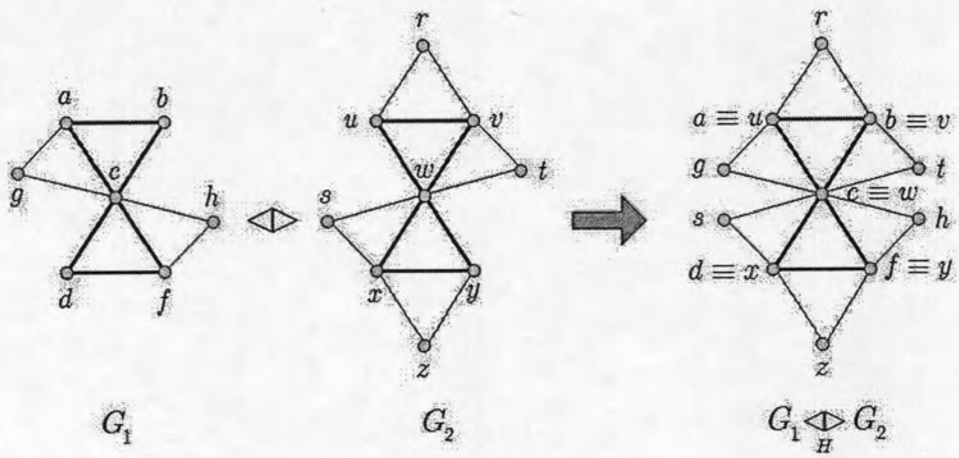

Figure 3.2.2: A glued graph at induced clone

It is evident that $cc(G_1) = 4$, $cc(G_2) = 6$, $cc(H) = 2$ and $cc(G_1 \underset{H}{\lhd\!\rhd} G_2) = 6$. Therefore $cc(G_1 \underset{H}{\lhd\!\rhd} G_2) = 4 + 6 - 2(2) = cc(G_1) + cc(G_2) - 2cc(H)$. □