เซตของจุดตรึงและเซตแห่งการลู่เข้าของฟังก์ชันต่อเนื่อง

นายสมพงษ์ ฉุยสุริฉาย

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต

สาขาวิชาคณิตศาสตร์ ภาควิชาคณิตศาสตร์

คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย

ปีการศึกษา 2545

ISBN 974-17-9790-7

THE FIXED-POINT SET AND THE CONVERGENCE SET OF A CONTINUOUS FUNCTION

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A Thesis Submitted in Partial Fulfillment of the Requirements

for the Degree of Master of Science in Mathematics

Department of Mathematics

Faculty of Science

Chulalongkorn University

Academic Year 2002

ISBN 974-17-9790-7

Thesis Title	The fixed-point set and the convergence set of a continuous function
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Field of study	Mathematics
Thesis Advisor	Phichet Chaoha, Ph.D.

Accepted by the Faculty of Science, Chulalongkorn University in Partical Fulfillment of the Requirements for the Master's Degree

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สมพงษ์ ฉุยสุริฉาย : เซตของจุดตรึงและเซตแห่งการลู่เข้าของฟังก์ชันต่อเนื่อง. (THE FIXED-POINT SET AND THE CONVERGENCE SET OF A CONTINUOUS FUNCTION). อ.ที่ปรึกษา : อ.ดร.พิเชฐ ชาวหา, 21 หน้า. ISBN 974-17-9790-7

ให้ X เป็นปริภูมิเฮาส์ดอฟฟ์ และ $f : X \to X$ เป็นฟังก์ชันต่อเนื่อง เซตของจุดตรึงของ f (เขียนแทนด้วย $F(f)$) คือ เซตของจุด x ใน X ซึ่ง $f(x) = x$ และ เซตแห่งการลู่เข้าของ f (เขียนแทนด้วย $C(f)$) คือ เซตของจุด x ใน X ซึ่งลำดับ $(f^n(x))$ ลู่เข้าใน X เมื่อ $f^n = \overbrace{f \circ f \circ \cdots \circ f}^{n \text{ ครั้ง}}$ เรานิยามฟังก์ชัน $f^\infty : C(f) \to F(f)$ โดย $f^\infty(x) = \lim_{n \to \infty} f^n(x)$ สำหรับทุก $x \in C(f)$.

ในวิทยานิพนธ์นี้ เราศึกษาคุณสมบัติเบื้องต้นของ $F(f)$ และ $C(f)$ และแสดงว่า f^∞ เป็นฟังก์ชันต่อเนื่องก็ต่อเมื่อ f^∞ ต่อเนื่องที่ทุกจุดใน $F(f)$ นอกจากนี้เรายังพิสูจน์ว่า ถ้า X เป็นปริภูมิอิงระยะทาง และ f เป็นฟังก์ชันกึ่งไม่ขยายตัวแล้ว f^∞ จะเป็นฟังก์ชันต่อเนื่อง

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ปีการศึกษา 2545

4472432823 : MAJOR MATHEMATICS
KEY WORDS : FIXED POINT SET, CONVERGENCE SET

SOMPONG CHUYSURICHAY : THE FIXED-POINT SET AND THE CONVERGENCE SET OF A CONTINUOUS FUNCTION. THESIS ADVISOR: PHICHET CHAOHA, Ph.D., 21 pp.
ISBN 974-17-9790-7

Let X be a Hausdorff space and $f: X \to X$ a continuous function. The fixed-point set of f, denoted by $F(f)$, is the set of all points $x \in X$ such that $f(x) = x$. The convergence set of f, denoted by $C(f)$, is the set of all points $x \in X$ such that the sequence $(f^n(x))$ converges in X where $f^n = \overbrace{f \circ f \circ \cdots \circ f}^{n \ times}$. We define $f^\infty : C(f) \to F(f)$ by $f^\infty(x) = \lim_{n\to\infty} f^n(x)$ for all $x \in C(f)$.

In this thesis, we study basic properties of $F(f)$ and $C(f)$ and show that f^∞ is continuous if and only if f^∞ is continuous at each point in $F(f)$. Moreover, we prove that if X is a metric space and f is quasi-nonexpansive then f^∞ is continuous.

Department **Mathematics**	Student's signature......................................
Field of study **Mathematics**	Advisor's signature......................................
Academic year **2001**	

ACKNOWLEDGEMENTS

I am greatly indebted to Dr. Phichet Chaoha, my thesis advisor, for his thoughtful and helpful advice in preparing and writing my thesis. Sincere thanks and deep appreciation are also extended to Assistant Professor Dr. Imchit Termwuttipong, the chairman, and Associate Professor Dr. Kritsana Neammanee, the committee member, for their comments and suggestions.

I would like to thank all of the teachers who have taught me, for their valuable knowledge. I am also grateful to the Development and Promotion of Science and Technology Talents Project (DPST) for providing me support throughout my graduate study. Thanks to all of my friends for giving me happiness. Finally, I would like to express my greatly gratitude to my family for their love and encouragement during my study.

CONTENTS

CHAPTER I

INTRODUCTION

Most research in fixed point theory is dealing with the existence of a fixed point for various kinds of self-maps of spaces. Even though some mathematicians are also interested in geometric features of the fixed-point set itself, more geometric structures, such as metrizability and linearity, must be imposed on spaces to facilitate the study. For our research, we try to take a different approach by not adding those geometric structures to spaces and considering topological features of the fixed-point set instead. To do this, we introduce the convergence set of a continuous self-map of a Hausdorff space and study a relationship between the convergence set and the fixed-point set by considering a special function from the former set to the latter.

There are four chapters in this thesis. In chapter II, we introduce fundamental facts used throughout this work. In chapter III, we investigate definitions of the fixed-point set and the convergence set of a continuous self-map with some basic properties of them. In chapter IV, we develop some tools to determine the continuity of the function f^∞ corresponding to a continuous self-map f. Moreover, we encounter certain conditions on a continuous self-map f which make f^∞ continuous. As a result, we show that for a continuous quasi-nonexpansive self-map of a metric space, the fixed-point set is always a retract of the convergence set.

CHAPTER II

PRELIMINARIES

In this chapter, we review some fundamental facts that will be used throughout this work. A neighborhood is always open. For a set X and a function $f : X \to X$, $f^n = \overbrace{f \circ f \circ \cdots \circ f}^{n \text{ times}}$ $(n = 1, 2, \ldots)$ and f^0 is the identity function. If a function f is continuous, we will simply call it a map.

Definition 2.1. A topological space X is called a *Hausdorff space* if for each pair x_1 and x_2 of distinct points of X, there exist neighborhoods U_1 and U_2 of x_1 and x_2, respectively, such that $U_1 \cap U_2 = \varnothing$.

Definition 2.2. A sequence (x_n) in X is said to *converge* to a point x of X if for each neighborhood U of x, there exists a positive integer N such that x_i lies in U for all $i \geq N$. In this case, x is called a *limit* of the sequence (x_n), and we write $(x_n) \to x$.

Lemma 2.3. In a Hausdorff space, every convergent sequence has a unique limit and every subsequence of a convergent sequence is convergent.

Proof The proof can be found in [1]. ■

Lemma 2.4. (The sequence lemma). Let X be a topological space and $A \subseteq X$. If there is a sequence of points of A converging to x, then $x \in \overline{A}$.

Proof The proof can be found in [1]. ■

Definition 2.5. Suppose that one-point sets are closed in X. Then X is said to be *regular* if for each pair consisting of a point x and a closed set B disjoint from x, there exist disjoint open sets U and V containing x and B, respectively.

Lemma 2.6. Let X be a topological space such that one-point sets are closed. Then X is regular if and only if given a point x of X and a neighborhood U of x, there is a neighborhood V of x such that $\overline{V} \subseteq U$.

Proof The proof can be found in [1]. ■

Definition 2.7. Let X be a set and $f : X \to X$ a function. A point x in X is said to be a *fixed point* of f provided that $f(x) = x$.

Definition 2.8. Let (X, d) be a metric space and $f : X \to X$ a function. Then f is called

(1) a *contraction* if there is a constant $\alpha \in [0, 1)$ such that for each $x, y \in X$,

$$d(f(x), f(y)) \leq \alpha d(x, y).$$

(2) *shrinking* or *contractive* if for each $x, y \in X$ and $x \neq y$,

$$d(f(x), f(y)) < d(x, y).$$

(3) an *isometry* if for each $x, y \in X$,

$$d(f(x), f(y)) = d(x, y).$$

(4) *nonexpansive* if for each $x, y \in X$,

$$d(f(x), f(y)) \leq d(x, y).$$

(5) *quasi-nonexpansive* if for each fixed point x of f and $y \in X$,

$$d(f(y), x) \leq d(y, x).$$

Remarks 2.9. (1) From the above definitions, we have the implications

$$(1) \Rightarrow (2) \Rightarrow (4) \Rightarrow (5) \text{ and } (3) \Rightarrow (4).$$

(2) The condition in (4) implies that f is continuous, so (1), (2), and (3) are also continuous.

(3) A quasi-nonexpansive function need not be continuous as the following example shows, so (5) does not imply (4).

(4) Every function having no fixed point is always quasi-nonexpansive.

Example 2.10. Let $f : [0,1] \to [0,1]$ be defined by

$$f(x) = \begin{cases} x^2 & \text{if } 0 \leq x < 1 \\ 0 & \text{if } x = 1 \end{cases}$$

Then 0 is the unique fixed point of f. It is easy to see that f is not continuous. Let $x \in [0,1]$. If $x = 1$, then $|f(x) - 0| = 0 < |1 - 0| = |x - 0|$. If $0 \leq x < 1$, then $|f(x) - 0| = |x^2 - 0| = x^2 \leq x = |x - 0|$. Thus f is quasi-nonexpansive.

Theorem 2.11. (Banach's Contraction Principle). Let (X, d) be a nonempty complete metric space and $f : X \to X$ a contraction. Then f has a unique fixed point $z \in X$. Furthermore, for any $x \in X$, we have $\lim_{n\to\infty} f^n(x) = z$.

Proof If $\alpha = 0$, then f is a constant map. Thus the statement holds in this case. Now we assume that $\alpha > 0$. Let $a = \inf\{d(x, f(x)) | x \in X\}$. We will show that $a = 0$. Let $\epsilon > 0$ and $x \in X$ be such that $d(x, f(x)) < a + (\frac{1-\alpha}{\alpha})\epsilon$. Then

$$\begin{aligned} a &\leq d(f(x), f^2(x)) \\ &\leq \alpha d(x, f(x)) \\ &< \alpha(a + (\frac{1-\alpha}{\alpha})\epsilon) \\ &= \alpha a + (1-\alpha)\epsilon. \end{aligned}$$

Thus $(1-\alpha)a < (1-\alpha)\epsilon$, so $a < \epsilon$. Hence $a = 0$, as required.

For each $n \in \mathbb{N}$, let $M_n = \{x \in X | d(x, f(x)) \leq \frac{1}{n}\}$. Then M_n is closed and nonempty since the function $x \longmapsto d(x, f(x))$ is continuous and $a = 0$. For any $x, y \in M_n$, we have

$$\begin{aligned} d(x,y) &\leq d(x, f(x)) + d(f(x), f(y)) + d(f(y), y) \\ &\leq \frac{2}{n} + \alpha d(x,y). \end{aligned}$$

Thus $d(x,y) \leq \frac{2}{(1-\alpha)n}$ so that $\operatorname{diam}(M_n) \leq \frac{2}{(1-\alpha)n} \to 0$ as $n \to \infty$. Therefore $\lim\limits_{n\to\infty} \operatorname{diam}(M_n) = 0$. By Cantor's intersection theorem, $\bigcap\limits_{n=1}^{\infty} M_n = \{z\}$ for some $z \in X$. Thus $d(z, f(z)) \leq \frac{1}{n}$ for all $n \in \mathbb{N}$, so $f(z) = z$.

To show the uniqueness, let y be a fixed point of f. Then

$$d(y,z) = d(f(y), f(z)) \leq \alpha d(y,z).$$

Thus $d(y,z) = 0$ since $0 \leq \alpha < 1$. Hence $y = z$.

For each $x \in X$, we have

$$d(f^n(x), z) = d(f^n(x), f^n(z)) \leq \alpha^n d(x,z) \to 0 \text{ as } n \to \infty.$$

Thus $(f^n(x)) \to z$. This completes the proof. ■

Theorem 2.12. Let (X,d) be a nonempty compact metric space and $f : X \to X$ a shrinking map. Then f has a unique fixed point $z \in X$. Furthermore, for any $x \in X$, we have $\lim\limits_{n\to\infty} f^n(x) = z$.

Proof Let $\phi : X \to \mathbb{R}$ be defined by

$$\phi(x) = d(x, f(x)) \quad \text{for all } x \in X.$$

Then ϕ is continuous. Thus it attains the minimum, says at some $z \in X$, since X is compact. Next, we show that $f(z) = z$. Suppose that $f(z) \neq z$. Then

$$\phi(f(z)) = d(f(z), f^2(z)) < d(z, f(z)) = \phi(z).$$

This contradicts the minimality of ϕ at z. Therefore $f(z) = z$, as required.

To show the uniqueness, let y be a fixed point of f. Suppose that $y \neq z$. Then

$$d(y,z) = d(f(y), f(z)) < d(y,z),$$

which is a contradiction.

Finally, let $x \in X$ and $a_n = d(f^n(x), z)$ for each $n \in \mathbb{N}$. Then for each $n \in \mathbb{N}$,

$$\begin{aligned} a_{n+1} &= d(f^{n+1}(x), z) \\ &= d(f^{n+1}(x), f^{n+1}(z)) \\ &\leq d(f^n(x), f^n(z)) \\ &= d(f^n(x), z) \\ &= a_n. \end{aligned}$$

Thus $(a_n) \to a$ for some $a \geq 0$. We shall show that $a = 0$. Suppose that $a > 0$. Since X is compact, there is a subsequence $(f^{n_k}(x))$ of $(f^n(x))$ such that $(f^{n_k}(x)) \to y$ for some $y \in X$. Obviously, $a = d(y, z)$ since $a_{n_k} = d(f^{n_k}(x), z) \to d(y, z)$ as $n \to \infty$. Thus

$$\begin{aligned} a = \lim_{k \to \infty} d(f^{n_k+1}(x), z) &= d(f(y), z) \\ &= d(f(y), f(z)) < d(y, z) = a. \end{aligned}$$

This is a contradiction. Therefore $a = 0$ and hence $\lim\limits_{n \to \infty} f^n(x) = z$. This completes the proof. ■

Definition 2.13. A subspace A of a topological space X is said to be a *retract* of X if there is a map $r : X \to A$ such that $r(x) = x$ for all $x \in A$. The map r is called a *retraction* of X onto A.

For each $n \in \mathbb{N}$, let $D^n = \{x \in \mathbb{R}^n | \; \|x\| \leq 1\}$ and $S^n = \{x \in \mathbb{R}^{n+1} | \; \|x\| = 1\}$. D^n is called the *n-disk* (or *n-ball*) and S^n is called the *n-sphere* (of radius 1 and centered at the origin).

Theorem 2.14. If $n \geq 0$, then S^n is not a retract of D^{n+1}.

Proof The proof can be found in [2]. ■

CHAPTER III

FIXED POINT SETS AND CONVERGENCE SETS

Let X be a Hausdorff topological space and $f : X \to X$ a map (= a continuous function).

Definition 3.1. The *fixed-point set* of f, denoted by $F(f)$, is defined by

$$F(f) = \{x \in X | f(x) = x\}.$$

Definition 3.2. The *convergence set* of f, denoted by $C(f)$, is defined by

$$C(f) = \{x \in X | (f^n(x)) \text{ converges in } X\}.$$

Hence, by definitions above, we obtain a function $f^\infty : C(f) \to F(f)$ defined by

$$f^\infty(x) = \lim_{n\to\infty} f^n(x).$$

Clearly, $f^\infty(C(f)) = F(f)$ since for each $x \in C(f)$, we have

$$\begin{aligned} f(f^\infty(x)) &= f(\lim_{n\to\infty} f^n(x)) \\ &= \lim_{n\to\infty} f^{n+1}(x) \\ &= f^\infty(x). \end{aligned}$$

Example 3.3. Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by $f(x) = 1 + \frac{1}{x}$. Then $\omega = \frac{1+\sqrt{5}}{2}$ is the unique fixed point of f. For each $x \in \mathbb{R}^+$ and $n \geq 2$, we have

$$\begin{aligned} f^n(x) &= \frac{F_n x + F_{n-1}}{F_{n-1}x + F_{n-2}} \\ &= \frac{\frac{F_n}{F_{n-1}}x + 1}{x + \frac{F_{n-2}}{F_{n-1}}} \end{aligned}$$

where F_n is the n^{th} Fibonacci number. Notice that $\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \omega$.

Thus $(f^n(x))$ is convergent and

$$\begin{aligned} \lim_{n\to\infty} f^n(x) &= \lim_{n\to\infty} \frac{\frac{F_n}{F_{n-1}}x + 1}{x + \frac{F_{n-2}}{F_{n-1}}} \\ &= \frac{\omega x + 1}{x + \frac{1}{\omega}} \\ &= \omega. \end{aligned}$$

Therefore $C(f) = \mathbb{R}^+$ and $F(f) = \{\omega\}$. Note also that f is not a contraction.

Example 3.4. For $n \in \mathbb{N}$, let D^n be the unit disk in $\mathbb{R}^n$. Fix $i \in \{1, 2, ..., n\}$ and let e_i be the i^{th} standard basis element of $\mathbb{R}^n$ and π_i the projection of $\mathbb{R}^n$ onto the i^{th}-factor of $\mathbb{R}^n$. We define $f : D^n \to \mathbb{R}^n$ by

$$f(x) = x + (1 - \|x\|)e_i.$$

Clearly, f is continuous and $f(D^n) \subseteq D^n$ since for each $x \in D^n$, we have

$$\begin{aligned} \|f(x)\| &= \|x + (1 - \|x\|)e_i\| \\ &\leq \|x\| + (1 - \|x\|)\|e_i\| \\ &= \|x\| + (1 - \|x\|) \\ &= 1. \end{aligned}$$

We can easily see that $F(f) = S^{n-1}$, the unit sphere in $\mathbb{R}^n$.

Next, we will show that $C(f) = D^n$. Let $x \in D^n$. For each $k \in \mathbb{N}$, we have

$$\begin{aligned} f^k(x) &= f^{k-1}(x) + (1 - \|f^{k-1}(x)\|)e_i \\ &= f^{k-2}(x) + [2 - (\|f^{k-2}(x)\| + \|f^{k-1}(x)\|)]e_i \\ &\vdots \\ &= x + [k - (\|x\| + \|f(x)\| + \cdots + \|f^{k-1}(x)\|)]e_i \quad \ldots\ldots\ldots\ldots (1) \end{aligned}$$

and
$$\begin{aligned} \pi_i(f^k(x)) &= \pi_i(x + [k - (\|x\| + \|f(x)\| + \cdots + \|f^{k-1}(x)\|)]e_i) \\ &= \pi_i(x) + [k - (\|x\| + \|f(x)\| + \cdots + \|f^{k-1}(x)\|)]\pi_i(e_i) \\ &= \pi_i(x) + [k - (\|x\| + \|f(x)\| + \cdots + \|f^{k-1}(x)\|)]. \end{aligned}$$

Thus we obtain the relation

$$k - (\|x\| + \|f(x)\| + \cdots + \|f^{k-1}(x)\|) = \pi_i(f^k(x)) - \pi_i(x) \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

By substituting (2) in (1), we get

$$f^k(x) = x + (\pi_i(f^k(x)) - \pi_i(x))e_i \text{ for all } k \in \mathbb{N} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Since $\pi_i(f^k(x)) = \pi_i(f^{k-1}(x)) + (1 - \|f^{k-1}(x)\|)$, we have

$$\pi_i(f^k(x)) \geq \pi_i(f^{k-1}(x)) \text{ for all } k \in \mathbb{N}.$$

This implies that $(\pi_i(f^k(x)))$ is an increasing (and bounded) sequence of real numbers and hence it is convergent. Consequently, by (3), $(f^k(x))$ is also convergent. Therefore $C(f) = D^n$, as required. Note that f^∞ is not continuous since S^{n-1} is not a retract of D^n.

Definition 3.5. For $x \in F(f)$, the *convergence set of* x, denoted by $C_x(f)$, is defined to be the inverse image of x under f^∞; i.e.,

$$C_x(f) = \{y \in C(f) \mid f^\infty(y) = x\}.$$

Example 3.6. Let $f : [0,1] \to [0,1]$ be defined by $f(x) = x^{k+1}$ for $k \in \mathbb{N}$. Then we have $F(f) = \{0,1\}, C(f) = [0,1], C_0(f) = [0,1), C_1(f) = \{1\}$, and f^∞ is defined by

$$f^{\infty}(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{if } 0 \le x < 1 \end{cases}$$

Evidently, f^{∞} is not continuous.

The following lemma summarizes basic properties of notions discussed above.

Lemma 3.7. Let X be a Hausdorff topological space and $f : X \to X$ a map. Then the following statements hold:

(1) $F(f) \subseteq C(f)$.

(2) $F(f)$ is closed in X.

(3) $C(f)$ is nonempty iff $F(f)$ is nonempty.

(4) Both $F(f)$ and $C(f)$ are invariant under f; i.e., $f(F(f)) \subseteq F(f)$ and $f(C(f)) \subseteq C(f)$.

(5) $f^k f^{\infty} = f^{\infty} f^k = f^{\infty}$ for each $k \in \mathbb{N}$.

(6) $C(f) = \dot{\bigcup\limits_{x \in F(f)}} C_x(f)$, where $\dot{\bigcup}$ denotes the disjoint union.

(7) If f^{∞} is continuous, then $C_x(f)$ is closed for each $x \in F(f)$.

(8) If f^{∞} is continuous, then $F(f)$ is a retract of $C(f)$.

Proof

(1) Obvious.

(2) If $F(f) = X$, we are done. Now we assume that $F(f) \neq X$.

Let $x \in X \setminus F(f)$. Then $x \neq f(x)$, so there are disjoint open sets U and V containing x and $f(x)$, respectively. Thus $U \cap f^{-1}(V)$ is a neighborhood of x in X.

We claim that $U \cap f^{-1}(V) \subseteq X \setminus F(f)$. Let $y \in U \cap f^{-1}(V)$. Then $y \in U$ and $f(y) \in V$. Since $U \cap V = \varnothing$, we have $f(y) \neq y$. This shows that $U \cap f^{-1}(V) \subseteq X \setminus F(f)$. Therefore $X \setminus F(f)$ is open in X and hence $F(f)$ is closed in X.

(3) Follows from (1) and the definition of f^∞.

(4) Follows from Lemma 2.3

(5) Follows directly from Lemma 2.3.

(6) Follows directly from the definition of $C_x(f)$.

(7) Follows immediately from the continuity of f^∞ and the definition of $C_x(f)$.

(8) Follows immediately by the continuity of f^∞ and the definition of retract. ■

Example 3.8. Some maps f from the unit disk D^2, realized as a subspace of $\mathbb{C}$, into itself with the fixed-point set, convergence set, and continuity of f^∞.

$f : D^2 \to D^2$	$F(f)$	$C(f)$	Is f^∞ continuous?
1. $f :=$ a shrinking map	$\{*\}$	D^2	Yes
2. $f :=$ the identity map	D^2	D^2	Yes
3. $f(z) = e^{i\theta}z$; $\theta \in (0, 2\pi)$	$\{0\}$	$\{0\}$	Yes
4. $f(z) = \overline{z}$	$[-1, 1]$	$[-1, 1]$	Yes
5. $f(z) = \lvert z\rvert$	$[0, 1]$	D^2	Yes
6. $f(z) = z^{n+1}$; $n \in \mathbb{N}$	$\{0\} \cup \Omega_n^{\dagger}$	$\mathrm{Int}(D^2) \cup K^{\ddagger}$	No
7. $f(z) = \lvert z\rvert z$	$S^1 \cup \{0\}$	D^2	No
8. $f(z) = 1 - \lvert z\rvert + z$	S^1	D^2	No

† $\Omega_n =$ the set of all n-roots of unity.

‡ $K = \{e^{2\pi i\theta} \mid \theta = \frac{m+rn}{n(n+1)^k}$ for some $m \in \{0, ..., n-1\}$, $r \in \mathbb{Z}$, and $k \in \mathbb{N}\}$

Proposition 3.9. If X is a nonempty complete metric space and f is a contraction, then $C(f) = X$ and $F(f)$ is a one-point set.

Proof Follows directly from Banach's contraction principle. ■

Proposition 3.10. If X is a nonempty compact metric space and f is a shrinking map, then $C(f) = X$ and $F(f)$ is a one-point set.

Proof Follows directly from Theorem 2.12. ■

Proposition 3.11. Let (X, d) be a metric space. If f^N is an isometry for some $N \in \mathbb{N}$, then $C(f) = F(f)$.

Proof If $C(f) = \varnothing$, then $F(f) = \varnothing$. Assume that $C(f) \neq \varnothing$. Let $x \in C(f)$. We will show that $f^\infty(x) = x$. Suppose that $f^\infty(x) \neq x$. Then $d(x, f^\infty(x)) > 0$ and there is an $M \in \mathbb{N}$ such that $d(f^m(x), f^\infty(x)) < d(x, f^\infty(x))$ for all $m \geq M$. Thus $d(f^{MN}(x), f^\infty(x)) < d(x, f^\infty(x))$ and

$$\begin{aligned} d(x, f^\infty(x)) &= d(f^{MN}(x), f^{MN}(f^\infty(x))) \\ &= d(f^{MN}(x), f^\infty(x)) \\ &< d(x, f^\infty(x)), \end{aligned}$$

which is a contradiction. Hence, $f(x) = f(f^\infty(x)) = f^\infty(x) = x$ and $C(f) = F(f)$ as desired. ■

Example 3.12. Let $f : \mathbb{C} \smallsetminus \{0\} \to \mathbb{C} \smallsetminus \{0\}$ be defined by

$$f(z) = \frac{1}{z}.$$

Then f is not nonexpansive but $f^2 = I$ is nonexpansive. By Proposition 3.11, we have $C(f) = F(f) = \{-1, 1\}$. Thus f^∞ is continuous.

CHAPTER IV

THE CONTINUITY OF f^∞

As before, X is a Hausdorff topological space and $f : X \to X$ is a map (= a continuous function). In this chapter, we develop some tools to determine the continuity of the function f^∞. Certain classes of well-behaved fixed points are studied in this chapter.

Throughout this chapter, we will assume that $F(f)$ is nonempty, the domain of f^n $(n = 1, 2, ...)$ is restricted to $C(f)$, and a neighborhood of x is an <u>open subset of $C(f)$</u> containing x.

Proposition 4.1. $f^\infty = f$ on $C(f)$ if and only if $f = f^2$ on $C(f)$. In particular, if f is an idempotent map, then f^∞ is continuous.

Proof Assume that $f^\infty = f$ on $C(f)$. Then $f = f^\infty = f \circ f^\infty = f^2$ on $C(f)$.

Conversely, assume that $f = f^2$ on $C(f)$. Then the sequence $(f^n(x)) = (f(x))$ converges to $f(x)$ for all x in $C(f)$. Hence $f^\infty = f$ on $C(f)$, as required. ■

Remark 4.2. f^∞ is always an idempotent function.

Example 4.3. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 0 & \text{if } x \le 0 \\ x & \text{if } 0 < x < 1 \\ 1 & \text{if } x \ge 1 \end{cases}$$

Then f is idempotent with $F(f) = [0,1]$ and $C(f) = \mathbb{R}$. Thus, by Proposition 4.1, f^∞ is continuous.

Example 4.4. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x+2 & \text{if } x < -3 \\ -1 & \text{if } -3 \le x < -1 \\ x & \text{if } -1 \le x < 1 \\ 1 & \text{if } 1 \le x < 3 \\ x-2 & \text{if } x \ge 3 \end{cases}$$

Then $F(f) = [-1,1]$ and $C(f) = [-3,3]$. Clearly, f is not idempotent on $\mathbb{R}$. However, f is idempotent on $[-3,3] = C(f)$. Thus, by Proposition 4.1, f^∞ is continuous.

Theorem 4.5. (A Continuity Criterion for f^∞)

f^∞ is continuous iff f^∞ is continuous at each point in $F(f)$.

Proof ($\Rightarrow$) Obvious.

($\Leftarrow$) Let $x \in C(f)$ and U a neighborhood of $f^\infty(x)$. Since f^∞ is continuous at $f^\infty(x)$, there is a neighborhood V of $f^\infty(x)$ such that $f^\infty(V) \subseteq U$. Since the sequence $(f^n(x))$ converges to $f^\infty(x)$, there is an $N \in \mathbb{N}$ such that $f^N(x) \in V$. Thus $f^{-N}(V)$ is a neighborhood of x and

$$\begin{aligned} f^\infty(f^{-N}(V)) &= f^\infty(f^N(f^{-N}(V))) \\ &\subseteq f^\infty(V) \\ &\subseteq U. \end{aligned}$$

Hence f^∞ is continuous at x. ■

Now we see that $F(f)$ plays a crucial role in determining the continuity of f^∞ by the above theorem. Therefore, we will investigate some kinds of fixed points that are useful for our purpose.

Definition 4.6. The fixed point x of f is said to be *attractive* if there is a neighborhood U of x such that $f^\infty(U) = \{x\}$.

Theorem 4.7. The fixed point x of f is attractive iff $C_x(f)$ is open in $C(f)$.

Proof Let x be an attractive fixed point of f. Let $y \in C_x(f)$. Since x is attractive, there is a neighborhood U of x such that $f^\infty(U) = \{x\}$. Since the sequence $(f^n(y))$ converges to x, there is an $N \in \mathbb{N}$ such that $f^N(y) \in U$. Thus $f^{-N}(U)$ is a neighborhood of y and

$$\begin{aligned} f^\infty(f^{-N}(U)) &= f^\infty(f^N(f^{-N}(U))) \\ &\subseteq f^\infty(U) \\ &= \{x\}. \end{aligned}$$

This shows that $f^{-N}(U) \subseteq C_x(f)$ and hence $C_x(f)$ is open in $C(f)$.

Conversely, assume that $C_x(f)$ is open in $C(f)$. Then $C_x(f)$ is a neighborhood of x and $f^\infty(C_x(f)) = \{x\}$. Hence x is attractive. ■

Remark 4.8. If f has a unique fixed point, then its fixed point is immediately attractive.

Example 4.9. From example 3.6 we can see that 0 is an attractive fixed point of f since $C_0(f) = [0, 1)$ is open in $[0, 1]$ but 1 is not attractive since $C_1(f) = \{1\}$ is not open in $[0, 1]$.

Proposition 4.10. If $x \in F(f)$ is an attractive fixed point, then f^∞ is continuous at x.

Proof This is obvious by the definition of attractive fixed point. ■

Remark 4.11. The converse of Proposition 4.10 is not true in general as the identity map on $\mathbb{R}$ shows. However, the converse is true if $F(f)$ is finite as we can see later.

Definition 4.12. A fixed point x of f is said to be *isolated* if there is a neighborhood U of x such that $U \cap F(f) = \{x\}$.

Example 4.13. From example 3.6 both 0 and 1 are isolated since $[0, \frac{1}{2}) \cap F(f) = \{0\}$ and $(\frac{1}{2}, 1] \cap F(f) = \{1\}$.

Example 4.9 and Example 4.13 show that an isolated fixed point need not be attractive but the converse is always true.

Proposition 4.14. Every attractive fixed point of f is isolated.

Proof Follows from Theorem 4.7. ■

Proposition 4.15. If $F(f)$ is finite, then every fixed point of f is isolated.

Proof This is clear since $C(f)$ is Hausdorff. ■

Theorem 4.16. Suppose that $F(f)$ is finite. Then f^∞ is continuous iff every fixed point of f is attractive.

Proof Let $x \in F(f)$ and assume that f^∞ is continuous. Then, by Proposition 4.15, x is isolated. Thus there is a neighborhood U of x such that $U \cap F(f) = \{x\}$. By the continuity of f^∞, there is a neighborhood V of x such that $f^\infty(V) \subseteq U$. Thus $f^\infty(V) \subseteq U \cap F(f) = \{x\}$. Therefore x is attractive.

The converse is true by Proposition 4.10.

Another proof for the " if " part By Lemma 3.7(7), $C(f)$ is a disjoint union of $C_x(f)$ where $x \in F(f)$. By Lemma 3.7(6), $C_x(f)$ is closed for all $x \in F(f)$. Thus $C_x(f) = C(f) \smallsetminus \bigcup\limits_{y \neq x} C_y(f)$ is open for all $x \in F(f)$ since $F(f)$ is finite. This implies that every fixed point of f is attractive by Theorem 4.7. ■

Corollary 4.17. Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial map of degree greater than 1. Then f^∞ is continuous iff every fixed point of f is attractive.

Definition 4.18. Let $N \in \mathbb{N}$. The fixed point x of f is said to be N-*invariant* if for each neighborhood U of x, there is a neighborhood V of x such that $V \subseteq U$ and $f^N(V) \subseteq V$.

Theorem 4.19. If $C(f)$ is a regular topological space and $x \in F(f)$ is an N-invariant fixed point of f, then f^∞ is continuous at x.

Proof Let x be an N-invariant fixed point of f and U a neighborhood of x. By the regularity of $C(f)$, there is a neighborhood V of x such that $\overline{V} \subseteq U$. Since x is an N-invariant fixed point of f, there is a neighborhood W of x such that $W \subseteq V$ and $f^N(W) \subseteq W$. Thus $f^{kN}(W) \subseteq W$ for all $k \in \mathbb{N}$. For each $w \in W$, we have $(f^{kN}(w))$ converges to a point in $\overline{W}$ and consequently $(f^k(w))$ converges to a point in $\overline{W}$ as well. This gives $f^\infty(W) \subseteq \overline{W} \subseteq \overline{V} \subseteq U$. Therefore f^∞ is continuous at x. ■

Corollary 4.20. If every fixed point of f is N-invariant, then f^∞ is continuous.

Remark 4.21. It is clear that an N-invariant fixed point need not be attractive as the identity map on $\mathbb{R}$ shows.

We will end this chapter by discussing another useful condition on f that makes f^∞ continuous.

Theorem 4.22. Let (X, d) be a metric space and $f : X \to X$ a map. If $F(f)$ is nonempty and f^N is quasi-nonexpansive for some $N \in \mathbb{N}$, then every fixed point of f is N-invariant; hence, f^∞ is continuous.

Proof Let $x \in F(f)$ and let U be a neighborhood of x. Let $\epsilon > 0$ be such that $V = B(x; \epsilon) \cap C(f) \subseteq U$. For each $y \in V$, we have

$$d(f^N(y), x) \leq d(y, x) < \epsilon.$$

Thus $f^N(V) \subseteq V$. Therefore x is N-invariant. ■

Corollary 4.23. Let (X, d) be a metric space and $f : X \to X$ a map. If $F(f)$ is nonempty and f^N is quasi-nonexpansive for some $N \in \mathbb{N}$, then $F(f)$ is a retract of $C(f)$.

Example 4.24. There is no quasi-nonexpansive map $f : D^n \to D^n$ such that $F(f) = S^{n-1}$ and $C(f) = D^n$.

Example 4.25. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \frac{x}{2}\sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Then f is continuous and differentiable at every $x \neq 0$ and $F(f) = \{0\}$. Suppose that f is nonexpansive. Then for each $x \neq \frac{1}{\pi}$, we have $|f(x) - f(\frac{1}{\pi})| \leq |x - \frac{1}{\pi}|$. Thus $|f'(\frac{1}{\pi})| \leq 1$. This is impossible since $f'(\frac{1}{\pi}) = \frac{\pi}{2} > 1$. Therefore f is not nonexpansive. However, f is quasi-nonexpansive since for each $x \neq 0$, we have

$$\begin{aligned} |f(x) - 0| &= |\frac{x}{2}\sin(\frac{1}{x})| \\ &\leq |\frac{x}{2}| \\ &= \frac{1}{2}|x - 0| \\ &\leq |x - 0|. \end{aligned}$$

Thus, by Corollary 4.23, f^{∞} is continuous.

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