CHAPTER III

WEAKLY REGULAR Γ-SEMINEARRINGS

This chapter is separated into studying regular Γ-seminearrings in the first section and weakly regular Γ-seminearrings in the next.

3.1 Regular Γ-seminearrings

It is known that a ring R is called regular if and only if for each $x \in R$, there exists $a \in R$ such that $x = xax$. Accordingly, regularlities in the sense of Γ-seminearrings are defined.

Definition 3.1.1. A Γ-seminearring R is called *regular* if for all $x \in R$, there exist $r \in R$ and $\alpha, \beta \in \Gamma$ such that $x = x\alpha r\beta x$.

We exhibit some properties of regular Γ-seminearrings.

Theorem 3.1.1. *Let θ be a Γ-homomorphism from R into S. If R is regular, then* $\operatorname{im}\theta$ *is also regular.*

Proof. Recall from Proposition 2.3.1(iii) that $\operatorname{im}\theta$ is a sub Γ-seminearring of S. To show that $\operatorname{im}\theta$ is regular, let $x \in R$. Then $x = x\alpha r\beta x$ for some $r \in R$ and $\alpha, \beta \in \Gamma$ since R is regular. Thus $\theta(x) = \theta(x\alpha r\beta x) = \theta(x)\alpha\theta(r)\beta\theta(x)$. Hence $\operatorname{im}\theta$ is regular. □

Proposition 3.1.2. *Let θ be an one-to-one Γ-homomorphism from R into S. If* $\operatorname{im}\theta$ *is regular, then R is regular.*

Proof. Assume that $\text{im}\theta$ is regular. Let $x \in R$. Then $\theta(x) = \theta(x)\alpha\theta(r)\beta\theta(x)$ for some $r \in R$ and $\alpha, \beta \in \Gamma$ so that $\theta(x) = \theta(x\alpha r\beta x)$. Since θ is one-to-one, it follows that $x = x\alpha r\beta x$. Hence R is regular. □

Theorem 3.1.3. *Let R be a Γ-seminearring and I an ideal of the semigroup R.*

(i) If R is regular, then R/I is regular.

(ii) If R/I and I are both regular, then R is regular.

Proof. (i) Assume that R is regular. From Theorem 2.3.11, $R/I = \text{im}\,\varphi$ where φ is the natural Γ-homomorphism. Moreover, $\text{im}\,\varphi$ is regular by the above theorem. As a result, R/I is regular.

(ii) Assume that both R/I and I are regular. To show that R is regular, let $x \in R$. Then there exist $r \in R$ and $\alpha, \beta \in \Gamma$ such that $x + I = (x + I)\alpha(r + I)\beta(x + I)$ and then $x + I = (x\alpha r\beta x) + I$. This implies that $x = x\alpha r\beta x$ or $x, x\alpha r\beta x \in I$. If $x = x\alpha r\beta x$, then we are done. If $x, x\alpha r\beta x \in I$, then there exist $a \in I$ and $\delta, \sigma \in \Gamma$ such that $x = x\delta a\sigma x$ since I is regular. Therefore R is regular.

□

Theorem 3.1.4. *Let R be a regular Γ-seminearring. Then $I \cap J = J\Gamma I$ for any left ideal I of R and any right ideal J of R.*

Proof. Let I and J be a left ideal and a right ideal of R, respectively. Then $J\Gamma I \subseteq I$ and $J\Gamma I \subseteq J$ so $J\Gamma I \subseteq I \cap J$.

Next, let $x \in I \cap J$. Since R is regular, $x = x\alpha r\beta x$ for some $r \in R$ and $\alpha, \beta \in \Gamma$. Then $x = (x\alpha r)\beta x \in J\Gamma x$ because $x \in J$ and J is a right ideal of R. Since $x \in I$, this implies that $x \in J\Gamma I$ so that $I \cap J \subseteq J\Gamma I$.

Therefore $I \cap J = J\Gamma I$. □

3.2 Weakly regular Γ-seminearrings

We study one-sided weakly regular Γ-seminearrings in this section. Recall that a ring R is left (right) weakly regular if and only if $x \in (RxRx)\big(x \in (xRxR)\big)$ for all $x \in R$.

Definition 3.2.1. A Γ-seminearring R is called *left (right) weakly regular* if $x \in (R\Gamma x)^2\Big(x \in (x\Gamma R)^2\Big)$ for all $x \in R$ where $(R\Gamma x)^2$ and $(x\Gamma R)^2$ stand for $(R\Gamma x)\Gamma(R\Gamma x)$ and $(x\Gamma R)\Gamma(x\Gamma R)$, respectively.

The following result shows relationship between regular Γ-seminearrings and weakly regular Γ-seminearrings.

Theorem 3.2.1. *Let R be a Γ-seminearring. If R is regular and has a left (right) identity, then R is left (right) weakly regular.*

Proof. It is enough to assume that R is regular and has a left identity. Let $x \in R$. Since R is regular, there exist $r \in R$ and $\alpha, \beta \in \Gamma$ such that $x = x\alpha r\beta x$. Since R has a left identity, say e, it follows that for any $\gamma \in \Gamma$, $x = e\gamma x \in R\Gamma x$ so that $x = x\alpha(r\beta x) \in (R\Gamma x)\Gamma(R\Gamma x) = (R\Gamma x)^2$. Thus R is left weakly regular. $\square$

Theorem 3.2.2. *Let θ be a Γ-homomorphism from R into S. If R is left (right) weakly regular, then $\operatorname{im}\theta$ is also left (right) weakly regular.*

Proof. Assume that R is left weakly regular. Then $\operatorname{im}\theta$ is a sub Γ-seminearring of S by Proposition 2.3.1(iii). To show that $\operatorname{im}\theta$ is left weakly regular, let $x \in R$. Then $x \in (R\Gamma x)^2$ so that $x = \sum\big(\sum r_i\alpha_i x\big)\delta_k\big(\sum s_j\beta_j x\big)$ where $r_i, s_j \in R$ and

$\alpha_i, \beta_j, \delta_k \in \Gamma$ for all i, j, k. Then

$$\begin{aligned}\theta(x) &= \theta\Big(\sum\Big(\sum r_i\alpha_i x\Big)\delta_k\Big(\sum s_j\beta_j x\Big)\Big)\\ &= \sum\Big(\Big(\sum \theta(r_i\alpha_i x)\Big)\delta_k\Big(\sum \theta(s_j\beta_j x)\Big)\Big)\\ &= \sum\Big(\Big(\sum \theta(r_i)\alpha_i\theta(x)\Big)\delta_k\Big(\sum \theta(s_j)\beta_j\theta(x)\Big)\Big)\\ &\in \big(\theta(R)\Gamma\theta(x)\big)^2.\end{aligned}$$

Therefore $\operatorname{im}\theta$ is left weakly regular since $\operatorname{im}\theta = \theta(R)$.

The proof for the other case is obtained similarly. □

Proposition 3.2.3. *Let θ be an one-to-one Γ-homomorphism from R into S. If $\operatorname{im}\theta$ is left (right) weakly regular, then R is left (right) weakly regular.*

Proof. It is enough to assume that $\operatorname{im}\theta$ is left weakly regular. To show that R is left weakly regular, let $x \in R$. Since $\operatorname{im}\theta$ is left weakly regular, it follows that $\theta(x) = \sum\Big(\sum\theta(r_i)\alpha_i\theta(x)\Big)\delta_k\Big(\sum\theta(s_j)\beta_j\theta(x)\Big)$ where $r_i, s_j \in R$ and $\alpha_i, \beta_j, \delta_k \in \Gamma$ for all i, j, k and then $\theta(x) = \theta\Big(\sum\Big(\sum r_i\alpha_i x\Big)\delta_k\Big(\sum s_j\beta_j x\Big)\Big)$. Since θ is one-to-one, $x = \sum\Big(\sum r_i\alpha_i x\Big)\delta_k\Big(\sum s_j\beta_j x\Big)$ which contains in $(R\Gamma x)^2$. As a result, R is left weakly regular. □

Theorem 3.2.4. *Let R be a Γ-seminearring and I an ideal of the semigroup R.*

(i) If R is left (right) weakly regular, then R/I is left (right) weakly regular.

(ii) If R/I and I are both left (right) weakly regular, then R is left (right) weakly regular.

Proof. (i) Assume that R is left weakly regular. This follows from Theorem 2.3.11 that $R/I = \operatorname{im}\varphi$ where φ is the natural Γ-homomorphism and from Theorem 3.2.2. As a result, R/I is left weakly regular.

(ii) Assume that both R/I and I are left weakly regular. To show that R is left weakly regular, let $x \in R$. Then

$$x + I = \sum\Big(\sum(r_i + I)\alpha_i(x + I)\Big)\delta_k\Big(\sum(s_j + I)\beta_j(x + I)\Big)$$
$$= \Big(\sum\big(\sum r_i\alpha_i x\big)\delta_k\big(\sum s_j\beta_j x\big)\Big) + I$$

where $r_i, s_j \in R$ and $\alpha_i, \beta_j, \delta_k \in \Gamma$ for all i, j, k since R/I is regular. Thus $x = \sum\big(\sum r_i\alpha_i x\big)\delta_k\big(\sum s_j\beta_j x\big)$ or both of $\sum\big(\sum r_i\alpha_i x\big)\delta_k\big(\sum s_j\beta_j x\big)$ and x belong to I. If $x = \sum\big(\sum r_i\alpha_i x\big)\delta_k\big(\sum s_j\beta_j x\big)$, then it is clear that R is left weakly regular. If x, $\sum\big(\sum r_i\alpha_i x\big)\delta_k\big(\sum s_j\beta_j x\big) \in I$, then $x \in (I\Gamma x)^2$ because I is left weakly regular containing x.

As a result, R is left weakly regular.

The proofs for the case of right weakly regularities are obtained analogously.

□

Let I and J be an ideal and a right ideal of a Γ-seminearring R, respectively. We know from Theorem 2.2.5 (ii) and (iii) that both of $J\Gamma I$ and $I \cap J$ are right ideals of R. We show that both of them are identical because of being right weakly regular of R.

Theorem 3.2.5. *Let R be a Γ-seminearring. If R is right weakly regular, then $I \cap J = J\Gamma I$ for any ideal I and any right ideal J of R.*

Proof. Let I and J be an ideal and a right ideal of R, respectively. Then $J\Gamma I \subseteq I$ and $J\Gamma I \subseteq J$, and then $J\Gamma I \subseteq I \cap J$.

Next, let $x \in I \cap J$. Since R is right weakly regular, $x \in (x\Gamma R)^2 = (x\Gamma R)\Gamma(x\Gamma R)$. Then $x \in J\Gamma I$ since I and J are right ideals of R. Thus $I \cap J \subseteq J\Gamma I$.

Hence $I \cap J = J\Gamma I$. □

Theorem 3.2.6. *Let R be a Γ-seminearring. If R is left weakly regular, then $I \cap J = I\Gamma J$ for any ideal I of R and any left ideal J of R.*

Proof. The proof is obtained analogously to one of Theorem 3.2.5. □

We remark here that according to Theorem 3.2.6, eventhough $I \cap J$ and $I\Gamma J$ are exactly the same but, in fact, $I \cap J$ is a left ideal of R by Theorem 2.2.5 (iii) while $I\Gamma J$ is not necessary a left ideal of R.

However, if R is also distributively generated, then $I\Gamma J$ is a left ideal of R from Theorem 2.2.8 (i).

This chapter is ended by investigation of idempotent ideals of Γ-seminearrings.

Definition 3.2.2. Let R be a Γ-seminearring. An ideal A of R is *idempotent* if $A\Gamma A = A$.

Theorem 3.2.7. *Let R be a Γ-seminearring. If I is a right ideal of R, then any idempotent right ideal of I is also a right ideal of R.*

Proof. Assume that I is a right ideal of R. Let J be an idempotent right ideal of I. This implies that $J\Gamma I \subseteq J = J\Gamma J \subseteq J\Gamma I$ so that $J\Gamma I = J$. Then $J\Gamma R = (J\Gamma I)\Gamma R \subseteq J\Gamma(I\Gamma R) \subseteq J\Gamma I \subseteq J$. As a result, J is a right ideal of R. □

The following results require distibutively generated Γ-seminearrings.

Theorem 3.2.8. *Let R be a distibutively generated Γ-seminearring. If I is a left ideal of R, then any idempotent left ideal of I is also a left ideal of R.*

Proof. Assume that I is a left ideal of R. Let J be an idempotent left ideal of I. This implies that $I\Gamma J \subseteq J = J\Gamma J \subseteq I\Gamma J$ so that $I\Gamma J = J$. Being distibutively generated, $R\Gamma(I\Gamma J) \subseteq (R\Gamma I)\Gamma J$ holds. Then $R\Gamma J = R\Gamma(I\Gamma J) \subseteq (R\Gamma I)\Gamma J \subseteq I\Gamma J \subseteq J$. Hence, J is a left ideal of R. □

Theorem 3.2.9. *Let R be a distributively generated Γ-seminearring which has the identity. If $I \cap J = I\Gamma J$ for each ideal I and for each left ideal J of R, then any left ideal of R is idempotent.*

Proof. Assume that $I \cap J = I\Gamma J$ for any ideal I and any left ideal J of R. Let A be a left ideal of R. Then $A\Gamma A \subseteq A$. Let $x \in A$. Since R is d.g. and has the identity, this implies that $R\Gamma x\Gamma R$ is an ideal of R containing x by Corollary 2.2.9. Then $x \in \big(R\Gamma x\Gamma R\big) \cap A = \big(R\Gamma x\Gamma R\big)\Gamma A \subseteq (A\Gamma R)\Gamma A \subseteq A\Gamma(R\Gamma A) \subseteq A\Gamma A$. Consequently, $A\Gamma A = A$ as desired.

Therefore, A is idempotent. □

Finally, we study connections between one-sided weakly regular properties and idempotent ideals.

Theorem 3.2.10. *Let R be a Γ-seminearring. If R is left (right) weakly regular, then every left (right) ideal of R is idempotent.*

Proof. It is enough to assume that R is left weakly regular. Let L be a left ideal of R. Then $L\Gamma L \subseteq L$. Next, let $x \in L$. Since R is left weakly regular, $x \in (R\Gamma x)^2 = (R\Gamma x)\Gamma(R\Gamma x) \subseteq L\Gamma L$ since $R\Gamma x \subseteq L$. Thus $L\Gamma L = L$. As a result, L is idempotent. □

The converse of Theorem 3.2.10 in the case of right ideals holds provided a Γ-seminearring contains a right identity.

Theorem 3.2.11. *Let R be a Γ-seminearring which has a right identity. If every right ideal of R is idempotent, then R is right weakly regular.*

Proof. Assume that every right ideal of R is idempotent. Let $x \in R$. Since R has a right identity, $x \in x\Gamma R$. Since $x\Gamma R$ is a right ideal of R by applying Corollary 2.2.6, it follows that $x\Gamma R$ is idempotent. Then $x \in x\Gamma R = (x\Gamma R)\Gamma(x\Gamma R) = (x\Gamma R)^2$. Hence R is right weakly regular. □

However, the converse of Theorem 3.2.10 in the case of left ideals still does not hold eventhough a Γ-seminearring contains a left identity. We find that the distributively generated property must also be included.

Theorem 3.2.12. *Let R be a distibutively generated Γ-seminearring which has a left identity. If every left ideal of R is idempotent, then R is left weakly regular.*

Proof. This follows directly from Theorem 3.2.11 that $x \in R\Gamma x$ for each $x \in R$. Since R is distibutively generated, by Corollary 2.2.9 $R\Gamma x$ is a left ideal of R. By the assumption $R\Gamma x$ is idempotent, and then R is left weakly regular. □

Next, we show the use of idempotent ideals on one-sided weeakly regularities.

Theorem 3.2.13. *Let R be a right weakly regular Γ-seminearring. Then each ideal of R is right weakly regular.*

Proof. Let I be an ideal of R. Then I is a sub Γ-seminearring of R. To show that I is a right weakly regular, let $x \in I$. Then $x\Gamma I$ is a right ideal of R by Theorem 2.2.5(ii) and then $x\Gamma I$ is idempotent from Theorem 3.2.10. Since R is right weakly regular and $x \in R$, this implies that $x \in (x\Gamma R)^2$ and then $x \in (x\Gamma R)\Gamma(x\Gamma R) \subseteq (x\Gamma R)\Gamma I \subseteq x\Gamma(R\Gamma I) \subseteq x\Gamma I = (x\Gamma I)^2$ since $x\Gamma I$ is idempotent. Hence I is right weakly regular. □

Theorem 3.2.14. *Let R be a right weakly regular Γ-seminearring. If I is an ideal of R, then any right ideal of I is also a right ideal of R.*

Proof. Assume that I is an ideal of R. Let J be a right ideal of I. By Theorem 3.2.13, I is right weakly regular, and then J is idempotent from Theorem 3.2.10. Hence J is a right ideal of R by Theorem 3.2.7. $\square$

The following facts can be concluded from above theorems.

Note 3.2.1. Let R be a Γ-seminearring. If R is distibutively generated and has the identity, then the followings are equivalent:

(i) R is left weakly regular.

(ii) $I \cap J = I\Gamma J$ for any ideal I and for any left ideal J of R.

(iii) Every left ideal of R is idempotent.

Note 3.2.2. Let R be a Γ-seminearring. If R has a right identity, then the followings are equivalent:

(i) R is right weakly regular.

(ii) Every right ideal of R is idempotent.