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EXISTENCE OF GLOBAL SOLUTIONS FOR SEMILINEAR PSEUDOPARABOLIC EQUATIONS WITH UNBOUNDED COEFFICIENT

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ในวิทยานิพนธ์ฉบับนี้ สนใจผลเฉลยเปลี่ยนเครื่องหมายได้ของปัญหาโคชี $\partial_t u - \Delta\partial_t u = \alpha\Delta u + V(x)|u|^\sigma u$ ใน $\mathbb{R}^n \times (0,\infty)$, $u|_{t=0} = u_0$, โดยที่ $\alpha,\ \sigma > 0$ เป็นค่าคงตัวและ u_0, V เป็นฟังก์ชันที่กำหนดให้ โดยให้ข้อสมมุติอย่างอ่อนกับสัมประสิทธิ์ V ว่าสอดคล้องกับเงื่อนไข $|V(x)| \lesssim |x|^a$ เมื่อ $|x| \to \infty$ สำหรับค่าคงตัว $a \geq 0$ นั่นคือในกรณีเฉพาะ V สามารถมีขอบเขตได้ ปริภูมิฟังก์ชันที่สนใจ คือ ปริภูมิเลอเบกแบบถ่วงน้ำหนักด้วยพหุนามอันดับ b ที่แทนด้วยสัญลักษณ์ $L^{q,b}(\mathbb{R}^n)$ หลังจากพิสูจน์การมีขอบเขตของตัวดำเนินการที่เกี่ยวข้อง โดยเฉพาะตัวดำเนินการศักย์เบสเซลและตัวดำเนินการกรีน แล้วจึงพิสูจน์การมีอยู่จริงของผลเฉลยเฉพาะที่สำหรับปัญหาโคชี จากนั้นโดยใช้การประมาณค่าในช่วงบนปริภูมิเลอเบกแบบถ่วงน้ำหนักจะสามารถพิสูจน์การมีอยู่จริงของผลเฉลยวงกว้างสำหรับเงื่อนไขค่าเริ่มต้น u_0 ที่เล็กมากพอ

ภาควิชา คณิตศาสตร์และวิทยาการคอมพิวเตอร์ ลายมือชื่อนิสิต..........

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In this work, we are interested in sign-changing solutions of the Cauchy problem $\partial_t u - \triangle \partial_t u = \alpha \triangle u + V(x)|u|^\sigma u$ in $\mathbb{R}^n \times (0, \infty)$, $u|_{t=0} = u_0$, where $\alpha, \sigma > 0$ are constants and u_0, V are given functions. We put a rather mild assumption on the coefficient V that it satisfies $|V(x)| \lesssim |x|^a$ as $|x| \to \infty$ for a constant $a \geq 0$. Thus, in particular, it can be bounded. The function spaces considered are weighted Lebesgue spaces with a polynomial weight of order b, denoted by $L^{q,b}(\mathbb{R}^n)$. After proving the boundedness of relevant operators, especially, the Bessel potential and the Green operators, we can establish the local existence of solutions for the Cauchy problem. Then, employing a modified interpolation estimate on the weight Lebesgue spaces, we can also prove the global existence of solutions provided the initial function u_0 is sufficiently small.

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CONTENTS

CHAPTER I

INTRODUCTION

Sobelev type equations are equations of the form $Lu_t = Mu$, where $L : \mathcal{U} \to \mathcal{V}$ is a bounded linear operator, $M : dom(M) \subseteq \mathcal{U} \to \mathcal{V}$ is a closed operator, $\mathcal{U}, \mathcal{V}$ are Banach spaces (see Sviridyuk [17]). If L is invertible, Sobelev type equations are also called *pseudoparabolic equations* in the literatures (see Showalter and Ting [16]). They play a main role to model many scientific and natural phenomena, such as nonstationary in crystalline semiconductors [12], seepage of homogeneous fluids through a fissured rock [3], and etc. (see Al'shin et al. [2] for more examples).

In this work, we study sign-changing solutions $u = u(x,t)$ of a Cauchy problem

$$\begin{cases} \partial_t(u - \triangle u) - \alpha\triangle u = V(x)|u|^\sigma u, & x \in \mathbb{R}^n, t > 0 \\ u(x,0) = u_0(x), & x \in \mathbb{R}^n \end{cases} \tag{1.1}$$

where $\alpha, \sigma > 0$ are constants and $V, u_0 : \mathbb{R}^n \to \mathbb{R}$ are given functions. We assume a mild condition on the *potential* $V(x)$ that it satisfies

$$|V(x)| \leq C|x|^a \quad \text{as} \quad |x| \to \infty$$

for some constant $C > 0$ and $a \geq 0$. There are many studies on (1.1) when $\mathbb{R}^n$ is replaced by a bounded domain and the potential V is a constant or a bounded function. These studies are based mostly on the usual Lebesgue or Sobolev spaces.

A particularly important problem in the history of nonlinear PDE theory occurred when the *viscosity* term $\partial_t\triangle u$ is dropped from (1.1), $V \equiv 1$, and we are looking for positive solutions. For an arbitrary V, the problem in this case, is the

Cauchy problem of nonlinear heat equation with power nonlinearity:

$$\begin{cases} \partial_t u - \alpha \triangle u = V(x)|u|^\sigma u, & x \in \mathbb{R}^n, t > 0 \\ u(x,0) = u_0(x), & x \in \mathbb{R}^n. \end{cases} \tag{1.2}$$

Let us first consider the case where $V \equiv 1$. Fujita [6], in 1966, showed that (1.2) has a unique non-negative local solution for each $u_0 \in L^\infty(\mathbb{R}^n)$ and $u_0 \geq 0$. Furthermore, he found that

(1) if $0 < \sigma < 2/n$ and u_0 is not the zero function, the solution always blows up in a finite time;

(2) if $\sigma > 2/n$, however, the solution can exhibit either globally in time if u_0 is sufficiently small, or is blowing up if u_0 is sufficiently large.

In the above context, a solution u is called blowing up in a finite time if

$$\lim_{t \to T^-} \|u(\cdot,t)\|_{L^\infty} = \infty$$

for a finite $T > 0$. Later on, Hayakawa [10], in 1973, and Weissler [20], in 1981, proved that the critical exponent $\sigma = 2/n$ belongs to the blowing up case, when $n = 2$ and for arbitrary $n \geq 1$, respectively. Now, let us consider the nonlinear heat equation (1.2) with a variable V. If either $V(x) \sim |x|^a$ with $a > -2$ or $0 \lneq a(x) \lesssim |x|^{-2}$, Pinsky [14], in 1997, proved that

(1) if $0 < \sigma \leq (2+a)/n$ and $u_0 \not\equiv 0$, the problem (1.2) always blows up in a finite time where either $n \leq 2$ and $\sigma \in (2,\infty)$, or $n = 1$ and $\sigma \in (-1,\infty)$;

(2) if $\sigma > (2+a)/n$, the solution can be global, moreover, the result are also held when $n = 1$ and $\sigma \in [-2,-1]$.

We now turn back to the nonlinear pseudoparabolic equation (1.1). Basic linear theory of the pseudoparabolic equation, for both initial and initial boundary value problems, were developed by Showalter and Ting [16] in around 1970. The linear theory of the pseudoparabolic equations is quite new compared to that of the heat equations. There are certain similarities and dissimilarities for solutions to the pseudoparabolic equation and that of the heat equations, this is why (1.1) is called pseudoparabolic [15].

Nonlinear pseudoparabolic equations with power nonlinearity were studied by Kaikina et al. [9] in 2005, for the case V is a constant. It was shown that the problem (1.1) admits a unique global sign-changing solution for arbitrary u_0 when $n \geq 3$, and the same is true when $n \in \{1, 2\}$ provided that u_0 is sufficiently small. Later, Cao et al. [4], in 2009, studied the positive solutions of the Cauchy problem (1.1) by using the contration mapping method and the two-normed approach developed by Kato in his study of Navier-Stokes equations. The results in [4] are similar to that of the heat equation with the same source. They proved that if $0 < \sigma \leq 2/n$, every nontrivial solution always blows up in a finite time, whereas, if $\sigma > 2/n$, the solution can be both global (for small u_0) and blowing up (for large u_0) in a finite time.

Let us describe the plan and results of this work. We consider real value mild solutions of the Cauchy problem (1.1). The phase spaces for the solutions are chosen to be the weighted Lebesgue space $L^{q,a} := L^q(\mathbb{R}^n; \langle x \rangle^a)$, where $1 \leq q \leq \infty$ and $a \in \mathbb{R}$. Thus, our solutions are continuous paths within some $L^{q,a}$ that satisfy an integral equation in the Banach space $L^{q,a}$. The formal definition is given in Chapter 2. In the same chapter, some elementary results and inequalities important to our study are given. For the pseudoparabolic equation, there are two important linear operators, the *Bessel potential* $\mathcal{B}$ and *Green operators* $\mathcal{G}(t)(t > 0)$. In the last part of Chapter 2, we prove the boundedness of $\mathcal{B}$ on the spaces $L^{q,a}$. In Chapter 3, we give the first main result on the boundedness and interpolation estimate for the Green operator that we extend the results from [9]. (These results will be appeared in a recent work of Khomrutai in [11]). We will employ these results to establish the local and global existence of solutions. The local existence of solutions is given in Chapter 4. There, we prove that for all $n \in \mathbb{N}$, if $\sigma > 0$ and there is a positive constant C and $a \geq 0$ such that for any $x \in \mathbb{R}^n$

$$V(x) \leq C|x|^a,$$

then the Cauchy problem (1.1) has a unique mild solution

$$u \in C\left([0,T); C(\mathbb{R}^n) \cap L^{1,b}(\mathbb{R}^n) \cap L^{\infty,b}(\mathbb{R}^n)\right),$$

for some $T > 0$ and $b \in \mathbb{R}$ such that $b \geq a/\sigma$. Our last result on the global existence of solutions for small u_0 is presented in Chapter 5. Specifically, if we assume $n \in \mathbb{N}$ and a, σ satisfy

$$0 \leq a < \frac{\sigma}{\sigma+1} \quad \text{and} \quad \sigma > \frac{3}{n},$$

then the Cauchy problem (1.1) has a unique global solution

$$u \in C([0,\infty); C(\mathbb{R}^n) \cap L^{1,b}(\mathbb{R}^n) \cap L^{\infty,b}(\mathbb{R}^n)),$$

where $b = a/\sigma$ and provided that u_0 is sufficiently small. The results in this work extend parts of the result obtained in [9].

CHAPTER II

PRELIMINARIES

In this chapter, we introduce essential tools that will be used.

2.1 Notation

Let $n \in \mathbb{N}$, $1 \leq q \leq \infty$, and $a \in \mathbb{R}$. We define the *weighted Lebesgue norm* by

$$\|\psi\|_{L^{q,a}} = \|\langle\cdot\rangle^a \psi\|_{L^q} = \begin{cases} \left(\int_{\mathbb{R}^n} |\langle x\rangle^a \psi(x)|^q dx\right)^{1/q} & 1 \leq q < \infty, \\ \sup_{x\in\mathbb{R}^n} \langle x\rangle^a |\psi(x)| & q = \infty, \end{cases}$$

where $\langle\cdot\rangle := \sqrt{1+|\cdot|^2}$ is the *Japanese bracket* and the *weighted Lebesgue space*

$$L^{q,a}(\mathbb{R}^n) = \{\psi \in L^q(\mathbb{R}^n) : \|\psi\|_{L^{q,a}} < \infty\}.$$

Denote by $C(I;B)$ the space of continuous functions from a time interval $I = [0,T]$, where $0 < T \leq \infty$, into a Banach space B.

The Fourier transform $\mathcal{F}$ and the inverse Fourier transform $\mathcal{F}^{-1}$ are defined by

$$(\mathcal{F}\psi)(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\psi(x)dx,$$
$$(\mathcal{F}^{-1}\phi)(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\phi(\xi)d\xi.$$

for sufficiently regular functions ψ and ϕ. For a function $w(\cdot,t)$, we denote $w(\cdot,t)$, for fixed t, to be the function $x \mapsto w(x,t)$.

2.2 Basic Results

Theorem 2.1 (Cauchy-Schwarz inequality, [5]). *Let* $x_1, \dots, x_n, y_1 \dots, y_n \in \mathbb{R}$. *Then,*

$$\left|\sum_{i=1}^{n} x_i y_i\right| \leq \left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right).$$

Theorem 2.2 (Young's inequality, [5]). *Let* $1 \leq p, q, r \leq \infty$ *with* $1 + 1/r = 1/p + 1/q$. *Then,*

$$\|\psi * \varphi\|_{L^r} \leq \|\psi\|_{L^p}\|\varphi\|_{L^q}$$

for all $\psi \in L^p(\mathbb{R}^n)$ *and* $\varphi \in L^q(\mathbb{R}^n)$. *Here,* $(\psi * \varphi)(x) = \int_{\mathbb{R}^n} \psi(y)\varphi(x-y)dy$ *is the convolution.*

Theorem 2.3 (Hölder inequality, [5]). *Let* $1 \leq p, q \leq \infty$ *with* $1/p + 1/q = 1$. *Then,*

$$\|\psi\varphi\|_{L^1} \leq \|\psi\|_{L^p}\|\varphi\|_{L^q}$$

for all $\psi \in L^p(\mathbb{R}^n)$ *and* $\varphi \in L^q(\mathbb{R}^n)$.

Lemma 2.4 (Peetre's Inequality, [1]). *Let* $a \in \mathbb{R}$ *and* $x, y \in \mathbb{R}^n$. *Then,*

$$\langle x\rangle^a \leq 2^{|a|}\langle x-y\rangle^{|a|}\langle y\rangle^a.$$

Proof. Note that $|w||z| \leq |w|^2 + |z|^2$ for $w, z \in \mathbb{R}^n$.

If $a \geq 0$, then by the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\langle x\rangle^a &= (1 + |x-y+y|^2)^{\frac{a}{2}} \\
&= (1 + |x-y|^2 + 2(x-y)\cdot y + |y|^2)^{\frac{a}{2}} \\
&\leq (1 + |x-y|^2 + 2|x-y||y| + |y|^2)^{\frac{a}{2}} \\
&\leq (1 + 4|x-y|^2 + 4|y|^2)^{\frac{a}{2}} \\
&\leq (4 + 4|x-y|^2 + 4|x-y|^2|y|^2 + 4|y|^2)^{\frac{a}{2}} \\
&= 2^a(1 + |x-y|^2 + |x-y|^2|y|^2 + |y|^2)^{\frac{a}{2}}
\end{aligned}$$

$$= 2^a(1+|x-y|^2)^{\frac{a}{2}}(1+|y|^2)^{\frac{a}{2}}.$$

If $a < 0$, then $-a > 0$. By using the above case, we have

$$\langle x\rangle^{-a} \leq 2^{-a}\langle x-y\rangle^{-a}\langle y\rangle^{-a}. \tag{2.1}$$

We multiply $\langle x\rangle^a\langle y\rangle^a$ on the both side of (2.1), this implies

$$\langle y\rangle^a \leq 2^{-a}\langle x-y\rangle^{-a}\langle x\rangle^a.$$

□

Lemma 2.5. *Let $a \in \mathbb{R}$ and $x, y \in \mathbb{R}^n$. Then,*

$$\langle x\rangle^a \leq 2^{|a|}(\langle x-y\rangle^{|a|} + \langle y\rangle^a).$$

Proof. If $a \geq 0$, then by the Minkowski inequality ([5]), we obtain

$$\langle x\rangle \leq \sqrt{4+|y+(x-y)|^2}.$$

Thus,

$$\begin{aligned} \langle x\rangle^a &\leq (\sqrt{4+|y+(x-y)|^2})^a \\ &\leq (\langle x-y\rangle + \langle y\rangle)^a \\ &\leq 2^a(\max\{\langle x-y\rangle, \langle y\rangle\})^a \\ &\leq 2^a(\langle x-y\rangle^a + \langle y\rangle^a) \end{aligned}$$

If $a < 0$, then

$$\langle x\rangle^a \leq 1 \leq \langle x-y\rangle^{-a} + \langle y\rangle^a \leq 2^{-a}(\langle x-y\rangle^{-a} + \langle y\rangle^a).$$

□

Lemma 2.6. *Let $p > 0$ and $a, b \in \mathbb{R}$. Then,*

$$|a|a|^p - b|b|^p| \leq 2^{p+1}(\max\{|a|, |b|\})^p|a-b|.$$

Proof. If $a, b > 0$, then $|a| = a$ and $|b| = b$. Thus,

$$|a|a|^p - b|b|^p| = ||a|^{p+1} - |b|^{p+1}|$$

$$
\begin{aligned}
&= \left|(p+1)\int_a^b x^p dx\right| \\
&\leq (p+1)(\max\{a,b\})^p \left|\int_a^b dx\right| \\
&= (p+1)(\max\{a,b\})^p|a-b| \\
&\leq 2^{p+1}(\max\{a,b\})^p|a-b|,
\end{aligned}
$$

where we have used that $p+1 \leq 2^{p+1}$ for all $p > 0$. The case $a, b \leq 0$ follows similarly.

If $a \leq 0, b \geq 0$, then $|a| = -a$ and $|b| = b$. Thus,

$$
\begin{aligned}
|a|a|^p - b|b|^p| &= |a|^{p+1} + |b|^{p+1} \\
&\leq 2(\max\{|a|,|b|\})^{p+1} \\
&\leq 2(|a|+|b|)^{p+1} \\
&= 2(|a|+|b|)^p(|a|+|b|) \\
&\leq 2^{p+1}(\max\{|a|,|b|\})^p(|a|+|b|) \\
&\leq 2^{p+1}(\max\{|a|,|b|\})^p|a-b|
\end{aligned}
$$

and similarly, for the case $a \geq 0, b \leq 0$. □

Theorem 2.7 (Contraction mapping principle, [5]). *Let (X,d) be a non-empty complete metric space. Assume that $\mathcal{M}$ is a contraction mapping on X, i.e., $\mathcal{M} : X \to X$ and there exists $k \in (0,1)$ such that $d(\mathcal{M}(x), \mathcal{M}(y)) \leq kd(x,y)$ for $x, y \in X$. Then, there is a unique $w \in X$ such that $\mathcal{M}(w) = w$.*

Theorem 2.8 (Faá di Bruno's identity, [8]). *Let $f, g : \mathbb{R} \to \mathbb{R}$ be k times differentiable functions and $h(x) = (f \circ g)(x)$. Then,*

$$
\frac{d^k h}{dx^k} = \sum_{i=1}^{k}\left(\sum_{\beta=(\beta_1,\ldots,\beta_k)} \frac{k!}{\beta!(1!)^{\beta_1}\cdots(k!)^{\beta_k}}(g^{(1)}(x))^{\beta_1}\cdots(g^{(k)}(x))^{\beta_k}\right) f^{(i)}(g(x))
$$

where $\beta! = \beta_1!\ldots\beta_k!$ and the sum are taken over all $\beta_1,\ldots,\beta_k$ are nonnegative integers solutions of $\sum_{l=1}^{k} l\beta_l = k$ and $\sum_{l=1}^{k}\beta_l = i$.

Corollary 2.9. *Let $i \in \{1, \ldots, n\}, r \in \mathbb{R}$, and k be a positive integer. Then, there exist a positive constant $C = C(r,k)$ such that for any $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$*

$$|\partial_{\xi_i}^k \langle \xi \rangle^r| \leq C \langle \xi \rangle^{r-k},$$

where $\partial_{\xi_i}^k \langle \xi \rangle^r$ is the r times partial derivative of $\langle \xi \rangle^r$ with respect to ξ_i.

Proof. First, we let $g(\xi_i) = |\xi|^2$ and $f(x) = (1+x)^{r/2}$, then $(f \circ g)(\xi_i) = \langle \xi \rangle^r$. We know that $f^{(k)}(x) = (r/2)(r/2-1)\ldots(r/2-k+1)(1+x)^{r/2-k}$ and

$$\partial_{\xi_i}^k g(\xi_i) = \begin{cases} 2\xi_i^{2-k}, & k \in \{1,2\}, \\ 0 & k \geq 3. \end{cases}$$

Using the Faá di Bruno's identity, we obtain

$$\begin{aligned}
|\partial_{\xi_i}^k \langle \xi \rangle^r| &= \left| \sum_{\frac{k}{2} \leq l \leq k} C_{l,k} (2\xi_i)^{2l-k} 2^{k-l} \left(\frac{r}{2}\right) \ldots \left(\frac{r}{2} - k + 1\right) (1+|\xi|^2)^{\frac{r}{2}-l} \right| \\
&\leq C \sum_{\frac{k}{2} \leq l \leq k} |\xi_i|^{2l-k} (1+|\xi|^2)^{\frac{r}{2}-l} \\
&\leq C \sum_{\frac{k}{2} \leq l \leq k} \frac{|\xi_i|^{2l-k}}{(1+|\xi|^2)^{l-\frac{k}{2}}} \langle \xi \rangle^{r-k} \\
&\leq C \langle \xi \rangle^{r-k},
\end{aligned}$$

where we have used that $\beta = (2l-k, k-l)$ is the solution of algebraic system $\beta_1 + 2\beta_2 = k, \beta_1 + \beta_2 = l$, and that $|\xi_i|^{2l-k}/(1+|\xi|^2)^{l-k/2}$ is bounded for all $\xi_i \in \mathbb{R}$. Therefore, this proof is complete. □

Lemma 2.10. *Let $n \in \mathbb{N}$ and $q, \alpha > 0$. Then,*

$$\left\| e^{-\alpha|x|^2} \right\|_{L^q(\mathbb{R}^n)} = \left(\frac{\pi}{\alpha q} \right)^{\frac{n}{2q}}.$$

Proof. By a straightforward calculation and the fact that

$$\int_{-\infty}^{\infty} e^{-x_j^2} dx_j = \sqrt{\pi}$$

for all $j \in \{1, 2, 3, \ldots, n\}$, we get

$$\begin{aligned}
\left\|e^{-\alpha|x|^2}\right\|_{L^q(\mathbb{R}^n)} &= \left(\int_{\mathbb{R}^n} e^{-q\alpha|x|^2} dx\right)^{\frac{1}{q}} \\
&= \left(\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-q\alpha(x_1^2+\ldots+x_n^2)} dx_1 \ldots dx_n\right)^{\frac{1}{q}} \\
&= \left(\prod_{j=1}^{n} \int_{-\infty}^{\infty} e^{-q\alpha x_j^2} dx_j\right)^{\frac{1}{q}} \\
&= (\alpha q)^{-\frac{n}{2q}} \left(\prod_{j=1}^{n} \int_{-\infty}^{\infty} e^{-u_j^2} du_j\right)^{\frac{1}{q}} \qquad (2.2) \\
&= \left(\frac{\pi}{\alpha q}\right)^{\frac{n}{2q}},
\end{aligned}$$

where we have used the substitution $u_j = \sqrt{q\alpha x_j}$ in (2.2). □

2.3 Bessel potential and Green operator

In this section, we recall the linear theory for the pseudoparabolic equation. Consider the linear Cauchy problem

$$\begin{cases} \partial_t(u - \triangle u) - \alpha \triangle u = f(x,t) & x \in \mathbb{R}^n, t > 0, \\ u(x,0) = u_0(x) & x \in \mathbb{R}^n. \end{cases} \tag{2.3}$$

If u_0 and f are sufficiently regular functions, then by using the Fourier transform and the variation of parameter technique, we obtain

$$u(x,t) = \mathcal{G}(t)u_0(x) + \int_0^t \mathcal{G}(t-\tau)\mathcal{B}[f(x,\tau)]d\tau \qquad (x \in \mathbb{R}^n, t > 0),$$

where $\mathcal{G}(t)$ is the *Green operator* for the Cauchy problem above (2.3) and $\mathcal{B}$ is *Bessel potential operator*. Precisely, $\mathcal{G}(t)$ is given by

$$\mathcal{G}(t)\psi(x) = \mathcal{F}^{-1}\big(e^{-\alpha t|\xi|^2\langle\xi\rangle^{-2}}\widehat{\psi}\big) = e^{-\alpha t}\sum_{k=0}^{\infty}\frac{\alpha^k t^k}{k!}\mathcal{B}^k\psi(x) = \int_{\mathbb{R}^n} G(x-y,t)\psi(y)dy,$$

where

$$G(x,t) = e^{-\alpha t}\sum_{k=0}^{\infty}\frac{\alpha^k t^k}{k!}B_k(x),$$

and $\mathcal{B} = \mathcal{B}^1$. For any $s > 0$, $\mathcal{B}^s$ is the *generalized Bessel potential operator*

$$\mathcal{B}^s\psi(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} B_s(x-y)\psi(y)dy,$$

and $B_s(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\langle\xi\rangle^{-2s}d\xi$.

Here, we gives a boundedness of $\mathcal{B}$ on the weighted Lebesgue spaces.

Theorem 2.11 ([11])**.** *Let $s > 0, 1 \le q \le \infty$, and $a \in \mathbb{R}$. Then, there is a positive constant $C = C(n, s, q)$ such that*

$$\|\mathcal{B}^s\psi\|_{L^{q,a}} \le C\|\psi\|_{L^{q,a}}.$$

In this work, we are interested in a sign-changing mild solution of the Cauchy problem (1.1) which is defined as follows.

Definition 2.12. Let $\alpha, \sigma > 0$ be constants and $V, u_0 : \mathbb{R}^n \to \mathbb{R}$ be given functions. A function $u : \mathbb{R}^n \times [0, T) \to \mathbb{R}$ is said to be a *mild solution*, on $[0, T)$, of the Cauchy problem (1.1), if $u \in C([0, T); L^{q,a}(\mathbb{R}^n))$ for some $1 \le q \le \infty, a \in \mathbb{R}$ and it satisfies for any $x \in \mathbb{R}^n$ and $t \in (0, T)$

$$u(x,t) = \mathcal{G}(t)u_0(x) + \int_0^t \mathcal{G}(t-\tau)\mathcal{B}[V(x)|u(x,\tau)|^\sigma u(x,\tau)]d\tau.$$

If $T = \infty$, u is said to be a *global mild solution.*

CHAPTER III

GREEN OPERATOR

Theorem 3.1 (Boundedness of the Green operator)**.** *Let* $1 \le q \le \infty$ *and* $a, m \in \mathbb{R}$. *Then, there is a positive constant* $C = C(\alpha, q, a, m, n)$ *such that for any* $\psi \in L^{q,a}(\mathbb{R}^n)$ *and* $t > 0$

$$\|\mathcal{G}(t)\psi\|_{L^{q,a}} \le C\langle t\rangle^{\frac{|a|}{2}+m}\|\psi\|_{L^{q,a}}.$$

To prove this theorem, we express

$$G(x,t) = e^{-\alpha t}\sum_{k=0}^{\infty}\frac{\alpha^k t^k}{k!}B_k(x) = e^{-\alpha t}\sum_{k=0}^{\infty}\frac{\alpha^k t^k}{k!}\mathcal{F}^{-1}(\langle\xi\rangle^{-2k}).$$

Taking the Fourier transformation, we get

$$\begin{aligned}\widehat{G}(\xi,t) &= e^{-\alpha t}\sum_{k=0}^{\infty}\frac{\alpha^k t^k}{k!}\langle\xi\rangle^{-2k} = e^{-\alpha t}e^{t\alpha\langle\xi\rangle^{-2}}\\ &= e^{-\alpha t}\sum_{k=0}^{N}\frac{\alpha^k t^k}{k!}\langle\xi\rangle^{-2k} + \widehat{R}_N(\xi,t)\end{aligned}$$

where

$$\widehat{R}_N(\xi,t) = e^{-\alpha t|\xi|^2\langle\xi\rangle^{-2}} - e^{-\alpha t}\sum_{k=0}^{N}\frac{\alpha^k t^k}{k!}\langle\xi\rangle^{-2k}$$

and N is a positive integer to be specified.

Lemma 3.2. *Let* $l \in \{0,1,2,\ldots,N+2\}$ *and* $m \in \mathbb{R}$. *Then, there is a positive constant* $C = C(\alpha, N, l, m)$ *such that for any* $\xi \in \mathbb{R}^n, t > 0$ *and* $j \in \{1,2,3,\ldots,N\}$ *we have*

$$|\partial_{\xi_j}^l \widehat{R}_N(\xi,t)| \le Ct^{\frac{l}{2}}\langle t\rangle^m e^{-\frac{\alpha t}{2}|\xi|^2} + Ce^{-\frac{\alpha t}{4}}t^{N+1}\langle\xi\rangle^{-2N-2},$$

where $\partial_{\xi_j}^l \widehat{R}_N(\xi,t)$ *is the* l *times partial derivative of* $\widehat{R}_N(\xi,t)$ *with respect to* ξ_j.

Proof. First, we assume that $|\xi| \geq 1$ or $0 < t \leq 1$. Note that

$$\widehat{R}_N(\xi,t) = e^{-\alpha t}\bigg(e^{\alpha t\langle\xi\rangle^{-2}} - \sum_{k=0}^{N} \frac{\alpha^k t^k}{k!}\langle\xi\rangle^{-2k}\bigg).$$

Let $g(\xi_j) = \langle\xi\rangle^{-2}$ and $f(z) = e^{\alpha t z} - \sum_{k=0}^{N} \frac{\alpha^k t^k}{k!} z^k$, thus, $\widehat{R}_N = e^{-\alpha t} f \circ g$. By Corollary 2.9, there exists $C_i > 0$ such that $|g^{(i)}(\xi_j)| \leq C_i\langle\xi\rangle^{-2-i}$ for all $i \geq 0$. For a fixed $z_0 > 0$, we define $h : [0, z_0] \to \mathbb{R}$ by

$$h(z) = f(z) - \frac{(\alpha t z)^{N+1}}{(N+1)!} e^{\alpha t z_0}.$$

We can compute that

$$h^{(i)}(z) = (\alpha t)^i e^{\alpha t z} - \sum_{k=0}^{N-i} \frac{(\alpha t)^{k+i}}{k!} z^k - \frac{(\alpha t)^{N+1} z^{N+1-i}}{(N+1-i)!} e^{\alpha t z_0}$$

and $h^{(i)}(0) = 0$ for all $i \in \{0, 1, 2, \ldots, N\}$. Moreover, for $i = N+1$ we get $h^{(N+1)}(z) \leq 0$ for all $z \in [0, z_0]$. We have

$$\int_0^z h^{(i)}(u)du = h^{(i-1)}(z) - h^{(i-1)}(0) = h^{(i-1)}(z), \qquad 1 \leq i \leq N+1$$

by the fundamental theorem of calculus, and $h^{(N+1)}(z) \leq 0$ for all $z \in [0, z_0]$. Thus, $h^{(i)}(z) \leq 0$ for all $z \in [0, z_0]$ and $i \in \{0, \ldots, N\}$. Hence,

$$0 \leq f^{(i)}(z) \leq \frac{\alpha^{N+1} t^{N+1}}{(N+1-i)!} z^{N+1-i} e^{\alpha t z_0}$$

for all $z \in [0, z_0]$ and $i \in \{0, 1, 2, \ldots, N+1\}$.

For $i = N+2$, we differentiate f with respect z directly to get

$$f^{(N+2)}(z) = \alpha^{N+2} t^{N+2} e^{\alpha t z} > 0, \qquad 0 \leq z \leq z_0.$$

Note that $0 < \langle\xi\rangle^{-2} < 1$ for $|\xi| > 0$, thus by taking $z_0 = g(\xi_j) = z$ we get

$$|f^{(i)}(g(\xi_j))| \leq C t^{N+1} \langle\xi\rangle^{-2N-2+2i} e^{\alpha t\langle\xi\rangle^{-2}} \tag{3.1}$$

for $i \in \{0, 1, 2, \ldots, N+1\}$ and

$$|f^{(N+2)}(g(\xi_j))| \leq Ct^{N+2} e^{\alpha t \langle \xi \rangle^{-2}} \leq Ct^{N+2} \langle \xi \rangle^2 e^{\alpha t \langle \xi \rangle^{-2}}. \tag{3.2}$$

For $l \in \{0, 1, 2, \ldots, N+1\}$, by the Faá di Bruno's identity applying with $f \circ g$ and (3.1), we get

$$\begin{aligned}
|\partial_{\xi_j}^l \widehat{R}_N(\xi, t)| &\leq e^{-\alpha t} \sum_{i=1}^{l} \left(\sum_{\beta} C \langle \xi \rangle^{-\sum\limits_{m=1}^{l} (2+m)\beta_m} \right) f^{(i)}(g(\xi_j)) \\
&\leq e^{-\alpha t} \sum_{i=1}^{l} \left(\sum_{\beta} C \langle \xi \rangle^{-\sum\limits_{m=1}^{l} (2+m)\beta_m} \right) \left\{ t^{N+1} \langle \xi \rangle^{-2N-2+2i} e^{\alpha t \langle \xi \rangle^{-2}} \right\} \\
&\leq Ce^{-\alpha t} \sum_{i=1}^{l} \langle \xi \rangle^{-2i-l} t^{N+1} \langle \xi \rangle^{-2N-2+2i} e^{\alpha t \langle \xi \rangle^{-2}} \\
&\leq Ce^{-\frac{3\alpha}{4}t} t^{N+1} \langle \xi \rangle^{-2N-2} e^{\alpha t \langle \xi \rangle^{-2}}.
\end{aligned}$$

In particular, for $l = N+2$, we have by (3.2) and the Faá di Bruno's identity that

$$\begin{aligned}
|\partial_{\xi_j}^{N+2} \widehat{R}_N(\xi, t)| &\leq e^{-\alpha t} \sum_{i=1}^{N+1} \left(\sum_{\beta} C \langle \xi \rangle^{-\sum\limits_{m=1}^{N+2} (2+m)\beta_m} \right) f^{(i)}(g(\xi_j)) \\
&\quad + e^{-\alpha t} \left(C \langle \xi \rangle^{-\sum\limits_{m=1}^{N+2} (2+m)\beta_m} \right) f^{(N+2)}(g(\xi_j)) \\
&\leq e^{-\frac{3\alpha}{4}t} \sum_{i=1}^{N+1} \left(\sum_{\beta} C \langle \xi \rangle^{-\sum\limits_{m=1}^{N+2} (2+m)\beta_m} \right) \left\{ t^{N+1} \langle \xi \rangle^{-2N-2+2i} e^{\alpha t \langle \xi \rangle^{-2}} \right\} \\
&\quad + e^{-\frac{3\alpha}{4}t} \left(\sum_{\beta} C \langle \xi \rangle^{-\sum\limits_{m=1}^{N+2} (2+m)\beta_m} \right) \left\{ t^{N+1} \langle \xi \rangle^{2} e^{\alpha t \langle \xi \rangle^{-2}} \right\} \\
&= e^{-\frac{3\alpha}{4}t} \sum_{i=1}^{N+2} \left(\sum_{\beta} C \langle \xi \rangle^{-\sum\limits_{m=1}^{N+2} (2+m)\beta_m} \right) \left\{ t^{N+1} \langle \xi \rangle^{-2N-2+2i} e^{\alpha t \langle \xi \rangle^{-2}} \right\} \\
&\leq Ce^{-\frac{3\alpha}{4}t} \sum_{i=1}^{N+2} \langle \xi \rangle^{-2i-2N-2} t^{N+1} \langle \xi \rangle^{-2N-2+2i} e^{\alpha t \langle \xi \rangle^{-2}} \\
&\leq Ce^{-\frac{3\alpha}{4}t} \sum_{i=1}^{N+2} \langle \xi \rangle^{-2i} t^{N+1} \langle \xi \rangle^{-2N-2+2i} e^{\alpha t \langle \xi \rangle^{-2}} \\
&\leq Ce^{-\frac{3\alpha}{4}t} t^{N+1} \langle \xi \rangle^{-2N-2} e^{\alpha t \langle \xi \rangle^{-2}}.
\end{aligned}$$

If $t \leq 1$, then $t \langle \xi \rangle^{-2} \leq 1$. Thus,

$$|\partial_{\xi_j}^l \widehat{R}_N(\xi, t)| \leq Ce^{-\frac{3\alpha}{4}t} t^{N+1} \langle \xi \rangle^{-2N-2} e^{\alpha} \leq Ce^{-\frac{\alpha}{4}t} t^{N+1} \langle \xi \rangle^{-2N-2} \tag{3.3}$$

and if $|\xi| \geq 1$, then $\langle\xi\rangle^{-2} \leq 1/2$. These imply that

$$|\partial_{\xi_j}^l \widehat{R}_N(\xi,t)| \leq Ce^{-\frac{3\alpha}{4}t}t^{N+1}\langle\xi\rangle^{-2N-2}e^{\frac{\alpha}{2}t} = Ce^{-\frac{\alpha}{4}t}t^{N+1}\langle\xi\rangle^{-2N-2}.$$

For $t \geq 1$ and $|\xi| \leq 1$. We rewrite $\widehat{R}_N = A - B$ with

$$A = e^{\alpha t|\xi|^2\langle\xi\rangle^{-2}} = e^{-\alpha t}e^{\alpha t\langle\xi\rangle^{-2}} \quad \text{and} \quad B = e^{-\alpha t}\sum_{k=0}^{N}\frac{\alpha^k t^k}{k!}\langle\xi\rangle^{-2k}.$$

For the term B, since $|\xi| \leq 1$, we have $\langle\xi\rangle^2 \in [1/2, 1]$. By Corollary 2.9 and $t \geq 1$, then we have

$$\begin{aligned}|\partial_{\xi_j}^l B| &\leq e^{-\alpha t}\sum_{k=0}^{N}\frac{\alpha^k t^k}{k!}|\partial_{\xi_j}^l\langle\xi\rangle^{-2k}| \leq Ce^{-\alpha t}\sum_{k=0}^{N}\frac{\alpha^k t^k}{k!}\langle\xi\rangle^{-2k-l} \\ &\leq Ce^{-\alpha t}\sum_{k=0}^{N}\frac{\alpha^k t^k}{k!} \leq Ce^{-\alpha t}t^{N+1} \\ &\leq Ce^{-\frac{\alpha t}{4}}t^{N+1}\langle\xi\rangle^{-2N-2}.\end{aligned}$$

For the term A, we write $A = e^{-\alpha t}p \circ q$, where $p(z) = e^{\alpha t/(1+z)}$ and $q(\xi_j) = |\xi|^2$.

Next, we consider $p(z)$. For convenience, we express $p(z)$ in the form $p = \eta \circ \gamma$ where $\eta(z) = e^{\alpha tz}$ and $\gamma(z) = 1/(1+z)$. Note that for each $i \in \mathbb{N} \cup \{0\}$,

$$\eta^{(i)}(z) = (\alpha t)^i e^{\alpha tz} \quad \text{and} \quad \gamma^{(i)}(z) = (-1)^i i!(1+z)^{-(i+1)}.$$

Using the Faá di Bruno's identity with $\eta \circ \gamma$, we obtain

$$\begin{aligned}\partial_z^k p(z) &= \sum_{i=0}^{k}\sum_{\beta} C_{k,\beta}(\gamma^{(1)})^{\beta_1}\cdots(\gamma^{(k)})^{\beta_k}\eta^{(i)}(\gamma(z)) \\ &= \sum_{i=1}^{k}\sum_{\beta}\frac{k!}{\beta!}(-1)^{\beta_1+\cdots\beta_k}(1+z)^{-(2\beta_1+\cdots+(k+1)\beta_k)}\alpha^i t^i e^{\frac{\alpha t}{1+z}} \\ &= \sum_{i=1}^{k}\sum_{\beta}\frac{k!}{\beta!}(-1)^k(1+z)^{-(k+i)}\alpha^i t^i e^{\frac{\alpha t}{1+z}} \\ &= (-1)^k e^{\frac{\alpha t}{1+z}}(1+z)^{-k}\sum_{i=1}^{k}\sum_{\beta}\frac{k!}{\beta!}\alpha^i\left(\frac{t}{1+z}\right)^i.\end{aligned}$$

It is easy to see that

$$\partial_{\xi_j}^i q(\xi_j) = \begin{cases} 2\xi_j^{2-i} & i \in \{1,2\}, \\ 0 & \text{otherwise,}\end{cases}$$

for $i \in \mathbb{N}$. Applying the Faá di Bruno's identity to $p \circ q$, we obtain

$$\begin{aligned}
\partial_{\xi_j}^l p \circ q(\xi_j) &= \sum_{k=1}^{l} \sum_{\beta} C_{l,\beta} (q^{(1)})^{\beta_1} \cdots (q^{(l)})^{\beta_l} p^{(k)}(q(\xi_j)) \\
&= \sum_{\frac{l}{2} \le k \le l} C_{l,k} \xi_j^{2k-l} \Big\{ (-1)^k e^{\frac{\alpha t}{1+|\xi|^2}} (1+|\xi|^2)^{-k} \sum_{i=1}^{k} C_{k,i} \alpha^i \Big(\frac{t}{1+|\xi|^2} \Big)^i \Big\} \\
&= e^{\frac{\alpha t}{1+|\xi|^2}} \sum_{\frac{l}{2} \le k \le l} C_{l,k} (-1)^k \xi_j^{2k-l} \Big\{ (1+|\xi|^2)^{-k} \sum_{i=1}^{k} C_{k,i} \alpha^i \Big(\frac{t}{1+|\xi|^2} \Big)^i \Big\}.
\end{aligned}$$

In above equation, we have used that the algebraic system $\beta_1 + \beta_2 = k, \beta_1 + 2\beta_2 = l$ has a unique solution $(\beta_1, \beta_2) = (2k-l, l-k)$ and $2k-l, l-k \ge 0$.

Since $t \ge 1$ and $1/2 \le 1/(1+|\xi|^2) < 1$, we have

$$0 < (1+|\xi|^2)^{-k} \sum_{i=1}^{k} C_{k,i} \alpha^i \Big(\frac{t}{1+|\xi|^2} \Big)^i \le C \sum_{i=1}^{k} t^i \le Ckt^k.$$

Next, we write $l = 2m + d$ with $d \in \{0, 1\}$ and $m \ge 0$. Using the above inequality, $t \ge 1$ and $|\xi| \le 1$, we can estimate $\partial_{\xi_j}^l A$ by

$$\begin{aligned}
|\partial_{\xi_j}^l A| &\le C e^{-\alpha t} e^{\frac{\alpha t}{1+|\xi|^2}} \sum_{\frac{l}{2} \le k \le l} |\xi_j|^{2k-l} t^k \\
&\le C e^{-\alpha t} e^{\frac{\alpha t}{1+|\xi|^2}} \sum_{k=m+d}^{l} |\xi|^{2k-l} t^k \\
&= C e^{-\alpha t} e^{\frac{\alpha t}{1+|\xi|^2}} \sum_{i=0}^{m} |\xi|^{2i+d} t^{i+m+d} \\
&= C e^{-\alpha t} e^{\frac{\alpha t}{1+|\xi|^2}} t^{\frac{l}{2}} (t|\xi|^2)^{\frac{d}{2}} \sum_{i=0}^{m} (t|\xi|^2)^i \\
&\le C e^{-\alpha t} e^{\frac{\alpha t}{1+|\xi|^2}} t^{\frac{l}{2}} (1 + t|\xi|^2)^{\frac{l}{2}}.
\end{aligned}$$

If $|\xi|^2 \le 2/3$, then $-1/(1+|\xi|^2) < -3/5$ and it follows that

$$\begin{aligned}
|\partial_{\xi_j}^l A| &\le C e^{-\frac{\alpha t |\xi|^2}{1+|\xi|^2}} t^{\frac{l}{2}} (1 + t|\xi|^2)^{\frac{l}{2}} \\
&\le C e^{-\frac{\alpha t}{2}|\xi|^2} t^{\frac{l}{2}} e^{-\frac{\alpha t}{10}|\xi|^2} (1 + t|\xi|^2)^{\frac{l}{2}} \\
&\le C e^{-\frac{\alpha t}{2}|\xi|^2} t^{\frac{l}{2}} \langle t \rangle^m
\end{aligned}$$

$$\leq Ce^{-\frac{\alpha t}{2}|\xi|^2} t^{\frac{l}{2}} \langle t \rangle^m + Ce^{-\frac{\alpha t}{4}} t^{N+1} \langle \xi \rangle^{-2N-2}.$$

On the other hand, if $2/3 \leq |\xi|^2 \leq 1$, then we have $|\xi|^2/(1+|\xi|^2) \geq 2/5$. Since $l \leq N+2$, $2/3 \leq |\xi|^2 \leq 1$ and $t \geq 1$, we can estimate $\partial_{\xi_j}^l A$ by

$$\begin{aligned}
|\partial_{\xi_j}^l A| &\leq Ce^{-\frac{\alpha t|\xi|^2}{1+|\xi|^2}} t^{\frac{l}{2}} (1+t|\xi|^2)^{\frac{l}{2}} \\
&\leq Ce^{-\frac{2\alpha t}{5}} t^{\frac{l}{2}} (1+t|\xi|^2)^{\frac{l}{2}} \\
&\leq Ce^{-\frac{2\alpha t}{5}} t^{N+2} \\
&\leq Ce^{-\frac{\alpha t}{4}} t^{N+1} \\
&\leq Ce^{-\frac{\alpha t}{4}} t^{N+1} \langle \xi \rangle^{-2N-2} \\
&\leq Ce^{-\frac{\alpha t}{2}|\xi|^2} t^{\frac{l}{2}} \langle t \rangle^m + Ce^{-\frac{\alpha t}{4}} t^{N+1} \langle \xi \rangle^{-2N-2}.
\end{aligned}$$

From the estimations of $\partial_{\xi_j}^l A$ and $\partial_{\xi_j}^l B$, we obtain

$$|\partial_{\xi_j}^l \widehat{R}_N(\xi,t)| \leq Ce^{-\frac{\alpha t}{2}|\xi|^2} t^{\frac{l}{2}} \langle t \rangle^m + Ce^{-\frac{\alpha t}{4}} t^{N+1} \langle \xi \rangle^{-2N-2}$$

as desired. □

Remark. It can be seen that the above lemma is true for all $l \in \mathbb{N} \cup \{0\}$ in general.

Now we start to prove Theorem 3.1.

Proof of Theorem 3.1. First, we can choose $N \in \mathbb{N}$ such that $N > n + |a| - 2$. Note that

$$R_N(x,t) = \mathcal{F}^{-1}(\widehat{R}_N(\xi,t)).$$

We consider two cases.

First, assume that $|x| \geq \sqrt{t}$. Then, $1 \leq |x|^2 t^{-1}$. Hence,

$$|x|^2 t^{-1} \leq 1 + |x|^2 t^{-1} \leq 2|x|^2 t^{-1}.$$

This implies that $|x| t^{-1/2} \leq \langle |x| t^{-1/2} \rangle \leq \sqrt{2} |x| t^{-1/2}$. Since $|x| \geq \sqrt{t} > 0$, we can choose $j \in \{1,2,3,\dots n\}$ such that $|x_j| = \max\{x_i : i \in \{1,2,3,\dots,n\}\} > 0$. Also note that

$$|x_j| \leq |x| \leq \sqrt{n}|x_j|.$$

By using the integration by parts with respect to ξ_j, we have

$$\begin{aligned} |R_N(x,t)| &= (2\pi)^{-\frac{n}{2}}\left|\int_{\mathbb{R}^n} e^{ix\cdot\xi}\widehat{R}_N(\xi,t)d\xi\right| \\ &= (2\pi)^{-\frac{n}{2}}|x_j|^{-1}\left|\int_{\mathbb{R}^n} e^{ix\cdot\xi}\partial_{\xi_j}\widehat{R}_N(\xi,t)d\xi\right| \\ &\leq \sqrt{n}(2\pi)^{-\frac{n}{2}}|x|^{-1}\left|\int_{\mathbb{R}^n} e^{ix\cdot\xi}\partial_{\xi_j}\widehat{R}_N(\xi,t)d\xi\right|. \end{aligned}$$

Integrating by parts $N+2$ times and use Lemma 3.2, we get

$$\begin{aligned} |R_N(x,t)| &\leq C|x|^{-N-2}\left|\int_{\mathbb{R}^n} e^{ix\cdot\xi}\partial_{\xi_j}^{N+2}\widehat{R}_N(\xi,t)d\xi\right| \\ &\leq C|x|^{-N-2}t^{\frac{N+2}{2}}\langle t\rangle^m\|e^{-\frac{\alpha t}{2}|\xi|^2}\|_{L^1_\xi} + C|x|^{-N-2}e^{-\frac{\alpha t}{4}}t^{N+1}\|\langle\xi\rangle^{-2N-2}\|_{L^1_\xi} \\ &\leq C(|x|t^{-\frac{1}{2}})^{-N-2}t^{-\frac{n}{2}}\langle t\rangle^m + C|x|^{-N-2}t^{\frac{N+2}{2}}t^{-\frac{n}{2}}\langle t\rangle^m \\ &\leq C(|x|t^{-\frac{1}{2}})^{-N-2}t^{-\frac{n}{2}}\langle t\rangle^m \\ &\leq C\langle|x|t^{-\frac{1}{2}}\rangle^{-N-2}t^{-\frac{n}{2}}\langle t\rangle^m. \end{aligned}$$

Now we assume that $|x| \leq \sqrt{t}$. Then, $|x|^2t^{-1} \leq 1$ and hence,

$$\langle|x|t^{-\frac{1}{2}}\rangle = \sqrt{1+|x|^2t^{-1}} \leq \sqrt{2}.$$

By Lemma 3.2, we have

$$\begin{aligned} |R_N(x,t)| &= (2\pi)^{-\frac{n}{2}}\left|\int_{\mathbb{R}^n} e^{ix\cdot\xi}\widehat{R}_N(\xi,t)d\xi\right| \\ &\leq C\|\widehat{R}_N(\xi,t)\|_{L^1_\xi} \\ &\leq C\langle t\rangle^m\|e^{-\frac{\alpha t}{2}|\xi|^2}\|_{L^1_\xi} + Ce^{-\frac{\alpha t}{4}}t^{N+1}\|\langle\xi\rangle^{-2N-2}\|_{L^1_\xi} \\ &\leq Ct^{-\frac{n}{2}}\langle t\rangle^m + Ct^{-\frac{n}{2}}\langle t\rangle^m \\ &= C\langle|x|t^{-\frac{1}{2}}\rangle^{-N-2}t^{-\frac{n}{2}}\langle t\rangle^m. \end{aligned}$$

Combining the preceeding two cases, we conclude that

$$|R_N(x,t)| \leq C\langle|x|t^{-\frac{1}{2}}\rangle^{-N-2}t^{-\frac{n}{2}}\langle t\rangle^m, \tag{3.4}$$

for all $x \in \mathbb{R}^n$ and $t > 0$.

We decompose

$$\mathcal{G}(t)\psi = G(\cdot,t) * \psi = e^{-\alpha t}\sum_{k=0}^{N} \frac{\alpha^k t^k}{k!}\mathcal{B}^k\psi + R(\cdot,t)*\psi.$$

For the first term, by using the Cauchy-Schwarz inequality, we can estimate its norm.

$$\begin{aligned}
\left\| e^{-\alpha t}\sum_{k=0}^{N} \frac{\alpha^k t^k}{k!}\mathcal{B}^k\psi\right\|_{L^{q,a}} &\le Ce^{-\alpha t}\sum_{k=0}^{N}\frac{\alpha^k t^k}{k!}\|\mathcal{B}^k\psi\|_{L^{q,a}} \\
&\le Ce^{-\alpha t}\sum_{k=0}^{N}\frac{\alpha^k t^k}{k!}\|\psi\|_{L^{q,a}} \\
&\le Ce^{-\alpha t}\left(\sum_{k=0}^{N}\frac{\alpha^{2k}}{k!^2}\right)^{\frac{1}{2}}\left(\sum_{k=0}^{N} t^{2k}\right)^{\frac{1}{2}}\|\psi\|_{L^{q,a}} \\
&\le Ce^{-\alpha t}\langle t\rangle^N\|\psi\|_{L^{q,a}} \\
&\le C\langle t\rangle^{\frac{|a|}{2}+m}\|\psi\|_{L^{q,a}}. \qquad (3.5)
\end{aligned}$$

For the second term, we use the Young's inequality, Lemma 2.4, and (3.4), to get

$$\begin{aligned}
\|R(\cdot,t)*\psi\|_{L^{q,a}} &\le \left\|\langle x\rangle^a\int_{\mathbb{R}^n} R_N(x-y)\psi(y)dy\right\|_{L^q_x} \\
&\le C\left\|\int_{\mathbb{R}^n}\langle x-y\rangle^{|a|}|R_N(x-y,t)|\langle y\rangle^a|\psi(y)|dy\right\|_{L^q_x} \\
&\le C\|R_N(\cdot,t)\|_{L^{1,|a|}}\|\psi\|_{L^{q,a}} \\
&\le Ct^{-\frac{n}{2}}\langle t\rangle^m\left\|\langle x\rangle^{|a|}\langle |x|t^{-\frac{1}{2}}\rangle^{-N-2}\right\|_{L^1_x}\|\psi\|_{L^{q,a}}.
\end{aligned}$$

It remains to estimate the term on the right hand side of the last inequality. Using the polar coordinates integration, we have

$$\begin{aligned}
&\left\|\langle x\rangle^{|a|}\langle |x|t^{-\frac{1}{2}}\rangle^{-N-2}\right\|_{L^1_x} \\
&\quad = \int_{\mathbb{R}^n}\langle x\rangle^{|a|}\langle |x|t^{-\frac{1}{2}}\rangle^{-N-2}dx \\
&\quad = \int_{|x|\le\sqrt{t}}\langle x\rangle^{|a|}\langle |x|t^{-\frac{1}{2}}\rangle^{-N-2}dx + \int_{|x|\ge\sqrt{t}}\langle x\rangle^{|a|}\langle |x|t^{-\frac{1}{2}}\rangle^{-N-2}dx
\end{aligned}$$

$$
\begin{aligned}
&= \int_{S^{n-1}} \left(\int_0^{\sqrt{t}} \langle r\rangle^{|a|} \langle rt^{-\frac{1}{2}}\rangle^{-N-2} r^{n-1} dr + \int_{\sqrt{t}}^{\infty} \langle r\rangle^{|a|} \langle rt^{-\frac{1}{2}}\rangle^{-N-2} r^{n-1} dr \right) d\sigma \\
&\leq \omega_n 2^{\frac{|a|}{2}} \left(\int_0^{\sqrt{t}} (\max\{1,\sqrt{t}\})^{|a|} \langle rt^{-\frac{1}{2}}\rangle^{-N-2} r^{n-1} dr \right. \\
&\qquad \left. + \int_{\sqrt{t}}^{\infty} (\max\{1,r\})^{|a|} \langle rt^{-\frac{1}{2}}\rangle^{-N-2} r^{n-1} dr \right) \\
&= \omega_n 2^{\frac{|a|}{2}} t^{\frac{n}{2}} \left((\max\{1,\sqrt{t}\})^{|a|} \int_0^1 w^{n-1} \langle w\rangle^{-N-2} dw \right. \\
&\qquad \left. + \int_1^{\infty} (\max\{1,\sqrt{t}w\})^{|a|} w^{n-1} \langle w\rangle^{-N-2} dw \right), \qquad (3.6)
\end{aligned}
$$

where in (3.6), we substitute $w = rt^{-1/2}$. There, $\omega_n := \int_{S^{n-1}} d\sigma$ and $d\sigma$ is a standard measure on the unit sphere S^{n-1}. It is easy to see that $\int_0^1 w^{n-1}\langle w\rangle^{-N-2} dw$ is finite since $n-1 \geq 0$. This implies that $w^{n-1}\langle w\rangle^{-N-2}$ is bounded on $[0,1]$.

Next, since $0 \leq w^{|a|+n-1}\langle w\rangle^{-N-2} \leq w^{|a|+n-N-3}$ for all $w \geq 1$ and $|a|+n-N-3 < -1$, it follows that $\int_1^\infty w^{|a|+n-N-3} dw$ is also finite. By the comparison test for integral, we conclude that $\int_1^\infty w^{|a|+n-1}\langle w\rangle^{-N-2} dw$ is convergent. If $t \leq 1$, then

$$\left\| \langle x\rangle^{|a|} \langle |x| t^{-\frac{1}{2}}\rangle^{-N-2} \right\|_{L^1_x} \leq \omega_n 2^{\frac{|a|}{2}} t^{\frac{n}{2}} \left(\int_0^1 w^{n-1}\langle w\rangle^{-N-2} + \int_1^\infty w^{|a|+n-1}\langle w\rangle^{-N-2} \right).$$

Thus,

$$\|R(\cdot,t) * \psi\|_{L^{q,a}} \leq Ct^{-\frac{n}{2}} t^{\frac{n}{2}} \langle t\rangle^m \|\psi\|_{L^{q,a}} \leq C\langle t\rangle^{\frac{|a|}{2}+m} \|\psi\|_{L^{q,a}}. \qquad (3.7)$$

If $t \geq 1$, then

$$\left\| \langle x\rangle^{|a|} \langle |x| t^{-\frac{1}{2}}\rangle^{-N-2} \right\|_{L^1_x} \leq \omega_n 2^{\frac{|a|}{2}} t^{\frac{n}{2}+\frac{|a|}{2}} \left(\int_0^1 w^{n-1}\langle w\rangle^{-N-2} + \int_1^\infty w^{|a|+n-1}\langle w\rangle^{-N-2} \right).$$

Therefore, we have

$$\|R(\cdot,t) * \psi\|_{L^{q,a}} \leq Ct^{-\frac{n}{2}} t^{\frac{n}{2}+\frac{|a|}{2}} \langle t\rangle^m \|\psi\|_{L^{q,a}} \leq C\langle t\rangle^{\frac{|a|}{2}+m} \|\psi\|_{L^{q,a}}, \qquad (3.8)$$

where we have used that $t \leq \langle t\rangle$ for all $t > 0$. Combining estimations (3.5), (3.7) and (3.8), we conclude that

$$\|\mathcal{G}(t)\psi\|_{L^{q,a}} \leq C\langle t\rangle^{\frac{|a|}{2}+m} \|\psi\|_{L^{q,a}},$$

which is our desired result. □

Theorem 3.3 (Interpolation)**.** *Let* $1 \le p \le q \le \infty, n \in \mathbb{N}, \alpha > 0$ *and* $a \in \mathbb{R}$. *Then, there is a positive constant* $C = C(\alpha, p, q, a, n)$ *such that for any* $t > 0$

$$\|\mathcal{G}(t)\psi\|_{L^{q,a}} \le Ce^{-\frac{\alpha t}{2}}\|\psi\|_{L^{q,a}} + C\langle t\rangle^{\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\|\psi\|_{L^{p,a}} + C\langle t\rangle^{\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)+\frac{|a|}{2}}\|\psi\|_{L^p}.$$

Proof. First, let $1 \le r < \infty$ be such that $1/p = 1/q - 1/r + 1$ and we choose $N \in \mathbb{N}$ such that $r(|a| - N - 2) + n < 0$. By the estimation (3.4) in the proof of Theorem 3.1, we immediately obtain

$$|R_N(x,t)| \le C\langle |x|t^{-\frac{1}{2}}\rangle^{-N-2}t^{-\frac{n}{2}} \qquad (x \in \mathbb{R}^n, t > 0).$$

Note that,

$$\mathcal{G}(t)\psi = G(\cdot,t) * \psi = e^{-\alpha t}\sum_{k=0}^{N}\frac{\alpha^k t^k}{k!}\mathcal{B}^k\psi + R(\cdot,t) * \psi.$$

We estimate the first term $\left\|e^{-\alpha t}\sum_{k=0}^{N}\frac{\alpha^k t^k}{k!}\mathcal{B}^k\psi\right\|_{L^{q,a}}$. By the proof of Theorem 3.1, we have

$$\left\|e^{-\alpha t}\sum_{k=0}^{N}\frac{\alpha^k t^k}{k!}\mathcal{B}^k\psi\right\|_{L^{q,a}} \le Ce^{-\alpha t}\langle t\rangle^N\|\psi\|_{L^{q,a}} \le Ce^{-\frac{\alpha t}{2}}\|\psi\|_{L^{q,a}}. \tag{3.9}$$

We estimate the second term by dividing into 2 cases. If $0 < t \le 1$, then by the estimation (3.3) in the proof of Lemma 3.2, we have

$$|\partial_{\xi_j}^l \widehat{R}_N(\xi,t)| \le Ce^{-\frac{\alpha}{4}t}t^{N+1}\langle\xi\rangle^{-2N-2}$$

for all $\xi \in \mathbb{R}^n$ and $l \in \{0,1,2,\ldots,N+2\}$. If $|x| \ge 1$, then $|x| \le \langle x\rangle \le \sqrt{2}|x|$ which implies that

$$\begin{aligned} |R_N(x,t)| &\le C|x|^{-N-2}\left|\int_{\mathbb{R}^n} e^{ix\cdot\xi}\partial_{\xi_j}^{N+2}\widehat{R}_N(\xi,t)d\xi\right| \\ &\le C|x|^{-N-2}e^{-\frac{\alpha}{4}t}t^{N+1}\|\langle\xi\rangle^{-2N-2}\|_{L^1_\xi} \le C\langle x\rangle^{-N-2}e^{-\frac{\alpha}{4}t}t^{N+1}. \end{aligned}$$

If $|x| \le 1$, then $1 \le 2^{(N+2)/2}\langle x\rangle^{-N-2}$. Hence,

$$|R_N(x,t)| \le C\left|\int_{\mathbb{R}^n} e^{ix\cdot\xi}\widehat{R}_N(\xi,t)d\xi\right|$$

$$\leq \ Ce^{-\frac{\alpha}{4}t}t^{N+1}\|\langle\xi\rangle^{-2N-2}\|_{L^1_\xi}$$
$$\leq \ C\langle x\rangle^{-N-2}e^{-\frac{\alpha}{4}t}t^{N+1}.$$

By the Young's inequality for convolution and Lemma 2.4, we get

$$\begin{aligned}\|R(\cdot,t)*\psi\|_{L^{q,a}} &\leq \left\|\langle x\rangle^a\int_{\mathbb{R}^n}R_N(x-y)\psi(y)dy\right\|_{L^q_x}\\ &\leq C\left\|\int_{\mathbb{R}^n}\langle x-y\rangle^{|a|}|R_N(x-y,t)|\langle y\rangle^a|\psi(y)|dy\right\|_{L^q_x}\\ &\leq C\|R_N(\cdot,t)\|_{L^{r,|a|}}\|\psi\|_{L^{p,a}}\\ &\leq Ct^{N+2}e^{-\frac{\alpha}{4}t}\left\|\langle x\rangle^{|a|-N-2}\right\|_{L^r_x}\|\psi\|_{L^{p,a}}\leq C\langle t\rangle^{\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\|\psi\|_{L^{p,a}},\end{aligned} \tag{3.10}$$

where we have used that

$$\begin{aligned}\left\|\langle x\rangle^{|a|-N-2}\right\|_{L^r_x} &= \left(\int_{\mathbb{R}^n}\langle x\rangle^{r(|a|-N-2)}dx\right)^{\frac{1}{r}}\\ &= \omega_n^{\frac{1}{r}}\left(\int_0^1\langle s\rangle^{r(|a|-N-2)}s^{n-1}ds+\int_1^\infty\langle s\rangle^{r(|a|-N-2)}s^{n-1}ds\right)^{\frac{1}{r}}\end{aligned}$$

which is finite because $n \geq 1$ and $r(|a|-N-2)+n<0$.

If $t \geq 1$, then $t \leq \langle t\rangle \leq \sqrt{2}t$. Then, by Lemma 2.5, the Young's inequality for convolution, and Lemma 2.4 again, we obtain

$$\begin{aligned}&\|R(\cdot,t)*\psi\|_{L^{q,a}}\\ &=\left\|\langle x\rangle^a\int_{\mathbb{R}^n}R_N(x-y)\psi(y)dy\right\|_{L^q_x}\\ &\leq C\left\|\int_{\mathbb{R}^n}\langle x-y\rangle^{|a|}R_N(x-y,t)\psi(y)dy\right\|_{L^q_x}+C\left\|\int_{\mathbb{R}^n}R_N(x-y,t)\langle y\rangle^a\psi(y)dy\right\|_{L^q_x}\\ &\leq C\|R_N(\cdot,t)\|_{L^{r,|a|}_x}\|\psi\|_{L^p}+\|R_N(\cdot,t)\|_{L^r_x}\|\psi\|_{L^{p,a}}\\ &\leq Ct^{-\frac{n}{2}}\left\|\langle x\rangle^{|a|}\langle|x|t^{-\frac{1}{2}}\rangle^{-N-2}\right\|_{L^r_x}\|\psi\|_{L^p}+Ct^{-\frac{n}{2}}\left\|\langle|x|t^{-\frac{1}{2}}\rangle^{-N-2}\right\|_{L^r_x}\|\psi\|_{L^{p,a}} &(3.11)\\ &\leq Ct^{-\frac{n}{2}}t^{\frac{n}{2r}+\frac{|a|}{2}}\|\psi\|_{L^p}+Ct^{-\frac{n}{2}}t^{\frac{n}{2r}}\|\psi\|_{L^{p,a}} &(3.12)\\ &\leq C\langle t\rangle^{\frac{n}{2}(\frac{1}{q}-\frac{1}{p})+\frac{|a|}{2}}\|\psi\|_{L^p}+C\langle t\rangle^{\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\|\psi\|_{L^{p,a}}. &(3.13)\end{aligned}$$

Notice that (3.12) follows from (3.11) by

$$
\begin{aligned}
&\left\| \langle x\rangle^{|a|}\langle |x|t^{-\frac{1}{2}}\rangle^{-N-2}\right\|_{L^r_x} \\
&\quad = \left(\int_{\mathbb{R}^n}\langle x\rangle^{r|a|}\langle |x|t^{-\frac{1}{2}}\rangle^{-r(N+2)}dx\right)^{\frac{1}{r}} \\
&\quad = \left(\int_{|x|\le\sqrt{t}}\langle x\rangle^{r|a|}\langle |x|t^{-\frac{1}{2}}\rangle^{-r(N+2)}dx + \int_{|x|\ge\sqrt{t}}\langle x\rangle^{r|a|}\langle |x|t^{-\frac{1}{2}}\rangle^{-r(N+2)}dx\right)^{\frac{1}{r}} \\
&\quad = \left(\int_{S^{n-1}}\left(\int_0^{\sqrt{t}}\langle s\rangle^{r|a|}\langle st^{-\frac{1}{2}}\rangle^{-r(N+2)}s^{n-1}ds\right.\right. \\
&\quad + \left.\left.\int_{\sqrt{t}}^{\infty}\langle s\rangle^{r|a|}\langle st^{-\frac{1}{2}}\rangle^{-r(N+2)}s^{n-1}ds\right)d\sigma\right)^{\frac{1}{r}} \\
&\quad \le \omega_n^{\frac{1}{r}}\left(\int_0^{\sqrt{t}}\max\{1,\sqrt{t}\}^{r|a|}\langle st^{-\frac{1}{2}}\rangle^{-r(N+2)}s^{n-1}ds\right. \\
&\qquad + \left.\int_{\sqrt{t}}^{\infty}\max\{1,s\}^{r|a|}\langle st^{-\frac{1}{2}}\rangle^{-r(N+2)}s^{n-1}ds\right)^{\frac{1}{r}} \\
&\quad = \omega_n^{\frac{1}{r}}t^{\frac{n}{2}}\left(\max\{1,\sqrt{t}\}^{r|a|}\int_0^1 w^{n-1}\langle w\rangle^{-r(N+2)}dw \qquad (w = st^{-\frac{1}{2}})\right. \\
&\qquad + \left.\int_1^{\infty}\max\{1,\sqrt{t}w\}^{r|a|}w^{n-1}\langle w\rangle^{-r(N+2)}dw\right)^{\frac{1}{r}} \\
&\quad \le \omega_n^{\frac{1}{r}}t^{\frac{n}{2}+\frac{|a|}{2}}\left(\int_0^1 w^{n-1}\langle w\rangle^{-r(N+2)}dw + \int_1^{\infty} w^{r|a|+n-1}\langle w\rangle^{-r(N+2)}dw\right)^{\frac{1}{r}},
\end{aligned}
$$

where the right hand side is finite since $n-1\ge 0$ and $r(|a|-N-2)+n<0$. Combined estimations (3.9),(3.10) and (3.13) we obtain the estimation for the Green operator.

$$
\|\mathcal{G}(t)\psi\|_{L^{q,a}} \le Ce^{-\frac{\alpha t}{2}}\|\psi\|_{L^{q,a}} + C\langle t\rangle^{\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\|\psi\|_{L^{p,a}} + C\langle t\rangle^{\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)+\frac{|a|}{2}}\|\psi\|_{L^p}
$$

as needed. □

We close this chapter with the following result which will be crucial in the study of global existence.

Lemma 3.4. *Let $n \in \mathbb{N}, k \geq 1, 1 \leq q \leq n/k$. Then, there is a positive constant $C = C(n,q,k)$ such that for any $t > 0$*

$$\|\psi\|_{L^q} \leq \begin{cases} \langle t\rangle^{-\frac{k}{2}} \|\psi\|_{L^{1,k}}^{\frac{k}{n}} \|\psi\|_{L^{\infty,k}}^{1-\frac{k}{n}} + C(1+\ln\langle t\rangle)^{\frac{k}{n}} \|\psi\|_{L^{\infty,k}} & ; n = qk \\ \\ \langle t\rangle^{-\frac{k}{2}} \|\psi\|_{L^{1,k}}^{\frac{1}{q}} \|\psi\|_{L^{\infty,k}}^{1-\frac{1}{q}} + C\langle t\rangle^{\frac{n}{2q}-\frac{k}{2}} \|\psi\|_{L^{\infty,k}} & ; n > qk. \end{cases}$$

Proof. By the definition of L^q norm and the Hölder inequality, we have

$$\begin{aligned} \|\psi\|_{L^q} &= \left(\int_{|x|\leq\sqrt{\langle t\rangle}} |\psi(x)|^q dx + \int_{|x|\geq\sqrt{\langle t\rangle}} |\psi(x)|^q dx \right)^{\frac{1}{q}} \\ &\leq \left(\int_{|x|\leq\sqrt{\langle t\rangle}} \langle x\rangle^{-qk} \langle x\rangle^{qk} |\psi(x)|^q dx \right)^{\frac{1}{q}} + \left(\int_{|x|\geq\sqrt{\langle t\rangle}} |x|^{-qk} |x|^{qk} |\psi(x)|^q dx \right)^{\frac{1}{q}} \\ &\leq \left(\int_{|x|\leq\sqrt{\langle t\rangle}} \langle x\rangle^{-qk} dx \right)^{\frac{1}{q}} \|\psi\|_{L^{\infty,k}} + \langle t\rangle^{-\frac{k}{2}} \left(\int_{|x|\geq\sqrt{\langle t\rangle}} \langle x\rangle^{qk} |\psi(x)|^q dx \right)^{\frac{1}{q}} \\ &\leq \left(\omega_n \int_0^{\sqrt{\langle t\rangle}} \langle r\rangle^{-qk} r^{n-1} dx \right)^{\frac{1}{q}} \|\psi\|_{L^{\infty,k}} + \langle t\rangle^{-\frac{k}{2}} \|\psi\|_{L^{q,k}} \\ &\leq \omega_n^{\frac{1}{q}} \left(\int_0^1 \langle r\rangle^{-qk} r^{n-1} + \int_1^{\sqrt{\langle t\rangle}} \langle r\rangle^{-qk} r^{n-1} \right)^{\frac{1}{q}} \|\psi\|_{L^{\infty,k}} + \langle t\rangle^{-\frac{k}{2}} \|\psi\|_{L^{1,k}}^{\frac{1}{q}} \|\psi\|_{L^{\infty,k}}^{1-\frac{1}{q}} \\ &\leq \omega_n^{\frac{1}{q}} \left(\int_0^1 \langle r\rangle^{-qk} r^{n-1} + \int_1^{\sqrt{\langle t\rangle}} r^{n-qk-1} \right)^{\frac{1}{q}} \|\psi\|_{L^{\infty,k}} + \langle t\rangle^{-\frac{k}{2}} \|\psi\|_{L^{1,k}}^{\frac{1}{q}} \|\psi\|_{L^{\infty,k}}^{1-\frac{1}{q}}. \end{aligned}$$

If $n = qk$, then

$$\int_1^{\sqrt{\langle t\rangle}} r^{n-qk-1} dr = \frac{1}{2} \ln\langle t\rangle$$

which implies that

$$\begin{aligned} \|\psi\|_{L^q} &\leq \omega_n^{\frac{1}{q}} \left(\int_0^1 \langle r\rangle^{-qk} r^{n-1} dr + \frac{1}{2} \ln\langle t\rangle \right)^{\frac{1}{q}} \|\psi\|_{L^{\infty,k}} + \langle t\rangle^{-\frac{k}{2}} \|\psi\|_{L^{1,k}}^{\frac{1}{q}} \|\psi\|_{L^{\infty,k}}^{1-\frac{1}{q}} \\ &\leq C(1+\ln\langle t\rangle)^{\frac{k}{n}} \|\psi\|_{L^{\infty,k}} + \langle t\rangle^{-\frac{k}{2}} \|\psi\|_{L^{1,k}}^{\frac{1}{q}} \|\psi\|_{L^{\infty,k}}^{1-\frac{1}{q}}. \end{aligned}$$

If $n > qk$, then $1 \leq \langle t\rangle^{n-qk}$ for all $t > 0$. Thus,

$$\int_1^{\sqrt{\langle t\rangle}} r^{n-qk-1} dr = \frac{\langle t\rangle^{\frac{n-qk}{2}} - 1}{n-qk}$$

and this implies

$$
\begin{aligned}
\|\psi\|_{L^q} &\le \omega_n^{\frac{1}{q}} \left(\int_0^1 \langle r\rangle^{-qk} r^{n-1} dr + \frac{\langle t\rangle^{\frac{n-qk}{2}} - 1}{n-qk} \right)^{\frac{1}{q}} \|\psi\|_{L^{\infty,k}} + \langle t\rangle^{-\frac{k}{2}} \|\psi\|_{L^{1,k}}^{\frac{1}{q}} \|\psi\|_{L^{\infty,k}}^{1-\frac{1}{q}} \\
&\le C \langle t\rangle^{\frac{n}{2q}-\frac{k}{2}} \|\psi\|_{L^{\infty,k}} + \langle t\rangle^{-\frac{k}{2}} \|\psi\|_{L^{1,k}}^{\frac{1}{q}} \|\psi\|_{L^{\infty,k}}^{1-\frac{1}{q}}. \qquad \square
\end{aligned}
$$

CHAPTER IV

EXISTENCE OF LOCAL SOLUTIONS

In this chapter, by adapting the approach used in [4] and [9], we can prove the local existence of solutions to (1.1). The results in both papers, however, are obtained under the assumption that V is a constant. Thus, the result in this chapter can be regarded as a non-trivial generalization of those papers.

Theorem 4.1 (The existence of local solutions)**.** *Let $\sigma > 0, a \geq 0$ be constants and $V : \mathbb{R}^n \to \mathbb{R}$ be given function. Assume that there exists a positive constant C such that for any $x \in \mathbb{R}^n$*

$$|V(x)| \leq C|x|^a.$$

If the initial condition $u_0 \in C(\mathbb{R}^n) \cap L^{1,b}(\mathbb{R}^n) \cap L^{\infty,b}(\mathbb{R}^n)$ where $b \geq a/\sigma$, then there exists $T > 0$ such that the Cauchy problem (1.1) has a unique mild solution

$$u \in C([0,T]; C(\mathbb{R}^n) \cap L^{1,b}(\mathbb{R}^n) \cap L^{\infty,b}(\mathbb{R}^n)).$$

Proof. Let $T > 0$ to be speified and

$$\mathcal{X} = \{w \in C([0,T]; C(\mathbb{R}^n) \cap L^{1,b}(\mathbb{R}^n) \cap L^{\infty,b}(\mathbb{R}^n)) : \|w\|_{\mathcal{X}} < \infty\}$$

with the mixed norm

$$\|w\|_{\mathcal{X}} := \sup_{t \in [0,T]} \left\{ \|w(\cdot,t)\|_{L^{1,b}(\mathbb{R}^n)} + \|w(\cdot,t)\|_{L^{\infty,b}(\mathbb{R}^n)} \right\}.$$

We define the operator $\mathcal{M}$ on $\mathcal{X}$ by

$$\mathcal{M}(w)(x,t) = \mathcal{G}(t)u_0(x) + \int_0^t \mathcal{G}(t-\tau)\mathcal{B}[V(x)|w(x,\tau)|^\sigma w(x,\tau)]d\tau$$

for all $w \in \mathcal{X}$. We will show that $\mathcal{M}$ maps $\mathcal{X}$ to itself. Let $w \in \mathcal{X}$. Then,

$$\|w(\cdot,t)\|_{L^{1,b}}, \|w(\cdot,t)\|_{L^{\infty,b}} \le \|w\|_{\mathcal{X}} < \infty, \qquad t > 0.$$

Note that $a \le b\sigma$ and $b \ge 0$. First, we consider $\|\mathcal{M}(w)(x,t)\|_{L^{\infty,b}}$. By Theorems 2.11 and 3.1 with $q = \infty$, we obtain

$$\begin{aligned}\|\mathcal{G}(t)\mathcal{B}[V(\cdot)|w(\cdot,\tau)|^\sigma w(\cdot,\tau)]\|_{L^{\infty,b}} &\le C\langle t\rangle^{\frac{b}{2}}\|\mathcal{B}[V(\cdot)|w(\cdot,\tau)|^\sigma w(\cdot,\tau)]\|_{L^{\infty,b}} \\ &\le C\langle t\rangle^{\frac{b}{2}}\|\langle\cdot\rangle^{b+b\sigma}w^{\sigma+1}(\cdot,\tau)\|_{L^\infty} \\ &= C\langle t\rangle^{\frac{b}{2}}\|w(\cdot,\tau)\|^{\sigma+1}_{L^{\infty,b}}. \end{aligned} \tag{4.1}$$

By Theorem 3.1 again and (4.1), we have

$$\begin{aligned}\|\mathcal{M}(w)(\cdot,t)\|_{L^{\infty,b}} &\le \|\mathcal{G}(t)u_0(\cdot)\|_{L^{\infty,b}} + \left\|\int_0^t \mathcal{G}(t-\tau)\mathcal{B}[V(\cdot)|w(\cdot,\tau)|^\sigma w(\cdot,\tau)]d\tau\right\|_{L^{\infty,b}} \\ &\le \|\mathcal{G}(t)u_0(\cdot)\|_{L^{\infty,b}} + \int_0^t \left\|\mathcal{G}(t-\tau)\mathcal{B}[V(\cdot)|w(\cdot,\tau)|^\sigma w(\cdot,\tau)]\right\|_{L^{\infty,b}}d\tau \\ &\le C\langle t\rangle^{\frac{b}{2}}\|u_0\|_{L^{\infty,b}} + C\int_0^t \langle t-\tau\rangle^{\frac{b}{2}}\|w(\cdot,\tau)\|^{\sigma+1}_{L^{\infty,b}}d\tau \\ &\le C\langle T\rangle^{\frac{b}{2}}\|u_0\|_{L^{\infty,b}} + C\langle T\rangle^{\frac{b}{2}}T\|w\|^{\sigma+1}_{\mathcal{X}}. \end{aligned} \tag{4.2}$$

This implies that $\mathcal{M}(w)(\cdot,t) \in L^{\infty,b}(\mathbb{R}^n)$ for $t \in [0,T]$. Next, we consider $\|\mathcal{M}(w)(\cdot,t)\|_{L^{1,b}}$. By Theorems 2.11 and 3.1 with $q = 1$ and the Hölder's inequality, we obtain

$$\begin{aligned}\|\mathcal{G}(t)\mathcal{B}[V(\cdot)|w(\cdot,\tau)|^\sigma w(\cdot,\tau)]\|_{L^{1,b}} &\le C\langle t\rangle^{\frac{b}{2}}\|\mathcal{B}[V(\cdot)|w(\cdot,\tau)|^\sigma w(\cdot,\tau)]\|_{L^{1,b}} \\ &\le C\langle t\rangle^{\frac{b}{2}}\|V(\cdot)|w(\cdot,\tau)|^\sigma w(\cdot,\tau)\|_{L^{1,b}} \\ &\le C\langle t\rangle^{\frac{b}{2}}\|\langle\cdot\rangle^{b+b\sigma}w^{\sigma+1}(\cdot,\tau)\|_{L^1} \\ &\le C\langle t\rangle^{\frac{b}{2}}\|\langle\cdot\rangle^{b}w(\cdot,\tau)\|_{L^1}\|\langle\cdot\rangle^{b\sigma}w^{\sigma}(\cdot,\tau)\|_{L^\infty} \\ &= C\langle t\rangle^{\frac{b}{2}}\|w(\cdot,\tau)\|_{L^{1,b}}\|w(\cdot,\tau)\|^{\sigma}_{L^{\infty,b}}. \end{aligned} \tag{4.3}$$

Finally, by Theorem 2.11 again and (4.3), we have

$$\|\mathcal{M}(w)(\cdot,t)\|_{L^{1,b}} \le \|\mathcal{G}(t)u_0(\cdot)\|_{L^{1,b}} + \left\|\int_0^t \mathcal{G}(t-\tau)\mathcal{B}[V(\cdot)|w(\cdot,\tau)|^\sigma w(\cdot,\tau)]d\tau\right\|_{L^{1,b}}$$

$$\leq \|\mathcal{G}(t)u_0(\cdot)\|_{L^{1,b}} + \int_0^t \left\|\mathcal{G}(t-\tau)\mathcal{B}[V(\cdot)|w(\cdot,\tau)|^\sigma w(\cdot,\tau)]\right\|_{L^{1,b}} d\tau$$
$$\leq C\langle t\rangle^{\frac{b}{2}}\|u_0\|_{L^{1,b}} + C\int_0^t \langle t-\tau\rangle^{\frac{b}{2}}\|w(\cdot,\tau)\|_{L^{1,b}}\|w(\cdot,\tau)\|^\sigma_{L^{\infty,b}} d\tau$$
$$\leq C\langle T\rangle^{\frac{b}{2}}\|u_0\|_{L^{1,b}} + C\langle T\rangle^{\frac{b}{2}} T\|w\|^{\sigma+1}_{\mathcal{X}}. \tag{4.4}$$

This implies that $\mathcal{M}(w)(\cdot,t) \in L^{1,b}(\mathbb{R}^n)$ for all $t \in [0,T]$. Also, $\mathcal{M}(w)(\cdot,t) \in C(\mathbb{R}^n)$. Moreover, by the semigroup property of Green's operator, we can conclude that $\mathcal{M}(w)(\cdot,t) \in C([0,T])$. Thus, we conclude that $\mathcal{M}(w) \in \mathcal{X}$ as required.

Next, let $\delta > 0$ be a sufficiently large number to be specified. We define

$$\mathcal{X}_\delta = \{w \in \mathcal{X} : \|w\|_{\mathcal{X}} \leq \delta\}.$$

By (4.2) and (4.4), we have, for $w \in \mathcal{X}_\delta$,

$$\|\mathcal{M}(w)\|_{\mathcal{X}} = \sup_{t\in[0,T]} \left\{\|\mathcal{M}(w)(\cdot,t)\|_{L^{1,b}(\mathbb{R}^n)} + \|\mathcal{M}(w)(\cdot,t)\|_{L^{\infty,b}(\mathbb{R}^n)}\right\}$$
$$\leq C_1\langle T\rangle^{\frac{b}{2}}(\|u_0\|_{L^{1,b}} + \|u_0\|_{L^{\infty,b}} + T\|w\|^{\sigma+1}_{\mathcal{X}})$$
$$\leq C_1\langle T\rangle^{\frac{b}{2}}(\|u_0\|_{L^{1,b}} + \|u_0\|_{L^{\infty,b}} + T\delta^{\sigma+1})$$
$$\leq \delta.$$

Therefore, we choose $\delta > 0$ and $T > 0$ such that

$$\max\{\|u_0\|_{L^{1,b}}, \|u_0\|_{L^{\infty,b}}\} \leq \frac{\delta}{4C_1\langle T\rangle^{\frac{b}{2}}} \quad \text{and} \quad C_1\langle T\rangle^{\frac{b}{2}} T \leq \frac{1}{2\delta^\sigma}.$$

Indeed, we fixed $T = 1$, then there exists $\delta > 0$ such that

$$C_1 2^{\frac{b}{4}+2}\max\{\|u_0\|_{L^{1,b}}, \|u_0\|_{L^{\infty,b}}\} \leq \delta.$$

Next, since $\lim_{t\to 0} t\langle t\rangle^{\frac{b}{2}} = 0$, there exists $T_1 > 0$ such that

$$C_1\langle T_1\rangle^{\frac{b}{2}} T_1 \leq \frac{1}{2\delta^\sigma}$$

and then we choose $T = \min\{1, T_1\}$. If $T = 1$, then

$$4C_1\langle T\rangle^{\frac{b}{2}}\max\{\|u_0\|_{L^{1,b}}, \|u_0\|_{L^{\infty,b}}\} = C_1 2^{\frac{b}{4}+2}\max\{\|u_0\|_{L^{1,b}}, \|u_0\|_{L^{\infty,b}}\} \leq \delta$$

and

$$C_1\langle T\rangle^{\frac{b}{2}}T \le C_1\langle T_1\rangle^{\frac{b}{2}}T_1 \le \frac{1}{2\delta^\sigma}.$$

If $T = T_1$, then

$$4C_1\langle T\rangle^{\frac{b}{2}}\max\{\|u_0\|_{L^{1,b}}, \|u_0\|_{L^{\infty,b}}\} \le C_1 2^{\frac{b}{4}+2}\max\{\|u_0\|_{L^{1,b}}, \|u_0\|_{L^{\infty,b}}\} \le \delta$$

and

$$C_1\langle T\rangle^{\frac{b}{2}}T = C_1\langle T_1\rangle^{\frac{b}{2}}T_1 \le \frac{1}{2\delta^\sigma}.$$

Our objective is to find $T > 0$ and $\delta > 0$ such that the map $\mathcal{M} : \mathcal{X} \mapsto \mathcal{X}$ is a contraction. Let $w_1, w_2 \in \mathcal{X}_\delta$. Then, $\|w_1(\cdot,t)\|_{L^{\infty,b}}, \|w_2(\cdot,t)\|_{L^{\infty,b}} \le \delta$ and

$$\begin{aligned}
&(\mathcal{M}(w_1) - \mathcal{M}(w_2))(x,t) \\
&\qquad = \int_0^t \mathcal{G}(t-\tau)\mathcal{B}[V(x)(|w_1(x,\tau)|^\sigma w_1(x,\tau) - |w_2(x,\tau)|^\sigma w_2(x,\tau))]d\tau \\
&\qquad := K(x,t)
\end{aligned}$$

for $x \in \mathbb{R}^n$ and $t \in [0,T]$. First, we consider $\|K(\cdot,t)\|_{L^{\infty,b}}$. By Lemma 2.6, Theorem 2.11, Theorem 3.1 and $a \le b\sigma$, we have

$$\begin{aligned}
\|K(\cdot,t)\|_{L^{\infty,b}} &\le \int_0^t \|\mathcal{G}(t-\tau)\mathcal{B}[V(\cdot)(|w_1(\cdot,\tau)|^\sigma w_1(\cdot,\tau) - |w_2(\cdot,\tau)|^\sigma w_2(\cdot,\tau))\|_{L^{\infty,b}}d\tau \\
&\le C\int_0^t \langle t-\tau\rangle^{\frac{b}{2}}\|\mathcal{B}[V(\cdot)(|w_1(\cdot,\tau)|^\sigma w_1(\cdot,\tau) - |w_2(\cdot,\tau)|^\sigma w_2(\cdot,\tau))\|_{L^{\infty,b}}d\tau \\
&\le C\int_0^t \langle t-\tau\rangle^{\frac{b}{2}}\|\langle\cdot\rangle^{b+b\sigma}(|w_1(\cdot,\tau)|^\sigma w_1(\cdot,\tau) - |w_2(\cdot,\tau)|^\sigma w_2(\cdot,\tau))\|_{L^\infty}d\tau \\
&\le C2^{\sigma+1}\int_0^t \langle t-\tau\rangle^{\frac{b}{2}}\|\langle\cdot\rangle^{b+b\sigma}(\max\{w_1,w_2\})^\sigma|w_1(\cdot,\tau) - w_2(\cdot,\tau)|\|_{L^\infty}d\tau \\
&\le C2^{\sigma+1}\langle T\rangle^{\frac{b}{2}}\delta^\sigma\int_0^t \|w_1(\cdot,\tau) - w_2(\cdot,\tau)\|_{L^{\infty,b}}d\tau. \qquad (4.5)
\end{aligned}$$

Next, we consider $\|K(\cdot,t)\|_{L^{1,b}}$. By Lemma 2.6, Theorem 2.11, Theorem 3.1, $a \le b\sigma$ and the Hölder inequality, we obtain

$$\|K(\cdot,t)\|_{L^{1,b}} \le \int_0^t \|\mathcal{G}(t-\tau)\mathcal{B}[V(\cdot)(|w_1(\cdot,\tau)|^\sigma w_1(\cdot,\tau) - |w_2(\cdot,\tau)|^\sigma w_2(\cdot,\tau))\|_{L^{1,b}}d\tau$$

$$
\begin{aligned}
&\le C\int_0^t \langle t-\tau\rangle^{\frac{b}{2}}\|\mathcal{B}[V(\cdot)(|w_1(\cdot,\tau)|^\sigma w_1(\cdot,\tau)-|w_2(\cdot,\tau)|^\sigma w_2(\cdot,\tau))\|_{L^{1,b}}d\tau\\
&\le C\int_0^t \langle t-\tau\rangle^{\frac{b}{2}}\|\langle\cdot\rangle^{b+b\sigma}(|w_1(\cdot,\tau)|^\sigma w_1(\cdot,\tau)-|w_2(\cdot,\tau)|^\sigma w_2(\cdot,\tau))\|_{L^1}d\tau\\
&\le C2^{\sigma+1}\int_0^t \langle t-\tau\rangle^{\frac{b}{2}}\|\langle\cdot\rangle^{b+b\sigma}(\max\{w_1,w_2\})^\sigma|w_1(\cdot,\tau)-w_2(\cdot,\tau)|\|_{L^1}d\tau\\
&\le C2^{\sigma+1}\langle T\rangle^{\frac{b}{2}}\delta^\sigma\int_0^t \|\langle\cdot\rangle^b\max\{w_1,w_2\}\|_{L^\infty}^\sigma\|w_1(\cdot,\tau)-w_2(\cdot,\tau)\|_{L^{1,b}}d\tau\\
&\le C2^{\sigma+1}\langle T\rangle^{\frac{b}{2}}\delta^\sigma\int_0^t \|w_1(\cdot,\tau)-w_2(\cdot,\tau)\|_{L^{1,b}}d\tau. \qquad (4.6)
\end{aligned}
$$

Combining (4.5) and (4.6) together, we obtain

$$
\begin{aligned}
\|(\mathcal{M}(w_1)-\mathcal{M}(w_2))\|_{\mathcal{X}} &= \sup_{t\in[0,T]}\left\{\|K(\cdot,t)\|_{L^{1,b}}+\|K(\cdot,t)\|_{L^{\infty,b}}\right\}\\
&\le C_2 2^{\sigma+1}\langle T\rangle^{\frac{b}{2}}\delta^\sigma\sup_{t\in[0,T]}\int_0^t\Big\{\|w_1(\cdot,\tau)-w_2(\cdot,\tau)\|_{L^{1,b}}\\
&\qquad\qquad +\|w_1(\cdot,\tau)-w_2(\cdot,\tau)\|_{L^{\infty,b}}\Big\}d\tau\\
&\le C_2 2^{\sigma+1}\langle T\rangle^{\frac{b}{2}}T\delta^\sigma\|w_1-w_2\|_{\mathcal{X}},
\end{aligned}
$$

where we choose $T>0$ satisfying $C_2 2^{\sigma+1}\langle T\rangle^{b/2}T\delta^\sigma<1$. Indeed, since $\lim_{t\to 0}\langle t\rangle^{b/2}t=0$, there exists $T_2>0$ such that $\langle T_2\rangle^{b/2}T_2\le 1/(C_2 2^{\sigma+2}\delta^\sigma)$. This implies

$$
C_2 2^{\sigma+1}\delta^\sigma\langle T_2\rangle^{\frac{b}{2}}T_2\le\frac{1}{2}<1.
$$

Therefore, we choose $T=\min\{1,T_1,T_2\}$ and it is easy to see that T satisfies all of previous conditions. Hence, $\mathcal{M}$ is a contraction mapping on $\mathcal{X}$. By the contraction mapping principle, there exists $u\in\mathcal{X}_\delta$ such that $\mathcal{M}(u)=u$. □

CHAPTER V

EXISTENCE OF GLOBAL SOLUTIONS

In this chapter, we prove the global existence of solutions to (1.1) by modifying the approach used in [4] and [9]. In the case V is a constant, this result found in [4] and [9].

Theorem 5.1 (The existence of global solutions). *Let $n \in \mathbb{N}$,*

$$0 \le a < \frac{\sigma}{\sigma+1}, \quad \sigma > \frac{3}{n} \quad \text{and} \quad b = \frac{a}{\sigma},$$

be constants and $V : \mathbb{R}^n \to \mathbb{R}$ be a given function. Assume that there exists a positive constant C such that for any $x \in \mathbb{R}^n$,

$$|V(x)| \le C|x|^a.$$

If $u_0 \in C(\mathbb{R}^n) \cap L^{1,b}(\mathbb{R}^n) \cap L^{\infty,b}(\mathbb{R}^n)$ with $\|u_0\|_{L^{1,b}} + \|u_0\|_{L^{\infty,b}}$ is sufficiently small, then the Cauchy problem (1.1) admits a unique global mild solution

$$u \in C([0,\infty); C(\mathbb{R}^n) \cap L^{1,b}(\mathbb{R}^n) \cap L^{\infty,b}(\mathbb{R}^n)).$$

Proof. Let $\mathcal{Z} = \{w \in C([0,\infty); C(\mathbb{R}^n) \cap L^{1,b}(\mathbb{R}^n) \cap L^{\infty,b}(\mathbb{R}^n)) : \|w\|_{\mathcal{Z}} < \infty\}$ with the mixed norm

$$\|w\|_{\mathcal{Z}} = \sup_{t>0} \left\{ \langle t \rangle^{-\frac{b}{2}} \|w(\cdot,t)\|_{L^{1,b}} + \langle t \rangle^{\gamma} \|w(\cdot,t)\|_{L^{\infty,b}} \right\}$$

for all $w \in \mathcal{Z}$ where $\gamma = n/2 - b/2$. Let $\delta > 0$ be to specified. We set

$$\mathcal{Z}_\delta = \{w \in \mathcal{Z} : \|w\|_{\mathcal{Z}} \le \delta\}$$

and the operator $\mathcal{M}$ on $\mathcal{Z}$ by

$$\mathcal{M}(w)(x,t) = \mathcal{G}(t)u_0(x) + \int_0^t \mathcal{G}(t-\tau)\mathcal{B}[V(x)|w(x,\tau)|^\sigma w(x,\tau)]d\tau$$

for all $w \in \mathcal{Z}$. First, we show that $\mathcal{M}$ maps $\mathcal{Z}_\delta$ to itself. Let $w \in \mathcal{Z}_\delta$. Then,

$$\langle t\rangle^\gamma \|w(\cdot,t)\|_{L^{\infty,b}}, \langle t\rangle^{-\frac{b}{2}} \|w(\cdot,t)\|_{L^{1,b}} \le \delta.$$

By Theorem 2.11, Lemma 3.4 where $q = 1, k = b = a/\sigma$ and the Hölder inequality, we obtain

$$\begin{aligned}
\|\mathcal{B}[V(\cdot)|w(\cdot,t)|^\sigma w(\cdot,t)]\|_{L^{1,b}} &\le C\|\langle\cdot\rangle^{b+b\sigma} w(\cdot,t)^{\sigma+1}\|_{L^1} \\
&\le C\|w(\cdot,t)\|^\sigma_{L^{\infty,b}} \|w(\cdot,t)\|_{L^{1,b}} \\
&\le C\delta^{\sigma+1}\langle t\rangle^{-\gamma\sigma+\frac{b}{2}} \\
&= C\delta^{\sigma+1}\langle t\rangle^{-\gamma(\sigma+1)+\frac{n}{2}}, \qquad (5.1)\\
\|\mathcal{B}[V(\cdot)|w(\cdot,t)|^\sigma w(\cdot,t)]\|_{L^{\infty,b}} &\le C\|\langle\cdot\rangle^{b+b\sigma} w(\cdot,t)^{\sigma+1}\|_{L^\infty} \le C\|w(\cdot,t)\|^{\sigma+1}_{L^{\infty,b}} \\
&\le C\delta^{\sigma+1}\langle t\rangle^{-\gamma(\sigma+1)} \qquad (5.2)
\end{aligned}$$

and

$$\begin{aligned}
\|\mathcal{B}[V(\cdot)|w(\cdot,t)|^\sigma w(\cdot,t)]\|_{L^1} &\le C\langle t\rangle^{-\frac{b}{2}} \|\mathcal{B}[V(\cdot)|w(\cdot,t)|^\sigma w(\cdot,t)]\|_{L^{1,b}} \\
&\quad + C\langle t\rangle^\gamma \|\mathcal{B}[V(\cdot)|w(\cdot,t)|^\sigma w(\cdot,t)]\|_{L^{\infty,b}} \\
&\le C\delta^{\sigma+1}(\langle t\rangle^{-\frac{b}{2}-\gamma\sigma+\frac{b}{2}} + \langle t\rangle^{\gamma-\gamma(\sigma+1)}) \\
&\le C\delta^{\sigma+1}\langle t\rangle^{-\gamma\sigma}. \qquad (5.3)
\end{aligned}$$

Next, from the conditions

$$0 \le a < \frac{\sigma}{\sigma+1} \quad \text{and} \quad \sigma > \frac{3}{n},$$

we have $b < 1/(\sigma+1)$, $\quad b/2 \le b \le a+b < 1$ and

$$\gamma\sigma = \frac{n\sigma}{2} - \frac{b\sigma}{2} > \frac{n\sigma}{2} - \frac{\sigma}{2(\sigma+1)} > \frac{n\sigma-1}{2} > 1.$$

First, we consider $\|\mathcal{M}(w)(\cdot,t)\|_{L^{\infty,b}}$. By the triangle inequality, we have

$$\|\mathcal{M}(w)(\cdot,t)\|_{L^{\infty,b}} \leq K_1 + K_2 + K_3,$$

where

$$\begin{aligned}
K_1 &= \|\mathcal{G}(t)u_0(\cdot)\|_{L^{\infty,b}},\\
K_2 &= \int_0^{\frac{t}{2}} \|\mathcal{G}(t-\tau)\mathcal{B}[V(\cdot)|w(\cdot,\tau)|^\sigma w(\cdot,\tau)]\|_{L^{\infty,b}} d\tau,
\end{aligned}$$

and

$$K_3 = \int_{\frac{t}{2}}^{t} \|\mathcal{G}(t-\tau)\mathcal{B}[V(\cdot)|w(\cdot,\tau)|^\sigma w(\cdot,\tau)]\|_{L^{\infty,b}} d\tau.$$

We estimate K_1 by using Theorem 3.3 where $p = 1$ and $q = \infty$, we have

$$\begin{aligned}
K_1 &\leq Ce^{-\frac{\alpha t}{2}}\|u_0\|_{L^{\infty,b}} + C\langle t\rangle^{-\frac{n}{2}}\|u_0\|_{L^{1,b}} + C\langle t\rangle^{-\gamma}\|u_0\|_{L^1}\\
&\leq C_1\langle t\rangle^{-\gamma}(\|u_0\|_{L^{\infty,b}} + \|u_0\|_{L^{1,b}}).
\end{aligned}$$

Next, we estimate K_2. By Theorem 3.3 with $p = 1, q = \infty$ and estimations (5.1), (5.2) and (5.3), we obtain

$$\begin{aligned}
K_2 &\leq C\int_0^{\frac{t}{2}} (e^{-\frac{\alpha}{2}(t-\tau)}\|\mathcal{B}[V(\cdot)|w(\cdot,\tau)|^\sigma w(\cdot,\tau)]\|_{L^{\infty,b}}\\
&\qquad + \langle t-\tau\rangle^{-\frac{n}{2}}\|\mathcal{B}[V(\cdot)|w(\cdot,\tau)|^\sigma w(\cdot,\tau)]\|_{L^{1,b}}\\
&\qquad + \langle t-\tau\rangle^{-\gamma}\|\mathcal{B}[V(\cdot)|w(\cdot,\tau)|^\sigma w(\cdot,\tau)]\|_{L^1}) d\tau\\
&\leq C\delta^{\sigma+1}\int_0^{\frac{t}{2}} (\langle t-\tau\rangle^{-\frac{n}{2}}\langle\tau\rangle^{-\gamma(\sigma+1)} + \langle t-\tau\rangle^{-\frac{n}{2}}\langle\tau\rangle^{-\gamma(\sigma+1)+\frac{n}{2}}\\
&\qquad + \langle t-\tau\rangle^{-\gamma}\langle\tau\rangle^{-\gamma\sigma}) d\tau\\
&\leq C\delta^{\sigma+1}\int_0^{\frac{t}{2}} (\langle t-\tau\rangle^{-\frac{n}{2}}\langle\tau\rangle^{-\gamma(\sigma+1)+\frac{n}{2}} + \langle t-\tau\rangle^{-\gamma}\langle\tau\rangle^{-\gamma\sigma}) d\tau\\
&\leq C\delta^{\sigma+1}\langle t\rangle^{-\gamma}\int_0^{\frac{t}{2}} (\langle t-\tau\rangle^{-\frac{b}{2}}\langle\tau\rangle^{-\gamma(\sigma+1)+\frac{n}{2}} + \langle\tau\rangle^{-\gamma\sigma}) d\tau\\
&\leq C\delta^{\sigma+1}\langle t\rangle^{-\gamma}\int_0^{\frac{t}{2}} \langle\tau\rangle^{-\gamma\sigma} d\tau
\end{aligned}$$

$$\le C_2\delta^{\sigma+1}\langle t\rangle^{-\gamma}.$$

For the term K_3, if $b>0$, by Theorem 2.11 and Lemma 3.4 with $q=n/b, k=b=a/\sigma$ and estimations (5.1), (5.2) and (5.3), we obtain

$$\begin{aligned}
&\|\mathcal{B}[V(\cdot)|w(\cdot,t)|^\sigma w(\cdot,t)]\|_{L^{\frac{n}{b}}} \\
&\quad\le \langle t\rangle^{-\frac{b}{2}}\|\mathcal{B}[V(\cdot)|w(\cdot,t)|^\sigma w(\cdot,t)]\|_{L^{1,b}}^{\frac{b}{n}}\|\mathcal{B}[V(\cdot)|w(\cdot,t)|^\sigma w(\cdot,t)]\|_{L^{\infty,b}}^{1-\frac{b}{n}} \\
&\qquad\quad + C(1+\ln\langle t\rangle)^{\frac{b}{n}}\|\mathcal{B}[V(\cdot)|w(\cdot,t)|^\sigma w(\cdot,t)]\|_{L^{\infty,b}} \\
&\quad\le C\delta^{\sigma+1}\langle t\rangle^{-\gamma(\sigma+1)} + C\delta^{\sigma+1}(1+\ln\langle t\rangle)^{\frac{b}{n}}\langle t\rangle^{-\gamma(\sigma+1)} \\
&\quad\le C\delta^{\sigma+1}\langle \ln\langle t\rangle\rangle^{\frac{b}{n}}\langle t\rangle^{-\gamma(\sigma+1)}. \qquad (5.4)
\end{aligned}$$

Moreover, we have the estimation for the weighted $L^{n/b}$ norm as shown below

$$\begin{aligned}
\|\mathcal{B}[V(\cdot)|w(\cdot,t)|^\sigma w(\cdot,t)]\|_{L^{\frac{n}{b},b}} &\le C\|\langle\cdot\rangle^{b+b\sigma}w(\cdot,t)^{\sigma+1}\|_{L^{\frac{n}{b}}} \\
&= C\left(\int_{\mathbb{R}^n}\langle x\rangle^{n(\sigma+1)-b}|w(x,t)|^{\frac{n}{b}(\sigma+1)-1}\langle x\rangle^b|w(x,t)|dx\right)^{\frac{b}{n}} \\
&\le C\|\langle\cdot\rangle^{b(\frac{n}{b}(\sigma+1)-1)}w(\cdot,t)^{\frac{n}{b}(\sigma+1)-1}\|_{L^\infty}^{\frac{b}{n}}\left(\int_{\mathbb{R}^n}\langle x\rangle^b|w(x,t)|dx\right)^{\frac{b}{n}} \\
&= C\|w(\cdot,t)\|_{L^{\infty,b}}^{\sigma+1-\frac{b}{n}}\|w(\cdot,t)\|_{L^{1,b}}^{\frac{b}{n}} \\
&\le C\delta^{\sigma+1}\langle t\rangle^{-\gamma(\sigma+1-\frac{b}{n})+\frac{b^2}{2n}} \\
&= C\delta^{\sigma+1}\langle t\rangle^{-\gamma(\sigma+1)+\frac{b}{2}} \qquad (5.5)
\end{aligned}$$

since $\gamma=n/2-b/2$. If $b=0$, then we use $q=\infty$. Thus, we have

$$\begin{aligned}
\|\mathcal{B}[V(\cdot)|w(\cdot,t)|^\sigma w(\cdot,t)]\|_{L^\infty} &\le \|\mathcal{B}[V(\cdot)|w(\cdot,t)|^\sigma w(\cdot,t)]\|_{L^{\infty,b}} \\
&\le C\delta^{\sigma+1}\langle t\rangle^{-\gamma(\sigma+1)} = C\delta^{\sigma+1}\langle\ln\langle t\rangle\rangle^{\frac{b}{n}}\langle t\rangle^{-\gamma(\sigma+1)}
\end{aligned}$$

and

$$\|\mathcal{B}[V(\cdot)|w(\cdot,t)|^\sigma w(\cdot,t)]\|_{L^{\infty,b}} \le C\delta^{\sigma+1}\langle t\rangle^{-\gamma(\sigma+1)} = C\delta^{\sigma+1}\langle t\rangle^{-\gamma(\sigma+1)+\frac{b}{2}}.$$

By Theorem 3.3 with $p = n/b, q = \infty$ and estimations (5.2), (5.4) and (5.5), we have the estimation for K_3 as follow.

$$
\begin{aligned}
K_3 \le C \int_{\frac{t}{2}}^{t} (e^{-\frac{\alpha(t-\tau)}{2}} &\|\mathcal{B}[V(\cdot)|w(\cdot,\tau)|^\sigma w(\cdot,\tau)]\|_{L^{\infty,b}} \\
&+ \langle t-\tau\rangle^{-\frac{b}{2}} \|\mathcal{B}[V(\cdot)|w(\cdot,\tau)|^\sigma w(\cdot,\tau)]\|_{L^{\frac{n}{b},b}} \\
&+ \|\mathcal{B}[V(\cdot)|w(\cdot,\tau)|^\sigma w(\cdot,\tau)]\|_{L^{\frac{n}{b}}}) d\tau \\
\le C\delta^{\sigma+1} \int_{\frac{t}{2}}^{t} (&\langle t-\tau\rangle^{-\frac{b}{2}} \langle\tau\rangle^{-\gamma(\sigma+1)} + \langle t-\tau\rangle^{-\frac{b}{2}} \langle\tau\rangle^{-\gamma(\sigma+1)+\frac{b}{2}} \\
&+ \langle \ln\langle\tau\rangle\rangle^{\frac{b}{n}} \langle\tau\rangle^{-\gamma(\sigma+1)}) d\tau \\
\le C\delta^{\sigma+1} \Big(\int_{\frac{t}{2}}^{t} &\langle t-\tau\rangle^{-\frac{b}{2}} \langle\tau\rangle^{-\gamma(\sigma+1)+\frac{b}{2}} d\tau + \int_{\frac{t}{2}}^{t} \langle \ln\langle\tau\rangle\rangle^{\frac{b}{n}} \langle\tau\rangle^{-\gamma(\sigma+1)} d\tau \Big) \\
\le C\delta^{\sigma+1} \Big(\int_{\frac{t}{2}}^{t} &\langle t-\tau\rangle^{-\frac{b}{2}} \langle\tau\rangle^{-\gamma(\sigma+1)+\frac{b}{2}} d\tau + \langle t\rangle^{-\gamma} \Big) \\
\le C\delta^{\sigma+1} \big(&\langle t\rangle^{-\gamma(\sigma+1)+\frac{b}{2}} \int_{\frac{t}{2}}^{t} \langle t-\tau\rangle^{-\frac{b}{2}} d\tau + \langle t\rangle^{-\gamma} \big) \\
\le C\delta^{\sigma+1} \langle t\rangle^{-\gamma} &\big(\langle t\rangle^{-\gamma\sigma+\frac{b}{2}} \int_{0}^{\frac{t}{2}} \langle s\rangle^{-\frac{b}{2}} ds + 1 \big) \qquad (5.6)
\end{aligned}
$$

since $t/2 \le \tau \le t$ implies $\langle t\rangle/2 \le \langle\tau\rangle \le \langle t\rangle$ and there is a constant $C > 0$ such that

$$\int_{\frac{t}{2}}^{t} \langle \ln\langle\tau\rangle\rangle^{\frac{b}{n}} \langle\tau\rangle^{-\gamma(\sigma+1)} d\tau \le C\langle t\rangle^{-\gamma}.$$

for all $t > 0$. Indeed,

$$\int_{\frac{t}{2}}^{t} \langle \ln\langle\tau\rangle\rangle^{\frac{b}{n}} \langle\tau\rangle^{-\gamma(\sigma+1)} d\tau \le C\langle t\rangle^{-\gamma} \int_{\frac{t}{2}}^{t} \langle \ln\langle\tau\rangle\rangle^{\frac{b}{n}} \langle\tau\rangle^{-\gamma\sigma} d\tau$$

and $\int_{t/2}^{t} \langle \ln\langle\tau\rangle\rangle^{b/n} \langle\tau\rangle^{-\gamma\sigma} d\tau$ is bounded on $[0,\infty)$ since

$$0 < \langle \ln\langle\tau\rangle\rangle^{\frac{b}{n}} \langle\tau\rangle^{-\gamma\sigma} \le C(\ln\sqrt{2}\tau)^{\frac{b}{n}} \tau^{-\gamma\sigma}$$

for $\tau \ge M$ where M is sufficiently large. Next, we choose $0 < k < n(\gamma\sigma - 1)/b$, thus, we have

$$\frac{(\ln\sqrt{2}\tau)^{\frac{b}{n}} \tau^{-\gamma\sigma}}{\tau^{-\gamma\sigma+\frac{kb}{n}}} = \frac{(\ln\sqrt{2}\tau)^{\frac{b}{n}}}{\tau^{\frac{kb}{n}}} \to 0 \quad \text{as} \quad \tau \to \infty$$

and $\int_M^\infty \tau^{-\gamma\sigma+kb/n}d\tau < \infty$. This implies

$$\int_0^\infty \langle \ln\langle\tau\rangle\rangle^{\frac{b}{n}} \langle\tau\rangle^{-\gamma\sigma} d\tau < \infty,$$

and we substitutes $s = t - \tau$, in (5.6). Note that $0 \le b < 1$, thus,

$$\int_0^{\frac{t}{2}} \langle s\rangle^{-\frac{b}{2}} ds \le \int_0^{\frac{t}{2}} s^{-\frac{b}{2}} ds = \frac{t^{1-\frac{b}{2}}}{2^{1-\frac{b}{2}}(1-\frac{b}{2})} \le C\langle t\rangle^{1-\frac{b}{2}}$$

and we have

$$K_3 \le C\delta^{\sigma+1}\langle t\rangle^{-\gamma}(\langle t\rangle^{-\gamma\sigma+1} + 1) \le C_3\delta^{\sigma+1}\langle t\rangle^{-\gamma}.$$

Next, we consider $\|\mathcal{M}(w)(\cdot,t)\|_{L^{1,b}}$. By using the triangle inequality, we obtain

$$\|\mathcal{M}(w)(\cdot,t)\|_{L^{1,b}} \le J_1 + J_2,$$

where $J_1 = \|\mathcal{G}(t)u_0(\cdot)\|_{L^{1,b}}$ and $J_2 = \int_0^t \|\mathcal{G}(t-\tau)\mathcal{B}[V(\cdot)|w(\cdot,\tau)|^\sigma w(\cdot,\tau)]\|_{L^{1,b}} d\tau$. By Theorem 3.3 with $p = q = 1$, we get

$$J_1 \le Ce^{-\frac{\alpha t}{2}}\|u_0\|_{L^{1,b}} + C\|u_0\|_{L^{1,b}} + C\langle t\rangle^{\frac{b}{2}}\|u_0\|_{L^1} \le C_4\langle t\rangle^{\frac{b}{2}}\|u_0\|_{L^{1,b}},$$

where C_4 is a positive constant. Next, we estimate J_2 by using Theorem 3.3 with $p = q = 1$ and estimations (5.1) and (5.3), we obtain

$$\begin{aligned} J_2 \le C &\int_0^t (e^{-\frac{\alpha(t-\tau)}{2}}\|\mathcal{B}[V(\cdot)|w(\cdot,\tau)|^\sigma w(\cdot,\tau)]\|_{L^{1,b}} \\ &+ \|\mathcal{B}[V(\cdot)|w(\cdot,\tau)|^\sigma w(\cdot,\tau)]\|_{L^{1,b}} + \langle t-\tau\rangle^{\frac{b}{2}}\|\mathcal{B}[V(\cdot)|w(\cdot,\tau)|^\sigma w(\cdot,\tau)]\|_{L^1}) d\tau \\ &\le C\delta^{\sigma+1}\int_0^t (\langle\tau\rangle^{-\gamma(\sigma+1)+\frac{n}{2}} + \langle t-\tau\rangle^{\frac{b}{2}}\langle\tau\rangle^{-\gamma\sigma}) d\tau \\ &\le C\delta^{\sigma+1}\langle t\rangle^{\frac{b}{2}}\int_0^t \langle\tau\rangle^{-\gamma\sigma} d\tau \\ &\le C_5\delta^{\sigma+1}\langle t\rangle^{\frac{b}{2}} \end{aligned}$$

since $0 \le \tau \le t$ implies $1 \le \langle\tau\rangle \le \langle t\rangle$ and $1 \le \langle t-\tau\rangle \le \langle t\rangle$. Moreover,

$$\int_0^t \langle\tau\rangle^{-\gamma\sigma} d\tau < \infty.$$

Thus, we choose $\delta > 0$ be such that $K\delta^\sigma \le 1/6$ and u_0 has a sufficiently small norm, i.e.,

$$\|u_0\|_{L^{\infty,b}} + \|u_0\|_{L^{1,b}} \le \frac{\delta}{6L},$$

where $K = \max\{C_1, C_2, C_3, C_4, C_5\}$ and $L = \max\{C_1, C_4\}$. Therefore, we combine K_1, K_2 and K_3 together, we have

$$\|\mathcal{M}(w)(\cdot,t)\|_{L^{\infty,b}} \le \frac{\delta}{6}\langle t\rangle^{-\gamma} + \frac{\delta}{6}\langle t\rangle^{-\gamma} + \frac{\delta}{6}\langle t\rangle^{-\gamma} \le \frac{\delta}{2}\langle t\rangle^{-\gamma}$$

for all $t > 0$. Similarly, we have

$$\|\mathcal{M}(w)(\cdot,t)\|_{L^{1,b}} \le \frac{\delta}{6}\langle t\rangle^{\frac{b}{2}} + \frac{\delta}{6}\langle t\rangle^{\frac{b}{2}} \le \frac{\delta}{4}\langle t\rangle^{\frac{b}{2}} + \frac{\delta}{4}\langle t\rangle^{\frac{b}{2}} \le \frac{\delta}{2}\langle t\rangle^{\frac{b}{2}}.$$

Thus,

$$\|\mathcal{M}(w)\|_{\mathcal{Z}} = \sup_{t>0}\left\{\langle t\rangle^{-\frac{b}{2}}\|\mathcal{M}(w)(\cdot,t)\|_{L^{1,b}} + \langle t\rangle^{\gamma}\|\mathcal{M}(w)(\cdot,t)\|_{L^{\infty,b}}\right\} \le \delta$$

and this implies that $\mathcal{M}(w) \in \mathcal{Z}_\delta$.

Next, we find $\delta > 0$ such that $\mathcal{M}$ is a contraction on $\mathcal{Z}_\delta$. Let $w_1, w_2 \in \mathcal{Z}_\delta$. Then, $\langle t\rangle^{-b/2}\|w_i(\cdot,t)\|_{L^{1,b}}, \langle t\rangle^{\gamma}\|w_i(\cdot,t)\|_{L^{\infty,b}} \le \delta$ for $i \in \{1,2\}$ and

$$\mathcal{K}(x,t) := (\mathcal{M}(w_1) - \mathcal{M}(w_2))(x,t) = \int_0^t \mathcal{G}(t-\tau)\mathcal{P}(x,\tau)d\tau,$$

where

$$\mathcal{P}(x,\tau) = \mathcal{B}[V(x)(|w_1(x,\tau)|^\sigma w_1(x,\tau) - |w_2(x,\tau)|^\sigma w_2(x,\tau))].$$

By Lemma 2.6, Theorem 2.11, $a = b\sigma$ and $\gamma = n/2 - b/2 > 0$, we have the estimation as follow.

$$\begin{aligned}
\|\mathcal{P}(\cdot,\tau)\|_{L^{1,b}} &\le C\|\langle\cdot\rangle^{b\sigma+b}(|w_1(\cdot,\tau)|^\sigma w_1(\cdot,\tau) - |w_2(\cdot,\tau)|^\sigma w_2(\cdot,\tau))\|_{L^1} \\
&\le C\|\langle\cdot\rangle^{b\sigma+b}(\max\{w_1,w_2\})^\sigma|w_1(\cdot,\tau) - w_2(\cdot,\tau)|\|_{L^1} \\
&\le C\|\langle\cdot\rangle^{b}\max\{w_1,w_2\}\|_{L^\infty}^\sigma\|w_1(\cdot,\tau) - w_2(\cdot,\tau)\|_{L^{1,b}}
\end{aligned}$$

$$
\begin{aligned}
&\leq C\delta^{\sigma}\langle\tau\rangle^{-\gamma\sigma}\|w_1(\cdot,\tau)-w_2(\cdot,\tau)\|_{L^{1,b}}\\
&\leq C\delta^{\sigma}\langle\tau\rangle^{-\gamma\sigma+\frac{b}{2}}\|w_1-w_2\|_{\mathcal{Z}}\\
&\leq C\delta^{\sigma}\langle\tau\rangle^{-\gamma\sigma+\frac{n}{2}}\|w_1-w_2\|_{\mathcal{Z}},\\
\|\mathcal{P}(\cdot,\tau)\|_{L^{\infty,b}} &\leq C\|\langle\cdot\rangle^{b\sigma+b}(|w_1(\cdot,\tau)|^{\sigma}w_1(\cdot,\tau)-|w_2(\cdot,\tau)|^{\sigma}w_2(\cdot,\tau))\|_{L^{\infty}}\\
&\leq C\|\langle\cdot\rangle^{b\sigma+b}(\max\{w_1,w_2\})^{\sigma}|w_1(\cdot,\tau)-w_2(\cdot,\tau)|\|_{L^{\infty}}\\
&\leq C\|\langle\cdot\rangle^{b}\max\{w_1,w_2\}\|_{L^{\infty}}^{\sigma}\|w_1(\cdot,\tau)-w_2(\cdot,\tau)\|_{L^{\infty,b}}\\
&\leq C\delta^{\sigma}\langle\tau\rangle^{-\gamma\sigma}\|w_1(\cdot,\tau)-w_2(\cdot,\tau)\|_{L^{\infty,b}}\\
&\leq C\delta^{\sigma}\langle\tau\rangle^{-\gamma(\sigma+1)}\|w_1-w_2\|_{\mathcal{Z}},
\end{aligned}
$$

and

$$
\begin{aligned}
\|\mathcal{P}(\cdot,\tau)\|_{L^{1}} &\leq C\langle\tau\rangle^{-\frac{b}{2}}\|\mathcal{P}(\cdot,\tau)\|_{L^{1,b}}+C\langle\tau\rangle^{\gamma}\|\mathcal{P}(\cdot,\tau)\|_{L^{\infty,b}}\\
&\leq C\delta^{\sigma}\langle\tau\rangle^{-\gamma\sigma}\|w_1-w_2\|_{\mathcal{Z}},\\
\|\mathcal{P}(\cdot,\tau)\|_{L^{\frac{b}{n}}} &\leq \langle\tau\rangle^{-\frac{b}{2}}\|\mathcal{P}(\cdot,\tau)\|_{L^{1,b}}^{\frac{b}{n}}\|\mathcal{P}(\cdot,\tau)\|_{L^{\infty,b}}^{1-\frac{b}{n}}+C(1+\ln\langle\tau\rangle)^{\frac{b}{n}}\|\mathcal{P}(\cdot,\tau)\|_{L^{\infty,b}}\\
&\leq C\delta^{\sigma}\langle\tau\rangle^{-\gamma\sigma}(\langle\tau\rangle^{-\frac{b}{2}}\|w_1(\cdot,\tau)-w_2(\cdot,\tau)\|_{L^{1,b}}^{\frac{b}{n}}\|w_1(\cdot,\tau)-w_2(\cdot,\tau)\|_{L^{\infty,b}}^{1-\frac{b}{n}}\\
&\qquad+C(1+\ln\langle\tau\rangle)^{\frac{b}{n}}\|w_1(\cdot,\tau)-w_2(\cdot,\tau)\|_{L^{\infty,b}})\\
&\leq C\delta^{\sigma}\langle\tau\rangle^{-\gamma(\sigma+1)}\langle\ln\langle\tau\rangle\rangle^{\frac{b}{n}}\|w_1-w_2\|_{\mathcal{Z}}.
\end{aligned}
$$

Consider $\|\mathcal{K}(\cdot,t)\|_{L^{\infty,b}}$, By the triangle inequality, we split $\|\mathcal{K}(\cdot,t)\|_{L^{\infty,b}}$ into 2 parts as below

$$
\|\mathcal{K}(\cdot,t)\|_{L^{\infty,b}} \leq \int_0^{\frac{t}{2}}\|\mathcal{G}(t-\tau)\mathcal{P}(\cdot,\tau)\|_{L^{\infty,b}}d\tau+\int_{\frac{t}{2}}^{t}\|\mathcal{G}(t-\tau)\mathcal{P}(\cdot,\tau)\|_{L^{\infty,b}}d\tau.
$$

As the calculation of K_2, K_3 and J_2, we have

$$
\left.\begin{aligned}
&\int_0^{\frac{t}{2}}\|\mathcal{G}(t-\tau)\mathcal{P}(\cdot,\tau)\|_{L^{\infty,b}}d\tau\\
&\int_{\frac{t}{2}}^{t}\|\mathcal{G}(t-\tau)\mathcal{P}(\cdot,\tau)\|_{L^{\infty,b}}d\tau
\end{aligned}\right\} \leq C_6\delta^{\sigma}\langle t\rangle^{-\gamma}\|w_1-w_2\|_{\mathcal{Z}}
$$

and

$$\int_0^t \|\mathcal{G}(t-\tau)\mathcal{P}(\cdot,\tau)\|_{L^{1,b}} d\tau \le C_7 \delta^\sigma \langle t \rangle^{\frac{b}{2}} \|w_1 - w_2\|_{\mathcal{Z}}.$$

where C_6 and C_7 are positive constants. Thus, we obtain

$$\begin{aligned} \|\mathcal{M}(w_1) - \mathcal{M}(w_2)\|_{\mathcal{Z}} &= \sup_{t>0} \left\{ \langle t \rangle^{-\frac{b}{2}} \|\mathcal{K}(\cdot,t)\|_{L^{1,b}} + \langle t \rangle^{\gamma} \|\mathcal{K}(\cdot,t)\|_{L^{\infty,b}} \right\} \\ &\le M\delta^\sigma \|w_1 - w_2\|_{\mathcal{Z}}, \end{aligned}$$

where $M = \max\{C_6, C_7\}$. Therefore, we choose $\delta > 0$ be such that $M\delta^\sigma \le 1/2$. Hence, $\mathcal{M}$ is a contraction mapping on $\mathcal{Z}_\delta$. By the contraction mapping principle, there exists $u \in \mathcal{Z}_\delta$ such that $\mathcal{M}(u) = u$ □

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