การแยกเชิงแฮมิลตันของไฮเพอร์กราฟบางชนิด

นางสาวศันสนีย์ เติมธนาสมบัติ

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรดุษฎีบัณฑิต
สาขาวิชาคณิตศาสตร์ ภาควิชาคณิตศาสตร์และวิทยาการคอมพิวเตอร์
คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย
ปีการศึกษา 2557

HAMILTONIAN DECOMPOSITIONS OF SOME HYPERGRAPHS

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A Dissertation Submitted in Partial Fulfillment of the Requirements
for the Degree of Doctor of Philosophy Program in Mathematics
Department of Mathematics and Computer Science
Faculty of Science
Chulalongkorn University
Academic Year 2014

Thesis Title	HAMILTONIAN DECOMPOSITIONS OF SOME HYPERGRAPHS
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ศันสนีย์ เติมธนาสมบัติ : การแยกเชิงแฮมิลตันของไฮเพอร์กราฟบางชนิด (HAMILTONIAN DECOMPOSITIONS OF SOME HYPERGRAPHS) อ. ที่ปรึกษาวิทยานิพนธ์หลัก : ดร.รตินันท์ บุญเคลือบ, อ. ที่ปรึกษาวิทยานิพนธ์ร่วม : ดร.ศิริรัตน์ สิงหันต์ , 43 หน้า.

วิทยานิพนธ์นี้ศึกษาการแยกเชิงแฮมิลตันของไฮเพอร์กราฟสองชนิด โดยพบว่าสามารถหาการแยกเชิงแฮมิลตันสำหรับไฮเพอร์กราฟปริซึมบนไฮเพอร์กราฟ 3-เอกรูปบริบูรณ์ $\mathrm{Prism}(K_n^{(3)})$ สำหรับ $n \in \{4,5,8\}$ และการแยกเชิงแฮมิลตันสำหรับไฮเพอร์กราฟ 3-เอกรูป สามส่วนบริบูรณ์ $K_{m,m,m}^{(3)}$ สำหรับทุกจำนวนเต็มบวก m ที่ $3|m$

ภาควิชา คณิตศาสตร์และวิทยาการคอมพิวเตอร์	ลายมือชื่อนิสิต ....
สาขาวิชา ....คณิตศาสตร์....................	ลายมือชื่อ อ.ที่ปรึกษาหลัก ....
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5373864623 : MAJOR MATHEMATICS

KEYWORDS : HYPERGRAPH / HAMILTONIAN CYCLE / HAMILTONIAN DECOMPOSITION / PRISM / TRIPARTITE

SANSANEE TERMTANASOMBAT : HAMILTONIAN DECOMPOSITIONS OF SOME HYPERGRAPHS. ADVISOR : RATINAN BOONKLURB, Ph.D., CO-ADVISOR : SIRIRAT SINGHUN, Ph.D., 43 pp.

This dissertation is involved in Hamiltonian decomposition of two families of hypergraphs. We found the Hamiltonian decompositions of the prism over a complete 3-uniform hypergraph Prism($K_n^{(3)}$) for $n \in \{4, 5, 8\}$ and the Hamiltonian decompositions of the complete tripartite 3-uniform hypergraph $K_{m,m,m}^{(3)}$ for all positive integer m such that $3 \mid m$.

Department : Mathematics and Computer Science	Student's Signature : ....
Field of Study : ....Mathematics...........	Advisor's Signature : ....
Academic Year : ....2014..................	Co-advisor's Signature : ....

ACKNOWLEDGEMENTS

I would like to thank my beloved advisors, Dr. Ratinan Boonklurb and Dr. Sirirat Singhun, who take care of the tidiness of this dissertation and my PhD life. Thank to Assistant Professor Chariya Uiyyasathian for her courses of graph theory and design theory which make me fascinate and be rigorous in this field of mathematics. Thank to many opportunities during studying at this department where I meet many expert mathematics professors and students around the world. Also a chance in CIMPA research school on Graphs, Codes and Designs at Ramkhamhaeng University with Professor Chis Rodger helps me solve some cases in this thesis. Thank to Science Achievement Scholarship of Thailand (SAST) for financial support.

Moreover, I might not complete the PhD without my precious friends: Korsuk Sirinukunwattana, Sutheerak Rerkdee, Natee Valunchapuk, Tanawat Wichianpaisarn, Veerapas Boonthong, Worawit Tepsaen, Supanut Chaidee and Teerapol Sukonvimolmarn who encourage me when I am in trouble. Lastly, the most important people in my life to thank is my family for their support whatever I do great or bad.

Thank to everyone who secretly supports me in the background. Thank to any readers who interest this work and develop this knowledge for the future of mathematics.

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CHAPTER I

INTRODUCTION

1.1 Definitions and Notations

Definition 1.1. [2] A *hypergraph* $\mathcal{H} = (V, \mathcal{E})$ consists of a nonempty finite set V of *vertices* with a family $\mathcal{E}$ of subsets of V, called *(hyper)edges*. The notation $V(\mathcal{H})$ and $\mathcal{E}(\mathcal{H})$ denote the set of vertices and the set of edges of a hypergraph $\mathcal{H}$, respectively. If each edge of $\mathcal{H}$ has size k, we say that $\mathcal{H}$ is a *k-uniform hypergraph*.

Definition 1.2. [2] The *complete k-uniform hypergraph* on n vertices, denoted by $K_n^{(k)}$, has all k-subsets of an n-set of vertices as edges.

A 2-uniform hypergraph is simply the ordinary graph and the hypergraph $K_n^{(2)}$ is the complete graph K_n.

Definition 1.3 (Katona [6])**.** A *cycle* of length ℓ in a k-uniform hypergraph is a cyclic ordering of its ℓ vertices such that each consecutive k-tuple of vertices is an edge. In other words, let $C = (v_0, v_1, \ldots, v_{\ell-1})$ be a cyclic ordering of ℓ vertices in a k-uniform hypergraph $\mathcal{H}$. Then C is a cycle of length ℓ in $\mathcal{H}$ if $\{v_i, v_{i+1}, v_{i+2}, \ldots, v_{i+k-1}\} \in \mathcal{E}(\mathcal{H})$ for all $i \in \{0, 1, \ldots, \ell-1\}$, where the addition of indices is calculated under the integer modulo ℓ.

A *path* of length $\ell - k + 1$ in a k-uniform hypergraph is a noncyclic ordering of its ℓ vertices such that each consecutive k-tuple of vertices is an edge. That is, if $P = [v_0, v_1, \ldots, v_{\ell-1}]$ is a path of length $\ell - k + 1$ in a k-uniform hypergraph $\mathcal{H}$, then $\{v_i, v_{i+1}, v_{i+2}, \ldots, v_{i+k-1}\} \in \mathcal{E}(\mathcal{H})$ for all $i \in \{0, 1, \ldots, \ell - k\}$.

If $\mathcal{H}$ has n vertices, a cycle of length n is called a *Hamiltonian cycle* and a path of length $n-k+1$ is called a *Hamiltonian path*.

Definition 1.4. [2] A *Hamiltonian decomposition* of a hypergraph is a partition of the set of edges into Hamiltonian cycles.

Definition 1.5. Let $\mathcal{H}=(V,\mathcal{E})$ be a k-uniform hypergraph and let $\widehat{\mathcal{H}}=(\widehat{V},\widehat{\mathcal{E}})$ be its copy. Let $\hat{v}\in\widehat{V}$ denote a copy of $v\in V$. The *prism* over $\mathcal{H}$, denoted by $\mathrm{Prism}(\mathcal{H})$, is a hypergraph obtained from $\mathcal{H}$ and $\widehat{\mathcal{H}}$ by adding a collection of edges

$$\mathcal{J}=\{\{v,\hat{v}\}\cup A : v\in V, A\subseteq (V\cup\widehat{V})\backslash\{v,\hat{v}\}, |A|=k-2\}.$$

A pair of vertices $\{v,\hat{v}\}$ is called a *transition node* and an edge containing a transition node is called a *junction*. An edge in $\mathcal{H}$ and $\widehat{\mathcal{H}}$ is called an *ordinary edge*. For concurrent notation, we define $\hat{\hat{v}}=v$.

Example 1.1. For $k=3$, let $\mathcal{H}$ denote the complete 3-uniform hypergraph $K_4^{(3)}$ whose 4 vertices are $1,2,3$ and 4. The following list shows all edges of $\mathrm{Prism}(\mathcal{H})$, where the first line and the second line are edges of $\mathcal{H}$ and $\widehat{\mathcal{H}}$, respectively, and the remaining edges are junctions.

$$\begin{array}{llll}
\{1,2,3\}, & \{1,2,4\}, & \{1,3,4\}, & \{2,3,4\},\\
\{\hat{1},\hat{2},\hat{3}\}, & \{\hat{1},\hat{2},\hat{4}\}, & \{\hat{1},\hat{3},\hat{4}\}, & \{\hat{2},\hat{3},\hat{4}\},\\
\{2,1,\hat{1}\}, & \{1,2,\hat{2}\}, & \{1,3,\hat{3}\}, & \{1,4,\hat{4}\},\\
\{3,1,\hat{1}\}, & \{3,2,\hat{2}\}, & \{2,3,\hat{3}\}, & \{2,4,\hat{4}\},\\
\{4,1,\hat{1}\}, & \{4,2,\hat{2}\}, & \{4,3,\hat{3}\}, & \{3,4,\hat{4}\},\\
\{1,\hat{1},\hat{2}\}, & \{2,\hat{2},\hat{1}\}, & \{3,\hat{3},\hat{1}\}, & \{4,\hat{4},\hat{1}\},\\
\{1,\hat{1},\hat{3}\}, & \{2,\hat{2},\hat{3}\}, & \{3,\hat{3},\hat{2}\}, & \{4,\hat{4},\hat{2}\},\\
\{1,\hat{1},\hat{4}\}, & \{2,\hat{2},\hat{4}\}, & \{3,\hat{3},\hat{4}\}, & \{4,\hat{4},\hat{3}\}.
\end{array}$$

Definition 1.6. A *complete tripartite k-uniform hypergraph* has the vertex set V partitioned into three subsets V_0, V_1 and V_2 and the edge set $\mathcal{E}$ such that $\mathcal{E} = \{e : e \subseteq V, |e| = k \text{ and } |e \cap V_i| < k \text{ for all } i \in \{0,1,2\}\}$, and denoted by $K^{(k)}_{m,m,m}$ when $|V_0| = |V_1| = |V_2| = m$.

Example 1.2. Let $V_0 = \{0,1,2\}$, $V_1 = \{\overline{0},\overline{1},\overline{2}\}$ and $V_2 = \{\overline{\overline{0}},\overline{\overline{1}},\overline{\overline{2}}\}$. For $k = 3$ and $m = 3$, the complete tripartite 3-uniform hypergraph $K^{(3)}_{3,3,3}$ whose vertex set is $V_0 \cup V_1 \cup V_2$ has edges as follows.

$$\{0,\overline{0},\overline{\overline{0}}\}, \{0,\overline{0},\overline{\overline{1}}\}, \{0,\overline{0},\overline{\overline{2}}\}, \{1,\overline{0},\overline{\overline{0}}\}, \{1,\overline{0},\overline{\overline{1}}\}, \{1,\overline{0},\overline{\overline{2}}\}, \{2,\overline{0},\overline{\overline{0}}\}, \{2,\overline{0},\overline{\overline{1}}\}, \{2,\overline{0},\overline{\overline{2}}\},$$
$$\{0,\overline{1},\overline{\overline{0}}\}, \{0,\overline{1},\overline{\overline{1}}\}, \{0,\overline{1},\overline{\overline{2}}\}, \{1,\overline{1},\overline{\overline{0}}\}, \{1,\overline{1},\overline{\overline{1}}\}, \{1,\overline{1},\overline{\overline{2}}\}, \{2,\overline{1},\overline{\overline{0}}\}, \{2,\overline{1},\overline{\overline{1}}\}, \{2,\overline{1},\overline{\overline{2}}\},$$
$$\{0,\overline{2},\overline{\overline{0}}\}, \{0,\overline{2},\overline{\overline{1}}\}, \{0,\overline{2},\overline{\overline{2}}\}, \{1,\overline{2},\overline{\overline{0}}\}, \{1,\overline{2},\overline{\overline{1}}\}, \{1,\overline{2},\overline{\overline{2}}\}, \{2,\overline{2},\overline{\overline{0}}\}, \{2,\overline{2},\overline{\overline{1}}\}, \{2,\overline{2},\overline{\overline{2}}\},$$
$$\{0,1,\overline{0}\}, \{0,2,\overline{0}\}, \{1,2,\overline{0}\}, \{0,1,\overline{1}\}, \{0,2,\overline{1}\}, \{1,2,\overline{1}\}, \{0,1,\overline{2}\}, \{0,2,\overline{2}\}, \{1,2,\overline{2}\},$$
$$\{0,1,\overline{\overline{0}}\}, \{0,2,\overline{\overline{0}}\}, \{1,2,\overline{\overline{0}}\}, \{0,1,\overline{\overline{1}}\}, \{0,2,\overline{\overline{1}}\}, \{1,2,\overline{\overline{1}}\}, \{0,1,\overline{\overline{2}}\}, \{0,2,\overline{\overline{2}}\}, \{1,2,\overline{\overline{2}}\},$$
$$\{\overline{0},\overline{1},0\}, \{\overline{0},\overline{2},0\}, \{\overline{1},\overline{2},0\}, \{\overline{0},\overline{1},1\}, \{\overline{0},\overline{2},1\}, \{\overline{1},\overline{2},1\}, \{\overline{0},\overline{1},2\}, \{\overline{0},\overline{2},2\}, \{\overline{1},\overline{2},2\},$$
$$\{\overline{0},\overline{1},\overline{\overline{0}}\}, \{\overline{0},\overline{2},\overline{\overline{0}}\}, \{\overline{1},\overline{2},\overline{\overline{0}}\}, \{\overline{0},\overline{1},\overline{\overline{1}}\}, \{\overline{0},\overline{2},\overline{\overline{1}}\}, \{\overline{1},\overline{2},\overline{\overline{1}}\}, \{\overline{0},\overline{1},\overline{\overline{2}}\}, \{\overline{0},\overline{2},\overline{\overline{2}}\}, \{\overline{1},\overline{2},\overline{\overline{2}}\},$$
$$\{\overline{\overline{0}},\overline{\overline{1}},0\}, \{\overline{\overline{0}},\overline{\overline{2}},0\}, \{\overline{\overline{1}},\overline{\overline{2}},0\}, \{\overline{\overline{0}},\overline{\overline{1}},1\}, \{\overline{\overline{0}},\overline{\overline{2}},1\}, \{\overline{\overline{1}},\overline{\overline{2}},1\}, \{\overline{\overline{0}},\overline{\overline{1}},2\}, \{\overline{\overline{0}},\overline{\overline{2}},2\}, \{\overline{\overline{1}},\overline{\overline{2}},2\},$$
$$\{\overline{\overline{0}},\overline{\overline{1}},\overline{0}\}, \{\overline{\overline{0}},\overline{\overline{2}},\overline{0}\}, \{\overline{\overline{1}},\overline{\overline{2}},\overline{0}\}, \{\overline{\overline{0}},\overline{\overline{1}},\overline{1}\}, \{\overline{\overline{0}},\overline{\overline{2}},\overline{1}\}, \{\overline{\overline{1}},\overline{\overline{2}},\overline{1}\}, \{\overline{\overline{0}},\overline{\overline{1}},\overline{2}\}, \{\overline{\overline{0}},\overline{\overline{2}},\overline{2}\}, \{\overline{\overline{1}},\overline{\overline{2}},\overline{2}\}.$$

Note that subsets $\{0,1,2\}, \{\overline{0},\overline{1},\overline{2}\}$ and $\{\overline{\overline{0}},\overline{\overline{1}},\overline{\overline{2}}\}$ are not edges in $K^{(3)}_{3,3,3}$.

1.2 History and Overview

The study of Hamiltonian decomposition was begun in graph theory before being generalized in hypergraph theory. In graph theory, it is well-known that a complete graph K_{2n+1} has a Hamiltonian decomposition by Walecki's construction [1]. Other graphs such as complete bipartite graphs and complete multipartite graphs were also studied. Next, the problem of finding Hamiltonian decompositions of hypergraphs is also questioned. However, a Hamiltonian cycle in a hypergraph can be defined in many ways. The first definition of a Hamiltonian cycle in a hypergraph was defined by Berge [3]. Berge's cycle of a hypergraph $\mathcal{H} = (V, \mathcal{E})$ is a sequence $(v_0, E_1, v_1, E_2, \ldots, v_{n-1}, E_n, v_0)$, where $\{v_0, v_1, \ldots, v_{n-1}\} = V$ and $E_1, E_2, \ldots, E_n$ are distinct edges in $\mathcal{E}$ such that $v_{i-1}, v_i \in E_i$ (i is of modulo n). By 1994, the problem of decomposing complete 3-uniform hypergraph $K_n^{(3)}$ into Hamiltonian cycles of Berge type was completely solved by Verrall [10]. Later, Katona and Kiearstead [6] introduced a stronger definition of Hamiltonian cycle in 1999 as we stated in Definition 1.3. Now, Katona-Kierstead's cycle is widely used in later publications. In 2010, Bailey and Stevens [2] decomposed $K_n^{(3)}$ into Hamiltonian cycles of Katona-Kiearstead type by computer programming using the clique finding method for $n \in \{7, 8\}$ and a difference method for $n \in \{10, 11, 16\}$. Along with Meszka and Rosa [9], they used an extension of difference method to obtain Hamiltonian decompositions of $K_n^{(3)}$ for all feasible $n \leq 32$. This work for $K_n^{(3)}$ leads us to begin new study for Prism($K_n^{(3)}$). Other hypergraphs such as complete bipartite 3-uniform hypergraphs $K_{n,n}^{(3)}$ defined by Definition 1.7 and complete k-uniform k-partite hypergraphs $K_{k\times m}^{(k)}$ defined by Definition 1.8 are also decomposable into Hamiltonian cycles of Katona-Kierstead type. In 2001, Jirimutu and Wang [5] found Hamiltonian decompositions of $K_{q,q}^{(3)}$ for every prime number q using ordered odd 2-splitting of an integer. Soon after, Xu and Wang [12] completely solved Hamiltonian decompositions of $K_{n,n}^{(3)}$ for all $n \geq 2$ with a simple idea of mixing two Hamiltonian cycles from two complete graphs. By 2013, Kuhl and Schroeder [7] studied Hamiltonian decomposition of $K_{k\times m}^{(k)}$ and got results for all m such that $k \mid m$ by classified each Hamiltonian cycle to a permissible k-tuple and partition the $\mathbb{Z}_m^{k-1}$ space. With these works, we extend the study from "bi"partite to "tri"partite which is different from Kuhl and Schroeder's in [7].

Definition 1.7 (Jirimutu [5])**.** A *complete bipartite k-uniform hypergraph* $K_{m,n}^{(k)}$ has two partite sets V_1 and V_2 such that $|V_1| = m$ and $|V_2| = n$ and the edge set $\mathcal{E}$ such that $\mathcal{E} = \{e \subseteq V_1 \cup V_2 : |e| = k, e \cap V_1 \neq \varnothing, e \cap V_2 \neq \varnothing\}$.

Definition 1.8 (Kuhl [7])**.** A *complete k-uniform k-partite hypergraph* $K_{k\times m}^{(k)}$ has k partite sets, $V_0, V_1, \ldots, V_{k-1}$, of equal size m, and each edge e of size k is in the form $e = \{v_0, v_1, \ldots, v_{k-1}\}$, where $v_i \in V_i$ for all $i \in \{0, 1, \ldots, k-1\}$.

Note that the complete 3-uniform 3-partite hypergraph defined by Kuhl and Schroeder in Definition 1.8 is a subhypergraph of the complete tripartite 3-uniform hypergraph $K_{m,m,m}^{(3)}$ defined by us in Definition 1.6.

At the beginning of this study, we usually calculate the necessary condition for the existence of a Hamiltonian decomposition by Proposition 1.1.

Proposition 1.1. *If a k-uniform hypergraph has a Hamiltonian decomposition, then the length of a Hamiltonian cycle divides the size of the edge set.*

We consider the number of overall edges in the target hypergraph in the formula of some parameters such as the number of vertices. This number will suggest the method to find Hamiltonian cycles in the decomposition. A collection of Hamiltonian cycles which is a candidate to be a Hamiltonian decomposition must gather the same number of overall edges in the hypergraph. There are two ways to prove that the candidate is the solution. The first way is to show that all edges in the collection are distinct. The other way is to show that each edge in the hypergraph is in unique Hamiltonian cycle.

The study of Hamiltonian decomposition of Prism($K_n^{(3)}$) and $K_{m,m,m}^{(3)}$ are separated into Chapter 2 and Chapter 3, respectively. The last chapter summarizes the results of the study and suggests some further works.

CHAPTER II

HAMILTONIAN DECOMPOSITIONS OF $\text{Prism}(K_n^{(3)})$

First, we must consider the necessary condition for the existence of Hamiltonian decomposition of $\text{Prism}(K_n^{(3)})$ which is calculated in Lemma 2.1. With fill-out method and cyclic method for finding the decomposition, we get some results on $n \in \{4, 5, 7, 8\}$ which is described in Section 2.2.

2.1 Preliminaries

We investigate some possible ways to find a Hamiltonian decomposition of $\text{Prism}(K_n^{(3)})$. Beginning with a familiar method in graph theory, a Hamiltonian decomposition of $\text{Prism}(K_n^{(3)})$ cannot be constructed in the same way. Next, we consider the number of junctions to find a suitable combination of Hamiltonian cycles in $\text{Prism}(K_n^{(3)})$ as we describe in Lemma 2.3.

Lemma 2.1. *If a Hamiltonian decomposition of* $\text{Prism}(K_n^{(3)})$ *exists, then* $3 \nmid n$.

Proof. Suppose that a Hamiltonian decomposition of $\text{Prism}(K_n^{(3)})$ exists. We count the number of edges of $\text{Prism}(K_n^{(3)})$. There are $\binom{n}{3}$ edges in $K_n^{(3)}$, as same as $\widehat{K_n^{(3)}}$, and $2n(n-1)$ junctions. Since the number of edges in a Hamiltonian cycle in $\text{Prism}(K_n^{(3)})$ is $2n$ and the number of edges of $\text{Prism}(K_n^{(3)})$ is $2\binom{n}{3} + 2n(n-1) = n(n-1)(n+4)/3$, there are $(n-1)(n+4)/6$ Hamiltonian cycles in the decomposition which must be an integer. Therefore, $n \equiv 1$ or $2 \pmod 3$, that is $3 \nmid n$. $\square$

First, consider a graph Prism(K_n). The complete graph K_{n+1} is decomposable into Hamiltonian cycles by Walecki's construction in [1] which is used to decompose the complete graph K_n into Hamiltonian paths where n is even. Thus, when n is even, Prism(K_n) is obviously decomposable into Hamiltonian cycles by joining two copies of each Hamiltonian path with edges at each end vertices. Figure 2.1 shows a prism over a complete graph K_4 and its Hamiltonian decomposition.

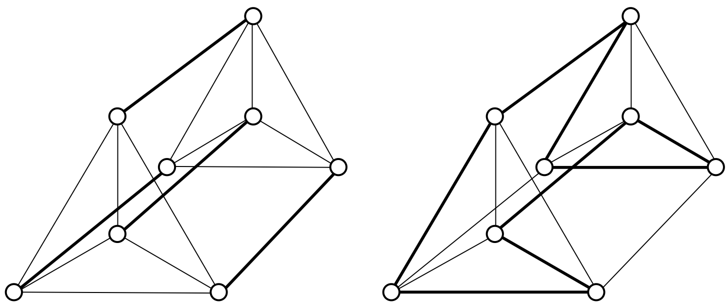

Figure 2.1: Left: Prism(K_4) constructed from two copies of K_4 and edges (drawn as bold line) connecting the copy of each vertex. Right: a Hamiltonian decomposition of Prism(K_4) consists of two Hamiltonian cycles drawn as bold line and light line.

Next, consider a hypergraph Prism($K_n^{(3)}$). We try to use Hamiltonian paths as same as the method in graph to find a Hamiltonian decomposition of a hypergraph Prism($K_n^{(3)}$). We use a Hamiltonian decomposition of $K_{n+1}^{(k)}$ and delete one vertex to retrieve Hamiltonian paths of $K_n^{(k)}$. The result is Lemma 2.2.

Lemma 2.2. *If $K_{n+1}^{(k)}$ is decomposable into Hamiltonian cycles, then $K_n^{(k)}$ is decomposable into Hamiltonian paths.*

Proof. Suppose that $K_{n+1}^{(k)}$ is decomposable into Hamiltonian cycles. Let $\mathcal{S}$ be a Hamiltonian decomposition of $K_{n+1}^{(k)}$. If we delete a vertex v together with all edges containing v in $K_{n+1}^{(k)}$, then each Hamiltonian cycle in $\mathcal{S}$ becomes noncyclic and is a Hamiltonian path in $K_{n+1}^{(k)} - \{v\}$ which is isomorphic to $K_n^{(k)}$. Hence, all edge-disjoint Hamiltonian cycles in $K_{n+1}^{(k)}$ become edge-disjoint Hamiltonian paths in $K_n^{(k)}$. Therefore, $K_n^{(k)}$ is decomposable into Hamiltonian paths. $\square$

Theorem 2.1 is the result of joining the Hamiltonian paths to decompose $\text{Prism}(K_n^{(3)})$ into Hamiltonian cycles. Unfortunately, this method is not complete the decomposition. However, we obtain a lower bound of the number of edge-disjoint Hamiltonian cycles in $\text{Prism}(K_n^{(3)})$.

Theorem 2.1. *If $K_{n+1}^{(3)}$ is decomposable into Hamiltonian cycles, then* $\text{Prism}(K_n^{(3)})$ *has at least $n(n-1)/6$ edge-disjoint Hamiltonian cycles.*

Proof. Assume that $K_{n+1}^{(3)}$ is decomposable into Hamiltonian cycles. Lemma 2.2 implies $K_n^{(3)}$ is decomposable into Hamiltonian paths. The number of Hamiltonian cycles in a Hamiltonian decomposition of $K_{n+1}^{(3)}$ is $\binom{n+1}{3}/(n+1) = n(n-1)/6 = \binom{n}{3}/(n-2)$ which is equal to the number of Hamiltonian paths in a decomposition of $K_n^{(3)}$ into Hamiltonian paths. Let $t = n(n-1)/6$ and $\mathcal{S} = \{P_1, P_2, \ldots, P_t\}$ be a decomposition of $K_n^{(3)}$ into Hamiltonian paths obtained by deleting a vertex v of $K_{n+1}^{(3)}$. We shall construct t edge-disjoint Hamiltonian cycles in $\text{Prism}(K_n^{(3)})$ in the following way. For each $P_i \in \mathcal{S}$, if $P_i = [v_1, v_2, \ldots, v_n]$, then $\widehat{P}_i = [\hat{v}_n, \hat{v}_{n-1}, \ldots, \hat{v}_1]$ is a Hamiltonian path in $\widehat{K_n^{(3)}}$. Next, define $C(P_i) = (v_1, v_2, \ldots, v_n, \hat{v}_n, \hat{v}_{n-1}, \ldots, \hat{v}_1)$. Thus, $C(P_i)$ is a Hamiltonian cycle in $\text{Prism}(K_n^{(3)})$ for all $i \in \{1, 2, \ldots, t\}$.

To show that any pair of Hamiltonian cycles obtained by this method are disjoint, we shall show that there is no repeated junction. Suppose that $C(P_i)$ and $C(P_j)$ have an identical junction T for some distinct path $P_i, P_j \in \mathcal{S}$. Without loss of generality, let $T = \{a, x, \hat{x}\}$, where $a, x \in K_n^{(3)}$ and $\hat{x}$ is the copy of x. Thus, $\{a, x\}$ are successive end vertices of P_i and also P_j. Since P_i and P_j are obtained by deleting a vertex v of $K_{n+1}^{(3)}$, an edge $\{a, x, v\}$ is in two disjoint Hamiltonian cycles in $K_{n+1}^{(3)}$, which is a contradiction. Hence, all junctions are different.

Since the total number of Hamiltonian cycles of $\text{Prism}(K_n^{(3)})$ if exists is $(n-1)(n+4)/6$, constructing $n(n-1)/6$ edge-disjoint Hamiltonian cycles is not complete the Hamiltonian decomposition of $\text{Prism}(K_n^{(3)})$. We can only conclude that $\text{Prism}(K_n^{(3)})$ has at least $n(n-1)/6$ edge-disjoint Hamiltonian cycles. □

To develop new method, we find some constraints stated in Lemma 2.3 to reduce the scope of finding a Hamiltonian decomposition of $\text{Prism}(K_n^{(3)})$.

Lemma 2.3. *Suppose that* $\text{Prism}(K_n^{(3)})$ *has a Hamiltonian decomposition. Let* x_i *be the number of Hamiltonian cycles in the decomposition with* $2i$ *transition nodes. Then* $\{x_1, x_2, \ldots, x_j\}$ *with* $j = \lfloor n/2 \rfloor$ *satisfies two equations:*

$$2x_1 + 4x_2 + 6x_3 + \ldots + 2jx_j = n(n-1) \tag{2.1}$$

$$x_1 + x_2 + x_3 + \ldots + x_j = (n-1)(n+4)/6 \tag{2.2}$$

Proof. A Hamiltonian cycle passes through every vertex in $\text{Prism}(K_n^{(3)})$, so it will transit from $K_n^{(3)}$ to its copy and come back. Thus, there are even number of transition nodes in each Hamiltonian cycle. Each transition in $\text{Prism}(K_n^{(3)})$ always requires two junctions, from $K_n^{(3)}$ and to its copy, so we can count the number just only one side of prism. The sum of the number of one-side junctions for each Hamiltonian cycle must equal a half of the number of junctions in $\text{Prism}(K_n^{(3)})$, as in Equation (2.1). Equation (2.2) refers to the sum of the number of Hamiltonian cycles. □

The number of edges used in Lemma 2.3 of $\text{Prism}(K_n^{(3)})$ with $n \in \{4, 5, 7, 8\}$ are shown in Table 2.1. By calculation, there are unique admissible $\{x_i\}$ for $n \in \{4, 5\}$ satisfying two equations in Lemma 2.3. However, for $n > 5$, there are many $\{x_i\}$'s satisfying these two equations as shown for the case $n = 7$ in Table 2.4. Table 2.2 shows some values of x_i associated to a Hamiltonian decomposition of Prism$(K_n^{(3)})$ for some n, where $n \in \{4, 5\}$ and $n \equiv 2 \pmod 6$. The case $n \equiv 2 \pmod 6$ is interesting because each x_i is divisible by $n - 1$.

n	$\binom{n}{3}$	$n(n-1)$	$(n-1)(n+4)/6$
4	4	12	4
5	10	20	6
7	35	42	11
8	56	56	14

Table 2.1: The number of one-side ordinary edges, one-side junctions, and Hamiltonian cycles associated to Prism$(K_n^{(3)})$

n	$(n-1)(n+4)/6$	x_1	x_2	x_3	x_4	x_5
4	4	2	2			
5	6	2	4			
8	14	7		7		
14	39	13	13		13	
20	76	19	19	19	19	
26	125	25	25	50	25	
32	186	62	31	31	31	31

Table 2.2: Some values of $\{x_i\}$'s associated to a Hamiltonian decomposition of Prism$(K_n^{(3)})$

2.2 Results

2.2.1 $n \in \{4, 5\}$

To find a Hamiltonian decomposition of Prism($K_n^{(3)}$), we use a *fill-out method* which is a trial-and-error method of drawing lines to connect vertices between two copies. The aim is try to draw distinct hyperedges which are distinct consecutive three vertices. We write vertices in two columns. Begin with choosing transition nodes for each cycle and then fill out junctions by connecting one vertex from the same side to each transition node. Try to make all junctions distinct. Next, draw lines to place each ordinary edge in a unique cycle. If it is not solved, then try another choice of junction and continue the process. Finally, we get the results for $n \in \{4, 5\}$.

Theorem 2.2. *A Hamiltonian decomposition of* Prism($K_4^{(3)}$) *exists.*

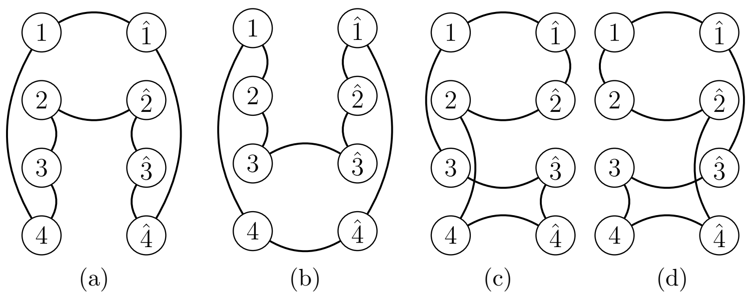

Figure 2.2: A Hamiltonian decomposition of Prism($K_4^{(3)}$)

Proof. Let the vertex set of $K_4^{(3)}$ be $\{1, 2, 3, 4\}$. The following is a Hamiltonian decomposition of Prism($K_4^{(3)}$) illustrated in Figure 2.2:

$$(2, 3, 4, 1, \hat{1}, \hat{4}, \hat{3}, \hat{2}), \quad (1, \hat{1}, \hat{2}, 2, 4, \hat{4}, \hat{3}, 3),$$

$$(3, 2, 1, 4, \hat{4}, \hat{1}, \hat{2}, \hat{3}), \quad (1, \hat{1}, \hat{3}, 3, 4, \hat{4}, \hat{2}, 2).$$

Check edge-disjoint Hamiltonian cycles in Figure 2.2 as follows.

1. Junctions containing $\{1, \hat{1}\}$ are in cycle (a), (c) and (d).
2. Junctions containing $\{2, \hat{2}\}$ are in cycle (a), (c) and (d).
3. Junctions containing $\{3, \hat{3}\}$ are in cycle (b), (c) and (d).
4. Junctions containing $\{4, \hat{4}\}$ are in cycle (b), (c) and (d).
5. Ordinary edges are in cycle (a) and (b).

We see that each edge is in a unique Hamiltonian cycle, so we get a Hamiltonian decomposition of $\text{Prism}(K_4^{(3)})$. □

Theorem 2.3. *A Hamiltonian decomposition of* $\text{Prism}(K_5^{(3)})$ *exists.*

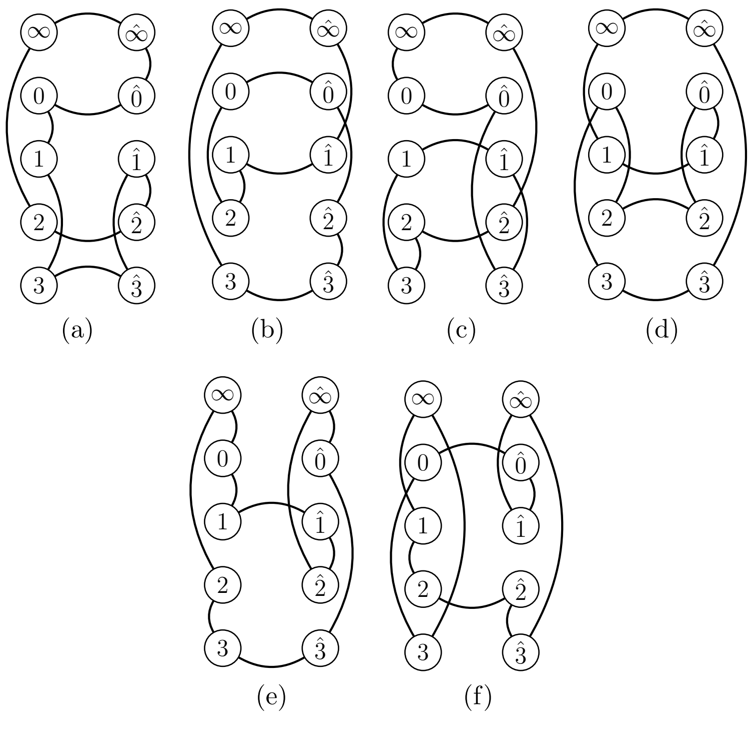

Figure 2.3: A Hamiltonian decomposition of $\text{Prism}(K_5^{(3)})$

Proof. Let the vertex set of $K_5^{(3)}$ be $\{\infty, 0, 1, 2, 3\}$. The following is a Hamiltonian decomposition of Prism($K_5^{(3)}$) illustrated in Figure 2.3:

$$(\infty, 2, \hat{2}, \hat{1}, \hat{3}, 3, 1, 0, \hat{0}, \hat{\infty}), \quad (1, 0, \infty, 2, 3, \hat{3}, \hat{0}, \hat{\infty}, \hat{2}, \hat{1}),$$
$$(\infty, 3, \hat{3}, \hat{2}, \hat{0}, 0, 2, 1, \hat{1}, \hat{\infty}), \quad (2, 1, \infty, 3, 0, \hat{0}, \hat{1}, \hat{\infty}, \hat{3}, \hat{2}),$$
$$(\infty, 0, \hat{0}, \hat{3}, \hat{1}, 1, 3, 2, \hat{2}, \hat{\infty}),$$
$$(\infty, 1, \hat{1}, \hat{0}, \hat{2}, 2, 0, 3, \hat{3}, \hat{\infty}).$$

Check edge-disjoint Hamiltonian cycles in Figure 2.3 as follows.

1. Junctions containing $\{\infty, \hat{\infty}\}$ are in cycle (a), (b), (c) and (d).
2. Junctions containing $\{0, \hat{0}\}$ are in cycle (a), (b), (c) and (f).
3. Junctions containing $\{1, \hat{1}\}$ are in cycle (b), (c), (d) and (e).
4. Junctions containing $\{2, \hat{2}\}$ are in cycle (a), (c), (d) and (f).
5. Junctions containing $\{3, \hat{3}\}$ are in cycle (a), (b), (d) and (e).
6. Ordinary edges containing ∞ or $\hat{\infty}$ are in cycle (e) and (f).
7. Ordinary edges with no ∞ or $\hat{\infty}$ are in cycle (a), (b), (c) and (d).

We see that each edge is in a unique Hamiltonian cycle, so we get a Hamiltonian decomposition of Prism($K_5^{(3)}$). Moreover, we found that a Hamiltonian decomposition of Prism($K_5^{(3)}$) has cyclic structure by developing two initial cycles modulo 4, where $\infty + i = \infty$ for all $i \in \mathbb{Z}_4$. □

2.2.2 $n \in \{7, 8\}$

In [2], Bailey and Stevens defined a difference pattern of triples and used it to partition the edges of $K_n^{(3)}$ into equivalence classes that are helpful to find a Hamiltonian decomposition of $K_n^{(3)}$ by computer search. We modify the definition of difference pattern to agree with edges of Prism($K_n^{(3)}$), as we see below.

Definition 2.1. Let the vertex set of $K_n^{(3)}$ be $\mathbb{Z}_{n-1} \cup \{\infty\}$ and let $T = \{a, b, c\}$ be an edge in $\text{Prism}(K_n^{(3)})$. The *difference pattern* of T, denoted by $\pi(T)$, is an equivalence class mapped T to the value as follows.

1. For $a, b, c \in \mathbb{Z}_{n-1}$, $\pi(T)$ is the cyclic ordering $(b-a, c-b, a-c)$ where $\{a, b, c\}$ are arranged in ascending order.

2. For $a, b \in \mathbb{Z}_{n-1}$ and $c = \infty$,
   $\pi(T) = (\min\{b - a \pmod{n-1}, a - b \pmod{n-1})\}, \infty)$.

3. For $a, b \in \mathbb{Z}_{n-1} \cup \{\infty\}$ and $c = \hat{b}$, $\pi(T) = b - a$.

4. Otherwise, T is an edge of the form $\{\hat{a}, \hat{b}, \hat{c}\}$ corresponding to one of three types above. Calculate $\pi(T)$ in the same way and put hat ($\hat{}$) over each difference value.

The differences are taken under modulo $n-1$, and it is assumed that $\infty - a = \infty$ and $b - \infty = -\infty$ for all $a, b \in \mathbb{Z}_{n-1}$.

For example, if $n = 8$, then we calculate under modulo 7. We have

1. $\pi(\{1, 0, 3\}) = \pi(\{0, 1, 3\}) = (1, 2, 4)$ which is equivalent to $(2, 4, 1)$ and $(4, 1, 2)$ but is not equivalent to $(1, 4, 2)$, $(4, 2, 1)$ nor $(2, 1, 4)$.

2. $\pi(\{\infty, 6, 5\}) = (1, \infty)$ and $\pi(\{1, \infty, 6\}) = (2, \infty)$.

3. $\pi(\{\hat{3}, 3, 2\}) = 1$, $\pi(\{\hat{2}, 2, 3\}) = 6$, $\pi(\{\hat{\infty}, \infty, 1\}) = \infty$ and $\pi(\{\hat{1}, 1, \infty\}) = -\infty$.

4. $\pi(\{\hat{0}, \hat{1}, \hat{3}\}) = (\hat{1}, \hat{2}, \hat{4})$, $\pi(\{\hat{\infty}, \hat{6}, \hat{5}\}) = (\hat{1}, \hat{\infty})$ and $\pi(\{3, \hat{3}, \hat{2}\}) = \hat{1}$.

Moreover, let C be a Hamiltonian cycle of $\text{Prism}(K_n^{(3)})$. The *difference type* of C, denoted by $\tau(C)$, is a collection of difference patterns of its edges.

For example, let $C = (\infty, 2, \hat{2}, \hat{1}, \hat{3}, 3, 1, 0, \hat{0}, \hat{\infty})$ which is a Hamiltonian cycle of $\mathrm{Prism}(K_5^{(3)})$ in Theorem 2.3. All edges of C are $\{\infty, 2, \hat{2}\}$, $\{2, \hat{2}, \hat{1}\}$, $\{\hat{2}, \hat{1}, \hat{3}\}$, $\{\hat{1}, \hat{3}, 3\}$, $\{\hat{3}, 3, 1\}$, $\{3, 1, 0\}$, $\{1, 0, \hat{0}\}$, $\{0, \hat{0}, \hat{\infty}\}$, $\{\hat{0}, \hat{\infty}, \infty\}$ and $\{\hat{\infty}, \infty, 2\}$. Thus,

$$\tau(C) = \{-\infty, \hat{1}, (\hat{1}, \hat{1}, \hat{2}), \hat{2}, 2, (1, 2, 1), 3, -\hat{\infty}, \hat{\infty}, \infty\}.$$

We see that the difference patterns in $\tau(C)$ are all distinct and the translations of C, $\{C + i \mid i \in \mathbb{Z}_4\}$ taken under modulo 4 and $\infty + i = \infty$, are edge-disjoint Hamiltonian cycles. For convenience, the translation of an initial cycle is called the *cyclic method* . The following lemma will show that this property is also satisfied for all $n \equiv 2 \pmod 6$, and lead to a construction of Hamiltonian decomposition of $\mathrm{Prism}(K_8^{(3)})$.

Lemma 2.4. *Suppose that* $\gcd(n-1, 6) = 1$*. Let the vertex set of* $K_n^{(3)}$ *be* $\mathbb{Z}_{n-1} \cup \{\infty\}$ *and let* C *be a Hamiltonian cycle of* $\mathrm{Prism}(K_n^{(3)})$*. If difference patterns in* $\tau(C)$ *are all distinct, then* $\{C + i \mid i \in \mathbb{Z}_{n-1}\}$ *taken under modulo* $n-1$*, where* $\infty + i = \infty$ *for all* $i \in \mathbb{Z}_{n-1}$*, are edge-disjoint Hamiltonian cycles.*

Proof. Let all translations be under modulo $n-1$. We first note that $\pi(T) = \pi(T')$ for some edges T and T' of $\mathrm{Prism}(K_n^{(3)})$ if and only if $T' = T + i$ for some $i \in \mathbb{Z}_{n-1}$. For any edge T of $\mathrm{Prism}(K_n^{(3)})$, we obtain the followings.

Case 1: If $\pi(T)$ is in the form $(b-a, c-b, a-c)$ and $n-1$ is not a multiple of 3, let $T = \{a, b, c\}$, where $a, b, c \in \mathbb{Z}_{n-1}$ are distinct. If $T = T + i$ for some $i \in \mathbb{Z}_{n-1}$, then $\{a, b, c\} = \{a+i, b+i, c+i\}$. Suppose that $T = T + i$ for some $i \in \mathbb{Z}_{n-1}$ and $i \neq 0$. Without loss of generality, we have $a \equiv b + i \pmod{n-1}$, $b \equiv c + i \pmod{n-1}$ and $c \equiv a + i \pmod{n-1}$. Thus, $a + b + c \equiv a + b + c + 3i \pmod{n-1}$. Deduce that $3i \equiv 0 \pmod{n-1}$. Since $i \neq 0$ and $i \in \mathbb{Z}_{n-1}$, we have $i = (n-1)/3$ is an integer. Thus, we must assume that $n-1$ is not a multiple of 3, so that $i = (n-1)/3$ is not an integer and $T = T + i$ only if $i = 0$. By this assumption, $\{T+i : i \in \mathbb{Z}_{n-1}\}$ are distinct.

Case 2: If $\pi(T)$ is in the form $(\min\{b-a, (n-1)-(b-a)\}, \infty)$ and $n-1$ is odd, let $T = \{a, b, \infty\}$, where $a, b \in \mathbb{Z}_{n-1}$ are distinct. If $T = T+i$ for some $i \in \mathbb{Z}_{n-1}$, then $\{a, b, \infty\} = \{a+i, b+i, \infty\}$. Suppose that $T = T+i$ for some $i \in \mathbb{Z}_{n-1}$ and $i \neq 0$. We have $a \equiv b+i \pmod{n-1}$ and $b \equiv a+i \pmod{n-1}$. Thus, $a+b \equiv a+b+2i \pmod{n-1}$. Deduce that $2i \equiv 0 \pmod{n-1}$. Since $i \neq 0$ and $i \in \mathbb{Z}_{n-1}$, we have $i = (n-1)/2$ is an integer. Thus, we must assume that $n-1$ is odd, so that $i = (n-1)/2$ is not an integer and $T = T+i$ only if $i = 0$. By this assumption, $\{T+i : i \in \mathbb{Z}_{n-1}\}$ are distinct.

Case 3: If $\pi(T)$ is in the form $b-a$, let $T = \{a, b, \hat{b}\}$, where $a, b \in \mathbb{Z}_{n-1} \cup \{\infty\}$ are distinct. If $T = T+i$ for some $i \in \mathbb{Z}_{n-1}$, then $\{a, b, \hat{b}\} = \{a+i, b+i, \widehat{b+i}\}$. Notice that $\hat{b}$ must be equal to $\widehat{b+i}$. Thus, $i = 0$ when $b \in \mathbb{Z}_{n-1}$. If $b = \infty$, then $a = a+i$ which implies $i = 0$. Therefore, $\{T+i : i \in \mathbb{Z}_{n-1}\}$ are distinct because $T = T+i$ only if $i = 0$.

Hence, when $\gcd(n-1, 6) = 1$, $\{T+i : i \in \mathbb{Z}_{n-1}\}$ are all distinct for each edge T.

Let C be a Hamiltonian cycle of $\mathrm{Prism}(K_n^{(3)})$ containing $T_1, T_2, \ldots, T_n$ as edges. Then $C+i$ contains $T_1+i, T_2+i, \ldots, T_n+i$ as edges. By assumption that difference patterns in $\tau(C)$ are all distinct, that is, $\pi(T_k) \neq \pi(T_{k'})$ for all $k \neq k'$, we get $T_k + i \neq T_{k'} + i'$ for all $k \neq k'$ or $i \neq i'$. Thus, $\{C+i \mid i \in \mathbb{Z}_{n-1}\}$ are edge-disjoint Hamiltonian cycles. □

From Table 2.2, one possible solution of $\{x_1, x_2, x_3\}$ for $\mathrm{Prism}(K_8^{(3)})$ is $\{7, 0, 7\}$. We shall construct a Hamiltonian decomposition of $\mathrm{Prism}(K_8^{(3)})$ by fixing one vertex as ∞. Next, construct two Hamiltonian cycles in which consist of six transition nodes and two transition nodes by fill-out the ordering. Try to fill-out edges of distinct difference patterns. Difference type of two cycles must be disjoint. Finally, develop two initial cycles under modulo 7 and get the result.

Theorem 2.4. *A Hamiltonian decomposition of* $\mathrm{Prism}(K_8^{(3)})$ *exists.*

Proof. Let the vertex set of $K_8^{(3)}$ be $\{\infty, 0, 1, 2, 3, 4, 5, 6\}$. A Hamiltonian decomposition of $\mathrm{Prism}(K_8^{(3)})$ is constructed by developing two initial cycles modulo 7, where $\infty + i = \infty$ for all $i \in \mathbb{Z}_7$. Those two initial cycles are

$$C_1 = (\infty, 3, \hat{3}, \hat{2}, 2, 0, 1, 4, \hat{4}, \hat{0}, \hat{1}, \hat{6}, 6, 5, \hat{5}, \hat{\infty}),$$

$$C_2 = (2, 4, 1, \infty, 6, 5, 3, 0, \hat{0}, \hat{5}, \hat{1}, \hat{\infty}, \hat{3}, \hat{4}, \hat{6}, \hat{2}).$$

$(1,1,5)$	$(1,2,4)$	$(1,3,3)$	$(1,4,2)$	$(2,2,3)$	$(1,\infty)$	$(2,\infty)$	$(3,\infty)$
$\{0,1,2\}C_1$	$\{0,1,3\}$	$\{0,1,4\}C_1$	$\{0,1,5\}$	$\{0,2,4\}$	$\{\infty,0,1\}$	$\{\infty,0,2\}$	$\{\infty,0,3\}$
$\{1,2,3\}$	$\{1,2,4\}C_2$	$\{1,2,5\}$	$\{1,2,6\}$	$\{1,3,5\}$	$\{\infty,1,2\}$	$\{\infty,1,3\}$	$\{\infty,1,4\}C_2$
$\{2,3,4\}$	$\{2,3,5\}$	$\{2,3,6\}$	$\{2,3,0\}$	$\{2,4,6\}$	$\{\infty,2,3\}$	$\{\infty,2,4\}$	$\{\infty,2,5\}$
$\{3,4,5\}$	$\{3,4,6\}$	$\{3,4,0\}$	$\{3,4,1\}$	$\{3,5,0\}C_2$	$\{\infty,3,4\}$	$\{\infty,3,5\}$	$\{\infty,3,6\}$
$\{4,5,6\}$	$\{4,5,0\}$	$\{4,5,1\}$	$\{4,5,2\}$	$\{4,6,1\}$	$\{\infty,4,5\}$	$\{\infty,4,6\}$	$\{\infty,4,0\}$
$\{5,6,0\}$	$\{5,6,1\}$	$\{5,6,2\}$	$\{5,6,3\}C_2$	$\{5,0,2\}$	$\{\infty,5,6\}C_2$	$\{\infty,5,0\}$	$\{\infty,5,1\}$
$\{6,0,1\}$	$\{6,0,2\}$	$\{6,0,3\}$	$\{6,0,4\}$	$\{6,1,3\}$	$\{\infty,6,0\}$	$\{\infty,6,1\}C_2$	$\{\infty,6,2\}$
6	5	4	3	2	1	$-\infty$	∞
$\{\hat{0},0,1\}$	$\{\hat{0},0,2\}$	$\{\hat{0},0,3\}C_2$	$\{\hat{0},0,4\}$	$\{\hat{0},0,5\}$	$\{\hat{0},0,6\}$	$\{\hat{0},0,\infty\}$	$\{\hat{\infty},\infty,0\}$
$\{\hat{1},1,2\}$	$\{\hat{1},1,3\}$	$\{\hat{1},1,4\}$	$\{\hat{1},1,5\}$	$\{\hat{1},1,6\}$	$\{\hat{1},1,0\}$	$\{\hat{1},1,\infty\}$	$\{\hat{\infty},\infty,1\}$
$\{\hat{2},2,3\}$	$\{\hat{2},2,4\}C_2$	$\{\hat{2},2,5\}$	$\{\hat{2},2,6\}$	$\{\hat{2},2,0\}C_1$	$\{\hat{2},2,1\}$	$\{\hat{2},2,\infty\}$	$\{\hat{\infty},\infty,2\}$
$\{\hat{3},3,4\}$	$\{\hat{3},3,5\}$	$\{\hat{3},3,6\}$	$\{\hat{3},3,0\}$	$\{\hat{3},3,1\}$	$\{\hat{3},3,2\}$	$\{\hat{3},3,\infty\}C_1$	$\{\hat{\infty},\infty,3\}C_1$
$\{\hat{4},4,5\}$	$\{\hat{4},4,6\}$	$\{\hat{4},4,0\}$	$\{\hat{4},4,1\}C_1$	$\{\hat{4},4,2\}$	$\{\hat{4},4,3\}$	$\{\hat{4},4,\infty\}$	$\{\hat{\infty},\infty,4\}$
$\{\hat{5},5,6\}C_1$	$\{\hat{5},5,0\}$	$\{\hat{5},5,1\}$	$\{\hat{5},5,2\}$	$\{\hat{5},5,3\}$	$\{\hat{5},5,4\}$	$\{\hat{5},5,\infty\}$	$\{\hat{\infty},\infty,5\}$
$\{\hat{6},6,0\}$	$\{\hat{6},6,1\}$	$\{\hat{6},6,2\}$	$\{\hat{6},6,3\}$	$\{\hat{6},6,4\}$	$\{\hat{6},6,5\}C_1$	$\{\hat{6},6,\infty\}$	$\{\hat{\infty},\infty,6\}$
$(\hat{1},\hat{1},\hat{5})$	$(\hat{1},\hat{2},\hat{4})$	$(\hat{1},\hat{3},\hat{3})$	$(\hat{1},\hat{4},\hat{2})$	$(\hat{2},\hat{2},\hat{3})$	$(\hat{1},\hat{\infty})$	$(\hat{2},\hat{\infty})$	$(\hat{3},\hat{\infty})$
$\{\hat{0},\hat{1},\hat{2}\}$	$\{\hat{0},\hat{1},\hat{3}\}$	$\{\hat{0},\hat{1},\hat{4}\}C_1$	$\{\hat{0},\hat{1},\hat{5}\}C_2$	$\{\hat{0},\hat{2},\hat{4}\}$	$\{\hat{\infty},\hat{0},\hat{1}\}$	$\{\hat{\infty},\hat{0},\hat{2}\}$	$\{\hat{\infty},\hat{0},\hat{3}\}$
$\{\hat{1},\hat{2},\hat{3}\}$	$\{\hat{1},\hat{2},\hat{4}\}$	$\{\hat{1},\hat{2},\hat{5}\}$	$\{\hat{1},\hat{2},\hat{6}\}$	$\{\hat{1},\hat{3},\hat{5}\}$	$\{\hat{\infty},\hat{1},\hat{2}\}$	$\{\hat{\infty},\hat{1},\hat{3}\}C_2$	$\{\hat{\infty},\hat{1},\hat{4}\}$
$\{\hat{2},\hat{3},\hat{4}\}$	$\{\hat{2},\hat{3},\hat{5}\}$	$\{\hat{2},\hat{3},\hat{6}\}$	$\{\hat{2},\hat{3},\hat{0}\}$	$\{\hat{2},\hat{4},\hat{6}\}C_2$	$\{\hat{\infty},\hat{2},\hat{3}\}$	$\{\hat{\infty},\hat{2},\hat{4}\}$	$\{\hat{\infty},\hat{2},\hat{5}\}$
$\{\hat{3},\hat{4},\hat{5}\}$	$\{\hat{3},\hat{4},\hat{6}\}C_2$	$\{\hat{3},\hat{4},\hat{0}\}$	$\{\hat{3},\hat{4},\hat{1}\}$	$\{\hat{3},\hat{5},\hat{0}\}$	$\{\hat{\infty},\hat{3},\hat{4}\}C_2$	$\{\hat{\infty},\hat{3},\hat{5}\}$	$\{\hat{\infty},\hat{3},\hat{6}\}$
$\{\hat{4},\hat{5},\hat{6}\}$	$\{\hat{4},\hat{5},\hat{0}\}$	$\{\hat{4},\hat{5},\hat{1}\}$	$\{\hat{4},\hat{5},\hat{2}\}$	$\{\hat{4},\hat{6},\hat{1}\}$	$\{\hat{\infty},\hat{4},\hat{5}\}$	$\{\hat{\infty},\hat{4},\hat{6}\}$	$\{\hat{\infty},\hat{4},\hat{0}\}$
$\{\hat{5},\hat{6},\hat{0}\}$	$\{\hat{5},\hat{6},\hat{1}\}$	$\{\hat{5},\hat{6},\hat{2}\}$	$\{\hat{5},\hat{6},\hat{3}\}$	$\{\hat{5},\hat{0},\hat{2}\}$	$\{\hat{\infty},\hat{5},\hat{6}\}$	$\{\hat{\infty},\hat{5},\hat{0}\}$	$\{\hat{\infty},\hat{5},\hat{1}\}C_2$
$\{\hat{6},\hat{0},\hat{1}\}C_1$	$\{\hat{6},\hat{0},\hat{2}\}$	$\{\hat{6},\hat{0},\hat{3}\}$	$\{\hat{6},\hat{0},\hat{4}\}$	$\{\hat{6},\hat{1},\hat{3}\}$	$\{\hat{\infty},\hat{6},\hat{0}\}$	$\{\hat{\infty},\hat{6},\hat{1}\}$	$\{\hat{\infty},\hat{6},\hat{2}\}$
$\hat{6}$	$\hat{5}$	$\hat{4}$	$\hat{3}$	$\hat{2}$	$\hat{1}$	$-\hat{\infty}$	$\hat{\infty}$
$\{0,\hat{0},\hat{1}\}$	$\{0,\hat{0},\hat{2}\}$	$\{0,\hat{0},\hat{3}\}$	$\{0,\hat{0},\hat{4}\}$	$\{0,\hat{0},\hat{5}\}C_2$	$\{0,\hat{0},\hat{6}\}$	$\{0,\hat{0},\hat{\infty}\}$	$\{\infty,\hat{\infty},\hat{0}\}$
$\{1,\hat{1},\hat{2}\}$	$\{1,\hat{1},\hat{3}\}$	$\{1,\hat{1},\hat{4}\}$	$\{1,\hat{1},\hat{5}\}$	$\{1,\hat{1},\hat{6}\}$	$\{1,\hat{1},\hat{0}\}$	$\{1,\hat{1},\hat{\infty}\}$	$\{\hat{\infty},\hat{\infty},\hat{1}\}$
$\{2,\hat{2},\hat{3}\}C_1$	$\{2,\hat{2},\hat{4}\}$	$\{2,\hat{2},\hat{5}\}$	$\{2,\hat{2},\hat{6}\}C_2$	$\{2,\hat{2},\hat{0}\}$	$\{2,\hat{2},\hat{1}\}$	$\{2,\hat{2},\hat{\infty}\}$	$\{\infty,\hat{\infty},\hat{2}\}$
$\{3,\hat{3},\hat{4}\}$	$\{3,\hat{3},\hat{5}\}$	$\{3,\hat{3},\hat{6}\}$	$\{3,\hat{3},\hat{0}\}$	$\{3,\hat{3},\hat{1}\}$	$\{3,\hat{3},\hat{2}\}C_1$	$\{3,\hat{3},\hat{\infty}\}$	$\{\infty,\hat{\infty},\hat{3}\}$
$\{4,\hat{4},\hat{5}\}$	$\{4,\hat{4},\hat{6}\}$	$\{4,\hat{4},\hat{0}\}C_1$	$\{4,\hat{4},\hat{1}\}$	$\{4,\hat{4},\hat{2}\}$	$\{4,\hat{4},\hat{3}\}$	$\{4,\hat{4},\hat{\infty}\}$	$\{\infty,\hat{\infty},\hat{4}\}$
$\{5,\hat{5},\hat{6}\}$	$\{5,\hat{5},\hat{0}\}$	$\{5,\hat{5},\hat{1}\}$	$\{5,\hat{5},\hat{2}\}$	$\{5,\hat{5},\hat{3}\}$	$\{5,\hat{5},\hat{4}\}$	$\{5,\hat{5},\hat{\infty}\}C_1$	$\{\infty,\hat{\infty},\hat{5}\}C_1$
$\{6,\hat{6},\hat{0}\}$	$\{6,\hat{6},\hat{1}\}C_1$	$\{6,\hat{6},\hat{2}\}$	$\{6,\hat{6},\hat{3}\}$	$\{6,\hat{6},\hat{4}\}$	$\{6,\hat{6},\hat{5}\}$	$\{6,\hat{6},\hat{\infty}\}$	$\{\infty,\hat{\infty},\hat{6}\}$

Table 2.3: The list of edges of $\mathrm{Prism}(K_8^{(3)})$ classified by their difference patterns. Edges that appear in C_1 and C_2 in Theorem 2.4 are labeled with C_1 and C_2.

Table 2.3 shows difference patterns for all edges in C_1 and C_2 and their translations. We see that both $\tau(C_1)$ and $\tau(C_2)$ contain distinct difference patterns and $\tau(C_1) \cap \tau(C_2) = \varnothing$. By Lemma 2.4, $\{C_1 + i \mid i \in \mathbb{Z}_7\} \cup \{C_2 + i \mid i \in \mathbb{Z}_7\}$ are 14 edge-disjoint Hamiltonian cycles and form a Hamiltonian decomposition of Prism$(K_8^{(3)})$. □

For Prism$(K_7^{(3)})$, actually, there are many choices to potentially form a Hamiltonian decomposition as shown in Table 2.4.

x_1	x_2	x_3
1	10	0
2	8	1
3	6	2
4	4	3
5	2	4
6	0	5

Table 2.4: All possible admissible $\{x_i\}$'s for a Hamiltonian decomposition of Prism$(K_7^{(3)})$

Let the vertex set of $K_7^{(3)}$ be $\{1, 2, 3, 4, 5, 6, 7\}$. Using $\{x_1, x_2, x_3\} = \{6, 0, 5\}$ and fixing vertices 6 and 7, we almost get a Hamiltonian decomposition. However, we get 10 Hamiltonian cycles, two cycles of length 5 from ordinary edges, and one cycle from four junctions, as follows:

$(5, 6, 2, 1, 3, 7, 4, \hat{4}, \hat{3}, \hat{7}, \hat{1}, \hat{6}, \hat{2}, \hat{5})$, $(2, 5, \hat{5}, \hat{1}, \hat{3}, 3, 4, \hat{4}, \hat{7}, 7, 1, 6, \hat{6}, \hat{2})$ $(1, 3, 5, 2, 4)$,

$(1, 6, 3, 2, 4, 7, 5, \hat{5}, \hat{4}, \hat{7}, \hat{2}, \hat{6}, \hat{3}, \hat{1})$, $(3, 1, \hat{1}, \hat{2}, \hat{4}, 4, 5, \hat{5}, \hat{7}, 7, 2, 6, \hat{6}, \hat{3})$, $(\hat{1}, \hat{2}, \hat{3}, \hat{4}, \hat{5})$,

$(2, 6, 4, 3, 5, 7, 1, \hat{1}, \hat{5}, \hat{7}, \hat{3}, \hat{6}, \hat{4}, \hat{2})$, $(4, 2, \hat{2}, \hat{3}, \hat{5}, 5, 1, \hat{1}, \hat{7}, 7, 3, 6, \hat{6}, \hat{4})$, $(6, \hat{6}, \hat{7}, 7)$,

$(3, 6, 5, 4, 1, 7, 2, \hat{2}, \hat{1}, \hat{7}, \hat{4}, \hat{6}, \hat{5}, \hat{3})$, $(5, 3, \hat{3}, \hat{4}, \hat{1}, 1, 2, \hat{2}, \hat{7}, 7, 4, 6, \hat{6}, \hat{5})$,

$(4, 6, 1, 5, 2, 7, 3, \hat{3}, \hat{2}, \hat{7}, \hat{5}, \hat{6}, \hat{1}, \hat{4})$, $(1, 4, \hat{4}, \hat{5}, \hat{2}, 2, 3, \hat{3}, \hat{7}, 7, 5, 6, \hat{6}, \hat{1})$.

For $n \equiv 2 \pmod 6$, we notice that each x_i in Table 2.2, is a multiple of $n-1$. We suggest that the cyclic method can be used to construct a Hamiltonian decomposition of Prism$(K_n^{(3)})$ for all $n \equiv 2 \pmod 6$. In conclusion, we have done just the case of $n \in \{4, 5, 8\}$.

CHAPTER III

HAMILTONIAN DECOMPOSITIONS OF $K^{(3)}_{m,m,m}$

The necessary condition for the existence of Hamiltonian decompositions of $K^{(3)}_{m,m,m}$ is $3 \mid m$ since the number of edges in $K^{(3)}_{m,m,m}$, which is $\binom{3m}{3} - 3\binom{m}{3}$, must be divisible by the length of a Hamiltonian cycle which is equal to $3m$. We will show that $K^{(3)}_{m,m,m}$ is Hamiltonian decomposable for all $m \equiv 0 \pmod 3$. First, we study the structure of $K^{(3)}_{m,m,m}$ and find possible Hamiltonian cycles in Section 3.1. Then a collection of Hamiltonian cycles can be constructed from some particular initial cycles by using one of the four algorithms given in Section 3.2. Finally, the combinations of various Hamiltonian cycles form Hamiltonian decompositions of $K^{(3)}_{m,m,m}$ in Section 3.3.

3.1 Preliminaries

The complete tripartite 3-uniform hypergraph $K^{(3)}_{m,m,m}$ defined in Definition 1.6 contains two types of edges, *abc*-edges and *xxy*-edges.

- An *abc*-edge has vertices from each of three partite sets.
- An *xxy*-edge has two vertices from the same partite set and a vertex from another partite set.

We may describe these in other words as follows.

- An *abc*-edge is an edge $\{u_a, u_b, u_c\}$ where $u_a \in V_0$, $u_b \in V_1$ and $u_c \in V_2$.

- An xxy-edge is an edge $\{u_x, u'_x, u_y\}$ where $u_x, u'_x \in V_i$ and $u_y \in V_j$ with $i \neq j$, where $i, j \in \{0, 1, 2\}$.

Since there are three partite sets, we have 6 variations of xxy denoted by aab, aac, bba, bbc, cca and ccb. Now, we investigate possible Hamiltonian cycles in $K^{(3)}_{m,m,m}$.

Due to Definition 1.3 of Katona-Kierstead cycle, a Hamiltonian cycle in $K^{(3)}_{m,m,m}$, which is in the form of cyclic ordering, requires that every consecutive three vertices in the ordering must not come from the same partite set. Let a, b and c indicate three different partite sets where vertices in the ordering come from. We find some possible patterns of indicators a, b and c to form a Hamiltonian cycle in $K^{(3)}_{m,m,m}$. For each 3-consecutive indicators, we must avoid aaa, bbb and ccc to be occurred in the pattern and expect that all six variations of xxy occur evenly in the pattern. Thus, we get four interesting patterns of a Hamiltonian cycle in $K^{(3)}_{m,m,m}$ as follows:

1. $(abccabbca)^*$, where $m = 3t$,
2. $(aabbcc)^*abc$, where $m = 2t + 1$,
3. $(aabbcc)^*$, where $m = 2t$, and
4. $(abc)^*$, where $m = t$.

The syntax $(\cdots)^*$ means repeated pattern in the parentheses t times. Each pattern has a corresponding construction in Section 3.2 summarized in Table 3.1.

For convenience, we denote the vertices of $K^{(3)}_{m,m,m}$ with

$$\begin{aligned} V_0 =& \{0, 1, \ldots, m-1\}, \\ V_1 =& \{\overline{0}, \overline{1}, \ldots, \overline{m-1}\}, \\ V_2 =& \{\overline{\overline{0}}, \overline{\overline{1}}, \ldots, \overline{\overline{m-1}}\}. \end{aligned}$$

By these notation, abc-edges are in the form $\{u, \overline{v}, \overline{\overline{w}}\}$, where $u, v, w \in \mathbb{Z}_m$. For xxy-edges with six variations aab, aac, bba, bbc, cca and ccb, we fix the form of these edges by $\{u, v, \overline{w}\}, \{u, v, \overline{\overline{w}}\}, \{\overline{u}, \overline{v}, w\}, \{\overline{u}, \overline{v}, \overline{\overline{w}}\}, \{\overline{\overline{u}}, \overline{\overline{v}}, w\}$ and $\{\overline{\overline{u}}, \overline{\overline{v}}, \overline{w}\}$, respectively, where $u, v, w \in \mathbb{Z}_m$. All variations of xxy-edges are determined to the form $\{x, x', y\}$, where $x, x' \in \mathbb{Z}_m$ are under the same partite set and $y \in \mathbb{Z}_m$ is under another partite set. Throughout this chapter, for $u, v \in \mathbb{Z}_m$, we define $||u - v||$ by

$$||u - v|| = \min\{(u - v) \pmod{m}, (v - u) \pmod{m}\}.$$

3.2 Hamiltonian Cycles Constructions

We provide four constructions of Hamiltonian cycles in $K^{(3)}_{m,m,m}$ as follows:

1. $C(i, j)$,
2. $C'(i, j)$,
3. $C_M(i)$ and $C'_M(i)$,
4. $h(x, y)$.

Pattern	Condition	Construction	Indices
$(abccabbca)^*$	$m \equiv 0 \pmod 3$	$C(i, j)$	$i, j \in \mathbb{Z}_m$
$(aabbcc)^*abc$	m is odd	$C'(i, j)$	$i, j \in \mathbb{Z}_m$
$(aabbcc)^*$	m is even	$C_M(i), C'_M(i)$	$i \in \mathbb{Z}_{m/2}$
$(abc)^*$	–	$h(x, y)$	$(x, y) \in \mathbb{Z}_m^2$

Table 3.1: Hamiltonian cycle patterns and constructions in $K^{(3)}_{m,m,m}$

3.2.1 $C(i,j)$

For $m \equiv 0 \pmod 3$, define a Hamiltonian cycle $C(i,j)$ of $K^{(3)}_{m,m,m}$ by

$$\begin{aligned}C(i,j) = (&a_0+i, \overline{b_0+j}, \overline{\overline{c_0+i+j}}, \overline{\overline{c_1+i+j}}, a_1+i, \overline{b_1+j}, \overline{b_2+j}, \overline{\overline{c_2+i+j}}, a_2+i,\\ &a_3+i, \overline{b_3+j}, \overline{\overline{c_3+i+j}}, \overline{\overline{c_4+i+j}}, a_4+i, \overline{b_4+j}, \overline{b_5+j}, \overline{\overline{c_5+i+j}}, a_5+i,\\ &\ldots, a_{m-3}+i, \overline{b_{m-3}+j}, \overline{\overline{c_{m-3}+i+j}}, \overline{\overline{c_{m-2}+i+j}}, a_{m-2}+i, \overline{b_{m-2}+j},\\ &\overline{b_{m-1}+j}, \overline{\overline{c_{m-1}+i+j}}, a_{m-1}+i),\end{aligned}$$

where $i,j \in \mathbb{Z}_m$, $\{a_0, a_1, \ldots, a_{m-1}\} = \mathbb{Z}_m$, $\{b_0, b_1, \ldots, b_{m-1}\} = \mathbb{Z}_m$, and $\{c_0, c_1, \ldots, c_{m-1}\} = \mathbb{Z}_m$.

Lemma 3.1. *Let $m \equiv 0 \pmod 3$. Suppose $C(0,0)$ has properties that $c_k - b_k = c_{k'} - b_{k'}$ for all $k,k' \in \mathbb{Z}_m$ with $k \neq k'$, and $||a_{3k-1} - a_{3k}|| \neq ||a_{3k'-1} - a_{3k'}||, ||b_{3k+1} - b_{3k+2}|| \neq ||b_{3k'+1} - b_{3k'+2}||, ||c_{3k} - c_{3k+1}|| \neq ||c_{3k'} - c_{3k'+1}||$ for all $k,k' \in \{0,1,\ldots,\frac{m}{3}-1\}$ with $k \neq k'$. Then $\{C(i,j) : i,j \in \mathbb{Z}_m\}$ is a set of m^2 edge-disjoint Hamiltonian cycles of $K^{(3)}_{m,m,m}$.*

Proof. For abc-edges, we will show that if $\{a_k+i, \overline{b_k+j}, \overline{\overline{c_k+i+j}}\} = \{a_{k'}+i', \overline{b_{k'}+j'}, \overline{\overline{c_{k'}+i'+j'}}\}$, then $i=i'$, $j=j'$ and $k=k'$.

Suppose that $\{a_k+i, \overline{b_k+j}, \overline{\overline{c_k+i+j}}\} = \{a_{k'}+i', \overline{b_{k'}+j'}, \overline{\overline{c_{k'}+i'+j'}}\}$ for some $i,i',j,j',k,k' \in \mathbb{Z}_m$. Then

$$\begin{aligned} a_k+i &\equiv a_{k'}+i' \pmod m,\\ b_k+j &\equiv b_{k'}+j' \pmod m,\\ c_k+i+j &\equiv c_{k'}+i'+j' \pmod m.\end{aligned}$$

Since $c_k - b_k = c_{k'} - b_{k'}$, we get $i=i'$. Thus, $a_k = a_{k'}$, so $k=k'$. Hence, $j=j'$.

For aab-edges, we will show that if $\{a_{3k-1}+i, a_{3k}+i, \overline{b_{3k}+j}\} = \{a_{3k'-1}+i', a_{3k'}+i', \overline{b_{3k'}+j'}\}$, then $i=i'$, $j=j'$ and $k=k'$.

Suppose that $\{a_{3k-1}+i, a_{3k}+i, \overline{b_{3k}+j}\} = \{a_{3k'-1}+i', a_{3k'}+i', \overline{b_{3k'}+j'}\}$ for some $i, i', j, j' \in \mathbb{Z}_m$ and $k, k' \in \{0, 1, \ldots, \frac{m}{3}-1\}$. Then

$$a_{3k-1}+i \equiv a_{3k'-1}+i' \pmod{m},$$

$$a_{3k}+i \equiv a_{3k'}+i' \pmod{m},$$

$$b_{3k}+j \equiv b_{3k'}+j' \pmod{m},$$

or

$$a_{3k-1}+i \equiv a_{3k'}+i' \pmod{m},$$

$$a_{3k}+i \equiv a_{3k'-1}+i' \pmod{m},$$

$$b_{3k}+j \equiv b_{3k'}+j' \pmod{m}.$$

$C(0,0)$ has property that $||a_{3k-1}-a_{3k}|| \neq ||a_{3k'-1}-a_{3k'}||$ for all $k \neq k'$, but we have $a_{3k-1}-a_{3k} \equiv a_{3k'-1}-a_{3k'} \pmod{m}$ or $a_{3k-1}-a_{3k} \equiv a_{3k'}-a_{3k'-1} \pmod{m}$. Thus, we can conclude that $k=k'$. It follows immediately that $i=i'$ and $j=j'$.

For other xxy-edges: aac, bba, bbc, cca and ccb, we can prove the same result in a similar manner. Thus, all $3m \times m^2$ edges of $\{C(i,j) : i,j \in \mathbb{Z}_m\}$ are distinct and $\{C(i,j) : i,j \in \mathbb{Z}_m\}$ is a set of m^2 edge-disjoint Hamiltonian cycles of $K^{(3)}_{m,m,m}$. $\square$

Lemma 3.2. *Let $m \equiv 0 \pmod 3$. Let $c_i = b_i = x_i$ and $a_i = x_{i+1}$ for all $i \in \mathbb{Z}_m$, where*

$$x_{3k} = \begin{cases} 3k/2 & \text{if } k \text{ is even,} \\ (3k+1)/2 & \text{if } k \text{ is odd,} \end{cases}$$

$$x_{3k+1} = 3k+1,$$

$$x_{3k+2} = \begin{cases} \lceil m/2 \rceil + 3k/2 & \text{if } k \text{ is even,} \\ \lceil m/2 \rceil + (3k+1)/2 & \text{if } k \text{ is odd,} \end{cases}$$

and $k \in \{0, 1, \ldots, \frac{m}{3}-1\}$. Then $C(0,0)$ has properties as in Lemma 3.1. Moreover, $||x-x'|| \equiv 1$ or $2 \pmod 3$ for all xxy-edges of the form $\{x, x', y\}$ in $C(0,0)$.

Proof. By this setting, we have $c_k - b_k = c_{k'} - b_{k'} = 0$ for all $k, k' \in \mathbb{Z}_m$ with $k \neq k'$. For $k \in \{0, 1, \ldots, \frac{m}{3} - 1\}$,

$$||a_{3k-1} - a_{3k}|| = ||x_{3k} - x_{3k+1}|| = \begin{cases} (3k+2)/2 & \text{if } k \text{ is even,} \\ (3k+1)/2 & \text{if } k \text{ is odd,} \end{cases}$$

$$||b_{3k+1} - b_{3k+2}|| = ||x_{3k+1} - x_{3k+2}|| = \begin{cases} \lceil m/2 \rceil - (3k+2)/2 & \text{if } k \text{ is even,} \\ \lceil m/2 \rceil - (3k+1)/2 & \text{if } k \text{ is odd,} \end{cases}$$

$$||c_{3k} - c_{3k+1}|| = ||x_{3k} - x_{3k+1}|| = \begin{cases} (3k+2)/2 & \text{if } k \text{ is even,} \\ (3k+1)/2 & \text{if } k \text{ is odd.} \end{cases}$$

Thus, $||a_{3k-1} - a_{3k}|| \neq ||a_{3k'-1} - a_{3k'}||$, $||b_{3k+1} - b_{3k+2}|| \neq ||b_{3k'+1} - b_{3k'+2}||$, $||c_{3k} - c_{3k+1}|| \neq ||c_{3k'} - c_{3k'+1}||$ for all $k, k' \in \{0, 1, \ldots, \frac{m}{3} - 1\}$ with $k \neq k'$ and $||x - x'|| \equiv 1$ or $2 \pmod 3$ for all xxy-edges of the form $\{x, x', y\}$. □

Example 3.1. Let $m = 6$. The cycle $C(0,0)$ in Lemma 3.2 is

$$C(0,0) = (1, \overline{0}, \overline{\overline{0}}, \overline{\overline{1}}, 3, \overline{1}, \overline{3}, \overline{\overline{3}}, 2, 4, \overline{2}, \overline{\overline{2}}, \overline{\overline{4}}, 5, \overline{4}, \overline{5}, \overline{\overline{5}}, 0).$$

3.2.2 $C'(i,j)$

For an odd integer m, define a Hamiltonian cycle $C'(i,j)$ of $K^{(3)}_{m,m,m}$ by

$$\begin{aligned} C'(i,j) = (&a_0 + j, a_1 + j, \overline{b_0 + i + j}, \overline{b_1 + i + j}, \overline{\overline{c_0 + 2i + j}}, \overline{\overline{c_1 + 2i + j}}, \\ &a_2 + j, a_3 + j, \overline{b_2 + i + j}, \overline{b_3 + i + j}, \overline{\overline{c_2 + 2i + j}}, \overline{\overline{c_3 + 2i + j}}, \ldots, \\ &a_{m-3} + j, a_{m-2} + j, \overline{b_{m-3} + i + j}, \overline{b_{m-2} + i + j}, \\ &\overline{\overline{c_{m-3} + 2i + j}}, \overline{\overline{c_{m-2} + 2i + j}}, \\ &a_{m-1} + j, \overline{b_{m-1} + i + j}, \overline{\overline{c_{m-1} + 2i + j}}), \end{aligned}$$

where $i, j \in \mathbb{Z}_m$, $\{a_0, a_1, \ldots, a_{m-1}\} = \mathbb{Z}_m$, $\{b_0, b_1, \ldots, b_{m-1}\} = \mathbb{Z}_m$, and $\{c_0, c_1, \ldots, c_{m-1}\} = \mathbb{Z}_m$.

A similar argument as in the proof of Lemma 3.1 can be used to prove Lemma 3.3.

Lemma 3.3. *For an odd integer m, suppose $C'(0,0)$ has properties that $a_0 + c_{m-1} \neq a_{m-1} + c_{m-2} \pmod{m}$ and $||a_{2k+1} - a_{2k}|| \neq ||a_{2k'+1} - a_{2k'}||, ||b_{2k+1} - b_{2k}|| \neq ||b_{2k'+1} - b_{2k'}||, ||c_{2k+1} - c_{2k}|| \neq ||c_{2k'+1} - c_{2k'}||$ for all $k, k' \in \{0, 1, \ldots, \frac{m-1}{2} - 1\}$ with $k \neq k'$. Then $\{C'(i,j) : i, j \in \mathbb{Z}_m\}$ is a set of m^2 edge-disjoint Hamiltonian cycles of $K^{(3)}_{m,m,m}$.*

Proof. For abc-edges, we will show that if $\{a_k + j, \overline{b_{m-1} + i + j}, \overline{\overline{c_\ell + 2i + j}}\} = \{a_{k'} + j', \overline{b_{m-1} + i' + j'}, \overline{\overline{c_{\ell'} + 2i' + j'}}\}$, then $i = i'$, $j = j'$, $k = k'$ and $\ell = \ell'$.

Suppose that $\{a_k + j, \overline{b_{m-1} + i + j}, \overline{\overline{c_\ell + 2i + j}}\} = \{a_{k'} + j', \overline{b_{m-1} + i' + j'}, \overline{\overline{c_{\ell'} + 2i' + j'}}\}$ for some $i, i', j, j' \in \mathbb{Z}_m$, and $(k, \ell), (k', \ell') \in \{(0, m-1), (m-1, m-1), (m-1, m-2)\}$. Then

$$a_k + j \equiv a_{k'} + j' \pmod{m},$$

$$b_{m-1} + i + j \equiv b_{m-1} + i' + j' \pmod{m},$$

$$c_\ell + 2i + j \equiv c_{\ell'} + 2i' + j' \pmod{m}.$$

If $k = k'$, then $j = j'$, $i = i'$ and $\ell = \ell'$. Suppose that $k \neq k'$. Without loss of generality, assume that $k = 0$ and $k' = m-1$. We will make a contradiction. By the values of k and k', $\ell = m-1$ and $\ell' = m-1$ or $m-2$. If $\ell = \ell' = m-1$, then $i = i'$, $j = j'$ and $a_0 = a_{m-1}$, a contradiction. Next, let $\ell' = m-2$. We get $a_0 + c_{m-1} + 2i + 2j \equiv a_{m-1} + c_{m-2} + 2i' + 2j' \pmod{m}$. Thus, $a_0 + c_{m-1} \equiv a_{m-1} + c_{m-2} \pmod{m}$. This contradicts to the assumption that $a_0 + c_{m-1} \neq a_{m-1} + c_{m-2} \pmod{m}$.

For aab-edges, we will show that if $\{a_{2k} + j, a_{2k+1} + j, \overline{b_{2k} + i + j}\} = \{a_{2k'} + j', a_{2k'+1} + j', \overline{b_{2k'} + i' + j'}\}$, then $i = i'$, $j = j'$ and $k = k'$.

Suppose that $\{a_{2k} + j, a_{2k+1} + j, \overline{b_{2k} + i + j}\} = \{a_{2k'} + j', a_{2k'+1} + j', \overline{b_{2k'} + i' + j'}\}$

for some $i, i', j, j' \in \mathbb{Z}_m$ and $k, k' \in \{0, 1, \ldots, \frac{m-1}{2} - 1\}$. Then

$$a_{2k} + j \equiv a_{2k'} + j' \pmod m,$$

$$a_{2k+1} + j \equiv a_{2k'+1} + j' \pmod m,$$

$$b_{2k} + i + j \equiv b_{2k'} + i' + j' \pmod m,$$

or

$$a_{2k} + j \equiv a_{2k'+1} + j' \pmod m,$$

$$a_{2k+1} + j \equiv a_{2k'} + j' \pmod m,$$

$$b_{2k} + i + j \equiv b_{2k'} + i' + j' \pmod m.$$

$C'(0,0)$ has property that $||a_{2k+1} - a_{2k}|| \neq ||a_{2k'+1} - a_{2k'}||$ for all $k \neq k'$, but we have $a_{2k+1} - a_{2k} \equiv a_{2k'+1} - a_{2k'} \pmod m$ or $a_{2k+1} - a_{2k} \equiv a_{2k'} - a_{2k'+1} \pmod m$. Thus, we can conclude that $k = k'$. It follows immediately that $j = j'$ and $i = i'$.

For other xxy-edges: aac, bba, bbc, cca and ccb, we can prove the same result in a similar manner. Thus, all $3m \times m^2$ edges of $\{C'(i,j) : i, j \in \mathbb{Z}_m\}$ are distinct and $\{C'(i,j) : i, j \in \mathbb{Z}_m\}$ is a set of m^2 edge-disjoint Hamiltonian cycles of $K^{(3)}_{m,m,m}$. □

Lemma 3.4. *For an odd integer m, let $a_i = b_i = x_i$ for all $i \in \mathbb{Z}_m$, $c_{m-3} = x_0$, $c_{m-2} = x_1$, $c_{m-1} = x_{m-1}$ and $c_i = x_{i+2}$ for all $i \in \{0, 1, \ldots, m-4\}$, where*

$$x_{m-1} = 1,$$

$$x_{2k} = m - k,$$

$$x_{2k+1} = k + 2,$$

and $k \in \{0, 1, \ldots, \frac{m-1}{2} - 1\}$. Then $C'(0,0)$ has properties as in Lemma 3.3. Moreover, $b_{m-1} - a_{m-1} = 0, b_{m-1} - a_0 = 1, c_{m-1} - b_{m-1} = 0, c_{m-2} - b_{m-1} = 1$.

Proof. By this setting, we have $a_0 + c_{m-1} = 1$ and $a_{m-1} + c_{m-2} = 3$.
For $k \in \{0, 1, \ldots, \frac{m-1}{2} - 1\}$,

$$\begin{aligned} ||a_{2k+1} - a_{2k}|| = ||b_{2k+1} - b_{2k}|| &= ||x_{2k+1} - x_{2k}|| \\ &= \min\{2k+2, m - (2k+2)\}. \end{aligned}$$

For $k \in \{0, 1, \ldots, \frac{m-1}{2} - 2\}$,

$$\begin{aligned} ||c_{2k+1} - c_{2k}|| &= ||x_{2k+3} - x_{2k+2}|| \\ &= \min\{2k+4, m-(2k+4)\} \end{aligned}$$

and $c_{m-2} - c_{m-3} = x_1 - x_0 = 2$.

Since m is odd, $\{||x_{2k+1} - x_{2k}|| : k \in \{0, 1, \ldots, \frac{m-1}{2} - 1\}\} = \{1, 2, \ldots, \frac{m-1}{2}\}$. Thus, $a_0 + c_{m-1} \neq a_{m-1} + c_{m-2} \pmod{m}$ and $||a_{2k+1} - a_{2k}|| \neq ||a_{2k'+1} - a_{2k'}||$, $||b_{2k+1} - b_{2k}|| \neq ||b_{2k'+1} - b_{2k'}||$, $||c_{2k+1} - c_{2k}|| \neq ||c_{2k'+1} - c_{2k'}||$ for all $k, k' \in \{0, 1, \ldots, \frac{m-1}{2} - 1\}$ with $k \neq k'$. □

Example 3.2. Let $m = 9$. The cycle $C'(0,0)$ in Lemma 3.4 is

$$C'(0,0) = (0, 2, \overline{0}, \overline{2}, \overline{\overline{8}}, \overline{\overline{3}}, 8, 3, \overline{8}, \overline{3}, \overline{\overline{7}}, \overline{\overline{4}}, 7, 4, \overline{7}, \overline{4}, \overline{\overline{6}}, \overline{\overline{5}}, 6, 5, \overline{6}, \overline{5}, \overline{\overline{0}}, \overline{\overline{2}}, 1, \overline{1}, \overline{\overline{1}}).$$

3.2.3 $C_M(i)$ and $C'_M(i)$

For an even integer m, we will construct a family of Hamiltonian cycles $C_M(i)$ and $C'_M(i)$ which contain no *abc*-edges. It requires the knowledge of 1-factors and orthogonal quasigroups.

Definition 3.1. [4] Let G be a graph. A *1-factor* of G is a subgraph of G in which every vertex has degree 1. A *1-factorization* of G is a partition of an edge set of G into 1-factors.

Definition 3.2. [8] $(\mathbb{Z}_n, \circ)$ is a *quasigroup* if

(1) $i \circ j \in \mathbb{Z}_n$ for all $i, j \in \mathbb{Z}_n$ and

(2) $i \circ j \neq i \circ j'$ and $i \circ j \neq i' \circ j$ for all $i, j \in \mathbb{Z}_n$ with $i \neq i', j \neq j'$.

Note that the multiplication table of $(\mathbb{Z}_n, \circ)$ is a *Latin square.*

Definition 3.3. [8] $(\mathbb{Z}_n, \circ_1)$ and $(\mathbb{Z}_n, \circ_2)$ are *orthogonal* if for $(i,j) \neq (i',j') \in \mathbb{Z}_n^2$, $i \circ_1 j = i' \circ_1 j'$ implies $i \circ_2 j \neq i' \circ_2 j'$.

The following Lemma is the well-known result concerning Latin squares.

Lemma 3.5. *[8] There exists a pair of mutually orthogonal Latin squares of order n for every $n \neq 2$ or 6.*

For an even integer m, let $M = \{x_0x_1, x_2x_3, x_4x_5, \dots, x_{m-2}x_{m-1}\}$ be a 1-factor of a graph with $\mathbb{Z}_m = \{x_0, x_1, \dots, x_{m-1}\}$ as a vertex set. By Lemma 3.5, there exists a pair of orthogonal quasigroups, $(\mathbb{Z}_{m/2}, \circ_1)$ and $(\mathbb{Z}_{m/2}, \circ_2)$ for $m \neq 4$ or 12. For $i \in \mathbb{Z}_{m/2}$, define Hamiltonian cycles of $K^{(3)}_{m,m,m}$, $C_M(i)$ and $C'_M(i)$, by

$$\begin{aligned} C_M(i) = (&x_0, x_1, \overline{x_{2(i\circ_1 0)}}, \overline{x_{2(i\circ_1 0)+1}}, \overline{\overline{x_{2(i\circ_2 0)}}}, \overline{\overline{x_{2(i\circ_2 0)+1}}}, \\ &x_2, x_3, \overline{x_{2(i\circ_1 1)}}, \overline{x_{2(i\circ_1 1)+1}}, \overline{\overline{x_{2(i\circ_2 1)}}}, \overline{\overline{x_{2(i\circ_2 1)+1}}}, \dots, \\ &x_{m-2}, x_{m-1}, \overline{x_{2(i\circ_1 \frac{m-2}{2})}}, \overline{x_{2(i\circ_1 \frac{m-2}{2})+1}}, \overline{\overline{x_{2(i\circ_2 \frac{m-2}{2})}}}, \overline{\overline{x_{2(i\circ_2 \frac{m-2}{2})+1}}}) \end{aligned}$$

and

$$\begin{aligned} C'_M(i) = (&x_1, x_0, \overline{x_{2(i\circ_1 0)+1}}, \overline{x_{2(i\circ_1 0)}}, \overline{\overline{x_{2(i\circ_2 0)+1}}}, \overline{\overline{x_{2(i\circ_2 0)}}}, \\ &x_3, x_2, \overline{x_{2(i\circ_1 1)+1}}, \overline{x_{2(i\circ_1 1)}}, \overline{\overline{x_{2(i\circ_2 1)+1}}}, \overline{\overline{x_{2(i\circ_2 1)}}}, \dots, \\ &x_{m-1}, x_{m-2}, \overline{x_{2(i\circ_1 \frac{m-2}{2})+1}}, \overline{x_{2(i\circ_1 \frac{m-2}{2})}}, \overline{\overline{x_{2(i\circ_2 \frac{m-2}{2})+1}}}, \overline{\overline{x_{2(i\circ_2 \frac{m-2}{2})}}}). \end{aligned}$$

Example 3.3. Let $m = 6$. The multiplication tables of orthogonal quasigroups $(\mathbb{Z}_3, \circ_1)$ and $(\mathbb{Z}_3, \circ_2)$ are as follows.

$\circ_1$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$\circ_2$	0	1	2
0	0	1	2
1	2	0	1
2	1	2	0

Let $M = \{x_0x_1, x_2x_3, x_4x_5\} = \{03, 14, 25\}$. Then

$$C_M(0) = (0, 3, \overline{0}, \overline{3}, \overline{\overline{0}}, \overline{\overline{3}}, 1, 4, \overline{1}, \overline{4}, \overline{\overline{1}}, \overline{\overline{4}}, 2, 5, \overline{2}, \overline{5}, \overline{\overline{2}}, \overline{\overline{5}}),$$

$$C_M(1) = (0, 3, \overline{1}, \overline{4}, \overline{\overline{2}}, \overline{\overline{5}}, 1, 4, \overline{2}, \overline{5}, \overline{\overline{0}}, \overline{\overline{3}}, 2, 5, \overline{0}, \overline{3}, \overline{\overline{1}}, \overline{\overline{4}}),$$

$$C_M(2) = (0, 3, \overline{2}, \overline{5}, \overline{\overline{1}}, \overline{\overline{4}}, 1, 4, \overline{0}, \overline{3}, \overline{\overline{2}}, \overline{\overline{5}}, 2, 5, \overline{1}, \overline{4}, \overline{\overline{0}}, \overline{\overline{3}}),$$

$$C'_M(0) = (3, 0, \overline{3}, \overline{0}, \overline{\overline{3}}, \overline{\overline{0}}, 4, 1, \overline{4}, \overline{1}, \overline{\overline{4}}, \overline{\overline{1}}, 5, 2, \overline{5}, \overline{2}, \overline{\overline{5}}, \overline{\overline{2}}),$$

$$C'_M(1) = (3, 0, \overline{4}, \overline{1}, \overline{\overline{5}}, \overline{\overline{2}}, 4, 1, \overline{5}, \overline{2}, \overline{\overline{3}}, \overline{\overline{0}}, 5, 2, \overline{3}, \overline{0}, \overline{\overline{4}}, \overline{\overline{1}}),$$

$$C'_M(2) = (3, 0, \overline{5}, \overline{2}, \overline{\overline{4}}, \overline{\overline{1}}, 4, 1, \overline{3}, \overline{0}, \overline{\overline{5}}, \overline{\overline{2}}, 5, 2, \overline{4}, \overline{1}, \overline{\overline{3}}, \overline{\overline{0}}).$$

For $m = 4$ and 12, there are no orthogonal quasigroups $(\mathbb{Z}_{m/2}, \circ_1)$ and $(\mathbb{Z}_{m/2}, \circ_2)$. Therefore, $C_M(i)$ and $C'_M(i)$ will be constructed by the following.

For $m = 4$, let $M = \{x_0x_1, x_2x_3\}$. Then

$$C_M(0) = (x_0, x_1, \overline{x_0}, \overline{x_1}, \overline{\overline{x_0}}, \overline{\overline{x_1}}, x_2, x_3, \overline{x_2}, \overline{x_3}, \overline{\overline{x_2}}, \overline{\overline{x_3}}),$$

$$C_M(1) = (x_0, x_1, \overline{x_3}, \overline{x_2}, \overline{\overline{x_3}}, \overline{\overline{x_2}}, x_2, x_3, \overline{x_1}, \overline{x_0}, \overline{\overline{x_1}}, \overline{\overline{x_0}}),$$

$$C'_M(0) = (x_1, x_0, \overline{x_1}, \overline{x_0}, \overline{\overline{x_2}}, \overline{\overline{x_3}}, x_3, x_2, \overline{x_3}, \overline{x_2}, \overline{\overline{x_0}}, \overline{\overline{x_1}}),$$

$$C'_M(1) = (x_1, x_0, \overline{x_2}, \overline{x_3}, \overline{\overline{x_1}}, \overline{\overline{x_0}}, x_3, x_2, \overline{x_0}, \overline{x_1}, \overline{\overline{x_3}}, \overline{\overline{x_2}}).$$

For $m = 12$, let $(\mathbb{Z}_{m/4}, \circ_3)$ and $(\mathbb{Z}_{m/4}, \circ_4)$ be orthogonal quasigroups. For $i \in \mathbb{Z}_{m/2}$, define $C_M(i)$ and $C'_M(i)$ by

$$\begin{aligned} C_M(i) = (&x_0, x_1, \overline{b_0(i)}, \overline{b_1(i)}, \overline{\overline{c_0(i)}}, \overline{\overline{c_1(i)}}, \\ &x_2, x_3, \overline{b_2(i)}, \overline{b_3(i)}, \overline{\overline{c_2(i)}}, \overline{\overline{c_3(i)}}, \ldots, \\ &x_{m-2}, x_{m-1}, \overline{b_{m-2}(i)}, \overline{b_{m-1}(i)}, \overline{\overline{c_{m-2}(i)}}, \overline{\overline{c_{m-1}(i)}}) \end{aligned}$$

and

$$C'_M(i) = (x_1, x_0, \overline{b'_0(i)}, \overline{b'_1(i)}, \overline{\overline{c'_0(i)}}, \overline{\overline{c'_1(i)}},$$
$$x_3, x_2, \overline{b'_2(i)}, \overline{b'_3(i)}, \overline{\overline{c'_2(i)}}, \overline{\overline{c'_3(i)}}, \ldots,$$
$$x_{m-1}, x_{m-2}, \overline{b'_{m-2}(i)}, \overline{b'_{m-1}(i)}, \overline{\overline{c'_{m-2}(i)}}, \overline{\overline{c'_{m-1}(i)}}),$$

where for $j, k \in \{0, 1, \ldots, \frac{m}{4} - 1\}$,

$$b_{2k}(j) = b_{\frac{m}{2}+2k+1}(j + \tfrac{m}{4}) = b'_{2k+1}(j) = b'_{\frac{m}{2}+2k}(j + \tfrac{m}{4}) = x_{2(j \circ_3 k)},$$
$$c_{2k}(j) = c_{\frac{m}{2}+2k+1}(j + \tfrac{m}{4}) = c'_{2k+1}(j + \tfrac{m}{4}) = c'_{\frac{m}{2}+2k}(j) = x_{2(j \circ_4 k)},$$
$$b_{2k+1}(j) = b_{\frac{m}{2}+2k}(j + \tfrac{m}{4}) = b'_{2k}(j) = b'_{\frac{m}{2}+2k+1}(j + \tfrac{m}{4}) = x_{2(j \circ_3 k)+1},$$
$$c_{2k+1}(j) = c_{\frac{m}{2}+2k}(j + \tfrac{m}{4}) = c'_{2k}(j + \tfrac{m}{4}) = c'_{\frac{m}{2}+2k+1}(j) = x_{2(j \circ_4 k)+1},$$
$$b_{2k+1}(j + \tfrac{m}{4}) = b_{\frac{m}{2}+2k}(j) = b'_{2k}(j + \tfrac{m}{4}) = b'_{\frac{m}{2}+2k+1}(j) = x_{\frac{m}{2}+2(j \circ_3 k)},$$
$$c_{2k+1}(j + \tfrac{m}{4}) = c_{\frac{m}{2}+2k}(j) = c'_{2k}(j) = c'_{\frac{m}{2}+2k+1}(j + \tfrac{m}{4}) = x_{\frac{m}{2}+2(j \circ_4 k)},$$
$$b_{2k}(j + \tfrac{m}{4}) = b_{\frac{m}{2}+2k+1}(j) = b'_{2k+1}(j + \tfrac{m}{4}) = b'_{\frac{m}{2}+2k}(j) = x_{\frac{m}{2}+2(j \circ_3 k)+1},$$
$$c_{2k}(j + \tfrac{m}{4}) = c_{\frac{m}{2}+2k+1}(j) = c'_{2k+1}(j) = c'_{\frac{m}{2}+2k}(j + \tfrac{m}{4}) = x_{\frac{m}{2}+2(j \circ_4 k)+1}.$$

Lemma 3.6. *For an even integer m, given a 1-factor M of a graph with $\mathbb{Z}_m$ as a vertex set, $\{C_M(i), C'_M(i) : i \in \mathbb{Z}_{m/2}\}$ is a set of m edge-disjoint Hamiltonian cycles of $K^{(3)}_{m,m,m}$.*

Proof. Let $M = \{x_0x_1, x_2x_3, \ldots, x_{m-2}x_{m-1}\}$. Consider the case where $m \notin \{4, 12\}$. For aab-edges, we will show that if $\{x_{2k}, x_{2k+1}, \overline{x_{2(i \circ_1 k)+j}}\} = \{x_{2k'}, x_{2k'+1}, \overline{x_{2(i' \circ_1 k')+j'}}\}$, then $i = i'$, $j = j'$ and $k = k'$.

Suppose that $\{x_{2k}, x_{2k+1}, \overline{x_{2(i \circ_1 k)+j}}\} = \{x_{2k'}, x_{2k'+1}, \overline{x_{2(i' \circ_1 k')+j'}}\}$ for some i, i', $k, k' \in \mathbb{Z}_{m/2}$ and $j, j' \in \{0, 1\}$. Then

$$2k = 2k',$$
$$2(i \circ_1 k) + j = 2(i' \circ_1 k') + j'.$$

That is $k = k'$, $j = j'$ and $i \circ_1 k = i' \circ_1 k$. Since $(\mathbb{Z}_{m/2}, \circ_1)$ is a quasigroup, $i = i'$.

The proof for aac-edges can be done in the same way.

For bbc-edges, we will show that if $\{\overline{x_{2(i\circ_1 k)}}, \overline{x_{2(i\circ_1 k)+1}}, \overline{\overline{x_{2(i\circ_2 k)+j}}}\} = \{\overline{x_{2(i'\circ_1 k')}}, \overline{x_{2(i'\circ_1 k')+1}}, \overline{\overline{x_{2(i'\circ_2 k')+j'}}}\}$, then $i = i'$, $j = j'$ and $k = k'$.

Suppose that $\{\overline{x_{2(i\circ_1 k)}}, \overline{x_{2(i\circ_1 k)+1}}, \overline{\overline{x_{2(i\circ_2 k)+j}}}\} = \{\overline{x_{2(i'\circ_1 k')}}, \overline{x_{2(i'\circ_1 k')+1}}, \overline{\overline{x_{2(i'\circ_2 k')+j'}}}\}$ for some $i, i', k, k' \in \mathbb{Z}_{m/2}$ and $j, j' \in \{0, 1\}$. Then

$$i \circ_1 k = i' \circ_1 k',$$

$$i \circ_2 k = i' \circ_2 k',$$

$$j = j'.$$

Since $(\mathbb{Z}_{m/2}, \circ_1)$ and $(\mathbb{Z}_{m/2}, \circ_2)$ are orthogonal quasigroups, we have $i = i'$ and $k = k'$.

The proof for other xxy-edges: bba, cca and ccb can also be done in the same way. Thus, all $3m \times m$ edges of $\{C_M(i), C'_M(i) : i \in \mathbb{Z}_{m/2}\}$ are distinct and $\{C_M(i), C'_M(i) : i \in \mathbb{Z}_{m/2}\}$ is a set of m edge-disjoint Hamiltonian cycles of $K^{(3)}_{m,m,m}$.

For $m = 4$, it is easy to see that $C_M(0)$, $C_M(1)$, $C'_M(0)$ and $C'_M(1)$ as shown before are mutually edge-disjoint Hamiltonian cycles of $K^{(3)}_{m,m,m}$.

For $m = 12$, consider aab-edges: $e_1 = \{x_{2k}, x_{2k+1}, \overline{x_i}\}$ and $e_2 = \{x_{\frac{m}{2}+2k}, x_{\frac{m}{2}+2k+1}, \overline{x_i}\}$, where $k \in \mathbb{Z}_{m/4}$ and $i \in \mathbb{Z}_m$. Note that $\{2(j \circ_3 k), 2(j \circ_3 k) + 1, \frac{m}{2} + 2(j \circ_3 k), \frac{m}{2} + 2(j \circ_3 k) + 1 : j, k \in \mathbb{Z}_{m/4}\} = \mathbb{Z}_m$ by means of a quasigroup.

If $i = 2(j \circ_3 k)$, then $e_1 \in C_M(j)$ and $e_2 \in C'_M(j + \frac{m}{4})$.

If $i = 2(j \circ_3 k) + 1$, then $e_1 \in C'_M(j)$ and $e_2 \in C_M(j + \frac{m}{4})$.

If $i = \frac{m}{2} + 2(j \circ_3 k)$, then $e_1 \in C'_M(j + \frac{m}{4})$ and $e_2 \in C_M(j)$.

If $i = \frac{m}{2} + 2(j \circ_3 k) + 1$, then $e_1 \in C_M(j + \frac{m}{4})$ and $e_2 \in C'_M(j)$.

Thus, each aab-edge is in a unique Hamiltonian cycle. We can also use this argument to show the same result for aac-edges.

For bbc-edges: $\{\overline{x_{2(j\circ_3 k)}}, \overline{x_{2(j\circ_3 k)+1}}, \overline{\overline{x_i}}\}$ (or $\{\overline{x_{\frac{m}{2}+2(j\circ_3 k)}}, \overline{x_{\frac{m}{2}+2(j\circ_3 k)+1}}, \overline{\overline{x_i}}\}$), we will show that if $\{\overline{x_{2(j\circ_3 k)}}, \overline{x_{2(j\circ_3 k)+1}}, \overline{\overline{x_i}}\} = \{\overline{x_{2(j'\circ_3 k')}}, \overline{x_{2(j'\circ_3 k')+1}}, \overline{\overline{x_{i'}}}\}$, then $i = i'$, $j = j'$ and $k = k'$.

Suppose that $\{\overline{x_{2(j\circ_3 k)}}, \overline{x_{2(j\circ_3 k)+1}}, \overline{\overline{x_i}}\} = \{\overline{x_{2(j'\circ_3 k')}}, \overline{x_{2(j'\circ_3 k')+1}}, \overline{\overline{x_{i'}}}\}$ for some j, j', $k, k' \in \mathbb{Z}_{m/4}$ and $i \in \mathbb{Z}_m$. Then

$$j \circ_3 k = j' \circ_3 k',$$
$$i = i'.$$

There are four possibilities for i: $2(j\circ_4 k)$, $2(j\circ_4 k)+1$, $\frac{m}{2}+2(j\circ_4 k)$ or $\frac{m}{2}+2(j\circ_4 k)+1$ (also for i': $2(j'\circ_4 k')$, $2(j'\circ_4 k')+1$, $\frac{m}{2}+2(j'\circ_4 k')$ or $\frac{m}{2}+2(j'\circ_4 k')+1$). Since $i = i'$, in any cases, we have $j \circ_4 k = j' \circ_4 k'$. The orthogonality of $(\mathbb{Z}_{m/4}, \circ_3)$ and $(\mathbb{Z}_{m/4}, \circ_4)$ implies $j = j'$ and $k = k'$.

Other xxy-edges: bba, cca and ccb can be showed by using the same technique. This completes the proof. □

3.2.4 $h(x, y)$

For $(x, y) \in \mathbb{Z}_m^2$, define a Hamiltonian cycle of $K^{(3)}_{m,m,m}$, $h(x, y)$ by

$$h(x,y) = (0, \overline{x}, \overline{\overline{x+y}}, m-1, \overline{m-1+x}, \overline{\overline{m-1+x+y}}, \ldots, 1, \overline{1+x}, \overline{\overline{1+x+y}}).$$

We lend this construction of Hamiltonian cycle of $K^{(3)}_{m,m,m}$ from Kuhl and Schroeder in [7]. They assign the *difference type* $(v-u, w-v)$ to each abc-edges $\{u, \overline{v}, \overline{\overline{w}}\}$. In $K^{(3)}_{m,m,m}$, there are m^2 distinct difference types and there are m edges with a specific difference type. Each $h(x, y)$ has $3m$ abc-edges and contains all abc-edges of difference types (x, y), $(x+1, y)$ and $(x, y+1)$.

Lemma 3.7. *[7] Let* $m \equiv 0 \pmod 3$ *and* $\mathcal{A}_0 = \{(x,y) \in \mathbb{Z}_m^2 : x-y \equiv 0 \pmod 3\}$. *Then* $\{h(x,y) : (x,y) \in \mathcal{A}_0\}$ *is a set of* $m^2/3$ *edge-disjoint Hamiltonian cycles of* $K^{(3)}_{m,m,m}$.

Example 3.4. From Lemma 3.7, in the case of $m = 6$, we obtain $\mathcal{A}_0 = \{(0,0), (1,1), (2,2), (3,3), (4,4), (5,5), (0,3), (1,4), (2,5), (3,0), (4,1), (5,2)\}$ and $h(x,y)$, where $(x,y) \in \mathcal{A}_0$ are listed below.

$$
\begin{aligned}
h(0,0) &= (0,\bar{0},\bar{\bar{0}},5,\bar{5},\bar{\bar{5}},4,\bar{4},\bar{\bar{4}},3,\bar{3},\bar{\bar{3}},2,\bar{2},\bar{\bar{2}},1,\bar{1},\bar{\bar{1}}),\\
h(1,1) &= (0,\bar{1},\bar{\bar{2}},5,\bar{0},\bar{\bar{1}},4,\bar{5},\bar{\bar{0}},3,\bar{4},\bar{\bar{5}},2,\bar{3},\bar{\bar{4}},1,\bar{2},\bar{\bar{3}}),\\
h(2,2) &= (0,\bar{2},\bar{\bar{4}},5,\bar{1},\bar{\bar{3}},4,\bar{0},\bar{\bar{2}},3,\bar{5},\bar{\bar{1}},2,\bar{4},\bar{\bar{0}},1,\bar{3},\bar{\bar{5}}),\\
h(3,3) &= (0,\bar{3},\bar{\bar{0}},5,\bar{2},\bar{\bar{5}},4,\bar{1},\bar{\bar{4}},3,\bar{0},\bar{\bar{3}},2,\bar{5},\bar{\bar{2}},1,\bar{4},\bar{\bar{1}}),\\
h(4,4) &= (0,\bar{4},\bar{\bar{2}},5,\bar{3},\bar{\bar{1}},4,\bar{2},\bar{\bar{0}},3,\bar{1},\bar{\bar{5}},2,\bar{0},\bar{\bar{4}},1,\bar{5},\bar{\bar{3}}),\\
h(5,5) &= (0,\bar{5},\bar{\bar{4}},5,\bar{4},\bar{\bar{3}},4,\bar{3},\bar{\bar{2}},3,\bar{2},\bar{\bar{1}},2,\bar{1},\bar{\bar{0}},1,\bar{0},\bar{\bar{5}}),\\
h(0,3) &= (0,\bar{0},\bar{\bar{3}},5,\bar{5},\bar{\bar{2}},4,\bar{4},\bar{\bar{1}},3,\bar{3},\bar{\bar{0}},2,\bar{2},\bar{\bar{5}},1,\bar{1},\bar{\bar{4}}),\\
h(1,4) &= (0,\bar{1},\bar{\bar{5}},5,\bar{0},\bar{\bar{4}},4,\bar{5},\bar{\bar{3}},3,\bar{4},\bar{\bar{2}},2,\bar{3},\bar{\bar{1}},1,\bar{2},\bar{\bar{0}}),\\
h(2,5) &= (0,\bar{2},\bar{\bar{1}},5,\bar{1},\bar{\bar{0}},4,\bar{0},\bar{\bar{5}},3,\bar{5},\bar{\bar{4}},2,\bar{4},\bar{\bar{3}},1,\bar{3},\bar{\bar{2}}),\\
h(3,0) &= (0,\bar{3},\bar{\bar{3}},5,\bar{2},\bar{\bar{2}},4,\bar{1},\bar{\bar{1}},3,\bar{0},\bar{\bar{0}},2,\bar{5},\bar{\bar{5}},1,\bar{4},\bar{\bar{4}}),\\
h(4,1) &= (0,\bar{4},\bar{\bar{5}},5,\bar{3},\bar{\bar{4}},4,\bar{2},\bar{\bar{3}},3,\bar{1},\bar{\bar{2}},2,\bar{0},\bar{\bar{1}},1,\bar{5},\bar{\bar{0}}),\\
h(5,2) &= (0,\bar{5},\bar{\bar{1}},5,\bar{4},\bar{\bar{0}},4,\bar{3},\bar{\bar{5}},3,\bar{2},\bar{\bar{4}},2,\bar{1},\bar{\bar{3}},1,\bar{0},\bar{\bar{2}}).
\end{aligned}
$$

3.3 Results

We will show that $K^{(3)}_{m,m,m}$ is decomposable into Hamiltonian cycles for all $m \equiv 0 \pmod 3$. We separate the cases of m into odd and even, that is $m \equiv 0 \pmod 6$ and $m \equiv 3 \pmod 6$, and add a special case of $m = 3$.

Let H be a subhypergraph of $K^{(3)}_{m,m,m}$. Let $n_1(H)$ and $n_2(H)$ denote the number of abc-edges and xxy-edges in H, respectively. Each Hamiltonian cycle H in Section 3.2, $C(i,j)$, $C'(i,j)$, ($C_M(i)$ and $C'_M(i)$), and $h(x,y)$ can be regarded as a subhypergraph of $K^{(3)}_{m,m,m}$. The number of edges in $K^{(3)}_{m,m,m}$, $\binom{3m}{3} - 3\binom{m}{3}$ which is equal to $4m^3 - 3m^2$ can be divided into m^3 *abc*-edges and $3m^3 - 3m^2$ *xxy*-edges. We count $n_1(H)$ and $n_2(H)$ as shown in Table 3.2. If $\mathcal{H}$ is a Hamiltonian decomposition of $K^{(3)}_{m,m,m}$, then $\sum_{H\in\mathcal{H}} n_1(H) = m^3$ and $\sum_{H\in\mathcal{H}} n_2(H) = 3m^3 - 3m^2$.

H	$n_1(H)$	$n_2(H)$	Condition
$K^{(3)}_{m,m,m}$	m^3	$3m^3 - 3m^2$	–
$C(i,j)$	m	$2m$	$m \equiv 0 \pmod 3$
$C'(i,j)$	3	$3m - 3$	m is odd
$C_M(i), C'_M(i)$	0	$3m$	m is even
$h(x,y)$	$3m$	0	–

Table 3.2: The number of *abc*-edges, $n_1(H)$, and *xxy*-edges, $n_2(H)$, in H

Let $C(0,0)$ be a Hamiltonian cycle in Lemma 3.2 and $C'(0,0)$ be a Hamiltonian cycle in Lemma 3.4. We obtain several results as follows.

3.3.1 $m \equiv 0 \pmod 6$

For $m \equiv 0 \pmod 6$, we have two families $\mathcal{H}_1$ and $\mathcal{H}_2$ of Hamiltonian cycles forming two Hamiltonian decompositions of $K^{(3)}_{m,m,m}$.

First, let $\mathcal{F}_1$ be a 1-factorization of a graph G with $V(G) = \mathbb{Z}_m = [\bar{0}] \cup [\bar{1}] \cup [\bar{2}]$, the union of the classes of modulo 3, and $E(G) = \{uv : u, v \in \mathbb{Z}_m, ||u - v|| \equiv 0 \pmod 3\}$. G is isomorphic to $3K_{m/3}$, three copies of $K_{m/3}$. Each component consists of vertices in the same class of modulo 3. An example of G is shown in Figure 3.1. Then

$$\mathcal{H}_1 = \{C(i,j) : i, j \in \mathbb{Z}_m\} \cup \{C_M(i), C'_M(i) : i \in \mathbb{Z}_{m/2}, M \in \mathcal{F}_1\}.$$

Next, let $\mathcal{A}_0 = \{(x,y) \in \mathbb{Z}_m^2 : x - y \equiv 0 \pmod 3\}$ and $\mathcal{F}_2$ is a 1-factorization of

K_m with $\mathbb{Z}_m$ as a vertex set. Then

$$\mathcal{H}_2 = \{h(x,y) : (x,y) \in \mathcal{A}_0\} \cup \{C_M(i), C'_M(i) : i \in \mathbb{Z}_{m/2}, M \in \mathcal{F}_2\}.$$

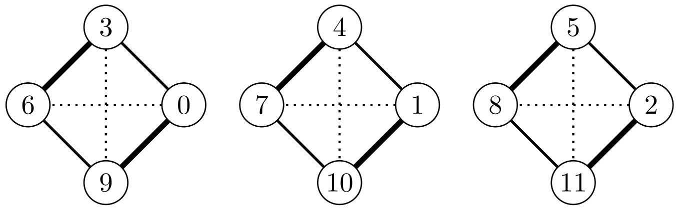

Figure 3.1: A graph G with $V(G) = \mathbb{Z}_{12}$ and its 1-factorization $\mathcal{F}_1$ indicated each 1-factor in the same edge style

Since K_{2n} is factorizable into $2n-1$ 1-factors [11], we have $|\mathcal{F}_1| = \frac{m}{3} - 1$ and $|\mathcal{F}_2| = m - 1$. Then we calculate the number of edges in $\mathcal{H}_1$, $\sum_{H \in \mathcal{H}_1} n_1(H) = m^2 \times m = m^3$ and $\sum_{H \in \mathcal{H}_1} n_2(H) = m^2 \times 2m + m(\frac{m}{3} - 1) \times 3m = 3m^3 - 3m^2$ and the number of edges in $\mathcal{H}_2$, $\sum_{H \in \mathcal{H}_2} n_1(H) = \frac{m^2}{3} \times 3m = m^3$ and $\sum_{H \in \mathcal{H}_2} n_2(H) = m(m-1) \times 3m = 3m^3 - 3m^2$.

Here are some facts that we can easily obtain.

F1 For any two 1-factors M and M' in K_m, we see that if M and M' are disjoint, then $C_M(i)$ and $C_{M'}(i)$ are also disjoint for all $i \in \mathbb{Z}_{m/2}$.

F2 For all xxy-edges $\{x, x', y\}$ in $\{C(i,j) : i, j \in \mathbb{Z}_m\}$, $||x - x'|| \equiv 1$ or $2 \pmod 3$ as in Lemma 3.2.

F3 For all xxy-edges $\{x, x', y\}$ in $\{C_M(i), C'_M(i) : i \in \mathbb{Z}_{m/2}, M \in \mathcal{F}_1\}$, $||x - x'|| \equiv 0 \pmod 3$ by the construction of $\mathcal{F}_1$.

F4 $\{h(x,y) : (x,y) \in \mathcal{A}_0\}$ contains only abc-edges.

F5 $\{C_M(i), C'_M(i) : i \in \mathbb{Z}_{m/2}, M \in \mathcal{F}_2\}$ contains only xxy-edges.

With F2 and F3, any Hamiltonian cycles in $\{C(i,j) : i,j \in \mathbb{Z}_m\}$ and any Hamiltonian cycles in $\{C_M(i), C'_M(i) : i \in \mathbb{Z}_{m/2}, M \in \mathcal{F}_1\}$ are disjoint because of distinct xxy-edges. By Lemma 3.1 and Lemma 3.6 with F1-3, we see that $\mathcal{H}_1$ is a Hamiltonian decomposition of $K^{(3)}_{m,m,m}$. With F4 and F5 along with Lemma 3.7 and F1, it is clear that $\mathcal{H}_2$ is a Hamiltonian decomposition of $K^{(3)}_{m,m,m}$.

Example 3.5. For $m = 6$, the Hamiltonian decomposition $\mathcal{H}_1$ consists of $C(i,j)$ where $i,j \in \mathbb{Z}_6$, with $C(0,0)$ given in Example 3.1 and $C_M(i)$, $C'_M(i)$, where $i \in \mathbb{Z}_3$, and $M = \{03, 14, 25\}$ given in Example 3.3. The Hamiltonian decomposition $\mathcal{H}_2$ consists of $\{h(x,y) : (x,y) \in \mathcal{A}_0\}$ given in Example 3.4 and $C_M(i)$, $C'_M(i)$, where $i \in \mathbb{Z}_3$, and $M \in \{\{01, 25, 34\}, \{02, 31, 45\}, \{03, 42, 51\}, \{04, 53, 12\}, \{05, 14, 23\}\}$.

3.3.2 $m \equiv 3 \pmod 6$

For $m \equiv 3 \pmod 6$, let

$$\mathcal{H}_3 = \{C'(i,j) : i,j \in \mathbb{Z}_m\} \cup \{h(x,y) : (x,y) \in \mathcal{A}_0, x \neq y\},$$

where $\mathcal{A}_0 = \{(x,y) \in \mathbb{Z}_m^2 : x - y \equiv 0 \pmod 3\}$. We calculate the number of edges in $\mathcal{H}_3$, $\sum_{H \in \mathcal{H}_3} n_1(H) = m^2 \times 3 + (\frac{m^2}{3} - m) \times 3m = m^3$ and $\sum_{H \in \mathcal{H}_3} n_2(H) = m^2 \times (3m - 3) = 3m^3 - 3m^2$.

To show that $\mathcal{H}_3$ is a Hamiltonian decomposition of $K^{(3)}_{m,m,m}$, we must show that $\{C'(i,j) : i,j \in \mathbb{Z}_m\}$ contains all abc-edges of difference types (x,y), $(x+1,y)$, and $(x,y+1)$ for all $x,y \in \mathbb{Z}_m$ with $x = y$.

Three abc-edges in $C'(i,j)$ for each $i,j \in \mathbb{Z}_m$ have difference types

$$(b_{m-1} - a_{m-1} + i, c_{m-2} - b_{m-1} + i) = (i, i+1)$$

for the edge $\{a_{m-1} + j, \overline{b_{m-1} + i + j}, \overline{\overline{c_{m-2} + 2i + j}}\}$,

$$(b_{m-1} - a_{m-1} + i, c_{m-1} - b_{m-1} + i) = (i, i)$$

for the edge $\{a_{m-1} + j, \overline{b_{m-1} + i + j}, \overline{\overline{c_{m-1} + +2i + j}}\}$,

$$(b_{m-1} - a_0 + i, c_{m-1} - b_{m-1} + i) = (i+1, i)$$

for the edge $\{a_0 + j, \overline{b_{m-1} + i + j}, \overline{\overline{c_{m-1} + 2i + j}}\}$.

Since each i,j corresponds to m different values of $\mathbb{Z}_m$, $\{C'(i,j) : i,j \in \mathbb{Z}_m\}$ contains all edges of difference type (i,i), $(i+1,i)$ and $(i,i+1)$ as desired.

Thus, $\mathcal{H}_3$ is a Hamiltonian decompositions of $K^{(3)}_{m,m,m}$, where $m \equiv 3 \pmod 6$.

Example 3.6. For $m = 9$, the Hamiltonian decomposition $\mathcal{H}_3$ consists of $C'(i,j)$, where $i,j \in \mathbb{Z}_9$, with $C'(0,0)$ given in Example 3.2 and $h(x,y)$ where $(x,y) \in \{(0,3), (1,4), (2,5), (3,6), (4,7), (5,8), (6,0), (7,1), (8,2), (0,6), (1,7), (2,8), (3,0), (4,1), (5,2), (6,3), (7,4), (8,5)\}$. See Figure 3.2 for the demographic of (x,y) for $h(x,y)$ in $\mathcal{H}_3$.

3.3.3 A special case, $m = 3$

The case of $m = 3$ is special because it is also in the case of $m \equiv 3 \pmod 6$ but $h(x,y)$ is not required, and we can get another solution as in the case of $m \equiv 0 \pmod 6$ but $C_M(i)$ and $C'_M(i)$ are not required. We count the number of edges $n_1(K^{(3)}_{3,3,3}) = 27$ and $n_2(K^{(3)}_{3,3,3}) = 54$. The sets of Hamiltonian cycles $\mathcal{C}_1 = \{C(i,j) : i,j \in \mathbb{Z}_m\}$ and $\mathcal{C}_2 = \{C'(i,j) : i,j \in \mathbb{Z}_m\}$ both have m^2 Hamiltonian cycles. The number of edges in $\mathcal{C}_1$, $\sum_{H \in \mathcal{C}_1} n_1(H) = m^3 = 27$ and $\sum_{H \in \mathcal{C}_1} n_2(H) = 2m^3 = 54$, and the number of edges in $\mathcal{C}_2$, $\sum_{H \in \mathcal{C}_2} n_1(H) = 3m^2 = 27$ and $\sum_{H \in \mathcal{C}_2} n_2(H) =$

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Figure 3.2: The coordinates (x, y) of $h(x, y)$ in Example 3.6 shown as black cells with $|$ showing $(x, y+1)$ and - showing $(x+1, y)$

$3m^3 - 3m^2 = 54$. By Lemma 3.1 and Lemma 3.3, we can conclude that $\mathcal{C}_1$ and $\mathcal{C}_2$ are both Hamiltonian decompositions of $K^{(3)}_{3,3,3}$.

Example 3.7. Let $C(0,0) = (0, \overline{0}, \overline{\overline{0}}, \overline{\overline{1}}, 1, \overline{1}, \overline{2}, \overline{\overline{2}}, 2)$. Then the Hamiltonian decomposition $\mathcal{C}_1$ of $K^{(3)}_{3,3,3}$ obtained from Section 3.2.1 is shown below.

$$C(0,0) = (0, \overline{0}, \overline{\overline{0}}, \overline{\overline{1}}, 1, \overline{1}, \overline{2}, \overline{\overline{2}}, 2),$$

$$C(0,1) = (0, \overline{1}, \overline{\overline{1}}, \overline{\overline{2}}, 1, \overline{2}, \overline{0}, \overline{\overline{0}}, 2),$$

$$C(0,2) = (0, \overline{2}, \overline{\overline{2}}, \overline{\overline{0}}, 1, \overline{0}, \overline{1}, \overline{\overline{1}}, 2),$$

$$C(1,0) = (1, \overline{0}, \overline{\overline{1}}, \overline{\overline{2}}, 2, \overline{1}, \overline{2}, \overline{\overline{0}}, 0),$$

$$C(1,1) = (1, \overline{1}, \overline{\overline{2}}, \overline{\overline{0}}, 2, \overline{2}, \overline{0}, \overline{\overline{1}}, 0),$$

$$C(1,2) = (1, \overline{2}, \overline{\overline{0}}, \overline{\overline{1}}, 2, \overline{0}, \overline{1}, \overline{\overline{2}}, 0),$$

$$C(2,0) = (2, \overline{0}, \overline{\overline{2}}, \overline{\overline{0}}, 0, \overline{1}, \overline{2}, \overline{\overline{1}}, 1),$$

$$C(2,1) = (2, \overline{1}, \overline{\overline{0}}, \overline{\overline{1}}, 0, \overline{2}, \overline{0}, \overline{\overline{2}}, 1),$$

$$C(2,2) = (2, \overline{2}, \overline{\overline{1}}, \overline{\overline{2}}, 0, \overline{0}, \overline{1}, \overline{\overline{0}}, 1).$$

Let $C'(0,0) = (0, 2, \overline{0}, \overline{2}, \overline{\overline{0}}, \overline{\overline{2}}, 1, \overline{1}, \overline{\overline{1}})$. The Hamiltonian decomposition $\mathcal{C}_2$ of $K^{(3)}_{3,3,3}$ follows from Section 3.2.2.

We summarize these results in Table 3.3, and Theorem 3.1 concludes all the results.

Decomposition	Condition	The number of			
		$C(i,j)$	$C'(i,j)$	$C_M(i), C'_M(i)$	$h(x,y)$
$\mathcal{H}_1$	$m \equiv 0 \pmod 6$	m^2	–	$m(\frac{m}{3} - 1)$	–
$\mathcal{H}_2$	$m \equiv 0 \pmod 6$	–	–	$m(m-1)$	$\frac{m^2}{3}$
$\mathcal{H}_3$	$m \equiv 3 \pmod 6$	–	m^2	–	$\frac{m^2}{3} - m$
$\mathcal{C}_1$	$m = 3$	9	–	–	–
$\mathcal{C}_2$	$m = 3$	–	9	–	–

Table 3.3: The combinations of Hamiltonian cycles in decompositions of $K^{(3)}_{m,m,m}$

Theorem 3.1. *$K^{(3)}_{m,m,m}$ is decomposable into Hamiltonian cycles if and only if $m \equiv 0 \pmod 3$.*

CHAPTER IV

CONCLUSION

We have investigated two families of hypergraphs: prisms over complete 3-uniform hypergraphs, Prism($K_n^{(3)}$), and complete tripartite 3-uniform hypergraphs, $K_{m,m,m}^{(3)}$.

For Prism($K_n^{(3)}$), the difficulty of finding its Hamiltonian decomposition is increasing rapidly as n is getting larger. We use the fill-out method to find solutions for $n \in \{4, 5\}$ and develop some knowledge of cyclic method from Bailey and Stevens [2] to find a Hamiltonian decomposition of Prism($K_8^{(3)}$). The question on a Hamiltonian decomposition of Prism($K_7^{(3)}$) is still unsolved. However, we see the possibility of the cyclic method for $n \equiv 2 \pmod 6$ by calculating number of Hamiltonian cycles satisfying Lemma 2.3. In conclusion, we have done only the case of $n \in \{4, 5, 8\}$.

For $K_{m,m,m}^{(3)}$, we have completely found its Hamiltonian decomposition for all necessary condition of m. We provide four patterns of partite set ordering that can be formed a Hamiltonian cycle in $K_{m,m,m}^{(3)}$. Next, we develop the constructions that produce the set of disjoint Hamiltonian cycles for each pattern. The constructions of $C(i, j)$ and $C'(i, j)$ give the sense of cyclic method in 2-dimension while $C_M(i)$ and $C'_M(i)$ use 1-factors and orthogonal Latin squares. With Kuhl and Schroeder's work in [7], the construction of $h(x, y)$ fulfills the hole of the decompositions. A Hamiltonian decomposition of $K_{m,m,m}^{(3)}$ is a combination of various sets of edge-disjoint Hamiltonian cycles. The cases where m is odd and m is even are solved separately. Finally, we can conclude that a Hamiltonian decomposition of $K_{m,m,m}^{(3)}$

exists for all integer m such that m is a multiple of 3.

Further Works

We suggest the following tasks to do beyond this dissertation.

1. Find initial cycles and use the cyclic method for finding Hamiltonian decompositions of Prism$(K_n^{(3)})$, where $n \equiv 2 \pmod 6$.

2. Find the Hamiltonian decomposition of $K_{m,m,m}^{(3)} - I$, where I is a 1-factor of $K_{m,m,m}^{(3)}$ and $3 \nmid m$.

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