# ระเบียบวิธีแลตทิสโบลทซ์มันน์สำหรับสมการน้ำตื้นที่มีรอยต่อเปียก-แห้ง

นายวีรศักดิ์ ดีอ่ำ



วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต
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# LATTICE BOLTZMANN METHOD FOR SHALLOW WATER EQUATIONS WITH WET - DRY INT ERFACE

Mr. Weerasak Dee-am



A Thesis Submitted in Partial Fulfillment of the Requirements
for the Degree of Master of Science Program in Applied Mathematics and Computational Science
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วีรศักดิ์ ดีอ่ำ : ระเบียบวิธีแลตทิสโบลทซ์มันน์สำหรับสมการน้ำตื้นที่มีรอยต่อเปียก-แห้ง (LATTICE BOLTZMANN METHOD FOR SHALLOW WATER EQUATIONS WITH WET - DRY INTERFACE) อ.ที่ปรึกษาวิทยานิพนธ์หลัก: ผศ. ดร. คำรณ เมฆฉาย, อ.ที่ปรึกษาวิทยานิพนธ์ร่วม: รศ. ดร. มนตรี มาลีวงศ์, 58 หน้า.

ในงานวิจัยนี้เราได้ศึกษาระเบียบวิธีแลตทิซโบลทซ์มันน์สำหรับการไหลของของไหลแบบน้ำตื้น โดยเฉพาะอย่างยิ่งในปัญหาที่มีรอยต่อเปียก-แห้ง ซึ่งหมายถึงการไหลของของไหลจากบริเวณที่มีของไหลอยู่แล้วไปยังบริเวณพื้นที่ที่ไม่มีของไหลอยู่หรือในทำนองกลับกัน จากการศึกษางานวิจัยที่เกี่ยวของทำให้พบว่าปัญหาดังกล่าวนั้นก่อให้เกิดความยากลำบากในการคำนวณได้ในหลายๆระเบียบวิธี เช่นเดียวกันกับระเบียบวิธีแลตทิซโบลทซ์มันน์แบบปกติ สมการแลตทิซโบลทซ์มันน์ซึ่งใช้ในการคำนวณนั้นมีพจน์ที่ต้องถูกหารด้วยส่วนสูงของของไหล ซึ่งจะทำให้การคำนวณมีปัญหาเมื่ออยู่ในแลตทิซที่ส่วนสูงของน้ำมีค่าเป็นศูนย์ ในการศึกษาครั้งนี้เราจึงเน้นที่จะประยุกต์แนวคิดในการใช้การกระจายเทย์เลอร์และกระบวนการชาพมาน – เอ็นสก๊อก พร้อมด้วยการใช้เทคนิคการระบุตำแหน่งของรอยต่อเปียก-แห้งมาจัดการกับปัญหานี้ ในการสร้างแบบจำลองนี้เราได้ทดสอบตัวอย่างเชิงตัวเลขหลายๆอันและเปรียบเทียบกับผลเฉลยที่แท้จริงและผลลัพธ์ในงานวิจัยที่เป็นที่ยอมรับ



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WEERASAK DEE-AM: LATTICE BOLTZMANN METHOD FOR SHALLOW WATER EQUATIONS WITH WET - DRY INTERFACE. ADVISOR: ASST. PROF. KHAMRON MEKCHAY, Ph.D., CO-ADVISOR: ASSOC. PROF. MONTRI MALEEWONG, Ph.D., 58 pp.

This research concerns the lattice Boltzmann method (LBM) for shallow water flow problems, especially in wet-dry interfaces, where fluid flows from a wet area to a dry area or vice versa. This problem of wet-dry interfaces is known in literature that causes difficulty for numerical methods, also for the standard LBM, the lattice Boltzmann equations consist of terms that are divided by fluid depth, which is zero at dry lattice. In this study, we implement the LBM based on the idea of using the Taylor's expansion and Chapman-Enskog procedure to handle dry lattices without any fake assumption purposed by Liu & Zhou, and combine with wet-dry tracking technique to handle this problem. The implementation of the code tested with several numerical examples, which is accurate as compared with other exact solution and results in literature.



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# CHAPTER I

# INTRODUCTION

## 1.1 Motivation

Long-wave phenomena are commonly exist in nature such as wave run-up, tsunami and solute transport in blood vessel, etc. One way to model these phenomena is to use the so - called shallow water equations (SWEs). The SWEs are a set of hyperbolic partial differential equations that describes the flow with the horizontal length scale is much greater than the water depth. Moreover, the conservation laws make them powerful and efficient to simulate long-wave flow phenomena. Many research have demonstrated how to solve the problem described by shallow water equations using conventional numerical methods – such as the finite difference method (FDM), the finite element method (FEM) and the finite volume method (FVM) – and to simulate those problems with complex topography. Nowadays, the results of the shallow water flow problem with any traditional method are somewhat good but the calculation of these methods is quite complicated and can cause many mistakes. By this reason, if there is a new method that is not complicated and is easy for program coding, this new method will be interesting in numerical field.

The new numerical method called lattice Boltzmann method (LBM) was developed and introduced in recent decades based on kinetic model [2], [3] and [4]. The LBM is the mesoscopic method with simple arithmetic of just one parameter, the distribution function which is the function that describes the fluids particles. Lattice Boltzmann equation is the key equation of this method consisting of two crucial steps, the collision step and the streaming step. In this equation the distribution function of the previous time is needed to calculating the distribution function of the present time. The link between the distribution function and the unknown variables which describe the real phenomena like the water velocity and the water depth is efficient. Because we solve this problem with one variable, the distribution function, instead two or three variable depended on the number of dimension. These give a better way to manage the code.

The shallow water flow problems with the wet – dry interface are the troublesome for all numerical methods. Also this problem can cause difficulty to the standard LBM. In order to overcome this drawback, the Taylor expansion and Chapman-Enskog procedure are considered to handle the distribution function without any artificial assumption. This new scheme was first introduced by Liu and Zhou in [1]. Moreover, the source terms, i.e. the bed slope and the bed friction, are included in the problem.

We study both the SWEs and LBM, then applying the lattice Boltzmann method to solve the shallow water flow problem with the wet - dry interface, which is considered by the new scheme for it. Moreover, we apply this method to solve the example of the shallow water flow problem. The benefit of the thesis is to study the new method that is easier to cope with our problem.

**1.2 Literature Reviews**

Fluid flow problems governed by SWEs are studied by well – known numerical methods such as FEM, FDM, FVM etc. These method were used many times for these kind of problems. They were applied and modified to handle special fluid flow problems such as flows with complex geography. The numerical results from these methods perform quite well in general when compared with exact solutions and experiments. However, implementing with these methods is quite difficult for beginners, which may introduce mistakes in coding due to complexity of the methods. To overcome this, the LBM is the method that is not complex but can handle fluid flow problems under some assumptions. Because LBM was introduced in recent years, researchers have been trying to apply the method to handle many problems.

The following are some advantages of LBM. Firstly, the programing is simple because LBM comprises easy calculations in two main steps. Secondary, the single variable is used instead of two or three unknown variables of SWEs in the calculation. Thirdly, it can easily be modified to apply parallel computing for faster computation. Lastly, the method can easily handle complex boundary conditions.

There are many results in literatures of the lattice Boltzmann method for shallow water flows. In 2013 Rakwongwan & Maleewong solved the dam – break problem described by shallow water equations using LBM without the source terms such as bed slope and bed friction [5]. The study can handle the dam break problem and gives numerical results that agree with the exact solution. However, this result does not include the problem with source terms and the result diverges, when the two levels of fluid surface are far enough (see experiment 1, Chapter 4). Zhou (2002) [6] presented the simulation of LBM for shallow water flow by including some simple source terms such as linear bed slope with small slope. However, for the source term with high order differential term, the numerical result does not agree well with the exact solution. In 2011, Zhou [7] overcame this difficulty by using the LBM for shallow water problem with the complex source terms by employing the idea of centered scheme to manage the order of the accuracy of the source term in, but only for problem of wet area. For the problem with the wet – dry interface, the standard LBM does not work in general in the calculation. However, some researches use artificial assumptions such as that from a thin film to extrapolation of unknown variables [8]. Eventually in 2014, Liu & Zhou [1] introduced the approach to solve this problem with wet – dry interface by using the Taylor expansion and Chapman – Enskog procedure.

In this thesis as proposed in early 2014, the research was planned to extend the result of [5], which was for wet-wet problem without source terms, to handle more general cases for having source terms and with wet-dry interface. Since the idea of wet-dry interface was introduced during the time in [1] , the research is therefore focusing on implementation based on [1] by including additional techniques such as wet-dry tracking, which is illustrated with some numerical experiments.

**1.3 Overview**

In this thesis, we study and present the derivation of the SWEs by considering the flow through the fixed control volume. Then the LBM and its important mathematical aspects are expressed. These two mathematical topics are the important mathematical background of this research and presented in CHAPTER II. LBM for the shallow water flow problem with wet - dry transition is exhibited in CHAPTER III. The numerical results are shown in the CHAPTER IV. Finally, the conclusion is given in CHAPTER V.

# CHAPTER II
# MATHEMATICAL KNOWLEDGE

This chapter composes of two important mathematical knowledge for this research. Firstly, we describe and present the derivation of the shallow water equations (SWEs). The shallow water equations are the set of equations that describe the long – wave flow phenomena. Secondly, we introduce the lattice Boltzmann method (LBM).

## 2.1. Shallow Water Equations

Shallow water equations (SWEs) consist of a set of partial differential equations that describe some kind of fluid flow problems. The word "shallow" doesn't mean the fluid need to be shallow, but this word presents the relation between the depth of fluid and the wave length. The fluid flow phenomena described by SWEs should have long wave length compared with its depth. This property allows us to assume that the vertical effects can be neglected when it is compared to horizontal effects. There are many phenomena that can be described by SWEs such as the flow in river, the flow in estuaries, the coastal areas phenomena like wave run – up and wave run – down, tsunami prediction, atmosphere flows, storm surge, solute transport in blood vessel, flows through porous media, etc [7].

Herein we derive the two – dimensional shallow water equations based on three important assumptions:

a) The fluid is incompressible, this implies that the fluid density does not change in time.

b) The hydrostatic pressure – the pressure due to the force of gravity – is included in this model.

c) This fluid flows without turbulence.

These equations consist of two crucial parts which are the conservation of mass and the conservation of momentum. Before we begin to derive them, we have to define one thing, the control volume.

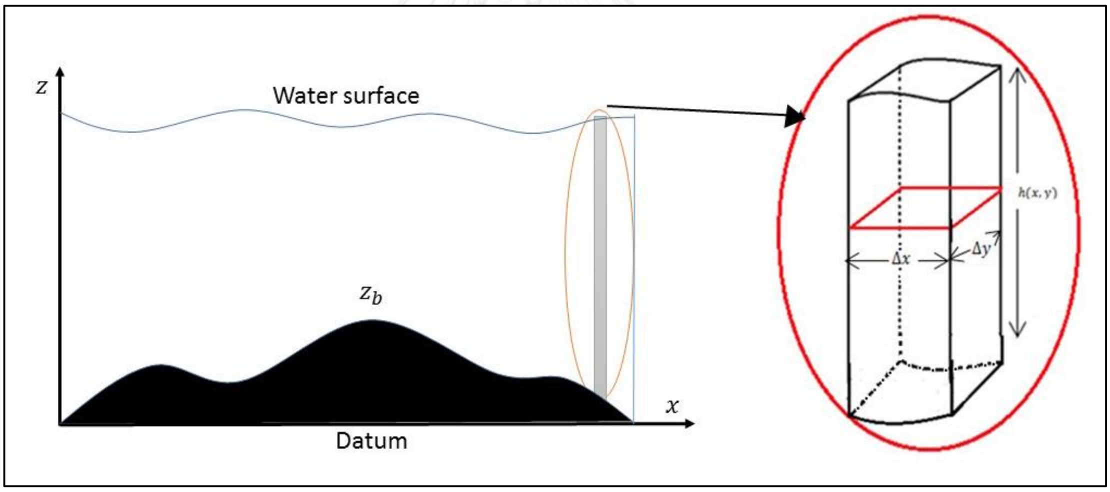



Figure 2. 1 The domain and its control volume $(V)$.

From figure 2.1, we subdivide the domain of fluid flow problem into small boxes with their fixed volume (V). These small boxes are as tall as their domain where they rest on. Their horizontal cross section has side of lengths $\Delta x$ and $\Delta y$ for $x$ direction and $y$ direction, respectively, where $\Delta x$ and $\Delta y$ are sufficiently small.

When we observe the control volume from the top side as figure 2.2, there are the velocity $u\left(x\left(t\right),y\left(t\right)\right)$ in the $x$ direction and $v\left(x\left(t\right),y\left(t\right)\right)$ in the $y$ direction, in which $t$ is time. Now, we could begin to derive the governing equation of shallow water flow problem, which composes of the two vital conservation laws.

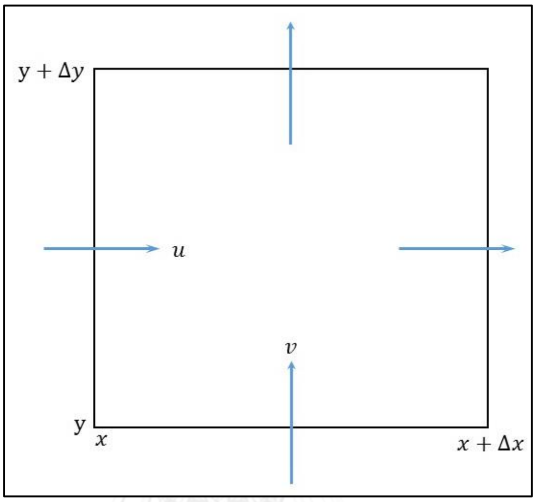



Figure 2. 2 The velocity through the control volume in each direction.

### 2.1.1 Conservation of Mass

From figure 2.2, we states that the rate of mass increasing within the control volume $(V)$ is equal to the net rate of mass flux entering the volume $V$. To consider this statement, we have to investigate for each direction of velocity.

In the $x$ direction, we can observe that mass flux entering into the control volume $(V)$ is the product of the fluid density $(\rho)$, the fluid velocity in the $x$ direction $\left(u\right)$ and the face area $\Delta yh$, resulting in

$$\left(\text{mass flux in}\right)_x = \rho uh\Delta y \qquad (2.1)$$

The mass leaving at the outlet have a little bit change of the fluid density $\rho$, the fluid depth $h$ and the fluid velocity in the $x$ direction $u$ due to the fluid passed through the volume. This gives

$$\left(\text{mass flux out}\right)_x = -\rho\left(x+\Delta x,\mathrm{y}\right)u\left(x+\Delta x,\mathrm{y}\right)h\left(x+\Delta x,\mathrm{y}\right)\Delta y \tag{2.2}$$

By the Taylor series expansion, the equations (2.2) is expanded and truncated up to the $1^{st}$ order in term of $\Delta x$, i.e., $O\left(\Delta x\right)$,

$$-\rho\left(x+\Delta x,\mathrm{y}\right)u\left(x+\Delta x,\mathrm{y}\right)h\left(x+\Delta x,\mathrm{y}\right)\Delta y = -\left(\rho uh\Delta y + \frac{\partial\left(\rho uh\right)}{\partial x}\Delta x\Delta y\right). \tag{2.3}$$

Similarly, in the $y$ direction, we obtain

$$\left(\text{mass flux in}\right)_y = \rho vh\Delta x \tag{2.4}$$

$$\left(\text{mass flux out}\right)_y = -\left(\rho vh\Delta x + \frac{\partial\left(\rho vh\right)}{\partial y}\Delta y\Delta x\right) \tag{2.5}$$

The summation of equations (2.1), (2.3), (2.4) and (2.5) – the net rate of mass flux entering the control volume $(V)$ – must equal to the retained mass in the volume $\Delta x\Delta yh$. The accumulated mass can be written as

$$\frac{\partial\left(\rho h\right)}{\partial t}\Delta x\Delta y \tag{2.6}$$

Therefore, we obtain

$$\frac{\partial\left(\rho h\right)}{\partial t}\Delta x\Delta y=\rho uh\Delta y+\rho vh\Delta x-\left(\rho uh\Delta y+\frac{\partial\left(\rho uh\right)}{\partial x}\Delta x\Delta y\right)$$

$$-\left(\rho vh\Delta x+\frac{\partial\left(\rho vh\right)}{\partial y}\Delta y\Delta x\right) \tag{2.7}$$

Simplifying the equation (2.7) leads to

$$\frac{\partial\left(\rho h\right)}{\partial t}\Delta x\Delta y=-\frac{\partial\left(\rho uh\right)}{\partial x}\Delta x\Delta y-\frac{\partial\left(\rho vh\right)}{\partial y}\Delta y\Delta x \tag{2.8}$$

Dividing the equation (2.8) by the area $\Delta x\Delta y$ yields

$$\frac{\partial\left(\rho h\right)}{\partial t}+\frac{\partial\left(\rho uh\right)}{\partial x}+\frac{\partial\left(\rho vh\right)}{\partial y}=0 \tag{2.9}$$

For the incompressible flow, the fluid density $\rho$ is constant. Then the conservation of mass can be written as



$$\frac{\partial h}{\partial t}+\frac{\partial\left(uh\right)}{\partial x}+\frac{\partial\left(vh\right)}{\partial y}=0 \tag{2.10}$$

**2.1.2 Conservation of Momentum**

Because the collision among the fluid particles always occur in the fluid flow phenomena, thus the momentum should be considered for these phenomena. When particles collide each other, the momentum before the collision should absolutely be equal to after collision. This is called the law of conservation of momentum. Momentum is a vector quantity which is product of mass and velocity. This means that, for two – dimensional problem, we have to derive them into the $x$ direction and the $y$ directions.

In the $x$ direction, we can observe that the rate of change of momentum within the control volume can be expressed as

$$\frac{\partial\left(\rho uh\right)}{\partial t}\Delta x\Delta y \tag{2.11}$$

The momentum flux entering into each face of the control volume is the product of the mass flux and the velocity in the $x$ direction. The momentum flux in x direction written as

$$\rho uuh\Delta y+\rho uvh\Delta x \tag{2.12}$$



The momentum flux for opposite side is

$$-\left(\rho uuh+\frac{\partial\left(\rho uuh\right)}{\partial x}\Delta x\right)\Delta y-\left(\rho uvh+\frac{\partial\left(\rho uvh\right)}{\partial y}\Delta y\right)\Delta x. \tag{2.13}$$

By the fact that the rate of change of the momentum within the control volume is equal to the net momentum flux entering the control volume plus with the sum of the force acting against on the control volume $\left(\sum F_x\right)$. This gives

$$\frac{\partial\left(\rho uh\right)}{\partial t}\Delta x\Delta y=\rho uuh\Delta y+\rho uvh\Delta x-\left(\rho uuh+\frac{\partial\left(\rho uuh\right)}{\partial x}\Delta x\right)\Delta y$$

$$-\left(\rho uvh+\frac{\partial\left(\rho uvh\right)}{\partial y}\Delta y\right)\Delta x+\sum F_x \qquad (2.14)$$

By simplifying the above equation (2.14) resulting in

$$\left(\frac{\partial\left(\rho uh\right)}{\partial t}+\frac{\partial\left(\rho uuh\right)}{\partial x}+\frac{\partial\left(\rho uvh\right)}{\partial y}\right)\Delta x\Delta y=\sum F_x \qquad (2.15)$$

Similarly in the $y$ direction, we obtain

$$\left(\frac{\partial\left(\rho vh\right)}{\partial t}+\frac{\partial\left(\rho uvh\right)}{\partial x}+\frac{\partial\left(\rho vvh\right)}{\partial y}\right)\Delta x\Delta y=\sum F_y. \qquad (2.16)$$

in which $\sum F_y$ is the sum of force acting on the control volume in the $y$ direction.



Now let's consider the force acting against the control volume. There are three crucial force acting on the control volume $V$; the gravity due to the bed slope $\left(F_g\right)$, friction $\left(F_f\right)$, and hydrostatic pressure force $\left(F_p\right)$. Those forces acting against on the control volume in $x$ direction as in figure 2.3.

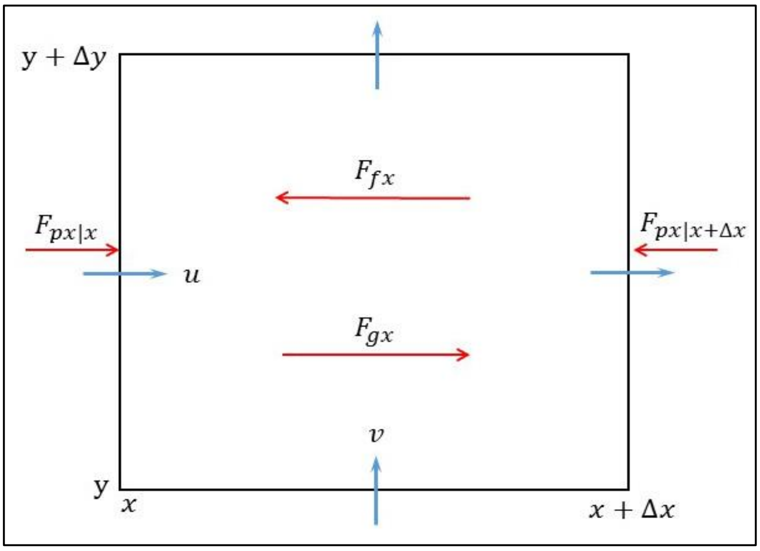



Figure 2. 3 The forces acting against on the control volume in $x$ direction.

In figure 2.3, $F_{gx}$ and $F_{fx}$ are the gravitational force due to the bed slope and the friction force in the $x$ direction, respectively. Their directions are depicted on the figure. $F_{px|x}$ and $F_{px|x+\Delta x}$ are the hydrostatic pressure forces, their directions point into the control volume as in the figure.

a) **Gravitational force due to the bed slope**

The gravity force due to the bed slope in $x$ direction, $F_{gx}$, can be expressed as

$$F_{gx} = mg\sin\theta = \rho gh\Delta x\Delta y\sin\theta, \qquad (2.17)$$

where $m$ is the fluid mass, $g$ is the gravitational acceleration and $\theta$ is the angle of the tangent of the bed slope which can be seen in the figure 2.4 .

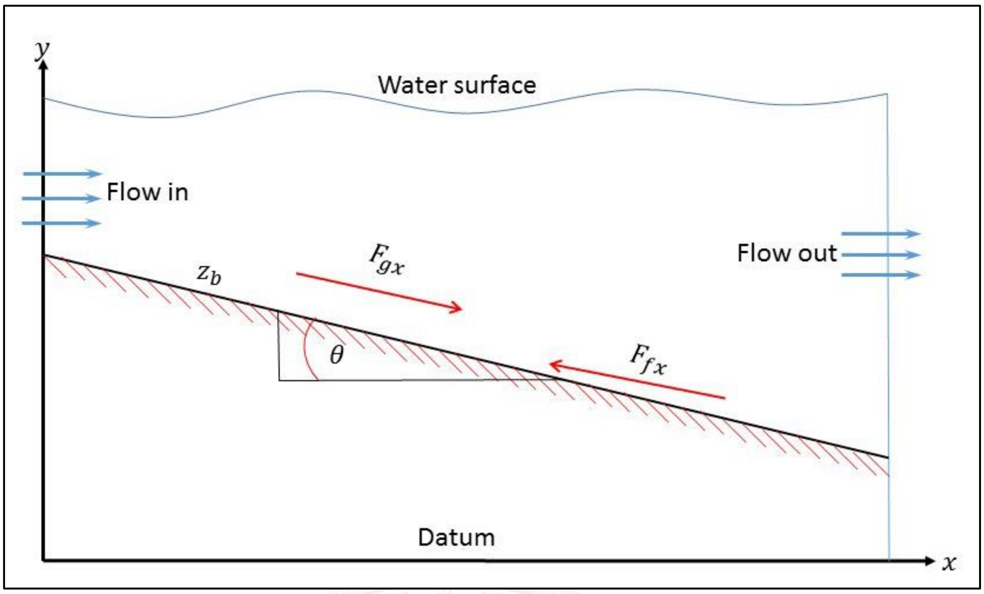



Figure 2. 4 The bed slope in $x - z$ plane.

The angle $\theta$ is small, this implies $\sin\theta \approx \tan\theta$ which is the bed slope. From the bed topography $z_b$, it is easy to find that the bed slope is $\frac{\partial z_b}{\partial x}$ and $\frac{\partial z_b}{\partial y}$ in the $x$ and $y$ directions, respectively. We obtain

$$F_{gx} = \rho g h \frac{\partial z_b}{\partial x} \Delta x \Delta y, \tag{2.18}$$

Similarly in the $y$ direction

$$F_{gy} = \rho g h \frac{\partial z_b}{\partial y} \Delta x \Delta y, \tag{2.19}$$

where, $F_{gy}$ is the bed slope in the $y$ direction.

b) **Frictional force**

In the $x$ direction, frictional force, $F_{fx}$, can be expressed as

$$F_{fx} = -\rho g h S_{fx} \Delta x \Delta y, \tag{2.20}$$

in which, $S_{fx}$ is the bed friction in the $x$ direction which is obtained from the Manning's equation, given as

$$S_{fx} = \frac{u n^2 \sqrt{u^2 + v^2}}{h^{4/3}}, \tag{2.21}$$

where, $n$ is the Manning's roughness coefficient.

It is similar for the frictional force in the $y$ direction

$$F_{fy} = -\rho g h S_{fy} \Delta x \Delta y, \tag{2.22}$$

in which, $S_{fy}$ is the bed friction in the $y$ direction, given as

$$S_{fy} = \frac{v n^2 \sqrt{u^2 + v^2}}{h^{4/3}}. \tag{2.23}$$

c) **Hydrostatic pressure force**

Firstly, we begin to consider the pressure force in the $x$ direction. We assume that $P$ is the hydrostatic pressure, $A$ is the area where the hydrostatic pressure force acting against, $h$ is the net depth of fluid ($h = h_s - z_b$) in which $h_s$ is the height of the water surface. The pressure force acting on the inlet in the $x$ direction can be written as

$$
\begin{aligned}
F_{px|x} &= \int P dA \\
&= \int_0^h \rho g \left(h - \eta\right) \Delta y d\eta \\
&= \Delta y \rho g \left. \frac{\left(h - \eta\right)^2}{2} \right|_h^0 \\
&= \Delta y \rho g \frac{h^2}{2}
\end{aligned}
\tag{2.24}
$$

Therefore, the net hydrostatic pressure force on the $x$ direction can be calculated as



$$
\begin{aligned}
F_{px|x} - F_{px|x+\Delta x} &= \Delta y \frac{\rho g h^2}{2} - \left( \Delta y \frac{\rho g h^2}{2} + \Delta y \Delta x \frac{\rho g}{2} \frac{\partial h^2}{\partial x} \right), \\
&= -\Delta y \Delta x \frac{\rho g}{2} \frac{\partial h^2}{\partial x}.
\end{aligned}
\tag{2.25}
$$

In the $y$ direction, the pressure force is

$$
F_{py|y} - F_{py|y+\Delta y} = -\Delta y \Delta x \frac{\rho g}{2} \frac{\partial h^2}{\partial y}. \tag{2.26}
$$

Substituting all of the force terms in the $x$ direction – the equation (2.18), (2.20) and (2.25) – into the conservation of momentum equation in the $x$ direction, the equation (2.15). This gives

$$\left(\frac{\partial\left(\rho uh\right)}{\partial t}+\frac{\partial\left(\rho uuh\right)}{\partial x}+\frac{\partial\left(\rho uvh\right)}{\partial y}\right)\Delta x\Delta y =$$

$$\rho gh\frac{\partial z_b}{\partial x}\Delta x\Delta y-\rho ghS_{fx}\Delta x\Delta y-\Delta y\Delta x\frac{\rho g}{2}\frac{\partial h^2}{\partial x}. \quad (2.27)$$

Dividing by $\Delta x\Delta y$ and $\rho$ yields

$$\frac{\partial\left(uh\right)}{\partial t}+\frac{\partial\left(uuh\right)}{\partial x}+\frac{\partial\left(uvh\right)}{\partial y}=-\frac{g}{2}\frac{\partial h^2}{\partial x}+gh\left(\frac{\partial z_b}{\partial x}-S_{fx}\right). \quad (2.28)$$

This is called the conservation of momentum in the $x$ direction.

Similarly in the $y$ direction, substituting the equation (2.19), (2.22) and (2.26) into the equation (2.16). We obtain the conservation of momentum in the $y$ direction as

$$\frac{\partial\left(vh\right)}{\partial t}+\frac{\partial\left(uvh\right)}{\partial x}+\frac{\partial\left(vvh\right)}{\partial y}=-\frac{g}{2}\frac{\partial h^2}{\partial y}+gh\left(\frac{\partial z_b}{\partial y}-S_{fy}\right). \quad (2.29)$$

Finally we obtain the shallow water equations for 2 – dimensional problem as in equations (2.10), (2.28) and (2.29). Moreover the SWEs for 1–dimensional problem can be written as

$$\frac{\partial h}{\partial t}+\frac{\partial\left(hu\right)}{\partial x}=0, \tag{2.30}$$

$$\frac{\partial\left(hu\right)}{\partial t}+u\frac{\partial u}{\partial x}=-\frac{g}{2}\frac{\partial h^2}{\partial x}+\nu\frac{\partial^2\left(hu\right)}{\partial x^2}-gh\frac{\partial z_b}{\partial x}-\frac{\tau_{bi}}{\rho}, \tag{2.31}$$

in which $\nu$ is the kinematic viscosity define by

$$\nu=C_s^2\Delta t\left(\tau-\frac{1}{2}\right), \tag{2.32}$$

and $C_s$ is sound speed in a lattice and $\tau$ is the relaxation time[1].

The bed shear stress[2], $\tau_{bi}$, demonstrates as

$$\tau_{bi}=\rho C_b u_i\sqrt{u_j u_j}, \tag{2.33}$$

where $C_b$ is the bed friction coefficient.

[1] The time required for a viscous substance to recover from shearing stress after flow has ceased.

[2] The way in which waves (or currents) transfer energy to the sea bed.

## 2.2. Lattice Boltzmann Method

### 2.2.1 Introduction

Shallow water flow problems have been being research numerously by the well – known methods such as the finite different method (FDM), the finite element method (FEM) and the finite volume method (FVM), etc. These methods somewhat work very well and give their good results but the way to manage these codes is quite complicated, and this may allow some mistakes to occur by the time we provide our code.

Lattice Boltzmann method (LBM) was developed and introduced in recent decades based on kinetic model in order to overcome the drawback of its ancestor method, the cellular automata [2]. Lattice Boltzmann method is the mesoscopic method, i.e. the method that consider both the microscopic world (molecular world) and the macroscopic world (real world). This method includes the simple arithmetic of just one parameter, the distribution function $f_\alpha$ , where $\alpha$ is index depending on the lattice model. The distribution function is the probability function that describes the behavior of the fluid particles.

This section will present the important concepts of the LBM. Firstly, lattice Boltzmann equation which is the key equation of this method consisting of two crucial steps – the collision step and the streaming step – would be explained. In this equation, the distribution function of the previous time is needed for calculating the distribution function at the present time. For each problem which is solved by lattice Boltzmann method, the process is almost the same except the equilibrium distribution function. The local equilibrium distribution function $f_\alpha^{eq}$ is the vital parameter that distinguishes the difference of problems. For the boundary condition, we can easily set them by simple method. It is no longer complicated like the conventional method. However, we have to provide the new scheme for the boundary condition of the shallow water flow problem with the wet – dry front in the CHAPTER III. Finally, the algorithm is introduced there.

In this work, this method is applied to solve the phenomena which is described by shallow water problem. Their domain must be subdivided into small square lattices. For the one – dimensional problem, D1Q3 (1 dimension and 3 nodes of particles) lattice model (see figure 2.5) is used and D2Q9 lattice model (see figure 2.6) is used for the two – dimensional problem.

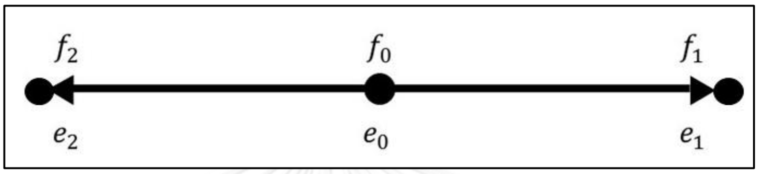



Figure 2. 5 The D1Q3 lattice model.

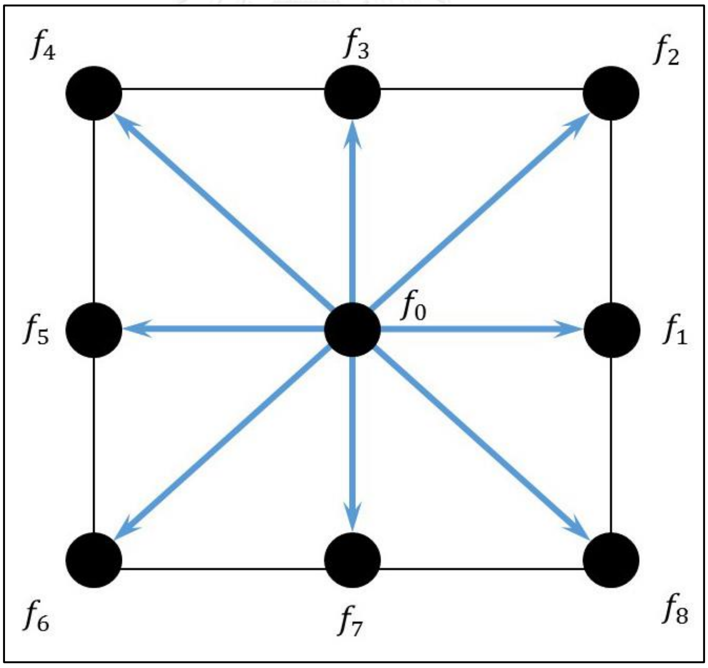



Figure 2. 6 The D2Q9 lattice model.

**2.2.2 Lattice Boltzmann equation**

As introduced by A. A. Mohummad [9] , the distribution function $f\left(r,e,t\right)$ is the number of the molecule at time $t$ located between the distance $r$ and $r+dr$ which have velocities between $e$ and $e+de$. There may be an external force $F$ acting on a fluid particle of unit mass and that would change the velocity and position of the particle. When the force $F$ acting on it, the particle will change its velocity from $e$ to $e+Fdt$ and changes its position from $r$ to $r+edt$.

When collisions come about between the fluid particles in the interval $drde$. The number of particles in that interval would change. The rate of change of the distribution function is called the collision operator $\Omega\left(f\right)$. We obtain

$$f\left(r+edt,e+Fdt,t+dt\right)-f\left(r,e,t\right)=\Omega\left(f\right)drdedt. \tag{2.34}$$

When $dt$ approaches $0$, this gives

$$\frac{df}{dt}=\Omega\left(f\right) \tag{2.35}$$

Because $f$ is a function of $r,e$ and $t$, the total differential and total derivative can be expressed and simplified as

$$\begin{aligned} df &= \nabla f \bullet dr + \frac{\partial f}{\partial e} \bullet de + \frac{\partial f}{\partial t} \bullet dt, \\ \frac{df}{dt} &= \nabla f \bullet \frac{dr}{dt} + \frac{\partial f}{\partial e} \bullet \frac{de}{dt} + \frac{\partial f}{\partial t}, \\ &= \nabla f \bullet e + \frac{\partial f}{\partial e} \bullet a + \frac{\partial f}{\partial t}, \end{aligned} \tag{2.36}$$

in which $a$ is the acceleration.

By the Newton's second law of motion, $a = \frac{F}{m}$, where $m$ is the mass, the Boltzmann transport equation is written as

$$\frac{\partial f}{\partial t} + \nabla f \bullet e + \frac{F}{m} \bullet \frac{\partial f}{\partial e} = \Omega\left(f\right). \quad (2.37)$$

For the problem without an external force, the particles move based on the Newton's first law. This means that the acceleration is zero. The equation could be expressed as

$$\frac{\partial f}{\partial t} + e \bullet \nabla f = \Omega\left(f\right), \quad (2.38)$$

where $e$ and $\nabla f$ are vectors.

In 1954, Bhatnagar, Gross and Krook (BGK) and Welender introduced in [10] the collision operator as follow

$$\Omega\left(f\right) = \frac{1}{\tau}\left(f^{eq} - f\right), \quad (2.39)$$

where $\tau$ is the relaxation time and $f^{eq}$ is the equilibrium distribution function.

The lattice Boltzmann equation (2.38) with the BGKW approximation can be discretized as,

$$\frac{f_\alpha\left(x, t + \Delta t\right) - f_\alpha\left(x, t\right)}{\Delta t} + e_\alpha \frac{f_\alpha\left(x + \Delta x, t + \Delta t\right) - f_\alpha\left(x, t + \Delta t\right)}{\Delta x}$$

$$= -\frac{1}{\tau}\left[f_\alpha\left(x, t\right) - f_\alpha^{eq}\left(x, t\right)\right]. \quad (2.40)$$

We know that $\Delta x = e_\alpha \Delta t$, then we obtain

$$f_\alpha\left(x+\Delta x, t+\Delta t\right)-f_\alpha\left(x,t\right)=-\frac{\Delta t}{\tau}\left[f_\alpha\left(x,t\right)-f_\alpha^{eq}\left(x,t\right)\right]. \quad (2.41)$$

Including with the source term, the lattice Boltzmann equation can be written as

$$f_\alpha\left(x+e_\alpha \Delta \mathrm{t}, t+\Delta t\right)=f_\alpha\left(x,t\right)-\frac{1}{\tau}\left[f_\alpha\left(x,t\right)-f_\alpha^{eq}\left(x,t\right)\right] + w_\alpha \frac{\Delta t}{C_s^2} e_{\alpha i} F_i\left(x,t\right), \quad (2.42)$$

in which $e_\alpha$ is the vector of the particle velocity at $\alpha$ direction, $e_{\alpha i}$ is the component of $e_\alpha$;

$w_\alpha$ is the weighing factor constant;

$F_i$ is source term in $i$ direction;

$\tau$ is the relaxation time which can be determined from (2.32).

In the procedure, we calculate the distribution function at the next time step by using the lattice Boltzmann equation. The lattice Boltzmann equation is separated into two steps of calculation as follow:

a) **The collision step**

The collision step is the calculation which represents the behavior of the fluid particles when they collide with each other. This action can affect the fluid particles, and then change their distribution function. This step can be stated as

$$f_\alpha\left(x,t\right)=f_\alpha\left(x,t\right)-\frac{1}{\tau}\left[f_\alpha\left(x,t\right)-f_\alpha^{eq}\left(x,t\right)\right]. \quad (2.43)$$

b) **The streaming step**

The streaming step is the reaction after the collision of the fluid particles. The fluid particles move in their new velocity direction. For the lattice Boltzmann method, this means that the distribution functions move to nearby neighboring lattices on their directions as shown in figure 2.7. This step is controlled by

$$f_{\alpha}\left(x+e_{\alpha}\Delta \mathrm{t},t+\Delta t\right)=f_{\alpha}\left(x,t\right)+w_{\alpha}\frac{\Delta t}{C_{s}^{2}}e_{\alpha i}F_{i}\left(x,t\right). \quad (2.44)$$

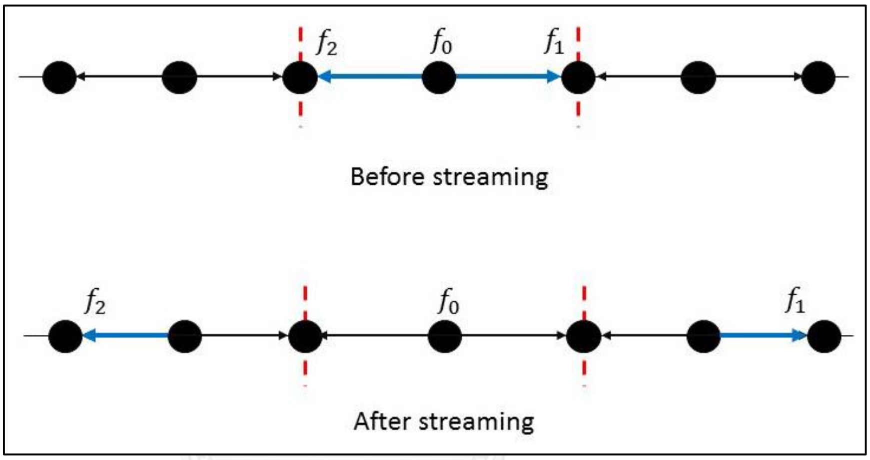



Figure 2. 7 The streaming step.



**2.2.3 Derivation of the Local Equilibrium Distribution Function**

Lattice Boltzmann method could be applied to solve many kinds of fluid problem such as  diffusion problem, fluid transport problem, porous media problem, shallow water flow problem, etc. The procedure of the calculation for each problems are rarely different. But the key thing that distinguishes each different problem is the local equilibrium distribution function. This means that different problems might use theirs specified equilibrium distribution function.

The equilibrium distribution function $f_{\alpha}^{eq}$ was initially derived from the normalized Maxwell's distribution function

$$f(u) = \frac{3\rho}{2\pi} e^{-\frac{3}{2}(c-u)^2} \tag{2.45}$$

It is the function of the velocity $u$ in which $\rho$ is the density and $c$ is the sound speed. By the Taylor expansion and the truncation at the order up to $2^{nd}$ order in the velocity, i.e. $O\left(u^2\right)$, we obtain

$$f_{\alpha}^{eq} = A_{\alpha} + B_{\alpha} e_{\alpha i} u_i + C_{\alpha} e_{\alpha i} e_{\alpha j} u_i u_j + D_{\alpha} u_i u_i, \tag{2.46}$$

in which $A_{\alpha}, B_{\alpha}, C_{\alpha}$ and $D_{\alpha}$ are constants, $\alpha$ is the index of the direction in a lattice.

Firstly, we consider the equilibrium distribution function for the one – dimensional problem or $f_{\alpha}^{eq}$ for D1Q3 lattice model. Because of the symmetry of the lattice model, we are able to suppose that

$$A_1 = A_2 = A, \qquad B_1 = B_2 = B,$$
$$C_1 = C_2 = C, \qquad D_1 = D_2 = D,$$

where $A, B, C$ and $D$ are also constants. Therefore, $f_{\alpha}^{eq}$ for the shallow water flows in D1Q3 lattice model can be stated as

$$f_{\alpha}^{eq} = \begin{cases} A_0 + D_0 u_i u_i, & \alpha = 0, \\ A + B e_{\alpha i} u_i + C e_{\alpha i} e_{\alpha j} u_i u_j + D u_i u_i, & \alpha = 1,2, \end{cases} \tag{2.47}$$

where $i, j$ is the Einstein's summation convention.

For the shallow water equations, the fluid motion is governed by the three crucial conservation laws [5], these are the conservation of mass, the conservation of momentum and the conservation of energy, which are expressed as

$$\sum_{\alpha} f_{\alpha}^{eq}\left(x,t\right) = h\left(x,t\right), \tag{2.48}$$

$$\sum_{\alpha} e_{\alpha i} f_{\alpha}^{eq}\left(x,t\right) = h\left(x,t\right) u_i\left(x,t\right), \tag{2.49}$$

$$\sum_{\alpha} e_{\alpha i} e_{\alpha j} f_{\alpha}^{eq}\left(x,t\right) = \frac{1}{2} g h^2\left(x,t\right)\delta_{ij} + h\left(x,t\right) u_i\left(x,t\right) u_j\left(x,t\right). \tag{2.50}$$

By substituting the equation (2.47) in the equations (2.48), (2.49) and (2.50), then by the comparison of the coefficient of $u$ and $h$. We obtain

$$A_0 = h - \frac{gh^2}{2e^2},\ D_0 = -\frac{h}{e^2},$$

$$A = \frac{gh^2}{4e^2},\ B = \frac{h}{2e^2},\ C = 0,\ D = \frac{h}{2e^2}.$$



By substituting the above coefficients into equation (2.47), we acquire

$$f_{\alpha}^{eq} = \begin{cases} h - \dfrac{gh^2}{2e^2} - \dfrac{hu^2}{e^2}, & \alpha = 0, \\ \dfrac{gh^2}{4e^2} + \dfrac{hu^2}{2e^2} + \dfrac{hu}{2e}, & \alpha = 1, \\ \dfrac{gh^2}{4e^2} + \dfrac{hu^2}{2e^2} - \dfrac{hu}{2e}, & \alpha = 2, \end{cases} \tag{2.51}$$

for the D1Q3 lattice model.

For two – dimensional problem, we use the D2Q9 model. Its equilibrium distribution function can be derived in the same way. This gives

$$f_\alpha^{eq} = \begin{cases} h - \dfrac{5gh^2}{6e^2} - \dfrac{2h}{3e^2} u_i u_i, & \alpha = 0, \\ \dfrac{gh^2}{6e^2} + \dfrac{h}{3e^2} e_{\alpha i} u_i + \dfrac{h}{2e^4} e_{\alpha i} e_{\alpha j} u_i u_j - \dfrac{h}{6e^2} u_i u_i, & \alpha = 1,3,5,7, \\ \dfrac{gh^2}{24e^2} + \dfrac{h}{12e^2} e_{\alpha i} u_i + \dfrac{h}{8e^4} e_{\alpha i} e_{\alpha j} u_i u_j - \dfrac{h}{24e^2} u_i u_i, & \alpha = 2,4,6,8. \end{cases} \tag{2.52}$$

**2.2.4 The relation between the LBM and macroscopic properties**

In the lattice Boltzmann method, all conservation laws absolutely holds true. Both conservation of mass and conservation of momentum still govern flows. Based on [11] and [9] , the law of conservation of mass is considered. It is true that the net mass at time $t$ and time $t + \Delta t$ equal each other. We can calculate the net mass by summing all of the distribution function at time $t$ and time $t + \Delta t$. By the conservation of mass, this gives

$$\sum_\alpha f_\alpha \left( x + e_\alpha \Delta t, t + \Delta t \right) = \sum_\alpha f_\alpha \left( x, t \right). \tag{2.53}$$

Substituting the equation (2.53) into the lattice Boltzmann equations (2.42), results in

$$\sum_\alpha f_\alpha \left( x, t \right) = \sum_\alpha f_\alpha^{eq} \left( x, t \right). \tag{2.54}$$

Therefore,

$$h\left( x, t \right) = \sum_\alpha f_\alpha^{eq} \left( x, t \right) = \sum_\alpha f_\alpha \left( x, t \right). \tag{2.55}$$

By conservation of momentum, multiplying $e_{\alpha i}$ into the equation (2.54), we obtain

$$u_i\left(x,t\right)=\frac{1}{h\left(x,t\right)}\sum_{\alpha}e_{\alpha i}f_{\alpha}^{eq}\left(x,t\right)=\frac{1}{h\left(x,t\right)}\sum_{\alpha}e_{\alpha i}f_{\alpha}\left(x,t\right). \quad (2.56)$$

**2.2.5 Boundary Conditions**

The fluid flow problem have one more thing that is significant point to govern themselves. They are the boundary conditions. This section discusses all about the suitable boundary conditions which is satisfied for the problem in this thesis and how to provide the suitable boundary conditions for each border of a problem. Similar to the lattice Boltzmann equation, the boundary conditions are presented in the form of the local distribution functions related to the frontier of the problem domain. Such boundary conditions are derived from the conservation law in order to preserve their real behavior.

Firstly, the boundary conditions for the solid wall, which includes both the slip boundary conditions and the no – slip boundary conditions would be shown in the following, then the inlet and outlet boundary condition are presented.

*2.2.5.1 The solid wall boundary*

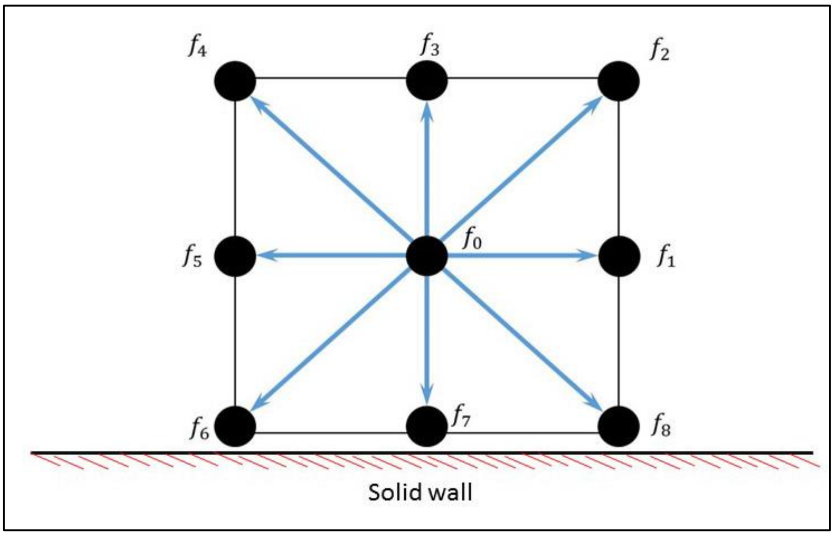



Figure 2. 8 The lattice at the solid wall.

a) **The slip boundary conditions**

From figure 2.8, the lattice near solid wall have three unknown distribution functions after the streaming steps. They are $f_2$, $f_3$ and $f_4$, which can be provided as

$$f_2 = f_8, \qquad f_3 = f_7, \qquad f_4 = f_6. \tag{2.57}$$

b) **The no – slip boundary conditions**

For this case, the bounce back scheme is used. It is the efficient way for the solid boundary. This makes lattice Boltzmann method competent to simulate flows with complex geometries domain. The idea of this scheme is obtained by the fact that the molecules which go towards the solids boundary would crash the wall and bounce back into the fluid. From figure 2.8, the unknown variables $f_2$, $f_3$ and $f_4$ can be represented by

$$f_2 = f_6, \quad f_3 = f_7, \qquad f_4 = f_8. \tag{2.58}$$

Moreover, the sum of the momentum at the solid wall is zero. This gives the velocity to be zero at the solid wall.

*2.2.5.2 The inflow and outflow boundary conditions*

After streaming, the distribution functions $f_1$, $f_2$ and $f_8$ of the lattice along the inlet border and the distribution functions $f_4$, $f_5$ and $f_6$ of the lattice along the outlet border are unknown variables (see figure 2.9). They need to be determined by the appropriate boundary conditions.

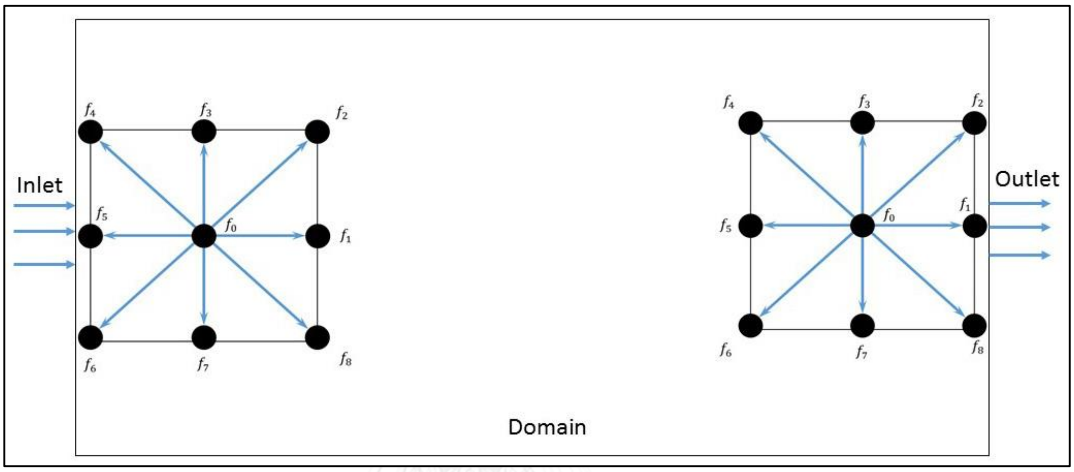



Figure 2. 9 The domain of the problem.

a) **The zero gradient**

For the inflow and the outflow boundaries whose depth and velocity are not given, many research point out that using zero gradient of the local distribution function along this boundary is the solution of this boundary. Thus the unknown distribution functions $f_1$, $f_2$ and $f_8$ at the inlet border can be computed by

$$f_\alpha\left(1, j\right) = f_\alpha\left(2, j\right), \quad \alpha{=}1{,}2{,}8. \tag{2.59}$$

And the unknown distribution functions $f_4$, $f_5$ and $f_6$ at the outlet are calculated by

$$f_\alpha\left(Lx, j\right) = f_\alpha\left(Lx - 1, j\right), \quad \alpha{=}4{,}5{,}6, \tag{2.60}$$

where $Lx$ is the total number of lattice node in the $x$ direction.

b) **Inlet and outlet boundary conditions with known depth and velocity**

For one – dimensional problem, the distribution function $f_1$ of the lattice at the inlet boundary and the distribution function $f_2$ of the lattice node at the outlet boundary are unknown variables (see figure 2.10). We have to calculate them from the relation in equations (2.55) and (2.56).


Figure 2. 10 The boundary for one – dimensional problem.

From the relation between microscopic and macroscopic variables in equations (2.55) and (2.56), we obtain subsequent relations for the one – dimensional problem of the inlet boundary condition

$$f_0 + f_1 + f_2 = h \quad \text{or} \quad ef_1 - ef_2 = hu,$$

then

$$f_1 = h - f_0 - f_2 \quad \text{or} \quad f_1 = \frac{hu + ef_2}{e}. \tag{2.61}$$

For the outlet boundary condition, we obtain

$$f_2 = h - f_0 - f_1 \quad \text{or} \quad f_2 = \frac{ef_1 - hu}{e}. \tag{2.62}$$

We can choose one of them to calculate the inlet or outlet boundary condition. These two choices always give the same result.

For two –dimensional problem, the unknown variable is as same as the case in 2.2.5.1. We introduce the derivation of their formula of the inlet boundary condition. We derive the desired formula from the relation in the previous section, the equations (2.55) and (2.56). We suppose the fluid depth $h$ and the fluid velocity $u$ and $v$ (where $\vec{u} = ui + vj$) are given.

At the inlet boundary condition, the unknown distribution functions are $f_1$, $f_2$ and $f_8$ (see figure 2.11). Our aim is to find the formulas for those unknown distribution functions by using the method which is introduced by Zou and He in [12].

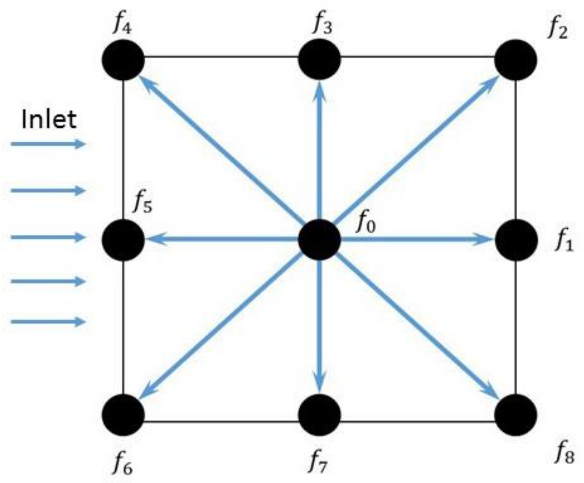



Figure 2. 11 The lattice at the inlet boundary.

From the equation (2.55), it can be rewritten as

$$f_1 + f_2 + f_8 = h - \left(f_0 + f_3 + f_4 + f_5 + f_6 + f_7\right). \qquad (2.63)$$

Also, the equation (2.56) can be separated into the $x$ component and the $y$ component of vector $e_\alpha$ result in

$$f_2 - f_8 = \frac{hv}{e} - f_3 - f_4 + f_6 + f_7, \qquad (2.64)$$

and

$$f_1 + f_2 + f_8 = \frac{hu}{e} + f_4 + f_5 + f_6. \tag{2.65}$$

Equations (2.63) and (2.65), gives

$$h = \frac{e}{e-u}\left[f_0 + f_3 + f_7 + 2\left(f_4 + f_5 + f_6\right)\right]. \tag{2.66}$$

This equation would be useful, if we did not know the water depth $h$. Next, we will determine the unknown distribution functions $f_1$, $f_2$ and $f_8$. But we have just 2 linear equations for 3 unknown variables. Certainly, it is impossible to find the exact values of these three unknown variables without other assumption. Zhou and He [12] suggested that using the bounce - back rule for the non – equilibrium part of the particle distribution [13] will work well. That is

$$f_1 - f_1^{eq} = f_5 - f_5^{eq}, \tag{2.67}$$

Thus,

$$f_1 = f_5 - f_5^{eq} + f_1^{eq}. \tag{2.68}$$

Then, substituting the equilibrium distribution function $f_1^{eq}$ and $f_5^{eq}$ into equation (2.68) results in

$$f_1 = f_5 - \left(\frac{gh^2}{6e^2} + \frac{h(-e)}{3e^2}u + \frac{h}{2e^4}(-e)(-e)u^2 - \frac{h}{6e^2}u^2 - \frac{h}{6e^2}v^2\right)$$

$$+\left(\frac{gh^2}{6e^2} + \frac{he}{3e^2}u + \frac{h}{2e^4}(-e)(-e)u^2 - \frac{h}{6e^2}u^2 - \frac{h}{6e^2}v^2\right),$$

$$= f_5 + \frac{2hu}{3e}. \tag{2.69}$$

Substituting equation (2.69) into equation (2.65), gives

$$f_2 + f_8 = \frac{hu}{3e} - f_4 + f_6 \tag{2.70}$$

Solve the equation (2.64) and equation (2.70) to get

$$f_2 = \frac{hu}{6e} + \frac{hv}{2e} + f_6 - f_4 + \frac{f_7 - f_3}{2}, \tag{2.71}$$

and

$$f_8 = \frac{hu}{6e} - \frac{hv}{2e} - \frac{f_7 - f_3}{2}. \tag{2.72}$$

Therefore, the inlet boundary condition at the left side nodes of a lattice are the equations (2.69), (2.71) and (2.72). For the other sides, we can derive it by the same idea.

### 2.2.6 Algorithm of the standard LBM

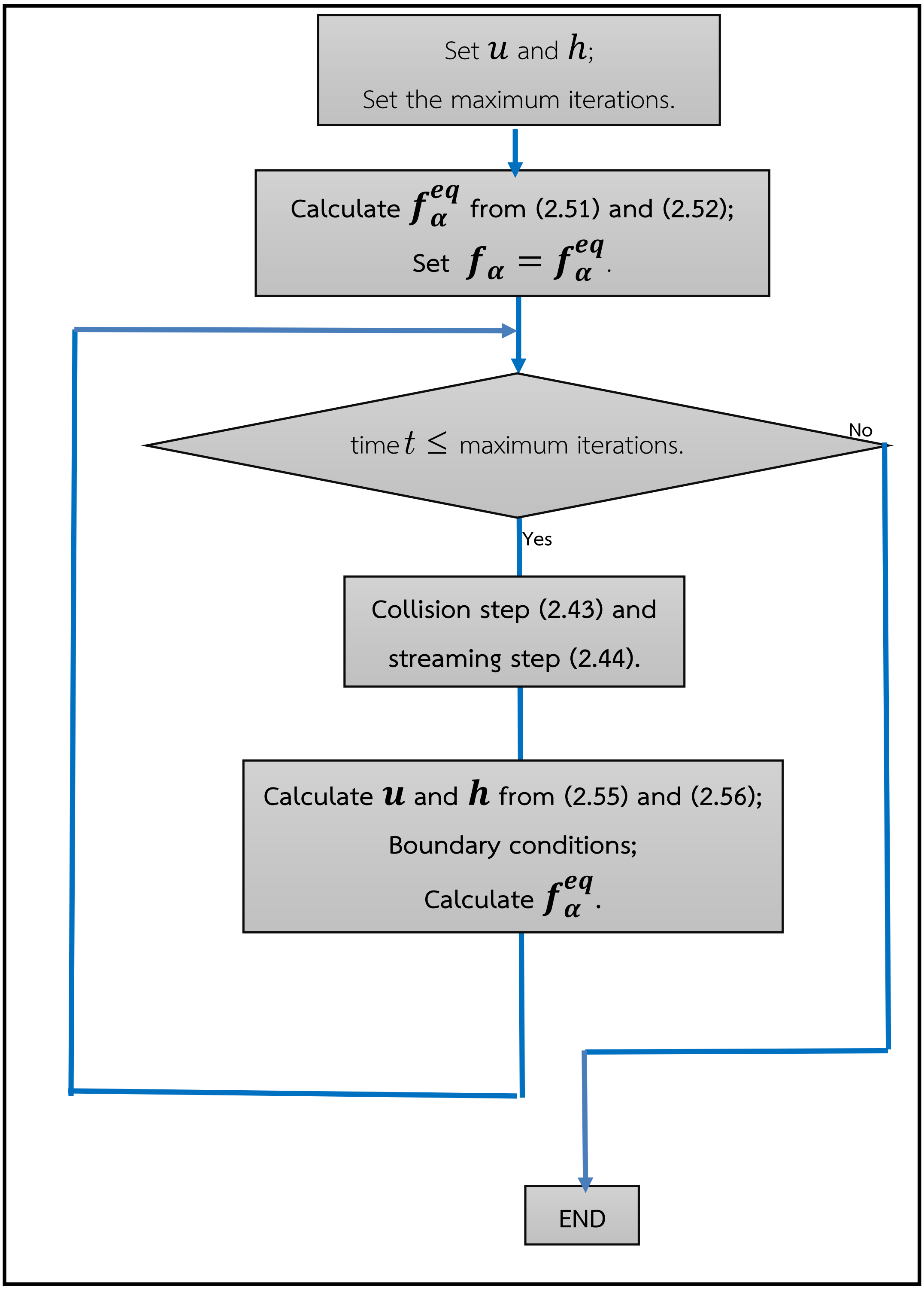



Figure 2. 12 Algorithm of the standard LBM.

# CHAPTER III
# THE SCHEME FOR
# THE PROBLEM WITH WET – DRY INTERFACE

## 3.1 One – Dimensional Problem

For the one – dimensional shallow water problems with the wet – dry interface, the common lattice Boltzmann method can cause difficulty, which are not based on conservation laws. In order to overcome this drawback, we have to modify something about the distribution function at the wet – dry front by some scheme. In 2014, Liu and Zhou introduced their scheme, which derived by using the Taylor expansion and the Chapman – Enskog procedure, for solving the shallow water flow problem with the wet – dry interface. Next, the derivation of this new scheme is explained.

Firstly, we set $\Delta t = \varepsilon$ and use the Taylor series to expand the left hand side of the lattice Boltzmann equation (2.42). This leads to

$$\varepsilon\left(\frac{\partial}{\partial t}+e_\alpha\frac{\partial}{\partial x}\right)f_\alpha+\frac{\varepsilon^2}{2}\left(\frac{\partial}{\partial t}+e_\alpha\frac{\partial}{\partial x}\right)^2 f_\alpha+O\left(\varepsilon^2\right)$$
$$=-\frac{1}{\tau}\left(f_\alpha-f_\alpha^{(0)}\right)+\frac{\varepsilon}{2e^2}e_\alpha F, \qquad (3.1)$$

in which $f_\alpha^{(0)}=f_\alpha^{eq}$.

Moreover, by using the Chapman – Enskog procedure, the distribution function $f_\alpha$ is expanded around $f_\alpha^{(0)}$ up to order $O\left(\varepsilon^2\right)$, we get

$$f_\alpha=f_\alpha^{(0)}+\varepsilon f_\alpha^{(1)}+O\left(\varepsilon^2\right). \qquad (3.2)$$

Substituting the equation (3.2) into the equation (3.1) results in

$$\varepsilon\left(\frac{\partial}{\partial t}+e_{\alpha}\frac{\partial}{\partial x}\right)\left(f_{\alpha}^{(0)}+\varepsilon f_{\alpha}^{(1)}\right)+\frac{\varepsilon^{2}}{2}\left(\frac{\partial}{\partial t}+e_{\alpha}\frac{\partial}{\partial x}\right)^{2}\left(f_{\alpha}^{(0)}+\varepsilon f_{\alpha}^{(1)}\right)$$
$$=-\frac{\varepsilon}{\tau}f_{\alpha}^{(1)}+\frac{\varepsilon}{2e^{2}}e_{\alpha}F. \tag{3.3}$$

The coefficient of order $\varepsilon$ in the equation (3.3) can be written as

$$\left(\frac{\partial}{\partial t}+e_{\alpha}\frac{\partial}{\partial x}\right)f_{\alpha}^{(0)}=-\frac{1}{\tau}f_{\alpha}^{(1)}+\frac{e_{\alpha}F}{2e^{2}}. \tag{3.4}$$

At the dry cell, the water depth and velocity should be zero, this gives

$$f_{\alpha}^{(0)}=f_{\alpha}^{eq}=0. \tag{3.5}$$

at every time.

So that

$$\frac{\partial f_{\alpha}^{(0)}}{\partial t}=0. \tag{3.6}$$

Now, the equation (3.4) can be rewritten as

$$f_{\alpha}^{(1)}=\tau\left(\frac{1}{2e^{2}}e_{\alpha}F-e_{\alpha}\frac{\partial f_{\alpha}^{(0)}}{\partial x}\right). \tag{3.7}$$

Substituting the equation (3.6), (3.7) and the source term $F$ into the equation (3.2), the distribution function at the dry cell can be demonstrated as

$$f_{\alpha}=f_{\alpha}^{(0)}+\varepsilon f_{\alpha}^{(1)}=\varepsilon f_{\alpha}^{(1)}$$
$$=\varepsilon\tau\left(\frac{1}{2e^{2}}e_{\alpha}\left(-gh\frac{\partial z_{b}}{\partial x}-\frac{\tau_{b}}{\rho}\right)-e_{\alpha}\frac{\partial f_{\alpha}^{(0)}}{\partial x}\right) \tag{3.8}$$

By the forward scheme, the terms $\dfrac{\partial z_b}{\partial x}$ and $\dfrac{\partial f_\alpha^{(0)}}{\partial x}$ can be approximated as

$$\frac{\partial z_b}{\partial x} = \frac{z_b\left(x + e_\alpha \Delta t\right) - z_b\left(x\right)}{e_\alpha \Delta t} \tag{3.9}$$

and

$$\frac{\partial f_\alpha^{(0)}}{\partial x} = \frac{f_\alpha^{(0)}\left(x + e_\alpha \Delta t\right) - f_\alpha^{(0)}\left(x\right)}{e_\alpha \Delta t}, \tag{3.10}$$

respectively.

Substituting the equations (3.9) and (3.10) into the equation (3.8), yields

$$\begin{aligned} f_\alpha = -\frac{gh\tau}{2e^2}\left(z_b\left(x + e_\alpha \Delta t\right) - z_b\left(x\right)\right) - \frac{\Delta t \tau}{2e^2} e_\alpha C_b u\left|u\right| \\ -\tau\left(f_\alpha^{(0)}\left(x + e_\alpha \Delta t, t\right) - f_\alpha^{(0)}\left(x, t\right)\right). \end{aligned} \tag{3.11}$$

The equation (3.11) can be used to determine the unknown distribution function at the dry lattice next to the wet lattice at the wet – dry interface.

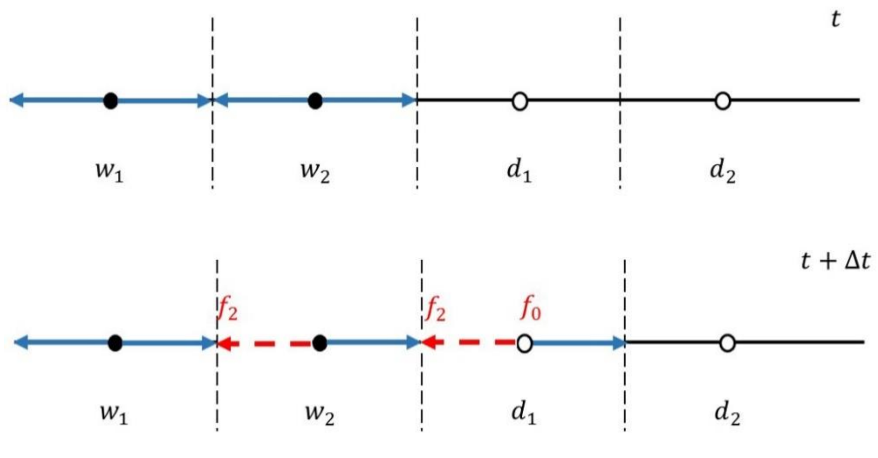



Figure 3. 1 The unknown $f_\alpha$ at the wet - dry interface.

From figure 3.1, the calculation is in the wet lattices. At time $t$ before the streaming step, the exact value of the distribution function $f_2$ at the dry lattice $d_1$ is not determined. This will be the problem when the undetermined distribution function $f_2\left(d_1\right)$ slides to the neighboring wet lattice $w_2$ in the streaming step. Because the distribution function $f_2\left(d_1\right)$ is on the dry lattice. Therefore, the value of it can be calculated by the equation (3.12). Then the streaming step can occur and all the distribution functions slide to the nearby lattice in their directions. After the streaming step, the distribution function $f_0$ and $f_2$ at the dry lattice $d_1$ are the unknown distribution function. It is easy to calculate the distribution function $f_2\left(d_1\right)$ by using the equation (3.12) again. Also, the way to obtain the value of the distribution function $f_0\left(d_1\right)$ is not difficult. It can be calculated by averaging the same distribution function of the neighboring lattices, so that

$$f_0\left(d_1\right) = \frac{f_0\left(d_2\right) + f_0\left(w_2\right)}{2}. \tag{3.12}$$

After summing all of the distribution functions for each lattice, we can calculate the water depth. Let's consider the dry lattice $d_1$, the $d_1$ lattice is still dry if $h \le 0$ and it changes to be wet when $h > 0$.These two equation (3.11) and (3.12) are very useful for the one – dimensional problem with the wet – dry interface. Moreover, these formulas help the standard lattice Boltzmann method to overcome the wet – dry problem.

**3.2 Two – Dimensional Problem**

In this case, we extend the one – dimensional scheme to the two – dimensional problem on the area domain. Because the source term on the lattice Boltzmann equation include a first derivative of the bed topography term $\dfrac{\partial z_b}{\partial x}$. So that we have to modify this troublesome derivative term by the idea proposed in [7] as follow:

$$f_\alpha\left(x+e_\alpha\Delta t,t+\Delta t\right)$$

$$=f_\alpha\left(x,t\right)-\frac{1}{\tau}\left(f_\alpha\left(x,t\right)-f_\alpha^{eq}\left(x,t\right)\right)-w_\alpha\frac{g\bar{h}}{C_s^2}\left(z_b\left(x+e_\alpha\Delta t\right)-z_b\left(x\right)\right)$$

$$+w_\alpha\frac{\Delta t}{C_s^2}e_{\alpha i}F_i, \quad (3.13)$$

in which $\bar{h}=\left(h\left(x+e_\alpha\Delta t,t+\Delta t\right)+h\left(x,t\right)\right)/2$.

Supposing that $\Delta t$ is small enough and $\varepsilon=\Delta t$, by substituting $\varepsilon$ into the equation (3.13), we obtain

$$f_\alpha\left(x+e_\alpha\varepsilon,t+\varepsilon\right)$$

$$=f_\alpha\left(x,t\right)-\frac{1}{\tau}\left(f_\alpha\left(x,t\right)-f_\alpha^{eq}\left(x,t\right)\right)-w_\alpha\frac{g\bar{h}}{C_s^2}\left(z_b\left(x+e_\alpha\varepsilon\right)-z_b\left(x\right)\right)$$

$$+w_\alpha\frac{\varepsilon}{C_s^2}e_{\alpha i}F_i. \quad (3.14)$$

By the Taylor's expansion of the left – hand side of the equation (3.14), we have

$$\varepsilon\left(\frac{\partial}{\partial t}+e_\alpha\frac{\partial}{\partial x_j}\right)f_\alpha+\frac{\varepsilon^2}{2}\left(\frac{\partial}{\partial t}+e_\alpha\frac{\partial}{\partial x_j}\right)^2 f_\alpha+O\left(\varepsilon^2\right)$$

$$=-\frac{1}{\tau}\left(f_\alpha\left(x,t\right)-f_\alpha^{eq}\left(x,t\right)\right)-w_\alpha\frac{g\bar{h}}{C_s^2}\left(z_b\left(x+e_\alpha\varepsilon\right)-z_b\left(x\right)\right)$$

$$+w_\alpha\frac{\varepsilon}{C_s^2}e_{\alpha i}F_i. \quad (3.15)$$

Considering the Chapman – Enskog procedure once more for this case, $f_\alpha$ is expanded around $f_\alpha^{(0)}$ up to order $O\left(\varepsilon^2\right)$, leads to

$$f_\alpha = f_\alpha^{(0)} + \varepsilon f_\alpha^{(1)} + \varepsilon^2 f_\alpha^{(2)} + O\left(\varepsilon^3\right). \tag{3.16}$$

The second term and the source term $F_i$ on the right hand side of the equation (3.14) can be expanded by the Taylor expansion as

$$w_\alpha \frac{g}{C_s^2}\left(h + \frac{\varepsilon}{2}\left(\frac{\partial h}{\partial t} + e_{\alpha j}\frac{\partial h}{\partial x_j}\right)\right)\left(\varepsilon e_{\alpha j}\frac{\partial z_b}{\partial x_j} + \frac{\varepsilon^2}{2}e_{\alpha i}e_{\alpha j}\frac{\partial^2 z_b}{\partial x_i \partial x_j}\right) + O\left(\varepsilon^3\right) \tag{3.17}$$

and

$$F_i = F_i\left(x + \frac{1}{2}e_\alpha \varepsilon, t + \frac{1}{2}\varepsilon\right) = F_i + \frac{\varepsilon}{2}\left(\frac{\partial F_i}{\partial t} + e_\alpha \frac{\partial F_i}{\partial x_i}\right) + O\left(\varepsilon^2\right), \tag{3.18}$$

where the source term $F_i$ is considered by the centred scheme [7].

Substituting the equation (3.16), (3.17) and (3.18) into (3.15), then the comparing with the coefficients of each order as follow $\varepsilon$ is used.

At order $\varepsilon^0$, we obtain

$$f_\alpha^{(0)} = f_\alpha^{eq}, \tag{3.19}$$

at order $\varepsilon^1$, we acquire

$$\left(\frac{\partial}{\partial t} + e_{\alpha j}\frac{\partial}{\partial x_j}\right) f_\alpha^{(0)} = -\frac{1}{\tau} f_\alpha^{(1)} - w_\alpha \frac{gh}{C_s^2} e_{\alpha j}\frac{\partial z_b}{\partial x_j} + w_\alpha \frac{e_{\alpha j}F_i}{C_s^2}, \tag{3.20}$$

and at order $\varepsilon^2$, we have

$$\left(\frac{\partial}{\partial t} + e_{\alpha j}\frac{\partial}{\partial x_j}\right) f_\alpha^{(1)} + \frac{1}{2}\left(\frac{\partial}{\partial t} + e_{\alpha j}\frac{\partial}{\partial x_j}\right)^2 f_\alpha^{(0)}$$

$$= -\frac{1}{\tau} f_\alpha^{(2)} - w_\alpha \frac{g e_{\alpha j}}{2C_s^2}\left(\frac{\partial h}{\partial t} + e_{\alpha i}\frac{\partial h}{\partial x_i}\right)\frac{\partial z_b}{\partial x_j} - w_\alpha \frac{gh e_{\alpha i}e_{\alpha j}}{2C_s^2}\frac{\partial^2 z_b}{x_i x_j}$$

$$+w_\alpha \frac{e_{\alpha i}}{2C_s^2}\left(\frac{\partial F_i}{\partial t}+e_{\alpha j}\frac{\partial F_i}{\partial x_i}\right). \tag{3.21}$$

Inserting the equation (3.20) into (3.21) and readjusting it, leads to

$$\left(1-\frac{1}{2\tau}\right)\left(\frac{\partial}{\partial t}+e_{\alpha j}\frac{\partial}{\partial x_j}\right)f_\alpha^{(1)}=-\frac{1}{\tau}f_\alpha^{(2)}. \tag{3.22}$$

Calculating $\sum_\alpha\Big((3.20)+\varepsilon\times(3.22)\Big)$ produces

$$\frac{\partial}{\partial t}\sum_\alpha f_\alpha^{(0)}+\frac{\partial}{\partial x_j}\sum_\alpha e_{\alpha j}f_\alpha^{(0)}=0. \tag{3.23}$$

By the relation between the microscopic variables and macroscopic variables in the equation (2.55), we can observe that the equation (3.24) is the conservation of mass. Similarly, calculating $\sum_\alpha e_{\alpha i}\Big((3.20)+\varepsilon\times(3.22)\Big)$, we can obtain conservation of momentum in another form, ie,

$$\frac{\partial}{\partial t}\sum_\alpha e_{\alpha i}f_\alpha^{(0)}+\frac{\partial}{\partial x_j}\sum_\alpha e_{\alpha i}e_{\alpha j}f_\alpha^{(0)}+\varepsilon\left(1-\frac{1}{2\tau}\right)\frac{\partial}{\partial x_j}\sum_\alpha e_{\alpha i}e_{\alpha j}f_\alpha^{(1)}$$
$$=F_i-gh\frac{\partial z_b}{\partial x_i}. \tag{3.24}$$

The formula for the unknown distribution function at dry lattice of the two – dimensional case is

$$f_\alpha=-\frac{w_\alpha gh\tau}{C_s^2}\Big(z_b\left(x+e_\alpha\Delta t\right)-z_b\left(x\right)\Big)-\frac{w_\alpha\Delta t\tau}{C_s^2}e_{\alpha i}C_b u_i\sqrt{u_j u_j}$$
$$-\tau\left(f_\alpha^{(0)}\left(x+e_\alpha\Delta t\right)-f_\alpha^{(0)}\left(x\right)\right). \tag{3.25}$$

Nevertheless, there may be other unknown distribution functions that are not facing to the wet nodes in their moving ways (see the figure 3.2 (a)). The above formula cannot be applied to these unknown distribution functions. However, they can be acquired by averaging values of its neighboring lattice as.

$$f_\alpha = \frac{1}{8}\sum_{\alpha=1}^{8} f_\alpha\left(x + e_\alpha \Delta t\right). \tag{3.26}$$

The equation (3.25) and (3.26) are used in 2 cases: $n < 4$ and $n \geq 4$, where $n$ is the number of the neighboring wet lattices of a dry lattice that is considering.

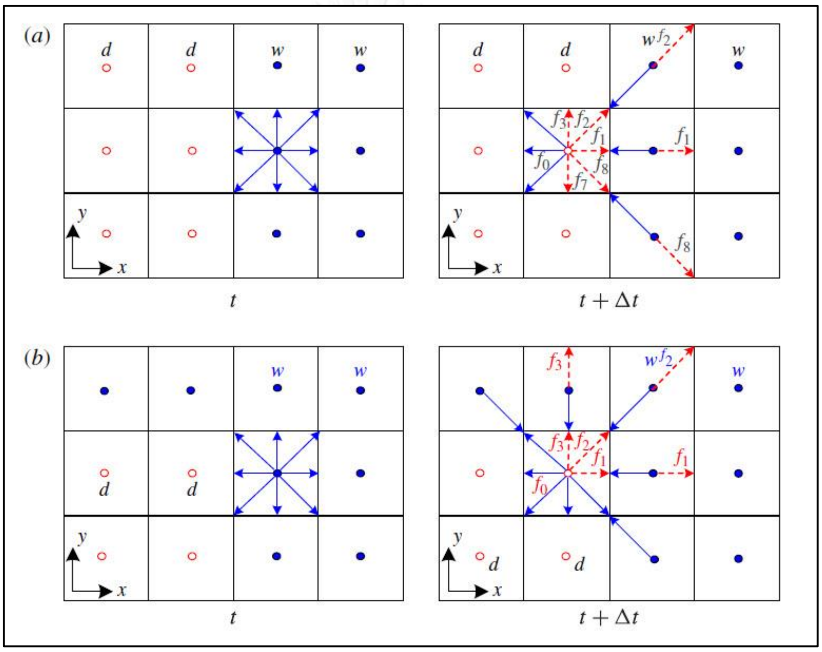



Figure 3. 2 The unknown $f_\alpha$ at the wet dry interface for the 2D problem from [1] .

**3.3 Wet – dry tracking technique**

At the initial step, we set the variable $w$ in order to retain the value about the wet state or dry state. If the fluid depth at the lattice which is interested is more than zero, it would be a wet lattice and set $w = 1$. Otherwise, if it is less than or equal to zero, it would be a dry lattice and set $w = 0$. Moreover we have to set the variable that retain the position of the wet-dry interface. It was called $dry\_node$.

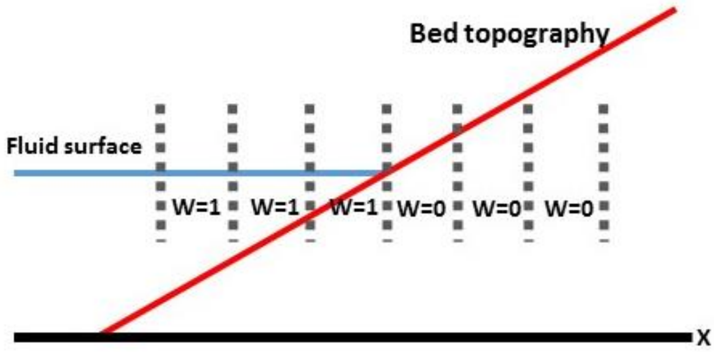



Figure 3. 3 The wet-dry interface.

After the streaming step, we get the new distribution function. We can some over them and get the new fluid depth at each lattice. The way to consider which lattice should be whether wet or dry is described as follow.

If the depth $h > 0.0001$, the lattice is wet. Otherwise, if the depth $h \leq 0.0001$ , this is the dry lattice and we set $h = 0$ . Therefore, the current interface is between wet lattice and day lattice.

### 3.4 Algorithm for LBM with the new scheme

1. Set the initial value of the fluid velocity $u$ and the water depth $h$.
2. Calculate $f_\alpha^{eq}$ from the equation (2.51) or (2.52), then set $f_\alpha = f_\alpha^{eq}$.
3. If $f_{\bar{\alpha}} > 0$, calculate $f_\alpha$ at dry cell from equations (3.11) and (3.12) for 1D problem or (3.25) and (3.26) for 2D problem.

   Otherwise, $f_\alpha = f_{\bar{\alpha}}$.
4. The collision step and the streaming step are computed and then $f_\alpha$ are updated. The wet-dry tracking technique is included in this step.
5. Use step 3 one more time to fill the unknown $f_\alpha$ at $t + \Delta t$ on dry lattice.
6. Update $u$ and $h$ by (2.55) and (2.56).
7. Repeat step 2 to step 6 until the results reach the desired point

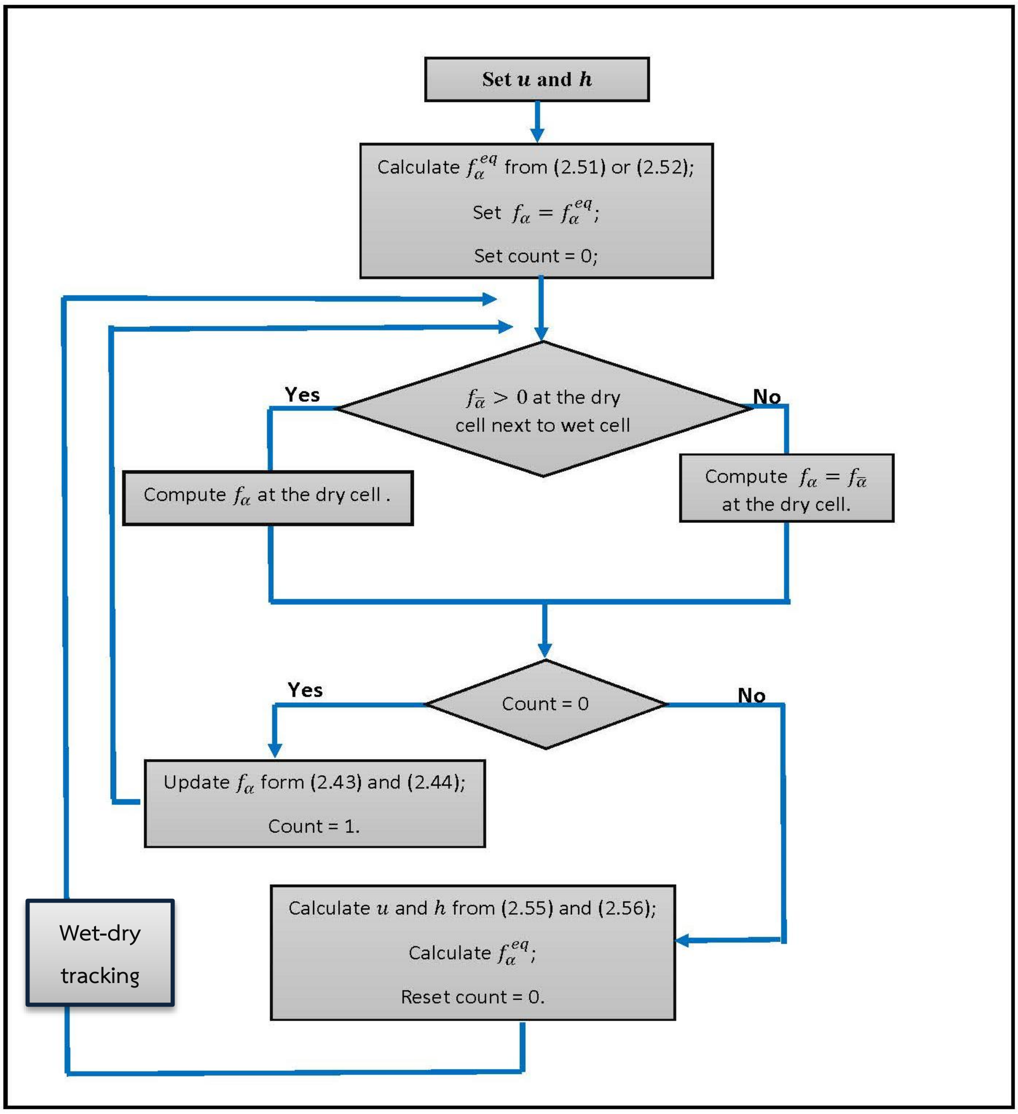



Figure 3. 4 Algorithm for LBM with the new scheme.

# CHAPTER IV
# NUMERICAL RESULTS

### 4.1 Dam break on a wet domain without friction

This problem which is governed by SWEs without friction is studied by LBM in [5]. The initial water depth is given via the following Riemann problem.

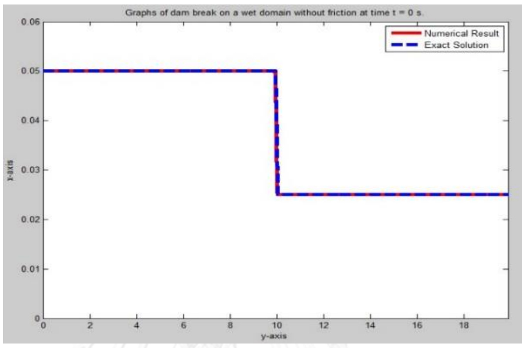



Figure 4. 1 Initial water levels.

$$h(x) = \begin{cases} h_l = 0.05 & for \quad 0 \le x \le x_0, \\ h_r = 0.025 & for \quad x_0 \le x \le L, \end{cases} \tag{4.1}$$

and the initial velocity $u(x) = 0$ m/s, $L = 20$ m and $x_0 = 10$ m. When $t \ge 0$ , the analytical solutions is proposed in [14] as

$$h(x) = \begin{cases} h_l & if \quad x \le x_A(t), \\ \dfrac{4}{9g}\left(\sqrt{gh_l} - \dfrac{x - x_0}{2t}\right)^2 & if \quad x_A(t) \le x \le x_B(t), \\ \dfrac{c_m^2}{g} & if \quad x_B(t) \le x \le x_C(t), \\ h_r & if \quad x_C(t) \le x, \end{cases} \tag{4.2}$$

and

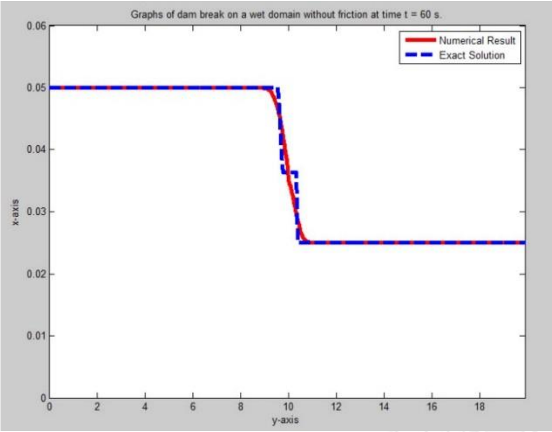
$$u(t,x)=\begin{cases} 0 & if \quad x \le x_A(t), \\ \frac{2}{3}\left(\frac{x-x_0}{t}+\sqrt{gh_l}\right) & if \quad x_A(t) \le x \le x_B(t), \\ 2\left(\sqrt{gh_l}-c_m\right) & if \quad x_B(t) \le x \le x_C(t), \\ 0 & if \quad x_C(t) \le x, \end{cases} \tag{4.3}$$

where $x_A(t) = x_0 - t\sqrt{gh_l}$, $x_B(t) = x_0 + t\left(2\sqrt{gh_l} - 3c_m\right)$ and

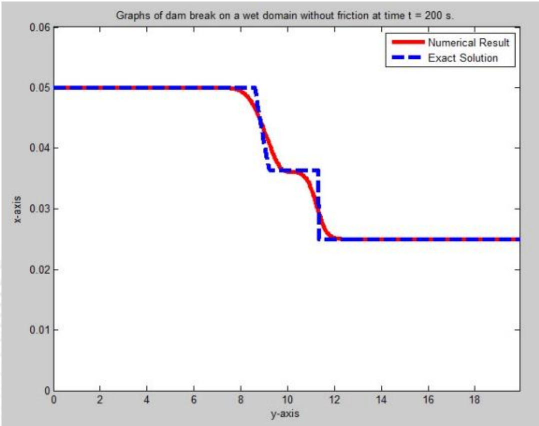
$x_C(t) = x_0 + t\dfrac{2c_m^2\left(\sqrt{gh_l} - c_m\right)}{c_m^2 - gh_r}$ with $c_m = \sqrt{gh_m}$ being the solution of

$$-8gh_r c_m^2\left(\sqrt{gh_l} - c_m\right)^2 + \left(c_m^2 - gh_r\right) + \left(c_m^2 + gh_r\right) = 0.$$

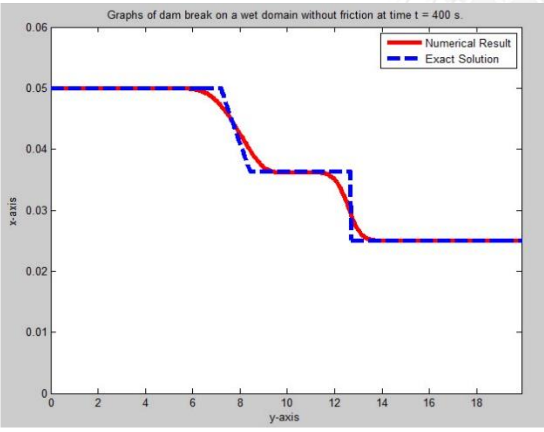
The numerical results (water depths) at $t = 60, 200, 400$ and $600$ s. are compared with the exact solution as shown in Figure 4.2, showing the agreement between these results. This concludes that the developed scheme can handle the dam break problem on a wet domain quite well agree with the exact solution and the result in [5].



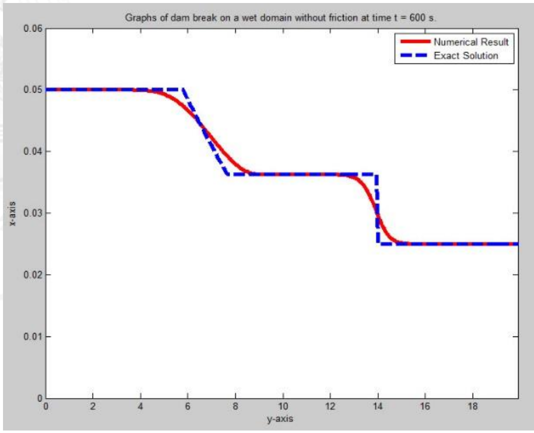

(a) $t = 60.0$ s




(b) $t = 200.0$ s



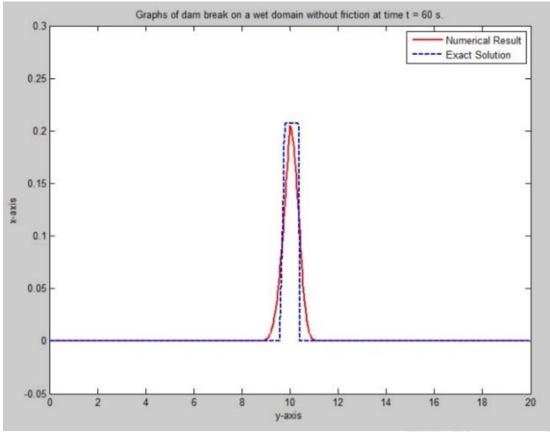

(c) $t = 400.0$ s



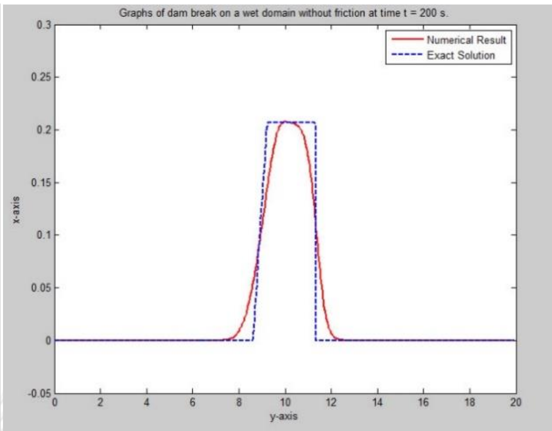

(d) $t = 600.0$ s

Figure 4. 2 Comparisons between the numerical results (water depths) and the exact solution for the dam break on wet domain at various times.

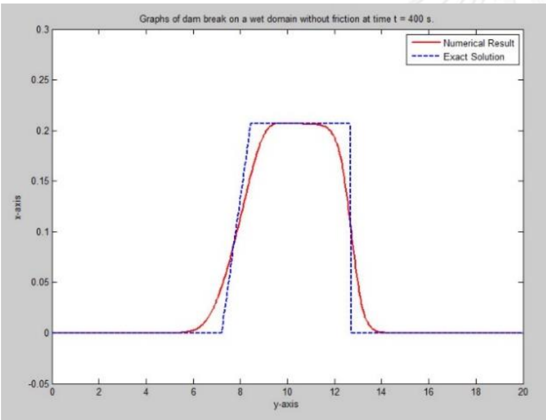
Similarly, the velocity profiles obtained from the numerical simulation also agree with the exact solution, as shown in Figure 4.3.



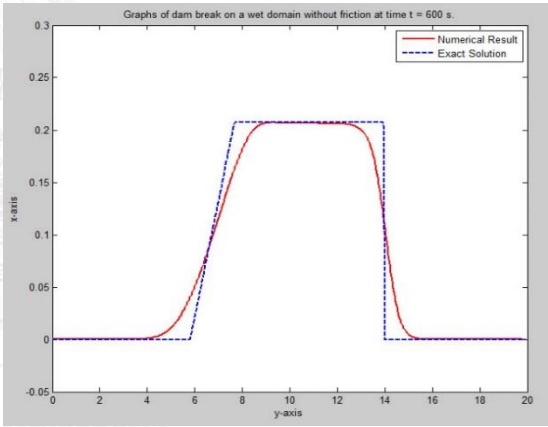

(a) $t = 60.0$ s




(b) $t = 200.0$ s




(c) $t = 400.0$ s




(d) $t = 600.0$ s

Figure 4. 3 Comparisons between the numerical velocity profiles and their exact solution for the dam break on wet domain at various times.

**4.2 Steady flow over a bump**

In this example, a 1 – dimensional subcritical flow in a 25 m long channel with a bump is simulated. This is a benchmark test case for dam break wave problems simulate by numerical methods. This problem was used by [6].

The bump is defined by

$$z_b(x) = \begin{cases} 0.2 - 0.05(x-10)^2 & 8 < x < 12; \\ 0 & otherwise; \end{cases} \tag{4.4}$$

The discharge per unit was specified at the inflow $q = 4.42m^2/s$; the flow height is $h = 2$ m; $e = 15$ m/s; and $\tau = 1.5$.

The velocity at inflow is always generated relating to the discharge per unit. The water depth at outflow is fixed at 2 m high. The result is shown in Figure 4.1. When the subcritical flow flows over the curved bump, the surface is dropping above the bump. The numerical result agrees quite well with the exact solution in [14].

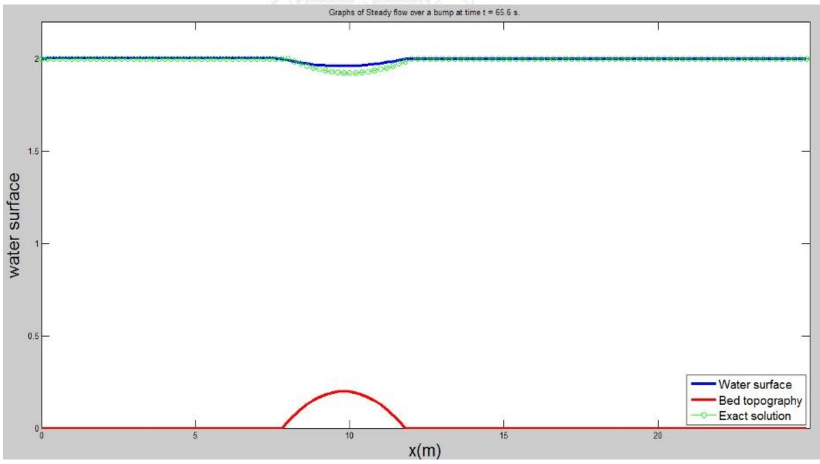


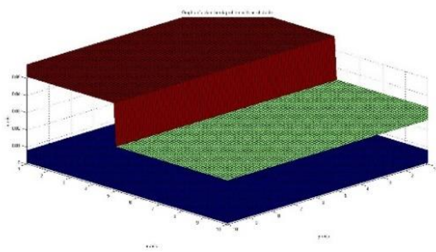

Figure 4. 4 The steady state.

### 4.3 Dam break flow over a spherical bump

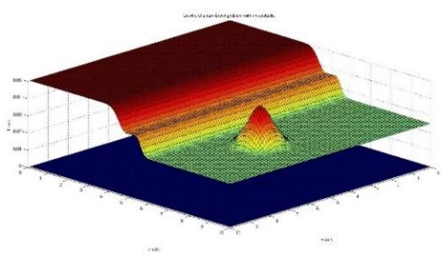
This experiment is performed to check the developed scheme for a dam break problem in 2D with a bump under the water surface. The computation is performed on the domain having 100x100 nodes with lattice size $\Delta x = 0.1$ , time step $\Delta t = 0.01$ , the kinetic viscosity $\nu = 0.01$, and the spherical bump defined by

$$z_b\left(x,y\right) = 0.005\left(x^2 - 15x + 56\right)\left(y^2 - 12y + 35.75\right). \quad (4.5)$$

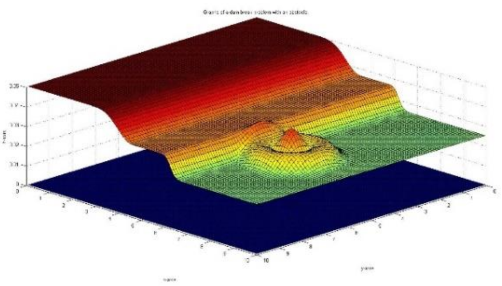
The water surface profiles at $t = 0, 1, 2, 2.5, 3$ and $4s$ are shown in Figure 4.5 displaying the waves generated by the bump.



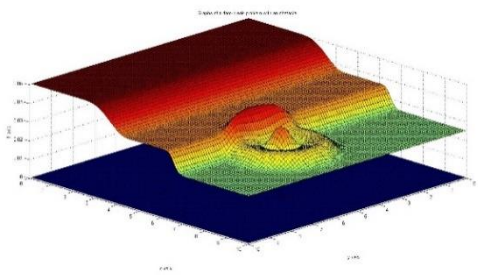
(a) $t = 0.0$ s



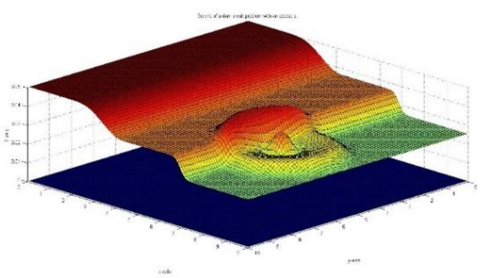
(b) $t = 1.0$ s



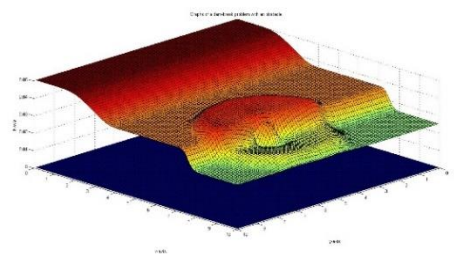
(c) $t = 2.0$ s



(d) $t = 2.5$ s



(e) $t = 3.0$ s



(f) $t = 4.0$ s

Figure 4. 5 A set of numerical results.

**4.4 Tidal wave over a variable sloping bed**

This problem was presented primarily by [15] and was introduced for LBM by [1] with the purpose of studying the shoreline movement over a bed topography defined in Table 4.1.

The computational domain is 500 m long with the bed slope as shown in Figure 4.6. The initial depth is the steady state fluid with the level of 1.75 m. The bounce back boundary condition is set at $x = 0$ m and the inlet boundary at $x = 500$ m is associated with the time – dependent water depth as,

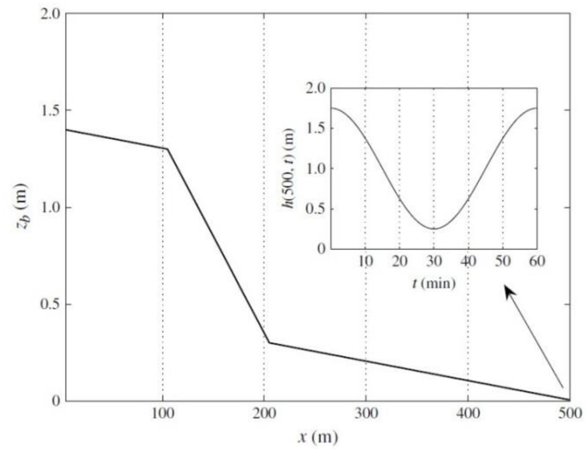



Figure 4. 6 Bed elevation and inlet boundary condition introduced by [1].

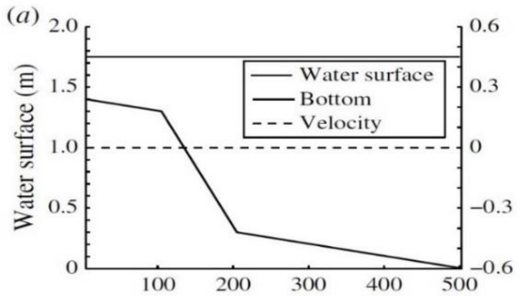
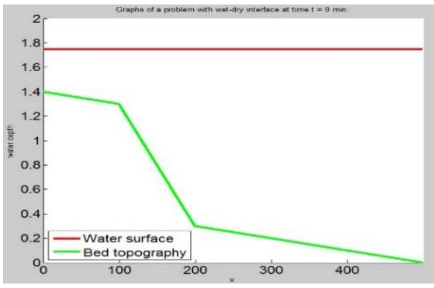
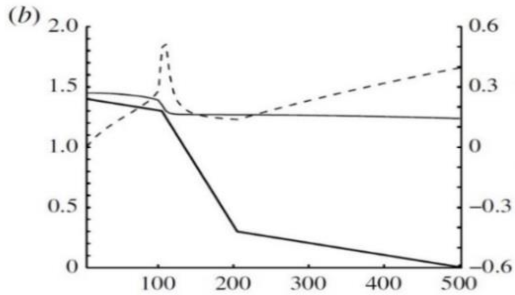
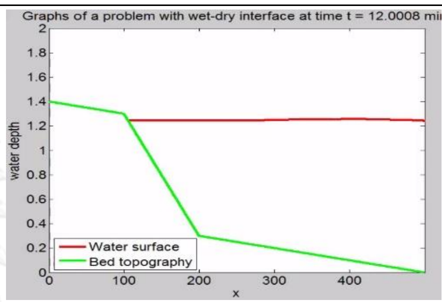
$$h\left(500, t\right) = h_0 + \lambda \cos\left(2\pi \frac{t}{T}\right), \qquad (4.6)$$

where $h_0 = 1$ , $\lambda = 0.75$, and $T = 3600$.

| $x\left(m\right)$ | 0 - 100 | 100 - 200 | 200 - 500 |
|---|---|---|---|
| slope | -0.001 | -0.01 | -0.001 |

Table 4. 1 **Bed slopes**

The results of the water surface obtained from the numerical experiment are shown in Figure 4.7 (right) and compared with the results from [1] (left).

<table>
<tr><th>Numerical results by [5]</th><th>Our numerical results</th></tr>
<tr><td>
</td><td>
</td></tr>
<tr><td>$t = 0$ min</td><td>$t = 0$ min</td></tr>
<tr><td>
</td><td>
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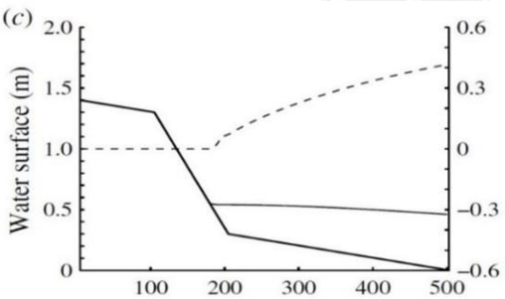
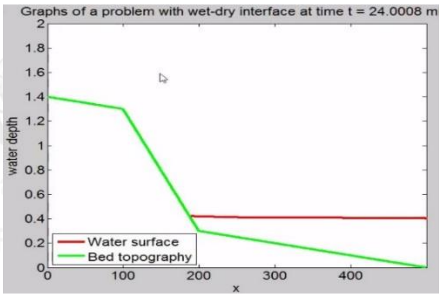
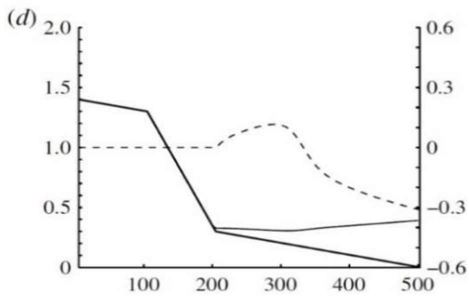
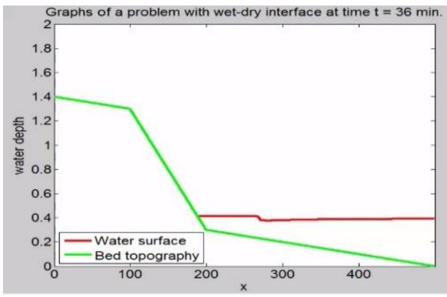
<tr><td>$t = 12$ min</td><td>$t = 12$ min</td></tr>
<tr><td>
</td><td>
</td></tr>
<tr><td>$t = 24$ min</td><td>$t = 24$ min</td></tr>
<tr><td>
</td><td>
</td></tr>
<tr><td>$t = 36$ min</td><td>$t = 36$ min</td></tr>
</table>

Figure 4. 7 Comparisons between our numerical results and the results from [1] at various times.

The results show agreement between our results and the results from [1]. This confirms that our developed scheme can handle wet dry problem quite well.

### 4.5 A 1D Wave run over a sloping bed

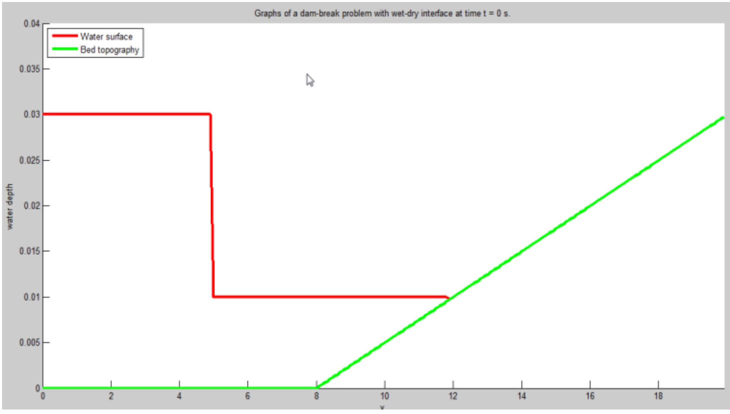



Figure 4. 8 The initial profile.

In this example, the fluid in two different levels is considered as a dam break problem with a sloping bed. The interface between the fluid and the dry bed node next to fluid is considered as a wet – dry boundary. The left and right boundaries are specified by bounce – back boundary.The channel is 20 m long with 200 lattices, the time step $\Delta t = 0.1s$, the initial fluid surface profiles are defined by

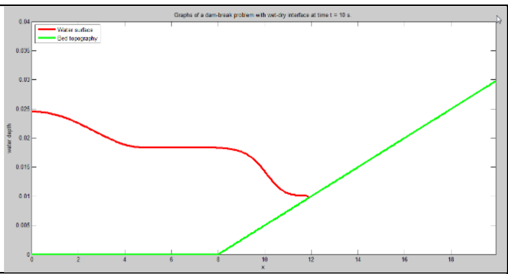
$$h(x) = \begin{cases} 0.03 & 0 < x < 5; \\ 0.01 & x \geq 5. \end{cases} \tag{4.7}$$

The bed topography is defined by

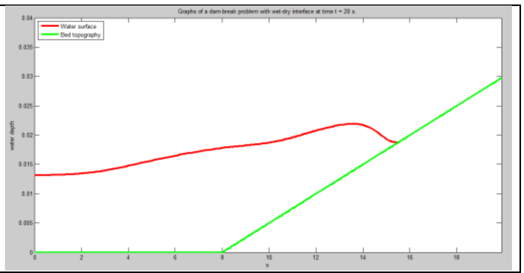
$$z_b(x) = \begin{cases} 0.0 & 0 < x < 8; \\ 0.0025(x-8) & x \geq 8. \end{cases} \tag{4.8}$$

The kinematic viscosity $\nu = 0.05$.

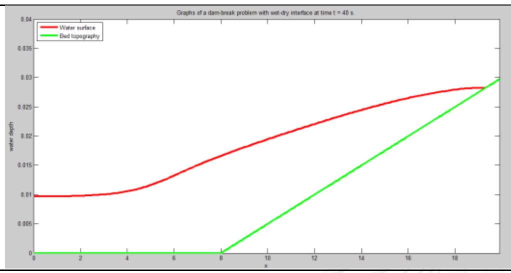
The numerical results are shown in the following.



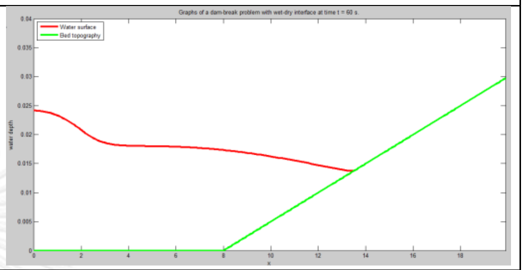
(a) $t = 10.0$ s



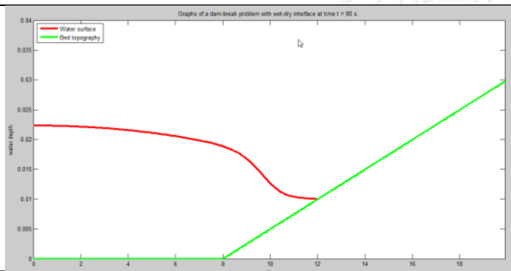
(b) $t = 20.0$ s



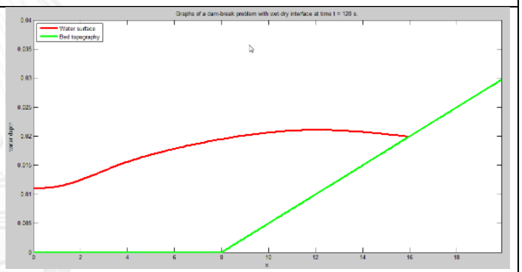
(c) $t = 40.0$ s



(d) $t = 60.0$ s



(e) $t = 80.0$ s



(f) $t = 120.0$ s

Figure 4. 9 Graphs of 1D-SWEs problem with wet –dry interface.

The different levels of fluid make it flows up to the sloping bed. Then it moves down. This phenomenon happens repeatedly until it reaches the steady state (see figure 4.10). We also check the volume of fluid at beginning and at the steady state. The volume a bit increases at the end.

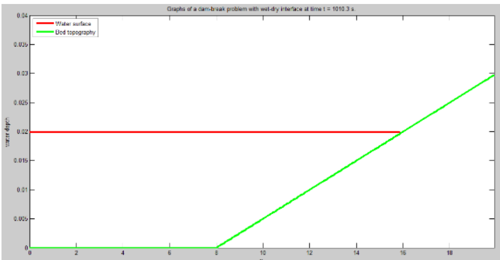


Figure 4. 10 The steady state.

# CHAPTER V
# CONCLUSION

The standard lattice Boltzmann method in general works quite well for the shallow water flow problem with the source terms such as the bed slope and the bed friction when the problem has no wet-dry interfaces. But when the problem has wet-dry interfaces, there will be some mistakes from the formula in the calculation at the dry lattice; some equations consist of terms that are divided by the water depth, which are zeros at dry lattices. Therefore, the standard LBM is not capable for this wet-dry interfaces problem. This difficulty is overcome as the result developed by [1], based on some modification of the distribution function at the dry lattice using the Taylor expansion and Chapman – Enskog procedure.

This work extended the result introduce in [5] for SWEs to handle wet-dry interface problems by employing the idea proposed by [1] . The work is focusing on the implementation of the LBM to handle wet-dry interfaces, by modifying the codes from [5] and combine with the idea of [1] to obtain distribution function at dry lattices, and together with wet-dry tracking technique in the calculation. The implemented program is tested with some numerical examples to validate the program, which show the results that agree well with exact solutions or results in literature in the experiments, see CHAPTER IV.

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