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PARTIAL REGULARITY OF HARMONIC MAPS TO SPHERES

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ให้ n,k เป็นจำนวนนับที่มากกว่าหรือเท่ากับ 2 และ Ω เป็นเซตเปิดที่มีขอบเขตในปริภูมิ $\mathbb{R}^n$ นิยามการส่งแบบฮาร์มอนิกไปยังผิวทรงกลม $u:\Omega \to S^{k-1}$ เป็นฟังก์ชันในปริภูมิ Sobolev $H^1(\Omega, S^{k-1})$ ที่สอดคล้องกับสมการเชิงอนุพันธ์ย่อยแบบไม่เชิงเส้น

$$-\Delta u = u|\nabla u|^2$$

และ u ทำให้การหาปริพันธ์ $E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2$ มีค่าน้อยที่สุดเทียบกับฟังก์ชันอื่น $w \in H^1(\Omega, S^{k-1})$ ซึ่ง $u = w$ บนย่านใกล้เคียงของ $\partial\Omega$

ทฤษฎีที่เป็นหลักในการแยกเยะจุดปรกติและจุดเอกฐาน ของการส่งแบบฮาร์มอนิกคือทฤษฎีการมีพลังงานน้อย ในวิทยานิพนธ์ฉบับนี้เราเสนอบทพิสูจน์อีกแบบสำหรับทฤษฎีดังกล่าว สำหรับการส่งแบบฮาร์มอนิกไปยังผิวของทรงกลม โดยอาศัยเทคนิกการประมาณแบบพินอลไลเซชัน

ภาควิชา **คณิตศาสตร์** ลายมือชื่อนิสิต..
สาขาวิชา **คณิตศาสตร์** ลายมือชื่ออาจารย์ที่ปรึกษา.......................................
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ABSTRACT

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Let n, k be integers greater than or equal to 2 and Ω a bounded open subset of the Euclidean space $\mathbb{R}^n$. A (minimizing) harmonic map to sphere $u : \Omega \to S^{k-1}$ is a function in the Sobolev space $H^1(\Omega, S^{k-1})$ which satisfies the following system of nonlinear partial differential equation

$$-\Delta u = u|\nabla u|^2,$$

and u minimizes the functional $E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx$ among those $w \in H^1(\Omega, S^{k-1})$ such that $u \equiv w$ in a neighborhood of $\partial\Omega$.

The key theorem for distingushing the singular and the regular points of harmonic maps is the small energy regularity theorem. In this thesis we give an alternative proof for this theorem when the target spaces are spheres via the penalty approximation technique.

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To my parents

CONTENTS

CHAPTER I

INTRODUCTION

The notion of harmonic maps between Riemannian manifolds can be viewed as the generalization of the classical notion of harmonic functions between Euclidean spaces. A *harmonic function* from a bounded open set $\Omega \subset \mathbb{R}^n$ into $\mathbb{R}^k$ is a C^2 function $u : \Omega \to \mathbb{R}^k$ which is a critical point of the "energy" functional $E(u) = \frac{1}{2}\int_\Omega |Du|^2 dx$. It then follows that u must satisfy the following Euler-Lagrange equation

$$\int_\Omega \sum_{j=1}^{n} D_j u \cdot D_j \zeta \, dx = 0 \qquad \text{(i)}$$

for all $\zeta \in C_0^\infty(\Omega, \mathbb{R}^k)$. This implies $-\triangle u = -\sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2} = 0$ on Ω.

By the theorem of H. Weyl that any weakly harmonic map (i.e. weakly differentiable function verified (i)) is harmonic, it motivates the following definition of harmonic maps. A *harmonic map* from a bounded open set $\Omega \subset \mathbb{R}^n$ into a smooth compact Riemannian manifold N, which can be assumed to be isometrically embedded in some $\mathbb{R}^k$, is a weakly differentiable function $u : \Omega \to \mathbb{R}^k$ such that $u(x) \in N$ for a.e. $x \in \Omega$ (with respect to the Lebesgue measure on Ω) and it is a critical point of the "energy" functional $E(u) = \frac{1}{2}\int_\Omega |Du|^2 dx$.

The Euler-Lagrange equation of harmonic maps are the following *nonlinear* partial differential equation

$$-\triangle u = \sum_{j=1}^{n} A_u(D_j u, D_j u),$$

where A_u is the second fundamental form of N in $\mathbb{R}^k$. In particular, for harmonic maps into spheres $u : \Omega \to S^{k-1}$, the Euler-Lagrange equation have the following form

$$-\triangle u = u|Du|^2. \tag{ii}$$

Here, u weakly solves (ii) in the sense that $\int_\Omega \sum_{j=1}^n D_j u \cdot D_j \zeta \, dx = \int_\Omega u \cdot \zeta |Du|^2 dx$ for all $\zeta \in C_0^\infty(\Omega, \mathbb{R}^k)$. The extension from $\Omega \subset \mathbb{R}^n$ to be arbitrary smooth compact Riemannian manifolds involves simple technical modifications.

Harmonic functions have very nice differentiable properties, in fact any harmonic functions are smooth. But in contrast, general harmonic maps do have singular points, that is they have only *partial regularity* properties. In some extreme cases, there is an example of everywhere discontinuous harmonic map due to T. Riviére. For general harmonic maps it is of great interest to study both analytic and geometric properties of the sets of singular points.

The harmonic maps that we defined above is often called *weakly* harmonic map. There are also the *stationary* and *minimizing* harmonic maps. In our work, we are mainly concerned with the minimizing harmonic maps. Though we also derive the variational identities for each types of harmonic maps in chapter 3.

The Existence Questions:

There is an important question, since the beginning of harmonic maps, about the existence of certain harmonic maps. The goal is to find whether there exists a harmonic map in a given homotopy class of a smooth map. A technique used to solve this question is to study the harmonic map flow equation. This is the heat equation associated with certain harmonic map equation.

The Regularity Questions:

This question involves both interior and boundary regularity properties. Since the equation of harmonic maps is of quasilinear elliptic type, the main tools for analyzing interior regularity come from the theory of elliptic partial differential equations. For interior regularity, the harmonic maps of stationary and minimizing types yield similar results. In fact they have quite similar criterion for a point to be regular, the *small energy regularity* theorem (also called ε-regularity theorem).

In this work, we shall mainly concerned with the interior regularity question. In addition, we only restrict our maps to the case that targets are the standard spheres. Our main result is an alternative proof of the small energy regularity theorem. The approach is to use penalized energy functional of the functional E *indirectly*. In the process we obtain a *smooth* minimizing harmonic map associated with a given minimizing harmonic map. Finally we get the theorem by applying the Bochner identity to the obtained smooth map and its corresponding gradient estimate. At the intermediate step before we can get the smooth minimizing map there is a question about the existence of the map minimizing the penalized energy subjected to certain boundary conditions. We will treat this point in the preliminary chapter.

Our work are organized into five chapters as follows.

In chapter II, we introduce fundamental facts from Functional Analysis, the theory of Sobolev spaces, and the Calculus of Variations. We begin formulating the harmonic maps problems in the chapter III. This include the derivation of the Euler-Lagrange equations for each types of harmonic maps. In chapter IV, we introduce the regularity theory of Elliptic Differential Equations. Chapter V contains our main results about the partial regularity of harmonic maps to spheres.

CHAPTER II

PRELIMINARIES

In this chapter we introduce various tools from the Theory of Partial Differential Equations and the Calculus of Variations which are needed in the sequel. Most propositions are supplied with proofs, since they can illustrate the techniques used in the subjects.

2.1 Introduction

In modern PDE theory, we tackle an equation in two steps, first determine the existence of solutions, then investigate the regularity of solutions. In the existence part, appropriate underlying function spaces must be chosen, and this turns out to be Sobolev spaces. We introduce an elementary theory of Sobolev spaces in the next section via the notion of weak partial derivatives. Three important properties of Sobolev functions: (1) Sobolev embedding theorem, (2) Rellich compactness Lemma and (3) Poincaré inequality are discuss in the section (2.3). They are essential tools in the regularity theory.

Since harmonic maps are defined to be minimizers (or critical points) of certain energy functionals (i.e. harmonic maps is a *Geometric Variational Problem*), we need to introduce some terminologies and results from the Calculus of Variation. We will discuss the direct method in the Calculus of Variations in section (2.4). In section (2.5) we propose the penalty approximation method for solving constrained minimization problems.

2.2 Sobolev Spaces

Nowadays, Sobolev spaces became the appropriate setting in modern theory of partial differential equations. The underlying success is the generalized notion of differentiation. So we begin with the definition of weak derivatives. (There is also an alternative approach via Fourier transformation.)

We assume throughout this work that Ω denotes an open subset of $\mathbb{R}^n$ for some $n \geq 2$. In what follows, unless otherwise explicitly stated, we use the n-dimensional Lebesgue measure for all measure-theoretic notions.

Weak Partial Derivatives

If $u \in L^1_{\text{loc}}(\Omega)$ we call $v_j \in L^1_{\text{loc}}(\Omega)$, $1 \leq j \leq n$, the j^{th}*-weak (partial) derivative* of u provided

$$\int_\Omega u\, D_j\varphi\, dx = -\int_\Omega v_j\, \varphi\, dx, \qquad \forall\, \varphi \in C_0^\infty(\Omega).$$

In this case we simply write $v_j = D_j u$ (more explicitly, $D_{x_j}u(x)$ if x_j is the j^{th} coordinate of x), so the identity take the familiar Gauss identity. It is easy to verify that such v_j is unique (almost everywhere).

Note that if $u \in C^1(\Omega)$ then the above identity, with v_j coincides with the j^{th} classical partial derivative, is a consequence of the integration by parts formula. So the notion of weak derivatives is indeed "weaker" than the classical notion.

In general, for a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$, $(\alpha_i \in \mathbb{N} \cup \{0\})$, we say that a function $v^\alpha \in L^1_{\text{loc}}(\Omega)$ is the α^{th}*-weak (partial) derivative* of a function $u \in L^1_{\text{loc}}(\Omega)$ provided

$$\int_\Omega u\, D^\alpha\varphi\, dx = (-1)^{|\alpha|} \int_\Omega v^\alpha \varphi\, dx, \qquad \forall\, \varphi \in C_0^\infty(\Omega),$$

here $|\alpha| = \sum_i \alpha_i$. Note carefully that $D^\alpha\varphi$ on the left is the α^{th} classical derivative of φ. As above, we denote $v^\alpha = D^\alpha u$ and if such v^α exists then it is unique. Also, if $u \in C^{|\alpha|}(\Omega)$ then v^α is just the classical derivative

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} = D_1^{\alpha_1} \cdots D_n^{\alpha_n} u.$$

Sobolev Spaces: Scalar-Valued

Let $m \in \mathbb{N}$ and $1 \leq p < \infty$. A function $u \in L^p(\Omega)$ is called a Sobolev function if $D^\alpha u$ exist, for all $|\alpha| \leq m$, and they lie in $L^p(\Omega)$. Let $W^{m,p}(\Omega)$ denote the space of all such functions u, and it is called a *Sobolev space*. As in L^p spaces, any two functions in $W^{m,p}(\Omega)$ are identified if they are equal almost everywhere.

In the following we list some basic facts about Sobolev spaces, and leave more involved concepts to the next section.

The space $W^{m,p}(\Omega)$ is a Banach space under the norm

$$\|u\|_{W^{m,p}(\Omega)} = \Big\{ \sum_{0 \leq |\alpha| \leq m} \int_\Omega |D^\alpha u|^p \, dx \Big\}^{1/p}.$$

Note that $u_i \to u$ in $W^{m,p}(\Omega)$ if and only if $D^\alpha u_i \to D^\alpha u$ in $L^p(\Omega)$ as $i \to \infty$ for all $0 \leq |\alpha| \leq m$. If $p = 2$ we particularly use $H^m(\Omega) = W^{m,2}(\Omega)$. $H^m(\Omega)$ is a Hilbert space under the inner product

$$(u, v)_{H^m(\Omega)} = \sum_{0 \leq |\alpha| \leq m} \int_\Omega D^\alpha u \, D^\alpha v \, dx.$$

We also use $W_0^{m,p}(\Omega)$ (and $H_0^m(\Omega)$) to denote the closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$ (and in $H^m(\Omega)$, respectively). The space $H_0^1(\Omega)$ has the following equivalent inner product

$$(u, v)_{H_0^1(\Omega)} = \sum_{j=1}^n \int_\Omega D_j u \, D_j v \, dx,$$

(for a proof see Rellich compactness lemma in the next section). By mollification, it can be shown that the smooth functions in $W^{m,p}(\Omega)$ are dense in the space $W^{m,p}(\Omega)$. Furthermore, $\zeta \in C_0^\infty(\Omega)$ and $u \in W^{m,p}(\Omega)$, then $\zeta u \in W_0^{m,p}(\Omega)$.

The space $L^p(\Omega)$ is separable if $1 \leq p < \infty$ and is reflexive if $1 < p < \infty$. $W^{m,p}(\Omega)$, which can be identified as a closed subspace of certain product space $[L^p(\Omega)]^N$, also has the inherited properties from $L^p(\Omega)$. Thus $W^{m,p}(\Omega)$ is separable if $1 \leq p < \infty$ and is reflexive if $1 < p < \infty$. In partial differential equations theory, we mainly interest in the case where $1 < p \leq n$ (see Sobolev embedding theorem).

Sobolev Spaces: Vector-Valued

Let $u : \Omega \to \mathbb{R}^k$, $u = (u^1, \ldots, u^k)$. We say that $u \in W^{m,p}(\Omega, \mathbb{R}^k)$ if $u^i \in W^{m,p}(\Omega)$ for all $1 \leq i \leq k$. $W^{m,p}(\Omega, \mathbb{R}^k)$ is called a Sobolev space. By the definition, $W^{m,p}(\Omega, \mathbb{R}^k)$ can be identified with the product space $[W^{m,p}(\Omega)]^k$.

For each $u \in W^{1,p}(\Omega, \mathbb{R}^k)$, we regard Du as an $k \times n$ matrix $(D_j u^i)$ with a matrix norm

$$|Du|^2 = \sum_{i=1}^{k} \sum_{j=1}^{n} (D_j u^i)^2.$$

We also define $|Du|^p = (|Du|^2)^{p/2}$. This notion can be generalized to higher order in a similar manner, i.e. for $u \in W^{m,p}(\Omega, \mathbb{R}^k)$, we regard $D^m u$ as an array $\{D^\alpha u^i\}$ over all $|\alpha| = m$, $1 \leq i \leq k$, and define $|D^m u|^2 = \sum_{i=1}^{k} \sum_{|\alpha|=m} (D^\alpha u^i)^2$.

We will use the vector notation $D_j u \cdot D_j v$ to denote $\sum_{i=1}^{k} D_j u^i \, D_j v^i$, and $D^\alpha u \cdot D^\alpha v = \sum_{i=1}^{k} D^\alpha u^i \, D^\alpha v^i$ for each multi-index α.

Many properties in the scalar case are carried over to this case by trivial modifications. For example, we define the norm on $W^{m,p}(\Omega, \mathbb{R}^k)$ by

$$\|u\|_{W^{m,p}(\Omega,\mathbb{R}^k)} = \Big\{ \sum_{i=1}^{k} \sum_{0 \leq |\alpha| \leq m} \int_\Omega |D^\alpha u^i|^p \, dx \Big\}^{1/p}.$$

It is easy to show that $W^{m,p}(\Omega, \mathbb{R}^k)$ is a Banach space under this norm. Similar to the real-valued Sobolev spaces, $H^m(\Omega, \mathbb{R}^k)$ is a Hilbert space under the inner product

$$(u, v)_{H^m(\Omega,\mathbb{R}^k)} = \sum_{0 \leq |\alpha| \leq m} \int_\Omega D^\alpha u \cdot D^\alpha v \, dx.$$

The smooth functions in $W^{m,p}(\Omega,\mathbb{R}^k)$ are dense in this space. There is also an equivalent inner product for $H_0^1(\Omega,\mathbb{R}^k)$ ($\cong \left[H_0^1(\Omega)\right]^k$) given by

$$(u,v)_{H_0^1(\Omega,\mathbb{R}^k)} = \sum_{j=1}^{n} \int_\Omega D_j u \cdot D_j v \, dx.$$

The separability and reflexivity of $W^{m,p}(\Omega,\mathbb{R}^k)$ are exactly the same as in scalar case. Lastly, if $\zeta \in C_0^\infty(\Omega)$ then $\zeta u \in W_0^{m,p}(\Omega,\mathbb{R}^k)$ for all $u \in W^{m,p}(\Omega,\mathbb{R}^k)$.

2.3 Properties of Sobolev Functions

In this section we introduce Sobolev embedding theorem, Rellich compactness lemma, and Poincaré inequality. These propositions are the essential tools in the theory of partial differential equations. See Chapter (4) for some applications.

Sobolev Embedding Theorem

Recall a normed linear space X is *continuously embedded* in a normed linear space Y, written $X \hookrightarrow Y$, if (1) X can be identified as a linear subspace of Y and (2) there is a constant C such that $\|u\|_Y \leq C\|u\|_X$ for all $u \in X$. By *compact embedding*, we mean that every bounded sequence in X contains a convergent subsequence in Y.

Theorem 2.1 (Sobolev Embedding Theorem)**.** Suppose Ω is a bounded open subset of $\mathbb{R}^n$ and it has a C^1 boundary. Let $m \in \mathbb{N}$ and $1 \leq p < \infty$.

1. If $mp < n$ then $W^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$ for all $1 \leq q \leq \frac{np}{n-mp}$. Moreover, it is a compact embedding if $1 \leq q < \frac{np}{n-mp}$.

2. If there is an integer $d \geq 0$ such that $d < m - \frac{n}{p} < d+1$ then $W^{m,p}(\Omega)$ is continuously embedded in $C^{d,\alpha}(\overline{\Omega})$ for all $0 \leq \alpha \leq m - \frac{n}{p} - l$ and the embedding is compact if $\alpha < m - \frac{n}{p} - l$.

This theorem says roughly that weakly differentiable L^p functions are in higher order L^q spaces or in Hölder spaces. By interpolation, it is well-known that any L^p function over a bounded domain is in lower order L^q spaces. Thus Sobolev embedding theorem gives a reverse interpolation for $W^{m,p}$ functions. The key for proving part 1 is the corollary of the following inequality.

Lemma 2.2 (Gagliardo-Nirenburg-Sobolev Inequality)**.** Let $1 \leq p < n$. There is a constant C depends only on n and p such that

$$\Big\{ \int_{\mathbb{R}^n} |u|^{p^*}\, dx \Big\}^{1/p^*} \leq C \Big\{ \int_{\mathbb{R}^n} |Du|^p\, dx \Big\}^{1/p},$$

for all $u \in C_0^1(\mathbb{R}^n)$, where $p^* = \frac{np}{n-p}$.

Proof. First we prove for $p = 1$, that

$$\int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}}\, dx \leq \Big\{ \int_{\mathbb{R}^n} |Du|\, dx \Big\}^{\frac{n}{n-1}}.$$

Fixed $\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_n)$. Since u has a compact support, $u(\tilde{x}) = \int_{-\infty}^{\tilde{x}_j} D_{x_j} u\, dx_j$. So $|u(\tilde{x})| \leq \int_{-\infty}^{+\infty} |Du|\, dx_j$ for all $1 \leq j \leq n$, and hence

$$|u(\tilde{x})|^{\frac{n}{n-1}} \leq \prod_{j=1}^{n} \Big(\int_{-\infty}^{+\infty} |Du|\, dx_j \Big)^{\frac{1}{n-1}}.$$

This holds for all $\tilde{x}$ so we can omit the tilde. Recall the following version of Hölder inequality: $\int |f_1 \cdots f_{n-1}|^{1/(n-1)} \leq \prod_{j=1}^{n-1} \{\int |f_j|\}^{1/(n-1)}$ for all measurable functions $f_1, \ldots, f_k$. Integrate over x_1 from $-\infty$ to $+\infty$ and note that $(\int_{-\infty}^{+\infty} |Du|\, dx_1)^{1/(n-1)}$ can be moved out of the integral, we then apply Hölder inequality to the remaining $n-1$ terms and we obtain that

$$\int_{-\infty}^{+\infty} |u|^{\frac{n}{n-1}}\, dx_1 \leq \Big(\int_{-\infty}^{+\infty} |Du|\, dx_1 \Big)^{\frac{1}{n-1}} \prod_{j=2}^{n} \Big(\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |Du|\, dx_1 dx_j \Big)^{\frac{1}{n-1}}.$$

Again, integrating over x_2 and note that the term $(\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |Du|\, dx_1 dx_2)^{1/(n-1)}$ can be moved out of the integral, we can apply Hölder inequality to the remaining

$n-1$ terms and so that

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}|u|^{\frac{n}{n-1}}\,dx_1dx_2 \le \Big(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}|Du|\,dx_1dx_2\Big)^{\frac{2}{n-1}}$$
$$\prod_{j=3}^{n}\Big(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}|Du|\,dx_1dx_2dx_j\Big)^{\frac{1}{n-1}}.$$

By continuing the procedure with $x_3,\dots,x_n$ respectively, we get the desired inequality.

Now consider for general $p>1$. Let $\lambda>1$ be a constant to be chosen later. Apply the result for $p=1$ to the function $|u|^\lambda$ one obtains that

$$\Big(\int_{\mathbb{R}^n}|u|^{\frac{\lambda n}{n-1}}\,dx\Big)^{\frac{n-1}{n}} \le \lambda\int_{\mathbb{R}^n}|u|^{\lambda-1}|Du|\,dx.$$

By Hölder's inequality, $\int_{\mathbb{R}^n}|u|^{\lambda-1}|Du|\,dx \le (\int_{\mathbb{R}^n}|u|^{\frac{(\lambda-1)p}{p-1}}\,dx)^{\frac{p-1}{p}}(\int_{\mathbb{R}^n}|Du|^p\,dx)^{\frac{1}{p}}$.

Now choose λ so that $\frac{\lambda n}{n-1}=\frac{(\lambda-1)p}{p-1}=p^*$. Then

$$\Big(\int_{\mathbb{R}^n}|u|^{p^*}\,dx\Big)^{\frac{n-1}{n}} \le C\Big(\int_{\mathbb{R}^n}|u|^{p^*}\,dx\Big)^{\frac{p-1}{p}}\Big(\int_{\mathbb{R}^n}|Du|^p\,dx\Big)^{\frac{1}{p}},$$

this implies the theorem. □

Corollary 2.3 (SOBOLEV INEQUALITY)**.** Let $1\le p<n$. Then there is a constant $C=C(n,p)$ such that

$$\Big\{\int_\Omega|u|^{p^*}dx\Big\}^{1/p^*} \le C\Big\{\int_\Omega|Du|^p\,dx\Big\}^{1/p},$$

for all $u\in W_0^{1,p}(\Omega)$.

Proof. Take a sequence $u_j\in C_0^\infty(\Omega)$ such that $u_j\to u$ in $W^{1,p}(\Omega)$. This implies $u_j\to u$ strongly in $L^p(\Omega)$ hence there is a subsequence $u_{j'}$ such that $u_{j'}\to u$ pointwise a.e. on Ω. Extend each $u_{j'}$ to a smooth function on $\mathbb{R}^n$ by setting $u_{j'}\equiv 0$ on Ω^c. By the previous lemma, $u_{j'}$ satisfies

$$\Big\{\int_\Omega|u_{j'}|^{p^*}dx\Big\}^{1/p^*} \le C(n,p)\Big\{\int_\Omega|Du_{j'}|^p\,dx\Big\}^{1/p}.$$

The right hand side converges to $C\|Du\|_{L^p(\Omega)}$. Apply Fatou's lemma to the left hand side, and we are done. □

Corollary 2.4. If $\Omega \subset \mathbb{R}^n$ is a bounded C^1 domain and $1 \le p < n$, then there is a constant C depends only on n, p and Ω such that

$$\|u\|_{L^{p^*}(\Omega)} \le C\|u\|_{W^{1,p}(\Omega)},$$

for all $u \in W^{1,p}(\Omega)$.

We omit the proof here. Roughly speaking, the idea is to extend $W^{1,p}$ functions on Ω to $W_0^{1,p}$ functions over $\mathbb{R}^n$. Under the smoothness assumption of Ω, we can apply the Sobolev inequality. Now by interpolation, $\|u\|_{L^q(\Omega)} \le C|\Omega|^{(\frac{1}{q}-\frac{1}{p^*})}\|u\|_{W^{1,p}(\Omega)}$, for all $1 \le q \le p^*$.

Additionally, if the domain Ω possesses certain symmetry, then we can remove the dependence on Ω out of the constant C. This is the case for domain that is a ball, i.e. $B_R(x_0) = x_0 + RB_1(0)$. For elliptic (interior) regularity theory, we can always restrict the domain to a ball. In particular, we have the following lemma.

Lemma 2.5. Let $R > 0$, $1 \le p < n$. Then there is a constant C depends only on n and p such that

$$\|u\|_{L^p(B_R(x_0))} \le CR\|u\|_{W^{1,p}(B_R(x_0))},$$

for all $u \in W^{1,p}(B_R(x_0))$.

Proof. By interpolation, there is C_0 such that

$$\Big\{\int_{B_1(0)} |w|^p dx\Big\}^{1/p} \le C_0|B_1(0)|^{\frac{1}{n}}\|w\|_{W^{1,p}(B_1(0))},$$

for all $w \in W^{1,p}(B_1(0))$. For each $u \in W^{1,p}(B_R(x_0))$, we define $w(x) = u(x_0 + Rx)$. Thus $w \in W^{1,p}(B_1(0))$ and satisfies the inequality above. By the change of variables $y = x_0 + Rx$, we see that

$$R^{-n/p}\Big\{\int_{B_R(0)} |u|^p dx\Big\}^{1/p} \le C_0|B_1(0)|^{\frac{1}{n}}R^{1-n/p}\Big\{\int_{B_R(0)} |Du|^p dx\Big\}^{1/p}.$$

This proves the assertion. Also, note that C_0 depends only on n and p. □

By the same argument as above, we can split the dependence on the domain of multiplier constant in Poincaré inequality for the domain that is a ball.

Rellich Compactness Lemma

First we introduce the equivalent norm for $W_0^{1,p}$ space. The proof is just the application of Sobolev inequality and interpolation of L^p functions.

Lemma 2.6. Suppose $\Omega \subset \mathbb{R}^n$ is open, $1 \leq p < n$. Then $W_0^{1,p}(\Omega)$ has the following equivalent norm

$$\|u\|_{W_0^{1,p}(\Omega)} = \Big\{ \sum_{j=1}^{n} \int_\Omega |D_j u|^p dx \Big\}^{1/p}.$$

Now we describe Rellich Compactness Lemma. It is a direct consequence of Sobolev Embedding Theorem and general theorems from Functional Analysis (see section(2.4) for the details definition of weak convergence "$\rightharpoonup$").

Lemma 2.7 (Rellich Compactness Lemma)**.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a C^1 boundary and $1 < p < n$. If $\{u_j\}$ is a bounded sequence in $W^{1,p}(\Omega)$ then there is a subsequence $\{u_{j'}\}$ and $u \in W^{1,p}(\Omega)$ such that

(1) $u_{j'} \rightharpoonup u$ in $W^{1,p}(\Omega)$, and

(2) $u_{j'} \to u$ in $L^p(\Omega)$,

(3) $\|Du\|_{L^p(\Omega)} \leq \liminf_{j' \to \infty} \|Du_{j'}\|_{L^p(\Omega)}$.

Proof. (1) follows from reflexivity of $W^{1,p}(\Omega)$. By Sobolev Embedding Theorem, $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact, so there is a subsequence of $\{u_{j'}\}$ that converges strongly in $L^p(\Omega)$ to some $\widetilde{u}$. But strong convergence implies weak convergence, thus $\widetilde{u} = u$. This proves (2) by choosing this subsequence of $u_{k'}$ instead. The assertion (3) follows from the previous lemma and the weak lower semi-continuity of norms. □

Poincaré Inequality

The Poincaré inequality gives the bound for the mean-square oscillation by norm of the gradient. Using it with Campanato lemma (see the next chapter) we can prove Hölder continuity of solutions of second order elliptic equations.

Lemma 2.8 (Poincaré Inequality)**.** Let $\Omega \subset \mathbb{R}^n$ be a bounded connected domain with a C^1 boundary. Then there is a constant C depends only on n and Ω such that every $u \in H^1(\Omega)$ satisfies

$$\int_\Omega |u - (u)_\Omega|^2 dx \leq C \int_\Omega |Du|^2 dx,$$

where $(u)_\Omega = |\Omega|^{-1} \int_\Omega u$. In particular, Ω is an open ball $B_R(x_0)$ we have the following more explicit estimate

$$\int_{B_R(x_0)} |u - (u)_{B_R(x_0)}|^2 dx \leq CR^2 \int_{B_R(x_0)} |Du|^2 dx,$$

for all $u \in H^1(B_R(x_0))$, where C depends only on n.

Proof. We prove by contradiction. Suppose the assertion fails, i.e. there is a sequence $\{u_j\} \subset H^1(\Omega)$ such that

$$\int_\Omega |u_j - (u_j)_\Omega|^2 dx > j \int_\Omega |Du_j|^2 dx,$$

for each j. Define $v_j = \{u_j - (u_j)_\Omega\}/\|u_j - (u_j)_\Omega\|_{L^2(\Omega)} \in L^2(\Omega)$ for each j. Then each v_j satisfies $(v_j)_\Omega = 0$, $\|v_j\|_{L^2(\Omega)} = 1$, and $\int_\Omega |Dv_j|^2 < \frac{1}{j}$. Apply Rellich Compactness Lemma to $\{v_j\}$, we can find $v_{j'} \rightharpoonup v$ in $H^1(\Omega)$ and $v_{j'} \to v$ in $L^2(\Omega)$ and $\|Dv\|_{L^2(\Omega)} \leq \liminf_{j' \to \infty} \|Dv_{j'}\|_{L^2(\Omega)} = 0$. The later implies v is constant a.e. on Ω, but the former implies $\|v\|_{L^2(\Omega)} = 1$ so $(v)_\Omega \neq 0$. By weak convergence, we get $(v)_\Omega = \lim_{j'\to\infty}(v_{j'})_\Omega = 0$, which is a contradiction. For the domain that is a ball, we apply the argument of lemma (2.5). □

2.4 The Direct Methods in The Calculus of Variations

A typical problem in the Calculus of Variations concerns with minimizing (or finding critical points) of a functional over a carefully chosen function space. The problem may subject to topological constraints (such as functions are subjected to certain boundary conditions) or geometric constraints (such as functions have image in manifolds). In this section we ourselves restrict to the unconstrained minimization problems and discuss the direct methods. Then in the next section we will consider the penalty approximation approach to the constrained problems.

Unconstrained Minimization Problems

Suppose we need to minimize a functional $E : X \to \mathbb{R}$ over a Banach space X. That is, we want to see whether there exists a $u \in X$ such that

$$E(u) = \inf_{w\in X} E(w).$$

The basic example is the following: given a function $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ and let

$$E(w) = \int_\Omega f(x, w(x), Dw(x))\, dx.$$

For instance, if $f = |Dw|^2$, we have the standard Dirichlet problem, and if $f = |Dw|^2 - G(x)w$ we obtain the Poisson problem. If we let $f = \sqrt{1 + |Dw|^2}$ we get the classical minimal surface problem.

The space X and the functional E must have certain properties, to guarantee the existence of such u.

Before proceeding further, we recall some basic facts and results from Functional Analysis.

1. If X is a normed linear space, then there is a canonical injection $i : X \to X^{**}$ given by $i(u)(v^*) = v^*(u)$ $(v^* \in X^*)$ for each $u \in X$. Clearly, i is a bounded linear operator. Moreover, $\|i(u)\| = \|u\|$ for all $u \in X$. The space X is said to be reflexive if i is an isometric isomorphism from X onto X^{**}.

2. Let X be a normed linear space. A sequence $\{u_j\}$ in X is said to converge weakly to $u \in X$, and write $u_j \rightharpoonup u$ in X, if for all $v^* \in X^*$, $v^*(u_j) \to v^*(u)$. For a reflexive Banach space X, every bounded sequence contains a weakly convergent subsequence.

3. Let $1 \leq p < \infty$ and $\Omega \subset \mathbb{R}^n$. Then $u_j \rightharpoonup u$ in $L^p(\Omega)$ if and only if $\int_\Omega u_j v \to \int_\Omega uv$ for all $v \in L^{p'}(\Omega)$, where $p' = p/(p-1)$. We say that $u_j \rightharpoonup u$ in $W^{1,p}(\Omega)$ if $u_j \rightharpoonup u$ in $L^p(\Omega)$ and $Du_j \rightharpoonup Du$ in $L^p(\Omega)$.

The Direct Methods

Definition. A functional $E : X \to \mathbb{R}$ on a normed linear space X is said to be *coercive* if for any sequence $\{u_j\}$ in X with $\|u_j\|_X \to \infty$ then $E(u_j) \to +\infty$.

Let $\{u_j\}$ be a *minimizing sequence*, i.e. $E(u_j) \to \inf E$ as $j \to \infty$. Since $\inf E < +\infty$, if E is coercive then $\{u_j\}$ must be bounded in X.

The next definition provides an additional condition on the functional E and the space X that guarantee the existence of minimizers.

Definition. A functional $E : X \to \mathbb{R}$ is said to be *(weakly) lower semi-continuous* if the following assertion holds: if u_j converges (weakly) to u in X then $E(u) \leq \liminf_{j\to\infty} E(u_j)$.

Note that if E is continuous with respect to the norm (or weak) topology of X then $\liminf_{j\to\infty} E(u_j) = E(u) = \limsup_{j\to\infty} E(u_j)$ if u_j converge (weakly, resp.)

to u in X. An example of weakly lower semi-continuous functional on any Banach space X is its norm. That is if $u_j \rightharpoonup u$ in X then $\|u\|_X \leq \liminf_{j\to\infty} \|u_j\|_X$.

Now we return to the discussion of the minimization problem. Given a reflexive Banach space X and assume that the functional E is coercive and weakly lower semi-continuous. Let $\{u_j\}$ be a minimizing sequence. By the boundedness of $\{u_j\}$, there is a subsequence $u_{j'} \rightharpoonup u$ in X for some $u \in X$. Hence, by weakly lower semi-continuous of E

$$E(u) \leq \lim_{j'\to\infty} E(u_{j'}) = \inf_{w\in X} E(w).$$

This established a minimizer u of E in X. For convenience, we state this result as a lemma.

Lemma 2.9. Let X be a reflexive Banach space. Let $E : X \to \mathbb{R}$ be a coercive, weakly lower semi-continuous functional on X. Then E attains at least one minimizer in X.

2.5 Penalized Approximation

In this section we introduce the penalty approximation technique to tackle minimization problems with constraints. To minimize a functional subjected to certain constraints are difficult. This means in particular that now we are minimizing the functional over nonlinear space.

The idea of penalization approximation technique is to replace a functional with constraints by approximated constraint-free functionals over a Banach space. Afterward, we solve the approximated minimization problems by the direct method, and expect that the sequence of the associated minimizers may converge to a minimizer of the original functional.

Constrained Minimization Problems

Assumptions: Given X is a reflexive Banach space, $E, G : X \to \mathbb{R}$ are nonnegative weakly lower semicontinuous functionals, E is coercive, and G attains zero in X. We wish to minimize the functional E subjected to the constraint G, that is to determine whether there exists $u \in X$ satisfying $G(u) = 0$ and

$$E(u) = \inf_{w \in X,\, G(w)=0} E(w).$$

Penalized Approximation Approach

For each $\varepsilon > 0$, we define the functional

$$E_\varepsilon(w) = E(w) + \frac{1}{\varepsilon}\, G(w),$$

and the corresponding relaxed minimization problem

$$E_\varepsilon(u_\varepsilon) = \inf_{w \in X} E_\varepsilon(w).$$

By the direct method, we immediately have the following conclusion: under the above assumptions on X, E and G, then there exists at least one minimizer $u_\varepsilon \in X$ for each E_ε.

Now choose a minimizer u_ε for each E_ε and let $\kappa(\varepsilon) = E_\varepsilon(u_\varepsilon)$. The map $\varepsilon \mapsto E_\varepsilon(w)$ is non-increasing for each $w \in X$. Also, we observe that $\varepsilon \mapsto \kappa(\varepsilon)$ is non-increasing. In fact, if $0 < \varepsilon_1 < \varepsilon_2$ then

$$\kappa(\varepsilon_2) \le E(w) + \frac{1}{\varepsilon_2} G(w) \le E(w) + \frac{1}{\varepsilon_1} G(w),$$

for all $w \in X$. Hence $\kappa(\varepsilon_2) \le \kappa(\varepsilon_1)$. Since $\kappa(\varepsilon)$ is locally Lipschitz except a null set, $\kappa(\varepsilon)$ is differentiable almost everywhere by Rademacher's theorem.

Theorem 2.10. Under the above assumption on X, E and G. Then $\kappa(\varepsilon)$ is bounded and there is a sequence $\varepsilon_j \to 0$ and minimizers u_{ε_j} for each E_{ε_j} and a minimizer $u \in X$ of E satisfying $G(u) = 0$ such that $u_{\varepsilon_j} \rightharpoonup u$ in X and

$$\varepsilon_j^{-1} G(u_{\varepsilon_j}) \leq C|\ln \varepsilon_j|^{-1},$$

for some constant $C > 0$.

Proof. We have $\kappa(\varepsilon) \leq E_\varepsilon(v) = E(v)$ for all v satisfying $G(v) = 0$ and all ε; in particular, $\kappa(\varepsilon) \leq \inf_{G(v)=0} E(v)$. Thus $\kappa(\varepsilon)$ is bounded.

Consider for each ε such that κ is differentiable, we have

$$\Big|\frac{\partial E_\varepsilon}{\partial \varepsilon}(u_\varepsilon)\Big| = \lim_{\varepsilon' \to \varepsilon^+} \frac{E_\varepsilon(u_\varepsilon) - E_{\varepsilon'}(u_\varepsilon)}{\varepsilon' - \varepsilon} \leq \lim_{\varepsilon' \to \varepsilon^+} \frac{E_\varepsilon(u_\varepsilon) - E_{\varepsilon'}(u_{\varepsilon'})}{\varepsilon' - \varepsilon} = |\kappa'(\varepsilon)|.$$

Note also that $\kappa' \leq 0$. If $0 < \varepsilon_1 < \varepsilon_2$ then

$$\kappa(\varepsilon_1) - \kappa(\varepsilon_2) = \int_{\varepsilon_1}^{\varepsilon_2} |\kappa'(\varepsilon)|\, d\varepsilon \geq \operatorname*{ess\,inf}_{\varepsilon_1 \leq \varepsilon \leq \varepsilon_2} \varepsilon|\kappa'(\varepsilon)| \int_{\varepsilon_1}^{\varepsilon_2} \frac{1}{\varepsilon}\, d\varepsilon.$$

The term on the left is bounded and the term $\int_{\varepsilon_1}^{\varepsilon_2} \frac{1}{\varepsilon}\, d\varepsilon = \ln(\varepsilon_2/\varepsilon_1)$. Thus after dividing by $|\ln \varepsilon_1|$ and sending $\varepsilon_1 \to 0$, we conclude that

$$C|\ln \varepsilon|^{-1} \geq \liminf_{\varepsilon \to 0} \varepsilon|\kappa'(\varepsilon)| \geq \liminf_{\varepsilon \to 0} \varepsilon \Big|\frac{\partial E_\varepsilon}{\partial \varepsilon}(u_\varepsilon)\Big| = \liminf_{\varepsilon \to 0} \varepsilon^{-1} G(u_\varepsilon).$$

So there is a sequence $\varepsilon_j \to 0$ such that

$$\varepsilon_j^{-1} G(u_{\varepsilon_j}) \leq C|\ln \varepsilon_j|^{-1},$$

for all minimizers u_{ε_j} of E_{ε_j}. In particular $\liminf_{j\to\infty} G(u_{\varepsilon_j}) = 0$.

By coercivity of E, since $E(u_{\varepsilon_j}) \leq E_{\varepsilon_j}(u_{\varepsilon_j}) = \kappa(\varepsilon_j)$ and the right hand side is bounded, $\{u_{\varepsilon_j}\}$ must be bounded. The reflexivity of X implies, by considering subsequences if necessary, that there is a sequence $\{u_{\varepsilon_j}\}$ and $u \in X$ such that $u_{\varepsilon_j} \rightharpoonup u$ in X. As E, G are weakly lower semicontinuous, we have $G(u) = 0$ and $E(u) \leq \liminf_{j\to\infty} E_{\varepsilon_j}(u_{\varepsilon_j}) \leq \inf_{G(v)=0} E(v)$. This completes the proof. $\square$

CHAPTER III

HARMONIC MAPS

This chapter is a detailed introduction to harmonic maps. They are purely variational objects. We study the case when the target spaces be any smooth compact Riemannian manifolds. We also derive the corresponding variational identities, i.e. the Euler-Lagrange equations. In the remaining section the penalized approximation of harmonic maps to spheres problem is formulated.

3.1 Introduction

Roughly speaking, a harmonic map between two spaces is a critical point of certain "energy" functional, among the maps with the same boundary values. In Physics, harmonic maps are viewed as the direct generalization of the usual Laplace or wave operators to nonlinear fields, i.e. fields that take values in curved manifold not in Euclidean space. We remark here that, in the recent years, there are many research papers related to harmonic maps in Mathematics or in Physics.

First of all we state our setting and give some conventions. We shall consider harmonic maps which have the domain Ω that is an open subset of a Euclidean space $\mathbb{R}^n$, where $n \geq 2$. The target space N of harmonic maps is assumed to be a smooth compact Riemannian manifold. By Nash isometric embedding theorem, N can be isometrically embedded in some Euclidean space $\mathbb{R}^k$ where $k \geq 2$. For a map u from Ω "into" N, we do not require $u(\Omega) \subset N$ but use the minimal assumption that $u(x) \in N$ for almost every $x \in \Omega$. This is done in order to put the harmonic maps problem into the framework of modern PDE theory. Thus by a map $u : \Omega \to N$, we mean $u = (u^1, \ldots, u^k) : \Omega \to \mathbb{R}^k$ such that $u(x) \in N$ for almost every $x \in \Omega$. Note that we have used the n-dimensional Lebesgue measure on Ω. We say $u \in H^1(\Omega, N)$ if $u : \Omega \to N$ in the above sense and $u^i \in H^1(\Omega)$ for all $1 \leq i \leq k$. Recall $H^1(\Omega)$ denotes the Sobolev space of those L^2 functions on Ω which have L^2 gradients. Finally, we say $u \in H^1_{\text{loc}}(\Omega, N)$ if $u \in H^1(\widetilde{\Omega}, N)$ for all open set $\widetilde{\Omega} \Subset \Omega$ (i.e. $\widetilde{\Omega}$ has compact support in Ω).

3.2 Classification of Harmonic Maps

Generally, harmonic maps can be classified into three types based on certain variational conditions.

For $u \in H^1_{\text{loc}}(\Omega, N)$ and $\widetilde{\Omega} \Subset \Omega$, we define the energy $E(u, \widetilde{\Omega})$ of u on $\widetilde{\Omega}$ by

$$E(u, \widetilde{\Omega}) = \frac{1}{2} \int_{\widetilde{\Omega}} |Du|^2 dx,$$

where $|Du|^2 = \sum_{i=1}^{k} \sum_{j=1}^{n} (D_j u^i)^2$.

1. Minimizing Harmonic Maps

A map $u \in H^1_{\text{loc}}(\Omega, N)$ is called an (energy) *minimizing* harmonic map if for any open set $\widetilde{\Omega} \Subset \Omega$ and all $w \in H^1_{\text{loc}}(\Omega, N)$ with $w \equiv u$ on $\Omega \setminus \widetilde{\Omega}$

$$E(u, \widetilde{\Omega}) \leq E(w, \widetilde{\Omega}).$$

To analyze interior regularity property, it is enough to assume that Ω is *bounded*, and the definition becomes simpler. We say that $u \in H^1(\Omega, N)$ is a minimizing harmonic map if

$$E(u, \Omega) \leq E(w, \Omega),$$

for all $w \in H^1(\Omega, N)$ such that $w \equiv u$ on a neighborhood of $\partial\Omega$.

2. Weakly Harmonic Maps

Suppose Ω is bounded. A map $u \in H^1(\Omega, N)$ is said to be a *weakly* harmonic map if it is a "weak" solution of the following system of nonlinear partial differential equation

$$\triangle u + \sum_{j=1}^{n} A_u(D_j u, D_j u) = 0 \qquad \text{on } \Omega,$$

where A_u denotes the second fundamental form at the point u of N in $\mathbb{R}^k$ (see the details below). By a weak solution u, we mean that u satisfies the following integral equation

$$\int_\Omega \sum_{j=1}^{n} \Big\{ D_j u \cdot D_j \zeta - \zeta \cdot A_u(D_j u, D_j u) \Big\} dx = 0,$$

for all $\zeta \in C_0^\infty(\Omega, \mathbb{R}^k)$, where $v \cdot w = \sum_{i=1}^{k} v^i w^i$ denotes the usual dot product of vectors v, w in $\mathbb{R}^k$.

We will show in the next section that the above equation of weakly harmonic maps is the Euler-Lagrange equation of the energy functional $E(u) = \int_\Omega |Du|^2 \, dx$ (under small perturbations on the image).

3. Stationary Harmonic Maps

Suppose Ω is bounded. A map $u \in H^1(\Omega, N)$ is said to be a *stationary* harmonic map if it is weakly harmonic and satisfies

$$\int_\Omega \sum_{i,j=1}^{n} \left\{ |Du|^2 \delta_{ij} - 2 D_i u \cdot D_j u \right\} D_i \zeta^j dx = 0,$$

for all $\zeta \in C_0^\infty(\Omega, \mathbb{R}^n)$. We will see that every minimizing harmonic map is stationary harmonic.

Being stationary gives the *monotonicity* formula for harmonic maps. We will also encounter the monotonicity identity in our penalized approximation setting. In the next section, we will show in full details how to derive this identity. Roughly speaking, it is the Euler-Lagrange equation of the energy functional $E(u) = \frac{1}{2} \int_\Omega |Du|^2 dx$ (under perturbations on the domain Ω).

Some Known Results of Harmonic Maps

Before proceeding further, let us discuss here some known results of harmonic maps about their regularity properties. For a good survey see ([5]).

In case the domain has dimension $n = 2$, every harmonic map is smooth. This was established for the minimizing harmonic case by C.E.B. Morrey in 1948, for the stationary harmonic case by R. Schoen in 1983, and for the weakly harmonic case by F. Hélein in 1991. Note here that by a map to be smooth, we do not mean that it is smooth almost everywhere, but it must have a smooth representative (in some Sobolev space).

Among the three types of harmonic maps, the weakly harmonic maps can have the worst regularity properties. In fact, in 1995, T. Riviére proposed an example of *everywhere discontinuous* weakly harmonic map from $B_1(0) \subset \mathbb{R}^3$ to S^2.

In general, for minimizing harmonic maps, a very important work of R. Schoen and K. Uhlenbeck (see [7]) shows that if the domain has dimension $n = 3$, then any minimizing harmonic maps are smooth away from a discrete set. If $n \geq 4$, they are smooth away from a (relatively) closed set of Hausdorff dimension $\leq n - 3$. The key ingredients in the proof are the minimality assumption of the harmonic maps and the so called *small energy regularity theorem*.

There is a similar small energy regularity theorem for stationary harmonic maps. In 1991, L.C. Evans proved the regularity in the case that targets are spheres. For general target manifold, it was developed by F. Bethuel in 1993.

3.3 The Euler-Lagrange Equations

In this section we will derive the variational equations which are the consequences of minimizing assumption on harmonic maps.

Assume Ω is a bounded open subset of $\mathbb{R}^n$ $(n \geq 2)$ and N is a manifold which is isometrically embedded in $\mathbb{R}^k$ $(k \geq 2)$. Recall $E(u) = \frac{1}{2}\int_\Omega |Du|^2 dx$ for all $u \in H^1(\Omega, N)$. A map $u \in H^1(\Omega, N)$ is a *critical* point of E if for any *variation* $u_s \in H^1(\Omega, N)$ $(-\varepsilon < s < \varepsilon)$ such that $u_0 = u$, $u_s \equiv u$ on $\partial\Omega$ for all s, the map $s \mapsto E(u_s)$ is critical at $s = 0$. Thus, we have

$$\frac{d}{ds}\Big|_{s=0} E(u_s) = 0,$$

provided the differentiation exists.

We begin with the identity for weakly harmonic maps. They are the result of variations in the target spaces.

The First Variational Identity

Let $u \in H^1(\Omega, N)$ be a minimizing harmonic map. Therefore u is a critical point of the functional E by the definition of minimizing harmonic maps.

Consider the variation $u_s \in H^1(\Omega, N)$ $(-\delta < s < \delta)$ defined by

$$u_s(x) = \Pi \circ (u(x) + s\zeta(x)) \qquad x \in \Omega,$$

for a given $\zeta \in C_0^\infty(\Omega, \mathbb{R}^k)$, where Π is the nearest point projection of $\mathbb{R}^k$ onto N. Since N is compact, Π is well-defined and smooth on some tabular neighborhood $U \subset \mathbb{R}^k$ of N, i.e. $U = \{x \in \mathbb{R}^k : \text{dist}(x, N) < \varepsilon\}$ for some $\varepsilon > 0$. It follows that $u_s \in H^1(\Omega, N)$ is a well-defined variation of u for each ζ, provided δ is small enough. By the assumption that u is energy minimizing, we have

$$\frac{d}{ds}E(u)\Big|_{s=0} = 0. \tag{i}$$

We will see that the differentiation exists in the weak sense. The derivation of the Euler-Lagrange equation will be split into several parts to make it easier to follow.

Step 1. We expand u_s with respect to the parameter s. By using the Taylor series and the chain rules, we get

$$u_s = u + s\frac{\partial u_s}{\partial s}\Big|_{s=0} + O(s^2) = u + s\, d\Pi_u \circ \zeta + O(s^2).$$

Thus

$$D_j u_s = D_j u + s\,\{d\Pi_u \circ D_j\zeta + D_j(d\Pi_u)\circ\zeta\} + O(s^2).$$

The term $D_j(d\Pi_u)$ is just the array $\{\partial^2\Pi_u^\alpha/\partial x^j\partial u^\beta\}$, where $1 \le \alpha, \beta \le k$. By the chain rule, each term equals

$$\sum_{\gamma=1}^{k} \frac{\partial^2\Pi_u^\alpha}{\partial u^\gamma \partial u^\beta}\frac{\partial u^\gamma}{\partial x^j} = \sum_{\gamma=1}^{k} D_j u^\gamma\, D_{u^\gamma u^\beta}\Pi_u^\alpha,$$

Therefore we obtain

$$D_j u_s = D_j u + s\big\{d\Pi_u \circ D_j\zeta + \operatorname{Hess}\Pi_u(D_j u, \zeta)\big\} + O(s^2),$$

where $\operatorname{Hess} f_x(u, v) := \sum_{i,j} u^i v^j D_{x_i x_j} f$.

Step 2. By simple geometric reasoning, we see that $d\Pi_y$ is the orthogonal projection from the tangent space $T_y\mathbb{R}^k \cong \mathbb{R}^k$ onto $T_{\Pi(y)}N$, for each $y \in U$. Recall that in the Euclidean space $\mathbb{R}^k$ we have the standard identification of tangent spaces by the parallel translation. Now each vector $v \in T_y\mathbb{R}^k$ can be written as $v = v^T + v^\perp$, where $v^T = d\Pi_y(v)$ and $v^\perp = v - v^T$. Thus $d\Pi_u \circ D_j\zeta = (D_j\zeta)^T$, i.e. the projection of $D_j\zeta$ on T_uN.

Step 3. Now from the equation (i), we have

$$0 = \int_\Omega \frac{\partial}{\partial s}\Big|_{s=0}|Du_s|^2 = \int_\Omega \sum_{j=1}^{n} D_j u \cdot \frac{\partial}{\partial s} D_j u_s\Big|_{s=0}.$$

Combine this with the first two steps, we obtain

$$\int_\Omega \sum_{j=1}^{n} \big\{D_j u \cdot (D_j\zeta)^T + D_j u \cdot \operatorname{Hess}\Pi_u(D_j u, \zeta)\big\} = 0. \qquad \text{(ii)}$$

Since $D_j u \in T_uN$, the first term on the left is equal $D_j u \cdot D_j\zeta$.

Step 4. Recall that the second fundamental form A of N in $\mathbb{R}^k$ is define for each $y \in N$ to be the symmetric bilinear map $A_y : T_yN \times T_yN \to T_y^\perp N$ (the space of vectors in $\mathbb{R}^k$ orthogonal to T_yN) which is given by

$$A_y(u^T, v^T) = -(D_{u^T} v^T)^\perp, \qquad u, v \in \mathbb{R}^k.$$

On the right hand side u, v are extended to C^∞ maps $\mathbb{R}^k \to \mathbb{R}^k$ that take the values u, v respectively at the point y. $D_{u^T} = u^T \cdot D$ is the directional derivative in the direction u^T. (It can be verified by direct computations that the definition of A_y is independent of choices of u, v and their extensions to smooth maps.) By simple calculation, we obtain the following two important identities

$$\operatorname{Hess}\Pi_y(u^T, v^T) = -A_y(u^T, v^T), \quad \text{and} \quad \operatorname{Hess}\Pi_y(u^T, v^\perp) = -v^\perp \cdot A_y(u^T, \cdot).$$

The second identity is just the short form of $w \cdot \operatorname{Hess}\Pi_y(u^T, v^\perp) = -v^\perp \cdot A_y(u^T, w^T)$ for all $w \in T_y\mathbb{R}^k$.

Step 5. The second term on the left of (ii) is equal to

$$D_ju \cdot \operatorname{Hess}\Pi_u(D_ju, \zeta^T) + D_ju \cdot \operatorname{Hess}\Pi_u(D_ju, \zeta^\perp) = \zeta \cdot \operatorname{Hess}\Pi_u(D_ju, D_ju).$$

Note that the first term vanishes since $D_ju \in T_uN$ and $\operatorname{Hess}\Pi_u(D_ju, \zeta^T) = -A_u(D_ju, \zeta^T) \in T_u^\perp N$. Therefore we conclude that

$$\int_\Omega \sum_{j=1}^n \left\{ D_ju \cdot D_j\zeta - \zeta \cdot A_u(D_ju, D_ju) \right\} = 0,$$

and this holds for all $\zeta \in C_0^\infty(\Omega, \mathbb{R}^k)$.

The Second Variational Identity

Again assume $u \in H^1(\Omega, N)$ is an energy minimizing map. Consider the variation $u_s \in H^1(\Omega, N)$ $(-\delta < s < \delta)$ defined by

$$u_s(x) = u(x + s\zeta(x)),$$

for a given $\zeta \in C_0^\infty(\Omega, \mathbb{R}^n)$. Clearly, $u_0 = u$ and $u_s \equiv u$ on a neighborhood of $\partial\Omega$. Note that if δ is small enough then u_s is admissible, for each such ζ. Now by the minimizing assumption of u, we have

$$\frac{d}{ds}E(u_s)\Big|_{s=0} = 0. \tag{i}$$

We split the derivation of the Euler-Lagrange equation into several steps.

Step 1. By chain rule we have

$$\begin{aligned} D_i u_s(x) &= \sum_{j=1}^{n} D_j u(x+s\zeta)\{\delta_{ij} + sD_i\zeta^j(x)\} \\ &= D_i u(x+s\zeta) + s\sum_{j=1}^{n} D_i\zeta^j(x)\, D_j u(x+s\zeta). \end{aligned}$$

Let $\xi = x + s\zeta$. Note that ξ gives a C^1 diffeomorphism on Ω if s is small enough. By the change of variables formula,

$$E(u_s) = \int_\Omega |Du_s(x)|^2 \left|\det\left(\frac{\partial x}{\partial \xi}\right)\right| d\xi.$$

Step 2. Applying the chain rule, we get

$$\begin{aligned} D_i\zeta^j(x) &= \frac{\partial \zeta^j}{\partial x^i} = \sum_{l=1}^{n} \frac{\partial \zeta^j}{\partial \xi^l}\frac{\partial \xi^l}{\partial x^i} \\ &= \sum_{l=1}^{n} D_l\zeta^j(\xi)(\delta_{il} + sD_i\zeta^l(x)) = D_i\zeta^j(\xi) + s\sum_{l=1}^{n} D_l\zeta^j(\xi)D_i\zeta^l(x), \end{aligned}$$

and hence

$$D_i u_s(x) = D_i u(\xi) + s\sum_{j=1}^{n} D_i\zeta^j(\xi)D_j u(\xi) + O(s^2).$$

Thus

$$|Du_s(x)|^2 = |Du(\xi)|^2 + 2s\sum_{i,j=1}^{n} D_i u(\xi)\cdot D_j u(\xi)\, D_i\zeta^j(\xi) + O(s^2).$$

Step 3. By Taylor series expansion,

$$\frac{\partial \xi^i}{\partial x^j} = \delta_{ij} + sD_j\zeta^i(x).$$

Using the previous step, we have

$$\left|\det\left(\frac{\partial\xi}{\partial x}\right)\right| = 1 + s\sum_{i=1}^{n} D_i\zeta^i(\xi) + O(s^2).$$

Therefore,

$$\left|\det\left(\frac{\partial x}{\partial\xi}\right)\right| = \left|\det\left(\frac{\partial\xi}{\partial x}\right)\right|^{-1} = 1 - s\sum_{i=1}^{n} D_i\zeta^i(\xi) + O(s^2).$$

Step 4. Finally, we have

$$\begin{aligned}|Du_s(x)|^2\left|\det\left(\frac{\partial x}{\partial\xi}\right)\right| = |Du(\xi)|^2 - s\Big\{&|Du(\xi)|^2 \operatorname{div}\zeta(\xi)\\ &- 2\sum_{i,j=1}^{n} D_iu(\xi)\cdot D_ju(\xi)\, D_i\zeta^j(\xi)\Big\} + O(s^2).\end{aligned}$$

Thus the equation (i) gives

$$\int_\Omega \sum_{i,j=1}^{n} \left\{|Du|^2\delta_{ij} - 2D_iu\cdot D_ju\right\} D_i\zeta^j = 0,$$

as required. Note that this holds for all $\zeta \in C_0^\infty(\Omega, \mathbb{R}^n)$.

3.4 Penalization Approach for Harmonic Maps to Spheres

In this section we derive the Euler-Lagrange equation for harmonic maps to spheres directly without using the result of previous section. We also consider the penalized approximation of harmonic maps to spheres.

The Euler-Lagrange Equation

Suppose Ω is a bounded open subset of $\mathbb{R}^n$ and S^{k-1} denotes the standard sphere in $\mathbb{R}^k$. Here we assume $n, k \geq 2$.

By definition, a harmonic map to sphere $u \in H^1(\Omega, S^{k-1})$ is a critical point of the energy integral

$$E(u) = \frac{1}{2}\int_\Omega |Du|^2 dx.$$

This means that for any (admissible) variation u_s $(-\delta < s < \delta)$ of u, the map $s \mapsto E(u_s)$ is critical at $s = 0$. That is $\frac{d}{ds}\big|_{s=0} E(u_s) = 0$ if the differentiation exists. Recall $u_s : \Omega \to \mathbb{R}^k$ $(-\delta < s < \delta)$ will be called an *admissible* variation of u if $u_s \in H^1(\Omega, S^{k-1})$ for each s, $u_0 = u$, and $u_s|_{\partial\Omega} \equiv u$ for all s.

Specifically, consider the 1-parameter family of maps $u_s = (u + s\zeta)/|u + s\zeta|$, where $\zeta \in C_0^\infty(\Omega, \mathbb{R}^k)$. Clearly, u_s is an admissible variation of u for each ζ if s is small enough. Thus

$$\frac{d}{ds}\Big|_{s=0} E(u_s) = \int_\Omega \frac{\partial}{\partial s}\Big|_{s=0} \left| D\left(\frac{u + s\zeta}{|u + s\zeta|}\right)\right|^2 dx = 0.$$

By straightforward computation, we have

$$\begin{aligned} D_i\left(\frac{u + s\zeta}{|u + s\zeta|}\right) &= \frac{(D_i u + sD_i\zeta)|u + s\zeta| - (u + s\zeta)|u + s\zeta|^{-1}(u + s\zeta)\cdot(D_i u + sD_i\zeta)}{|u + s\zeta|^2} \\ &= \frac{(D_i u + sD_i\zeta)}{|u + s\zeta|} - (u + s\zeta)\frac{(u + s\zeta)\cdot(D_i u + sD_i\zeta)}{|u + s\zeta|^3} := A_1 - A_2. \end{aligned}$$

Next, we compute

$$\frac{\partial}{\partial s}\Big|_{s=0} A_1 = D_i\zeta - D_i u(u\cdot\zeta).$$

By noting that $u \cdot D_i u = D_i(|u|^2/2) = 0$, we have

$$\begin{aligned} \frac{\partial}{\partial s}\Big|_{s=0} A_2 &= u(\zeta\cdot D_i u + u\cdot D_i\zeta) + \zeta(u\cdot D_i u) - u(u\cdot D_i u)(u\cdot\zeta) \\ &= u(\zeta\cdot D_i u + u\cdot D_i\zeta) = uD_i(u\cdot\zeta). \end{aligned}$$

Thus

$$\begin{aligned} \frac{\partial}{\partial s}\Big|_{s=0} \left| D\left(\frac{u + s\zeta}{|u + s\zeta|}\right)\right|^2 &= 2\sum_{i=1}^n D_i u \cdot \frac{\partial}{\partial s}\Big|_{s=0} D_i\left(\frac{u + s\zeta}{|u + s\zeta|}\right) \\ &= 2\sum_{i=1}^n D_i u\cdot(D_i\zeta - D_i u(u\cdot\zeta)). \end{aligned}$$

Therefore, we obtain

$$\int_\Omega \sum_{i=1}^n \left\{D_i u\cdot D_i\zeta - D_i u\cdot D_i u(u\cdot\zeta)\right\} = 0, \tag{i}$$

and this holds for all $\zeta \in C_0^\infty(\Omega, \mathbb{R}^k)$. In the terminology of partial differential equation theory, we call a function u which satisfies the equation (i), for all $\zeta \in C_0^\infty(\Omega, \mathbb{R}^k)$, a *weak solution* of the following equation

$$-\triangle u = u|Du|^2. \tag{ii}$$

Note that if $u \in C^2$ then (i) follows from (ii) by integration by parts. So we have the equation (ii) as a necessary condition for harmonic maps to spheres.

Penalization Approach

Suppose Ω is a bounded open subset of $\mathbb{R}^n$ and S^{k-1} the sphere in $\mathbb{R}^k$, where $n, k \geq 2$. For each $\varepsilon > 0$, we define for every $u \in H^1(\Omega, \mathbb{R}^k)$ the energy functional

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega |Du|^2 dx + \frac{1}{4\varepsilon^2}\int_\Omega (1 - |u|^2)^2 dx.$$

Let E denote the usual Dirichlet integral

$$E(u) = \frac{1}{2}\int_\Omega |Du|^2 dx,$$

and

$$G(u) = \int_\Omega (1 - |u|^2)^2 dx.$$

represented the deviation of images of u from the sphere S^{k-1}.

Let $g \in H^1(\Omega, S^{k-1})$ be given and u_ε be a minimizer of E_ε subjected to the Dirichlet boundary condition $u_\varepsilon|_{\partial\Omega} \equiv g$. We are now concerning with the minimization problem of E_ε over the closed affine subspace $g + H_0^1(\Omega, \mathbb{R}^k)$ of the Sobolev space $H^1(\Omega, \mathbb{R}^k)$. In this case the direct method still applicable. We will see later that E_ε is coercive and weakly lower semi-continuous. Therefore, there is at least one such minimizer u_ε.

By the minimality of u_ε, as $\varepsilon \to 0$, the term $(4\varepsilon^2)^{-1}G(u_\varepsilon)$ must be bounded. This in turn forces u_ε to have images close to S^{k-1}. The coercivity of E_ε implies that $u_{\varepsilon_j} \rightharpoonup u$ for some sequence u_{ε_j} and some $u \in H^1(\Omega, \mathbb{R}^k)$. By weak lower semi-continuity of E_ε, we conclude that $u \in H^1(\Omega, S^{k-1})$, $u|_{\partial\Omega} \equiv g$, and $E(u) \leq E(g)$.

The proof of above statements involve trivial modifications of the theorems in chapter 2. In the remaining, we will show that E_ε is coercive and weakly lower semi-continuous.

$\boldsymbol{E_\varepsilon}$ is coercive: Suppose $\{v_j\}$ is a sequence in $H^1(\Omega, \mathbb{R}^k)$ such that $\|v_j\|_{H^1} \to \infty$. Since $\|Dv_j\|_{L^2}$ goes to infinity as $j \to \infty$, we get

$$E_\varepsilon(v_j) \geq \frac{1}{2}\int_\Omega |Dv_j|^2 dx \to +\infty.$$

Hence E_ε is coercive as claimed.

$\boldsymbol{E_\varepsilon}$ is weakly lower semi-continuous: Suppose $v_j \rightharpoonup v$ in $H^1(\Omega, \mathbb{R}^k)$, i.e. $v_j \rightharpoonup v$ in $L^2(\Omega, \mathbb{R}^k)$ and $Dv_j \rightharpoonup Dv$ in $L^2(\Omega, \mathbb{R}^k)$. This implies that

$$\|v\|_{L^2} \leq \liminf_{j\to\infty} \|v_j\|_{L^2},$$

and

$$\|Dv\|_{L^2} \leq \liminf_{j\to\infty} \|Dv_j\|_{L^2}. \tag{i}$$

We also note that $\{v_j\}$ is bounded in H^1. Thus, by Rellich Compactness Lemma, there is a subsequence $v_{j'} \to v'$ strongly in L^2 for some $v' \in L^2(\Omega, \mathbb{R}^k)$. Since strong convergence implies weak convergence, we must have $v' = v$.

By considering a subsequence if necessary, we can assume that $v_{j'} \to v$ point-wise almost everywhere. By observing that

$$\int_\Omega (1 - |v|^2)^2 dx = \int_\Omega (1 - 2|v|^2 + |v|^4)\, dx,$$

and by Fatou's lemma, we have

$$\int_\Omega |v|^4 dx \leq \liminf_{j'\to\infty} \int_\Omega |v_{j'}|^4 dx.$$

Thus

$$\int_\Omega (1-|v|^2)^2 \leq \liminf_{j'\to\infty} \int_\Omega (1-|v_{j'}|^2)^2. \tag{ii}$$

Combine (i) and (ii), we conclude that E_ε is weakly lower semi-continuous.

CHAPTER IV

REGULARITY THEORY FOR ELLIPTIC SYSTEMS

This chapter shows some techniques used in the regularity theory for elliptic systems. Although the equation considered may not the most general one, but it illustrates various aspects of the theory.

4.1 Introduction

In this chapter we deal with the interior regularity theory of elliptic systems. The model elliptic systems are of the following forms

$$-\triangle u = f \qquad \text{in } \Omega,$$

in the sense that

$$\int_\Omega \sum_{j=1}^{n} D_j u \cdot D_j \varphi \, dx = \int_\Omega f \cdot \varphi \, dx \qquad \text{for all } \varphi \in H_0^1(\Omega, \mathbb{R}^k),$$

where $f \in L^q(\Omega, \mathbb{R}^k)$ is given for some $q \geq 1$. We assume that the solution $u : \Omega \to \mathbb{R}^k$ belongs to the Sobolev space $H^1(\Omega, \mathbb{R}^k)$ and $\Omega \subset \mathbb{R}^n$ is open. The solution u is often called a *weak solution.*

The main regularity result of this chapter is that if $q > n$ then, locally, u and its weak derivative Du are Hölder continuous with exponent γ for all $0 < \gamma \leq 1 - \frac{n}{q}$. In particular, this implies that u is C^1. We will use this result in the next chapter in order to show that the weak solutions for the penalized approximate equations of harmonic maps to spheres are smooth.

It is important to note here that for the general elliptic systems, in contrast to elliptic equations, the weak solutions are assumed only *partial regularity*; i.e.. they are regular on the whole domain except a small (relatively) closed set. This is particularly true for minimizing harmonic maps. We will show this phenomena in our main results.

4.2 Hölder Continuity

Let Ω be an open subset of $\mathbb{R}^n$, $k \in \mathbb{N}$, and $0 < \gamma \leq 1$. A function $u : \Omega \to \mathbb{R}^k$ is said to be *Hölder continuous* with exponent γ on Ω (or simply γ-Hölder continuous on Ω), provided there is a constant $C \geq 0$ such that

$$|u(x) - u(y)| \leq C|x - y|^\gamma, \qquad \text{for all } x, y \in \Omega. \tag{i}$$

The least such constant C (if exists) is denoted $[u]_{\gamma,\Omega}$. If $\gamma = 1$ we say that u is *Lipschitz continuous* on Ω. Note that Hölder continuity implies uniform continuity.

Let $C^{0,\gamma}(\overline{\Omega}, \mathbb{R}^k)$ be the space of all bounded γ-Hölder continuous functions $u : \Omega \to \mathbb{R}^k$. It is a Banach space with respect to the norm $\|u\|_{C^{0,\gamma}(\overline{\Omega})} = \|u\|_{L^\infty(\Omega)} + [u]_{\gamma,\Omega}$. This can be easily checked by using the Arzéla-Ascoli theorem. Similarly, we let $C^{l,\gamma}(\overline{\Omega}, \mathbb{R}^k)$ be the space of those $u : \Omega \to \mathbb{R}^k$ such that (1) $D^\alpha u \in L^\infty$ for all $|\alpha| \leq l$ and (2) $D^\alpha u$ are γ-Hölder continuous for all $|\alpha| = l$. It is a Banach space equipped with the following norm

$$\|u\|_{C^{l,\gamma}(\overline{\Omega})} = \sum_{0 \leq |\alpha| \leq l} \|D^\alpha u\|_{L^\infty(\Omega)} + \sum_{|\alpha| = l} [D^\alpha u]_{\gamma,\Omega}.$$

4.3 Campanato and Morrey Lemmas

The following lemmas are fundamental to prove Hölder continuity of solutions (and of their derivatives) for both elliptic equations and systems. They characterize those L^2 functions which are Hölder continuous by the decay of their oscillations.

For a domain $\Omega \subset \mathbb{R}^n$ with $|\Omega| \neq 0$, we set $(u)_\Omega := |\Omega|^{-1} \int_\Omega u\, dx$. Here $|\Omega|$ is the n-dimensional Lebesgue measure of Ω. In case of a ball $B_r(x) \subset \mathbb{R}^n$, we will use

$$u_{x,r} := (u)_{B_r(x_0)} = \frac{1}{|B_r(x)|} \int_{B_r(x)} u\, dx.$$

Note that $|B_r(x_0)| = \omega_n r^n$ where $\omega_n = |B_1(0)|$.

Lemma 4.1 (Campanato). Let $x_0 \in \mathbb{R}^n$ $(n \geq 2)$. Suppose $u \in L^2(B_{2r}(x_0), \mathbb{R}^k)$ and there are $\gamma \in (0,1]$ and a constant $M > 0$ such that

$$\int_{B_\rho(y)} |u - u_{y,\rho}|^2 dx \leq M^2 \rho^{n+2\gamma} \tag{i}$$

for all $y \in B_r(x_0)$ and $0 < \rho \leq r$. Then u is γ-Hölder continuous on $B_r(x_0)$, i.e. there is a γ-Hölder continuous function $\bar{u}$ on $B_r(x_0)$ such that $u = \bar{u}$ a.e. on $B_r(x_0)$. Moreover there is a constant $C = C(n, \gamma)$ such that $[u]_{\gamma, B_r(x_0)} \leq CM$.

Proof. By the inequality $|a-b|^2 \leq 2|c-a|^2 + 2|c-b|^2$, if $0 < \sigma < \rho \leq r$ then

$$\omega_n \sigma^n |u_{y,\rho} - u_{y,\sigma}|^2 \leq \int_{B_\sigma(y)} \Big\{ 2|u - u_{y,\rho}|^2 + 2|u - u_{y,\sigma}|^2 \Big\} dx \leq 4M^2 \rho^{n+2\gamma}$$

where $\omega_n = |B_1(0)|$. In particular if $\sigma = \rho/2$ then $|u_{y,\rho} - u_{y,\rho/2}| \leq 2^n \omega_n^{-1/2} M \rho^\gamma$. Let $\rho_k = r/2^k$, $k \in \{0, 1, 2, \ldots\}$. The simple estimate gives

$$|u_{y,\rho_k} - u_{y,\rho_{k+1}}| \leq C_0 M r^\gamma\, 2^{-k\gamma}, \tag{ii}$$

where $C_0 = 2^n \omega_n^{-1/2}$. This implies that u_{y,ρ_k} converges to a finite limit as $k \to \infty$. Let $\bar{u}(y)$ denote the limit. Furthermore, by (ii) we have

$$|u_{y,\rho_k} - \bar{u}(y)| \leq \sum_{i=k}^{\infty} |u_{y,\rho_i} - u_{y,\rho_{i+1}}| \leq C_1 M r^\gamma 2^{-k\gamma}, \tag{iii}$$

where $C_1 = C_1(n, \gamma)$.

Note that $\bar{u}(y)$ is independent of the choice $\rho_k \searrow 0$. In fact for $\sigma_l \searrow 0$, for each l take k such that $\rho_{k+1} < \sigma_l \leq \rho_k$ we obtain that $\omega_n \rho_{k+1}^n |u_{y,\rho_k} - u_{y,\sigma_l}|^2 \leq 4M^2 \rho_k^{n+2\gamma}$. Hence $|u_{y,\rho_k} - u_{y,\sigma_l}| \leq C_0 M r^\gamma\, 2^{-k\gamma}$. Thus $u_{y,\sigma_l} \to \bar{u}(y)$ as $l \to \infty$.

Next we show that $u = \bar{u}$ a.e. on $B_r(x_0)$. By (i) when $\rho = \rho_k$ and (iii), we have

$$\frac{1}{\rho_k^n}\int_{B_{\rho_k}(y)} |u - \bar{u}(y)|^2 dx \le 2(C_1^2+1)M^2 r^{2\gamma}\, 2^{-2k\gamma}. \tag{iv}$$

As $k \to \infty$, the right hand side goes to zero. Thus we can conclude that $u = \bar{u}$ a.e. on $B_r(x_0)$ by Lebesgue differentiation lemma.

Now we will show that $\bar{u}$ is γ-Hölder continuous on $B_r(x_0)$. First we assume that $x, y \in B_r(x_0)$ and $d = |x-y| \le r/4$. Take $k_0 \in \mathbb{N}$ such that $2^{-k_0-1}r < d \le 2^{-k_0}r$. By (iv) when $k = k_0 - 1$ and the fact that $B_{\rho_{k_0}}(x) \subset B_{\rho_{k_0-1}}(y)$, we get

$$\frac{1}{\rho_{k_0-1}^n}\int_{B_{\rho_{k_0}}(x)} |u - \bar{u}(y)|^2 dx \le C_2 M^2 r^{2\gamma}\, 2^{-2(k_0-1)\gamma} \le C_2 M^2 2^{4\gamma} d^{2\gamma},$$

where $C_2 = 2(C_1^2+1)$. Observe that the inequality still holds if we change the role of x, y. Thus by summing the two inequalities and applying $|a-b|^2 \le 2|c-a|^2 + 2|c-b|^2$, we obtain $\omega_n 2^n |\bar{u}(x) - \bar{u}(y)|^2 \le 4C_2 M^2 2^{4\gamma} d^{2\gamma}$. Therefore

$$|\bar{u}(x) - \bar{u}(y)| \le C_3 M d^\gamma = C_3 M |x-y|^\gamma,$$

where C_3 depends only on n, γ.

For arbitrary $x, y \in B_r(x_0)$, let z_i $(i = 0, \ldots, 8)$ be such that $z_0 = x, z_8 = y$, z_i lie on the line segment joining x, y and $|z_i - z_{i+1}| = |x-y|/8 \le r/4$. Applying the result above to each pair z_i, z_{i+1} and summing, we obtain

$$|\bar{u}(x) - \bar{u}(y)| \le \sum_{i=0}^{7} |z_i - z_{i+1}| \le 8C_3 8^{-\gamma} M|x-y|^\gamma = C(n,\gamma)M|x-y|^\gamma.$$

So $\bar{u}$ is Hölder continuous with exponent γ on $B_r(x_0)$ as claimed. □

Remark. It is easy to see that

$$\int_{B_\rho(y)} |u - u_{y,\rho}|^2 dx \le \int_{B_\rho(y)} |u - \lambda|^2 dx,$$

for all $\lambda \in \mathbb{R}^k$.

Combining the lemma of Campanato with Poincaré inequality, we obtain:

Lemma 4.2 (Morrey)**.** Suppose $u \in H^1\big(B_{2r}(x_0), \mathbb{R}^k\big)$ satisfies

$$\int_{B_\rho(y)} |Du|^2 dx \leq M^2 \rho^{n-2+2\gamma},$$

for all $y \in B_r(x_0)$ and all $0 < \rho \leq r$, where $\gamma \in (0,1]$ and $M > 0$ is a constant. Then u is γ-Hölder continuous on $B_r(x_0)$ and $[u]_{\gamma, B_r(x_0)} \leq CM$ for some constant C depending only on n, γ.

Proof. By Poincaré inequality and the lemma of Campanato, we get

$$\int_{B_\rho(y)} |u - u_{y,\rho}|^2 dx \leq C(n)\rho^2 \int_{B_\rho(y)} |Du|^2 dx \leq CM^2 \rho^{n+2\gamma}.$$

Thus this proves the assertion. □

4.4 Elliptic Estimation

First, we discuss in some details the concept of difference quotients. This is an indispensable tool in regularity theory of linear equations. Afterward, the $C^{1,\gamma}$ regularity of solutions for the equation $-\triangle u = f$ will be derived.

Difference Quotients

For most of partial differential equations, we cannot differentiate the equations directly, so we are forced to use difference quotient. It can be seen that taking difference quotient is *almost* the same as taking differentiation. So the ability to take difference quotient becomes very useful, especially for the linear equations. In such case, we can prove *higher* differentiability for weak solutions.

Let Ω be an open subset of $\mathbb{R}^n$, $h \neq 0$, $j \in \{1, 2, \dots, n\}$, and $x \in \Omega$ with $\text{dist}(x, \partial\Omega) > |h|$. Suppose $u : \Omega \to \mathbb{R}^k$. We define the *difference quotient* of u at x in the direction $e_j = (0, \dots, \overset{(j)}{1}, \dots, 0)$ by

$$D_j^h u(x) := \frac{u(x + he_j) - u(x)}{h}.$$

Clearly, it is a linear operator. Moreover, it has the following properties. Suppose $\varphi \in C_0^\infty(\Omega, \mathbb{R}^k)$ and $u \in L^1_{\text{loc}}(\Omega, \mathbb{R}^k)$. Then for all sufficiently small $|h| > 0$ (precisely, $\operatorname{dist}(\operatorname{supp}\varphi, \partial\Omega) > |h|$), we have

$$\int_\Omega (D_j^h u) \cdot \varphi \, dx = \int_\Omega u \cdot (D_j^{-h}\varphi) \, dx.$$

Lemma 4.3. Let $\overline{U} \subset \Omega$. If $u \in W^{1,p}(\Omega, \mathbb{R}^k)$ where $1 \le p < \infty$, then for all j and all $0 < h < \operatorname{dist}(U, \partial\Omega)$ there holds

$$\|D_j^h u\|_{L^p(U)} \le \|D_j u\|_{L^p(\Omega)}.$$

Proof. Without loss of generality assume $k = 1$. First assume $u \in C^\infty(\Omega)$. Since $D_j^h u(x) = \frac{1}{h} \int_0^h D_j u(x + te_j)\, dt$ for $(x \in U)$, by the Hölder inequality we get

$$|D_j^h u(x)|^p \le \frac{1}{h} \int_0^h |D_j u(x + te_j)|^p \, dt.$$

Integrating over U, we obtain

$$\int_U |D_j^h u(x)|^p \, dx \le \frac{1}{h} \int_0^h \int_{B_h(U)} |D_j u(x)|^p \, dx dt \le \int_\Omega |D_j u|^p \, dx,$$

where $B_h(U) = \bigcup_{x \in U} B_h(x)$.

Now for arbitrary u, let $u_i \to u$ in $W^{1,p}(\Omega)$ where each u_i are smooth. Note that $D_j^h u_i \to D_j^h u$ in $L^p(\Omega)$, and also in $L^p(U)$. Hence

$$\|D_j^h u\|_{L^p(U)} = \lim_{i\to\infty} \|D_j^h u_i\|_{L^p(\Omega)} \le \lim_{i\to\infty} \|D_j u_i\|_{L^p(\Omega)} = \|D_j u\|_{L^p(\Omega)}.$$

This proves the claim. □

Lemma 4.4. Let $u \in L^p(\Omega, \mathbb{R}^k)$, $1 < p < \infty$. If there are constants $C, h_0 > 0$ such that $\|D_j^h u\|_{L^p(U)} \le C$ for all $U \Subset \Omega$[1], $0 < h \le h_0$ satisfying $\operatorname{dist}(U, \partial\Omega) > h$, then the weak derivative $D_j u$ exists in $L^p(\Omega, \mathbb{R}^k)$; moreover $\|D_j u\|_{L^p(\Omega)} \le C$.

[1]this means $\overline{U}$ is a compact subset of Ω.

Proof. Assume without loss of generality $k = 1$. Extend any function $D_j^h u$ on U to Ω by setting its values outside U to be zero. For a sequence $\{D_j^{h_i} u\}$ where $h_i \searrow 0$, we can assume (by weak compactness of $L^p(\Omega)$) that $D_j^{h_i} u \rightharpoonup v$ in $L^p(\Omega)$ for some function $v \in L^p(\Omega)$. This implies that $\|v\|_{L^p(\Omega)} \leq C$ and

$$\int_\Omega (D_j^{h_i} u)\varphi \, dx \to \int_\Omega v\varphi \, dx,$$

for all $\varphi \in C_0^\infty(\Omega)$. Now for all large i such that $\operatorname{dist}(\operatorname{supp}\varphi, \partial\Omega) > h_i$, we have that

$$\int_\Omega (D_j^{h_i} u)\varphi \, dx = \int_\Omega u D_j^{-h_i} \varphi \, dx \to -\int_\Omega u \, D_j \varphi \, dx \quad \text{as } i \to \infty.$$

Thus $\int_\Omega u \, D_j \varphi \, dx = -\int_\Omega v\varphi \, dx$ for all $\varphi \in C_0^\infty(\Omega)$. So $D_j u = v$. □

Remark. As a consequence, we can say that the process of taking derivatives and of taking difference quotients on $W^{1,p}$ functions are the same thing. The point is that if a function u has the bounded difference quotients then it also has the weak derivatives.

$C^{1,\gamma}$ Regularity for the Equation $-\triangle u = f$

Let $\Omega \subset \mathbb{R}^n$ be open. We consider the following system of partial differential equation:

$$-\triangle u = f \qquad \text{on } \Omega, \tag{i}$$

where $u, f : \Omega \to \mathbb{R}^k$. This equation is just the Poisson equation if the function f is continuous. But now we assume only that $f \in L^q$ for some $q > n$ and $u \in H^1(\Omega, \mathbb{R}^k)$ is a "weak" solution of above equation in the following sense

$$\int_\Omega \sum_{i=1}^n D_i u \cdot D_i \varphi \, dx = \int_\Omega f \cdot \varphi \, dx \tag{ii}$$

for all $\varphi \in H_0^1(\Omega, \mathbb{R}^k)$. We will show that u is in fact $C^{1,\gamma}$ for all $0 < \gamma \leq 1 - \frac{n}{q}$.

The method of putting partial differential equations into "weak" form, as we dis in (ii), and analyzing in a priori the regularity of the weak solutions has shown to be very successful. The pioneering works of De Giorgi, independently by J. Nash, and putting forward by J. K. Moser, are very important to the later development.

First we derive the decay estimates for solutions of the homogeneous equation. We recall some elementary inequalities. Let $1 < p, q < \infty$ with $p^{-1} + q^{-1} = 1$.

(1) Young inequality with ε: if $a, b \geq 0$, $\varepsilon > 0$ then $ab \leq \varepsilon \frac{a^p}{p} + \varepsilon^{-\frac{q}{p}} \frac{b^q}{q}$.

(2) Hölder inequality: if $u, v : \Omega \to \mathbb{R}^k$ are (Lebesgue) measurable functions, then $\|u \cdot v\|_{L^1(\Omega)} \leq \|u\|_{L^p(\Omega)} \|v\|_{L^q(\Omega)}$. ("$u \cdot v$" means the dot product.)

Lemma 4.5. Let $\Omega \subset \mathbb{R}^n$. Suppose $u \in H^1(\Omega, \mathbb{R}^k)$ is a weak solution of the system: $-\triangle u = 0$, i.e.

$$\int_\Omega \sum_{i=1}^n D_i u \cdot D_i \varphi \, dx = 0, \qquad \forall \varphi \in H_0^1(\Omega, \mathbb{R}^k). \tag{i}$$

Then u satisfies

$$\int_{B_\rho(y)} |Du|^2 dx \leq C \Big(\frac{\rho}{\sigma}\Big)^n \int_{B_\sigma(y)} |Du|^2 dx,$$

and

$$\int_{B_\rho(y)} |Du - (Du)_{y,\rho}|^2 dx \leq C \Big(\frac{\rho}{\sigma}\Big)^{n+2} \int_{B_\sigma(y)} |Du - (Du)_{y,\sigma}|^2 dx,$$

for all $0 < \rho < \sigma$, $B_\sigma(y) \subset \Omega$, where C is a constant depending only on n, k.

Proof. Let $\widetilde{u}(x) = u(y + \sigma x)$. The above inequalities in terms of $\widetilde{u}$ are the same just by changing $\sigma \to 1$, $\rho \to \rho/\sigma$ and $y \to 0$. Therefore we will assume that $y = 0$ and $\sigma = 1$. Furthermore, we can assume $\rho \leq 1/2$ (otherwise choose $C = 2^{n+2}$).

First we prove that $u \in H^2(B_{1/2}, \mathbb{R}^k)$ ($B_{1/2} := B_{1/2}(0)$). Putting $D_j^{-h}\varphi$ ($h > 0$, $j \in \{1, \ldots, n\}$) for φ in (i), we have

$$\int_\Omega \sum_{i=1}^n D_i(D_j^h u) \cdot D_i \varphi \, dx = \int_\Omega \sum_{i=1}^n D_i u \cdot D_i(D_j^{-h}\varphi) dx = 0.$$

Let $\zeta \in C_0^\infty(B_1)$ with $h < \operatorname{dist}(\operatorname{supp}\zeta, \partial B_1)$. Now by setting $\varphi = \zeta^2 D_j^h u$, we get

$$\int_{B_1} \zeta^2 |D_j^h(Du)|^2 dx = -2\int_{B_1} \sum_{i=1}^n \zeta D_i(D_j^h u)\cdot(D_i\zeta\, D_j^h u)dx.$$

Using Schwartz inequality, $|\sum a_i b_i| \le (\sum a_i^2)^{1/2}(\sum b_i^2)^{1/2}$, and Young inequality with ε ($p = q = 2$) to the right, we find that

$$\int_{B_1} \zeta^2|D_j^h(Du)|^2 dx \le \varepsilon \int_{B_1} \zeta^2 |D_j^h(Du)|^2 dx + \varepsilon^{-1}\int_{B_1} |D\zeta|^2|D_j^h u|^2 dx.$$

Note that $\int_{B_1} |D\zeta|^2|D_j^h u|^2 dx \le \int_{B_1} |D\zeta|^2 |D_j u|^2 dx$. If ε is small enough, we have

$$\int_{B_1} \zeta^2 |D_j^h(Du)|^2 dx \le C_0 \int_{B_1} |D\zeta|^2|D_j u|^2 dx,$$

for some universal constant $C_0 > 0$. This inequality is often called a Caccioppoli type inequality. Now fix a ζ with $\zeta \equiv 1$ on $B_{1/2}$, then for all h small enough

$$\int_{B_{1/2}} |D_j^h(Du)|^2 dx \le C_0' \int_{B_1} |Du|^2 dx \equiv \text{const}.$$

Thus $Du \in H^1$ on $B_{1/2}$. Therefore, we can conclude that $u \in H^2$ on $B_{1/2}$ as claimed and $\|D^2 u\|_{L^2(B_{1/2})} \le \sqrt{n} C_0 \|Du\|_{L^2(B_1)}$.

If we restrict domain to the ball $B_{1/2}$, then by applying the same argument to each $D_j u$ (which now is in H^1 and weakly solves (i)), we can conclude that $Du \in H^2$. Hence $u \in H^3$, and $\|D^3 u\|_{L^2(B_{1/2})} \le n^{3/2} C_0^2 \|Du\|_{L^2(B_1)}$. Continuing in this way, we have that $u \in H^m(B_{1/2}, \mathbb{R}^k)$ for all $m \in \mathbb{N}$, and $\|D^m u\|_{L^2(B_{1/2})} \le C_1(n,m)\|Du\|_{L^2(B_1)}$.

By Sobolev Embedding Theorem, there is an integer m for which $H^m \hookrightarrow C^1$. Fixing this value of m, Du is C^1 on $B_{1/2}$ and we get

$$\|Du\|^2_{L^\infty(B_{1/2})} + \|D^2 u\|^2_{L^\infty(B_{1/2})} \le C_2 \|Du\|^2_{L^2(B_1)},$$

where $C_2 > 0$ is a constant depending only on n, k. Thus

$$\int_{B_\rho} |Du|^2 dx \le \omega_n \rho^n \|Du\|^2_{L^\infty(B_{1/2})} \le C_2 \omega_n \rho^n \int_{B_1} |Du|^2 dx,$$

where $\omega_n = |B_1(0)|$. This is the first inequality. By Poincaré inequality, we obtain that

$$\int_{B_\rho} |Du - (Du)_\rho|^2 dx \leq C(n)\rho^2 \int_{B_\rho} |D^2 u|^2 dx \leq C_2 C(n)\rho^{n+2} \int_{B_1} |Du|^2 dx.$$

Hence by using $Du - (Du)_1$ in place of Du and the remark after lemma (4.1), we get the second inequality. □

Remark. The lemma also proves that every "weakly" harmonic function, i.e. that $u \in H^1$ which solves the Laplace equation $-\triangle u = 0$ in the "weak" sense (i), is in fact a harmonic function. In fact, by proving that $u \in H^m$ for all m, the Sobolev Embedding Theorem implies that $u \in C^2$. Hence u satisfies the usual Laplace equation.

The next lemma will be employed in the proof of our main regularity result. It is also useful for the decay estimate for the general elliptic regularity theory.

Lemma 4.6 (A Technical Lemma)**.** Suppose $f : [0, r] \to [0, \infty)$ is nondecreasing and for any $0 < \rho < \sigma \leq r$ there holds

$$f(\rho) \leq A\Big[\Big(\frac{\rho}{\sigma}\Big)^\alpha + \varepsilon\Big] f(\sigma) + B\sigma^\beta, \qquad (\varepsilon > 0)$$

for some constants $A, B, \alpha, \beta > 0$ with $\beta < \alpha$. Then for each $\gamma \in [\beta, \alpha)$ there is an $\varepsilon_0 = \varepsilon_0(A, \alpha, \gamma) \geq 0$ such that if $\varepsilon \leq \varepsilon_0$ then for all $0 < \rho < \sigma \leq r$

$$f(\rho) \leq C\left[\Big(\frac{\rho}{\sigma}\Big)^\gamma f(\sigma) + B\rho^\beta\right],$$

where C is a constant depending only on A, α, γ, β.

Proof. Let $\tau \in (0, 1)$. For each $i \in \mathbb{N}$, $f(\tau^i \sigma) \leq A(\tau^\alpha + \varepsilon) f(\tau^{i-1}\sigma) + B(\tau^{i-1}\sigma)^\beta$, hence by induction, we have for each $k \in \mathbb{N}$

$$\begin{aligned} f(\tau^k \sigma) &\leq A^k (\tau^\alpha + \varepsilon)^k f(\sigma) + B(\tau^{k-1}\sigma)^\beta \sum_{i=0}^{k-1} A^i (\tau^\alpha + \varepsilon)^i \tau^{-\beta i} \\ &\leq A^k (\tau^\alpha + \varepsilon)^k f(\sigma) + B(\tau^{k-1}\sigma)^\beta \frac{1}{1 - A(\tau^\alpha + \varepsilon)\tau^{-\beta}}, \end{aligned}$$

provided $A(\tau^\alpha + \varepsilon)\tau^{-\beta} < 1$.

Let $\varepsilon_0 \leq \tau^\alpha$, where τ is chosen to satisfy $2A\tau^\alpha = \tau^\gamma/2$. Clearly τ and ε_0 depend only on A, α, γ. So, if $\varepsilon \leq \varepsilon_0$ then

$$f(\tau^k\sigma) \leq \tau^{k\gamma} f(\sigma) + \frac{1}{\tau^{2\beta}(1-\tau^{\gamma-\beta}/2)} B(\tau^{k+1}\sigma)^\beta.$$

For $0 < \rho < \sigma$, let $k \in \mathbb{N} \cup \{0\}$ be such that $\tau^{k+1}\sigma < \rho \leq \tau^k\sigma$. Thus

$$f(\rho) \leq f(\tau^k\sigma) \leq \frac{1}{\tau^\gamma}\Big(\frac{\rho}{\sigma}\Big)^\gamma f(\sigma) + \frac{2}{\tau^{2\beta}(2-\tau^{\gamma-\beta})} B\rho^\beta.$$

This proves the lemma, by letting $C \geq \max\Big\{\frac{1}{\tau^\gamma}, \frac{2}{\tau^{2\beta}(2-\tau^{\gamma-\beta})}\Big\}$. □

Now we come to the main regularity result of this chapter.

Theorem 4.7. Let $\Omega \subset \mathbb{R}^n$ be open. Suppose $f \in L^q(\Omega, \mathbb{R}^k)$ for some $q > n$ and $u \in H^1(\Omega, \mathbb{R}^k)$ is a weak solution of the system: $-\triangle u = f$, i.e.

$$\int_\Omega \sum_{j=1}^n D_j u \cdot D_j \varphi \, dx = \int_\Omega f \cdot \varphi \, dx, \qquad \forall \varphi \in H_0^1(\Omega, \mathbb{R}^k).$$

Then $u \in C^{1,\gamma}(\overline{B}_r(x_0), \mathbb{R}^k)$ for all $B_{2r}(x_0) \Subset \Omega$ and all $0 < \gamma \leq 1 - \frac{n}{q}$.

Proof. Fix a ball $B_{2r}(x_0) \Subset \Omega$. Assume $y \in B_r(x_0)$ and $0 < \rho < \sigma \leq r$. Let $X^1 := H^1(B_\sigma(y), \mathbb{R}^k)$ and $X_0^1 := H_0^1(B_\sigma(y), \mathbb{R}^k)$.

Step 1. First we prove that there is a unique $w \in X^1$ such that $w - u \in X_0^1$, i.e. $w = u - v$ for some $v \in X_0^1$, and

$$-\triangle w = 0, \qquad \text{on } B_r(x_0). \tag{i}$$

Note that $(g,h)_0 := \int_{B_\sigma(y)} \sum_{j=1}^n D_j g \cdot D_j h \, dx$ is an equivalent inner product of the Hilbert space X_0^1. Equation (i) is equivalent to that $(u - v, \varphi)_0 = 0$. Hence $(v, \varphi)_0 = \int_{B_\sigma(y)} f \cdot \varphi \, dx$, for all $\varphi \in X_0^1$. Since $\varphi \mapsto \int_{B_\sigma(y)} f \cdot \varphi \, dx$ is a bounded linear functional on X_0^1, by Riesz theorem[2] we can conclude that there exists a unique such function $v \in X_0^1$.

[2] Every bounded linear functional on a Hilbert space is an inner product by a fixed element.

Step 2. Next we employ lemma (4.5) to derive two preliminary estimates. Recall the inequality $(a+b)^2 \leq 2a^2 + 2b^2$ for all real numbers a, b. By direct computation and using lemma (4.5), we have

$$\begin{aligned}\int_{B_\rho(y)} |Du|^2 dx &\leq 2\int_{B_\rho(y)} |Dw|^2 dx + 2\int_{B_\rho(y)} |Dv|^2 dx \\ &\leq 2C\Big(\frac{\rho}{\sigma}\Big)^n \int_{B_\sigma(y)} |Dw|^2 dx + 2\int_{B_\sigma(y)} |Dv|^2 dx.\end{aligned}$$

Substituting $w = u - v$, we find that

$$\begin{aligned}\int_{B_\rho(y)} |Du|^2 dx &\leq 4C\Big(\frac{\rho}{\sigma}\Big)^n \int_{B_\sigma(y)} |Du|^2 dx + (2+4C)\int_{B_\sigma(y)} |Dv|^2 dx \\ &\leq C_1\Big\{\Big(\frac{\rho}{\sigma}\Big)^n \int_{B_\sigma(y)} |Du|^2 dx + \int_{B_\sigma(y)} |Dv|^2 dx\Big\}, \qquad \text{(ii)}\end{aligned}$$

where $C_1 = 2 + 4C$ depends only on n, k.

By noting that

$$\int_{B_\sigma(y)} |Dv - (Dv)_{y,\sigma}|^2 dx \leq \int_{B_\sigma(y)} |Dv|^2 dx$$

(see the remark after lemma (4.1)), in the same way as above we get

$$\begin{aligned}\int_{B_\rho(y)} |Du - (Du)_{y,\rho}|^2 dx \leq C_2\Big\{\Big(\frac{\rho}{\sigma}\Big)^{n+2} &\int_{B_\sigma(y)} |Du - (Du)_{y,\sigma}|^2 dx \\ &+ \int_{B_\sigma(y)} |Dv|^2 dx\Big\}, \qquad \text{(iii)}\end{aligned}$$

for some constant C_2 depending only on n, k.

Step 3. Next we estimate the term $\int_{B_\sigma(y)} |Dv|^2 dx$. Note that $v \in X_0^1$ also satisfies $\int_{B_\sigma(y)} \sum_{j=1}^n D_j v \cdot D_j \varphi \, dx = \int_{B_\sigma(y)} f \cdot \varphi \, dx$ for all $\varphi \in X_0^1$. Setting $\varphi = v$ and apply Hölder inequality and Sobolev inequality, we get

$$\begin{aligned}\int_{B_\sigma(y)} |Dv|^2 dx &\leq \|f\|_{L^q(B_\sigma(y))} \Big\{\int_{B_\sigma(y)} |v|^{2^*} dx\Big\}^{1/2^*} |B_\sigma(y)|^{(1-\frac{1}{q}-\frac{1}{2^*})} \\ &\leq C(n)\|f\|_{L^q(B_\sigma(y))} \Big\{\int_{B_\sigma(y)} |Dv|^2 dx\Big\}^{1/2} \sigma^{(\frac{n+2}{2}-\frac{n}{q})} \\ &\leq \delta \int_{B_\sigma(y)} |Dv|^2 dx + C(n,\delta)\|f\|^2_{L^q(B_\sigma(y))} \sigma^{(n-2\frac{n}{q}+2)}.\end{aligned}$$

Here we used Sobolev inequality in the second inequality and Young's inequality, $ab \leq \delta a^2 + \frac{1}{4\delta} b^2$ for all $a, b, \delta > 0$, in the last inequality. Choosing a small δ, then we obtain that

$$\int_{B_\sigma(y)} |Dv|^2 dx \leq C F^2 \sigma^{(n - \frac{2n}{q} + 2)}, \tag{iv}$$

where $F = \|f\|_{L^q(B_r(x_0))}$ and C is a constant depending only on n.

Step 4. By (ii) and (iv) together with the fact $-\frac{2n}{q} + 2 \geq -2 + 2\gamma$ for all $0 < \gamma \leq 1$, we have

$$\int_{B_\rho(y)} |Du|^2 dx \leq C_1 \left\{ \left(\frac{\rho}{\sigma}\right)^n \int_{B_\sigma(y)} |Du|^2 dx + C F^2 r^{n - 2\frac{n}{q} + 2} \left(\frac{\sigma}{r}\right)^{n-2+2\gamma} \right\},$$

for all $0 < \gamma < 1$, since $\sigma / r \leq 1$. Thus by the technical lemma we see that

$$\begin{aligned} \int_{B_\rho(y)} |Du|^2 dx &\leq C_1 \left\{ \frac{1}{\sigma^{n-2+2\gamma}} \int_{B_\sigma(y)} |Du|^2 dx + F^2 r^{4 - 2(\frac{n}{q} + \gamma)} \right\} \rho^{n-2+2\gamma} \\ &\leq C \left\{ \int_{B_r(x_0)} |Du|^2 dx + F^2 \right\} \rho^{n-2+2\gamma}, \end{aligned} \tag{v}$$

where $C = C(n, k, q, r)$. Therefore we can conclude that u is γ-Hölder continuous on $B_r(x_0)$ by Morrey's lemma.

Also, by (iii), (iv) and the argument as above, we have

$$\begin{aligned} \int_{B_\rho(y)} |Du - (Du)_{y,\rho}|^2 dx &\leq C_2 \Big\{ \frac{1}{\sigma^{n+2\gamma}} \int_{B_\sigma(y)} |Du - (Du)_{y,\sigma}|^2 dx \\ &\qquad + C F^2 r^{4 - 2(\frac{n}{q} + \gamma)} \Big\} \rho^{(n+2\gamma)} \\ &\leq C(n, k, q, r) \left\{ \int_{B_r(x_0)} |Du|^2 dx + F^2 \right\} \rho^{n+2\gamma}, \end{aligned} \tag{vi}$$

for all $0 < \gamma \leq 1 - \frac{n}{q}$. Thus $Du \in C^\gamma$ on $B_r(x_0)$ by Campanato's lemma. $\square$

Remark. If $f \in L^\infty(\Omega, \mathbb{R}^k)$ we can also verify that $u \in C^{1,1}(\overline{B}_r(x_0), \mathbb{R}^k)$ for all $B_{2r}(x_0) \Subset \Omega$. Further, we have by (vi) and Campanato's lemma that

$$[Du]_{1, B_r(x_0)} \leq C(n, k, r) \big\{ \|Du\|_{L^2(B_r(x_0))} + \|f\|_{L^\infty(B_r(x_0))} \big\}.$$

CHAPTER V

MAIN THEOREMS

This chapter contains the main results of our work. In the first theorem we show that any weak solutions of the penalized approximate equations are smooth. This nice property is used in the second theorem to prove the monotonicity identity (a similar analogue of the monotonicity identity occurred in stationary harmonic maps). Afterward, we give the gradient estimate for penalized approximate solutions in the third theorem. Finally, the last theorem is another proof of the *small energy regularity theorem* for minimizing harmonic maps.

5.1 Smoothness of Solutions for Penalized Approximate Equations

One advantage of the penalized approximation approach is that solutions to the approximate equations are smooth. More precisely, we will show in this section that any weak solution $u_\varepsilon \in H^1(\Omega, \mathbb{R}^k)$ of the following system of nonlinear PDE

$$-\triangle u_\varepsilon = \frac{1}{\varepsilon^2} u_\varepsilon \left(1 - |u_\varepsilon|^2\right) \qquad \text{in } \Omega,$$

is smooth. Recall that u_ε is a weak solution of the system if

$$\int_\Omega \sum_{j=1}^{n} D_j u_\varepsilon \cdot D_j \varphi \, dx = \frac{1}{\varepsilon^2} \int_\Omega u_\varepsilon \cdot \varphi (1 - |u_\varepsilon|^2) \, dx,$$

for all $\varphi \in H_0^1(\Omega, \mathbb{R}^k)$. Here Ω is an open subset of $\mathbb{R}^n$, $k \in \mathbb{N}$.

Also, recall that the difference quotient $D_j^h v$ for a function $v : \Omega \to \mathbb{R}^k$ has the following properties (see section (4.4) for details)

- if $v \in H^1(\Omega, \mathbb{R}^k)$ then $\|D_j^h v\|_{L^2(\Omega)} \le \|D_j v\|_{L^2(\Omega)}$ for all small $h > 0$;
- conversely, if $\|D_j^h v\|_{L^2(\Omega)} \le C$ for some fixed constant C for all small $h > 0$, then v has the weak derivative $D_j v$ in $L^2(\Omega, \mathbb{R}^k)$.

Theorem 5.1. If $u \in H^1(\Omega, \mathbb{R}^k)$ is a locally bounded weak solution of the following system

$$-\triangle u_\varepsilon = \frac{1}{\varepsilon^2}\, u_\varepsilon(1 - |u_\varepsilon|^2), \tag{$*$}$$

then u_ε is smooth.

Proof. Denote by f_0 the right hand side of $(*)$. Since f_0 is clearly bounded on $B_{2r}(x_0) \Subset \Omega$, the remark after theorem (4.7) gives $u_\varepsilon \in C^{1,1}$ on $B_r(x_0)$. This is true for all $B_{2r}(x_0) \Subset \Omega$, so u_ε is also C^1 on Ω.

We claim that $u_\varepsilon \in C^{m,1}$ on $B_r(x_0)$ for all $m \in \mathbb{N}$ and for all $B_{2r}(x_0) \Subset \Omega$. To this end we will use a "bootstrap" argument (. assuming by induction that u_ε is $C^{m,1}$ on $B_r(x_0)$ $(m \in \mathbb{N})$ for all $B_{2r}(x_0) \Subset \Omega$, we will show that $u_\varepsilon \in C^{m+1,1}$ on $B_r(x_0)$).

Fix $B_{2r}(x_0) \Subset \Omega$. Let $f_m^h = D_j^h(D^{m-1} f_0)$ and $v^h = D_j^h(D^{m-1} u_\varepsilon)$. Then v^h is a weak solution of the following equation

$$-\triangle v^h = f_m^h \qquad \text{in } B_{2r}(x_0).$$

By induction hypothesis u_ε is C^m on Ω. Hence $D^{m-1} f_0$ is C^1 on Ω. By the mean-value theorem, we have

$$\sup_{B_{2r}(x_0)} |f_m^h| \le \sup_{B_{2r+h_0}(x_0)} |D(D^{m-1} f_0)| := C_1 < \infty,$$

if $0 < h < h_0 < \operatorname{dist}(B_{2r}(x_0), \partial\Omega)$, where C_1 is independent of h.

By the remark after theorem (4.7), we conclude that v^h is $C^{1,1}$ on $B_r(x_0)$. Again by induction hypothesis u_ε is $C^{m,1}$ on $B_r(x_0)$. Hence on $B_{r+h_0}(x_0)$ for a small h_0. Therefore, we have that

$$\sup_{B_r(x_0)} |Dv^h| = \sup_{B_r(x_0)} |D_j^h(D^m u_\varepsilon)| \le [D^m u_\varepsilon]_{1,B_{r+h_0}(x_0)} := C_2 < +\infty,$$

if $0 < h < h_0 < r$, where C_2 is independent of h. Furthermore

$$\begin{aligned}[Dv^h]_{1,B_r(x_0)} &\le C(n,k,r)\left\{\|Dv^h\|_{L^2(B_r(x_0))} + \|f_m\|_{L^\infty(B_r(x_0))}\right\} \\ &\le C(n,k,r)\{\omega_n r^n C_2 + C_1\}.\end{aligned}$$

Thus for all sufficiently small $h > 0$, Dv^h are bounded and equicontinuous in $C(\overline{B}_r(x_0), \mathbb{R}^k)$. By Arzéla-Ascoli theorem, we can conclude that there is a sequence $h_i \searrow 0$ such that $w_j = \lim_{i\to\infty} D_j^{h_i}(D^m u_\varepsilon)$ is in $C(\overline{B}_r(x_0), \mathbb{R}^k)$ for each j. Therefore w_j is $C^{0,1}$ on $B_r(x_0)$. Now by Rademacher's theorem $D^m u_\varepsilon$ is differentiable almost everywhere, and $D_j(D^m u_\varepsilon) = w_j$ almost everywhere on $B_r(x_0)$. Therefore $u_\varepsilon \in C^{m+1,1}$ on $B_r(x_0)$. □

5.2 Monotonicity Identity

Theorem 5.2. Let $u_\varepsilon : \Omega \to \mathbb{R}^k$ be a smooth solution of the system

$$-\triangle u_\varepsilon = \frac{1}{\varepsilon^2} u_\varepsilon(1 - |u_\varepsilon|^2). \tag{i}$$

Then u_ε satisfies the following identity: if $\overline{B}_r(x_0) \subset \Omega$ and $0 < \rho < \sigma \le r$

$$\begin{aligned}\sigma^{2-n}\int_{B_\sigma(x_0)} e_\varepsilon(u_\varepsilon)\,dx - \rho^{2-n}\int_{B_\rho(x_0)} e_\varepsilon(u_\varepsilon)\,dx &= \int_{B_\sigma(x_0)\setminus B_\rho(x_0)} \xi^{2-n}\left|\frac{\partial u_\varepsilon}{\partial \xi}\right|^2 d\xi \\ &\quad + 2\int_\rho^\sigma t^{1-n}\int_{B_t(x_0)} \frac{1}{4\varepsilon^2}(1 - |u_\varepsilon|^2)^2 dx.\end{aligned}$$

Here $\xi = |x - x_0|$ and

$$e_\varepsilon(u_\varepsilon) = \frac{1}{2}|Du_\varepsilon|^2 + \frac{1}{4\varepsilon^2}(1 - |u_\varepsilon|^2)^2.$$

Proof. Without loss of generality, assume $x_0 = 0$. Let $B_t := B_t(0)$. Multiply (i) by $x_j D_j u_\varepsilon$ and integrate over $B_t(0)$, we have

$$-\int_{B_t} \sum_{i,j=1}^{n} x_j D_j u_\varepsilon \cdot (D_i D_i u_\varepsilon)\, dx = \frac{1}{\varepsilon^2} \int_{B_t} \sum_{j=1}^{n} x_j (D_j u_\varepsilon) \cdot u_\varepsilon (1 - |u_\varepsilon|^2)\, dx. \qquad \text{(ii)}$$

Integrating by parts, we see that the term on the left is

$$\int_{B_t} \sum_{i,j=1}^{n} D_i(x_j D_j u_\varepsilon) \cdot D_i u_\varepsilon\, dx - \int_{\partial B_t} \sum_{i,j=1}^{n} x_j D_j u_\varepsilon \cdot D_i u_\varepsilon \nu_i\, d\xi$$
$$= \int_{B_t} \sum_{i,j=1}^{n} D_i(x_j D_j u_\varepsilon) \cdot D_i u_\varepsilon\, dx - t \int_{\partial B_t} \Big|\frac{\partial u_\varepsilon}{\partial \xi}\Big|^2 d\xi$$

where $\nu = x/t$. The first term on the right can be expanded as follows. By differentiating and multiplying out we get

$$\int_{B_t} \sum_{i,j=1}^{n} D_i(x_j D_j u_\varepsilon) \cdot D_i u_\varepsilon\, dx = \int_{B_t} \sum_{i,j=1}^{n} (\delta_{ij} D_j u_\varepsilon + x_j D_i D_j u_\varepsilon) \cdot D_i u_\varepsilon\, dx$$
$$= \int_{B_t} |Du_\varepsilon|^2 dx + \int_{B_t} \sum_{i,j=1}^{n} (x_j D_j D_i u_\varepsilon) \cdot D_i u_\varepsilon\, dx.$$

Again by integration by parts, we obtain

$$\int_{B_t} |Du_\varepsilon|^2 dx + \int_{B_t} \sum_{j=1}^{n} x_j D_j \Big(\frac{|Du_\varepsilon|^2}{2}\Big) dx = \Big(1 - \frac{n}{2}\Big) \int_{B_t} |Du_\varepsilon|^2 dx + \frac{t}{2} \int_{\partial B_t} |Du_\varepsilon|^2 dx.$$

Therefore we have the left hand side of (ii) equals to

$$\Big(1 - \frac{n}{2}\Big) \int_{B_t} |Du_\varepsilon|^2 dx + \frac{t}{2} \int_{\partial B_t} |Du_\varepsilon|^2 dx - t \int_{\partial B_t} \Big|\frac{\partial u_\varepsilon}{\partial \xi}\Big|^2 d\xi. \qquad \text{(iii)}$$

Next we compute the term on the right hand side of (ii). Notice that it is equal to

$$-\frac{1}{\varepsilon^2} \int_{B_t} \sum_{j=1}^{n} x_j D_j \Big(\frac{1}{4}(1 - |u_\varepsilon|^2)^2\Big)\, dx = \frac{n}{4\varepsilon^2} \int_{B_t} (1 - |u_\varepsilon|^2)^2 dx - \frac{t}{4\varepsilon^2} \int_{\partial B_t} (1 - |u_\varepsilon|^2)^2 dx. \qquad \text{(iv)}$$

Combining (iii) and (iv), we obtain

$$t \int_{\partial B_t} e_\varepsilon(u_\varepsilon)\, dx = -(2 - n) \int_{B_t} e_\varepsilon(u_\varepsilon)\, dx + \frac{2}{4\varepsilon^2} \int_{B_t} (1 - |u_\varepsilon|^2)^2 dx + t \int_{\partial B_t} \Big|\frac{\partial u_\varepsilon}{\partial \xi}\Big|^2 d\xi.$$

This implies that

$$\begin{aligned}\frac{d}{dt}\Big(t^{2-n}\int_{B_t} e_\varepsilon(u_\varepsilon)\,dx\Big) &= (2-n)t^{1-n}\int_{B_t} e_\varepsilon(u_\varepsilon)\,dx + t^{2-n}\int_{\partial B_t} e_\varepsilon(u_\varepsilon)\,dx\\ &= \frac{2}{4\varepsilon^2}t^{1-n}\int_{B_t}(1-|u_\varepsilon|^2)^2dx + t^{2-n}\int_{\partial B_t}\Big|\frac{\partial u_\varepsilon}{\partial \xi}\Big|^2 d\xi.\end{aligned}$$

Finally, by integrating from ρ to σ, we obtain the identity. □

Remarks. 1. An immediate consequence of the theorem is the *monotonicity inequality* for the energy functional $\widetilde{E}_\varepsilon(u_\varepsilon, B_\rho(x_0)) = \rho^{2-n}\int_{B_\rho(x_0)} e_\varepsilon(u_\varepsilon)\,dx$. That is, if $0 < \rho < \sigma \le r$ ($B_r(x_0) \Subset \Omega$) and u_ε is a smooth solution of (i), then

$$\rho^{2-n}\int_{B_\rho(x_0)} e_\varepsilon(u_\varepsilon)\,dx \le \sigma^{2-n}\int_{B_\sigma(x_0)} e_\varepsilon(u_\varepsilon)\,dx.$$

Clearly $\lim_{\rho\to 0}\int_{B_\rho(x_0)} e_\varepsilon(u_\varepsilon)\,dx = 0$ for all $x_0 \in \Omega$ since u_ε is smooth. We say that u_ε has no *energy concentration* at any point $x_0 \in \Omega$.

2. A stationary (hence minimizing) harmonic map u also enjoys this phenomena. There is a monotonicity inequality for the energy functional $\widetilde{E}(u, B_\rho(x_0)) = \rho^{2-n}\int_{B_\rho(x_0)} |Du|^2 dx$, and a *regular point* $x_0 \in \Omega$ of u i.e. a point for which u is smooth in a neighborhood, is a point which u has no energy concentration.

5.3 Gradient Estimation

We begin with a Bochner type identity.

Lemma 5.3. Let $u_\varepsilon : \Omega \to \mathbb{R}^k$ be a smooth solution of the following system

$$-\triangle u_\varepsilon = \frac{1}{\varepsilon^2}\, u_\varepsilon(1-|u_\varepsilon|^2). \tag{i}$$

Then we have the following Bochner type inequality,

$$\triangle g_\varepsilon(u_\varepsilon) \ge 0,$$

where $g_\varepsilon(u_\varepsilon) = \frac{1}{2}|Du_\varepsilon|^2 dx - \frac{1}{4\varepsilon^2}(1-|u_\varepsilon|^2)^2$.

Proof. Differentiating (i) by D_j, multiplying by $D_j u_\varepsilon$ and summing over $j = 1, \ldots, n$, we have that

$$-\sum_{j=1}^{n} D_j u_\varepsilon \cdot \triangle D_j u_\varepsilon = \frac{1}{\varepsilon^2}|Du_\varepsilon|^2(1-|u_\varepsilon|^2) - \frac{2}{\varepsilon^2}\sum_{j=1}^{n}|u_\varepsilon \cdot D_j u_\varepsilon|^2. \qquad \text{(ii)}$$

Also, observe that

$$D_i\left(\frac{|Du_\varepsilon|^2}{2}\right) = \sum_{j=1}^{n} D_j u_\varepsilon \cdot D_i D_j u_\varepsilon,$$

and

$$\triangle\left(\frac{|Du_\varepsilon|^2}{2}\right) = \sum_{i,j=1}^{n}|D_i D_j u_\varepsilon|^2 + \sum_{j=1}^{n} D_j u_\varepsilon \cdot \triangle D_j u_\varepsilon. \qquad \text{(iii)}$$

On the other hand

$$D_j\left(\frac{1}{4\varepsilon^2}(1-|u_\varepsilon|^2)^2\right) = -\frac{1}{\varepsilon^2}u_\varepsilon \cdot D_j u_\varepsilon(1-|u_\varepsilon|^2),$$

and

$$\triangle\left(\frac{1}{4\varepsilon^2}(1-|u_\varepsilon|^2)^2\right) = -\left\{\frac{1}{\varepsilon^2}|Du_\varepsilon|^2(1-|u_\varepsilon|^2) - \frac{2}{\varepsilon^2}\sum_{j=1}^{n}|u_\varepsilon \cdot D_j u_\varepsilon|^2\right\}$$
$$-\frac{1}{\varepsilon^2}u_\varepsilon \cdot \triangle u_\varepsilon(1-|u_\varepsilon|^2).$$

Note that $\frac{1}{\varepsilon^2}u_\varepsilon \cdot \triangle u_\varepsilon(1-|u_\varepsilon|^2) = -|\triangle u_\varepsilon|^2$. By (ii), (iii), we find that

$$\triangle\left(\frac{|Du_\varepsilon|^2}{2}\right) = \sum_{i,j=1}^{n}|D_i D_j u_\varepsilon|^2 + \triangle\left(\frac{1}{4\varepsilon^2}(1-|u_\varepsilon|^2)^2\right) - |\triangle u_\varepsilon|^2.$$

By noting that $\sum_{i,j=1}^{n}|D_i D_j u_\varepsilon|^2 - |\triangle u_\varepsilon|^2 \geq 0$, we obtain the claim. $\square$

Theorem 5.4. Suppose Ω is an open subset of $\mathbb{R}^n$, $k \geq 2$. There exists $\delta_0 = \delta_0(n,k) > 0$ such that if $u_\varepsilon : \Omega \to \mathbb{R}^k$ is a smooth solution of the system

$$-\triangle u_\varepsilon = \frac{1}{\varepsilon^2}\, u_\varepsilon(1-|u_\varepsilon|^2) \qquad \text{in } \Omega,$$

and $r^{2-n}\int_{B_r(x_0)} e_\varepsilon(u_\varepsilon)\, dx < \delta_0$ $(B_r(x_0) \Subset \Omega)$, then

$$r \sup_{B_{r/2}(x_0)} |Du_\varepsilon| \leq C,$$

where $C = C(\delta_0) \to 0$ as $\delta_0 \to 0$.

Proof. First we transform the above equation into a more manageable one. Choose $0 < r_1 < r$ such that

$$\max_{0\le t\le r}\Big\{ (r-t)^2 \max_{B_t(x_0)} e_\varepsilon(u_\varepsilon)\Big\} = (r-r_1)^2 \max_{B_{r_1}(x_0)} e_\varepsilon(u_\varepsilon),$$

and $x_1 \in \overline{B}_{r_1}(x_0)$ to satisfy

$$\max_{B_{r_1}(x_0)} e_\varepsilon(u_\varepsilon) = e_\varepsilon(u_\varepsilon)(x_1).$$

Let $m^2 = e_\varepsilon(u_\varepsilon)(x_1)$. Since $B_{(\frac{r-r_1}{2})}(x_1) \subset B_{(\frac{r+r_1}{2})}(x_0)$, we have

$$\begin{aligned}\max_{B_{(\frac{r-r_1}{2})}(x_1)} e_\varepsilon(u_\varepsilon) &\le \max_{B_{(\frac{r+r_1}{2})}(x_0)} e_\varepsilon(u_\varepsilon)\\ &\le (r-r_1)^2 \max_{B_{r_1}(x_0)} e_\varepsilon(u_\varepsilon)\Big/\Big(r-\frac{r+r_1}{2}\Big)^2 = 4m^2.\end{aligned}$$

Let $R = (\frac{r-r_1}{2})m$. If we set $v(x) = u_\varepsilon(x_1 + \frac{x}{m})$ on $B_R(0)$ and $\varepsilon' = m\varepsilon$, then we find that the smooth function v satisfies

$$-\triangle v = \frac{1}{\varepsilon'^2} v(1-|v|^2) \qquad \text{on } B_R(0), \tag{i}$$

and

$$\max_{B_R(0)} e_{\varepsilon'}(v) \le 4, \qquad e_{\varepsilon'}(v)(0) = 1. \tag{ii}$$

Claim $R \le 1$: We prove the claim by contradiction, that is assume $R > 1$. We restrict the domain to the ball $B_1(0)$. We will show that this implies the existence of a universal constant $c > 0$, independent of v, ε' such that

$$1 \le c \int_{B_{B_1(0)}} e_{\varepsilon'}(v), \tag{iii}$$

for any pair v, ε' satisfying (i) and (ii).

Under the assumption $R > 1$, suppose that the assertion is false. We have that there are sequences $\varepsilon'_j \to 0$ and $\{v_j\}$ such that each ε'_j, v_j verify (i), (ii) on

$B_1(0)$ but

$$\int_{B_1(0)} e_{\varepsilon'_j}(v_j) \to 0, \qquad \text{as } j \to \infty.$$

By the previous lemma we conclude that $g_{\varepsilon'_j}(v_j)$ are subharmonic functions on $B_1(0)$ for all j. This implies, by Moser's Harnack inequality, that there is a constant $c > 0$ depending only on n, k such that

$$g_{\varepsilon'_j}(v_j)(0) \leq c \int_{B_1(0)} g_{\varepsilon'_j}(v_j)\, dx \leq c \int_{B_1(0)} e_{\varepsilon'_j}(v_j)\, dx$$

for all j. As $j \to \infty$, $\varepsilon'_j \to 0$, we have

$$g_{\varepsilon'_j}(v_j)(0) \to e_{\varepsilon'_j}(v_j)(0) = 1.$$

But $\int_{B_1(0)} e_{\varepsilon'_j}(v_j)\, dx \to 0$, so we get a contradiction. This prove (iii).

Next we will show that the assumption $R > 1$ leads to a contradiction. By (iii), we have a constant c such that

$$1 \leq c \int_{B_1(0)} e_{\varepsilon'}(v)\, dx.$$

Substituting $R = (\frac{r-r_1}{2})m$ back into the definition of v ,ε' and applying the monotonicity inequality, we get

$$\begin{aligned} 1 \leq c \int_{B_1(0)} e_{\varepsilon'}(v)\, dx &\leq cR^{2-n} \int_{B_R(0)} \left\{ \frac{1}{2}|Dv|^2 + \frac{1}{4m\varepsilon^2}(1-|v|^2)^2 \right\} dx \\ &= c\left(\frac{r-r_1}{2}\right)^{2-n} \int_{B_{(\frac{r-r_1}{2})}(x_0)} \left\{ \frac{1}{2}|Du_\varepsilon|^2 + \frac{1}{4\varepsilon^2}(1-|u_\varepsilon|^2)^2 \right\} dx \\ &\leq cr^{2-n} \int_{B_r(x_0)} e_\varepsilon(u_\varepsilon)\, dx. \end{aligned}$$

This cannot be true since $r^{2-n} \int_{B_r(x_0)} e_\varepsilon(u_\varepsilon)\, dx$ is assumed to have any arbitrary small value. So we have proved the claim. Now by the definition of r_1, x_1, we see that

$$\left(r - \frac{r}{2}\right)^2 \max_{B_{r/2}(x_0)} e_\varepsilon(u_\varepsilon) \leq (r - r_1)^2 m^2 = 4R^2.$$

Therefore we obtain $r\sup_{B_{r/2}(x_0)}|Du_\varepsilon| \le 16R^2$. Notice that the claim above also true if we replace 1 by any $0<\alpha<1$. Hence, for each $0<\alpha<1$, there is a $\beta>0$ such that $R\le\alpha$ if $\delta_0<\beta$. Thus $R\to 0$ if $\delta_0\to 0$. □

We will use this gradient estimate for proving the next theorem. Notice that the theorem asserts that if the energy $r^{2-n}\int_{B_r(x_0)} e_\varepsilon(u_\varepsilon)\,dx$ is small enough then the gradient Du in the smaller ball $B_{r/2}(x_0)$ is controlled by C/r.

5.4 Small Energy Regularity Theorem

In this section we give an alternative proof for the small energy regularity theorem (also called the ε-regularity theorem) for minimizing harmonic maps to spheres. We shall use the penalty approximation method developed before.

Theorem 5.5. Suppose Ω is an open subset of $\mathbb{R}^n$, $k\ge 2$. Then there is a constant $\delta_0=\delta_0(n,k)>0$ such that if $u\in H^1(\Omega,\mathbb{R}^k)$ is a minimizing harmonic map to the sphere S^{k-1} and $r^{2-n}\int_{B_r(x_0)}\frac{1}{2}|Du|^2dx<\delta_0$ ($B_r(x_0)\Subset\Omega$), then

$$r\sup_{B_{r/2}(x_0)}|Du|\le C,$$

where $C=C(\delta_0)\to 0$ as $\delta_0\to 0$.

Proof. By re-scaling we can assume that $r=1$ and $x_0=0$. For each $\varepsilon>0$, let $v_\varepsilon : B_1(0)\setminus B_{1/2}(0)\to\mathbb{R}^k$ be a minimizer of the penalized energy functional

$$E_\varepsilon(v_\varepsilon)=\int_T\frac{1}{2}|Dv_\varepsilon|^2dx+\frac{1}{4\varepsilon^2}\int_T(1-|v_\varepsilon|^2)^2dx,$$

over $H^1(T,\mathbb{R}^k)$ subjected to the Dirichlet boundary condition $v_\varepsilon\equiv u$ on ∂T. Here $T=B_1(0)\setminus B_{1/2}(0)$. The existence of v_ε follows from theorems of chapter 2 and the final section of chapter 3. In fact, the problem is reduced to minimize E_ε over the closed affine subspace $u+H^1_0(T,\mathbb{R}^k)$ of $H^1(T,\mathbb{R}^k)$.

We can also conclude that there are sequences $\varepsilon_j \to 0$ and $v_j = v_{\varepsilon_j} \in H^1(T, \mathbb{R}^k)$ such that $v_j \to v^*$ strongly in $H^1(T, \mathbb{R}^k)$ for some $v^* : T \to S^{k-1}$ where $v^*|_{\partial T} \equiv u$ and $E_\varepsilon(v^*) \leq E_\varepsilon(u)$.

We define the function $v : B_1(0) \to S^{k-1}$ by

$$v(x) = \begin{cases} u(x) & \text{if } x \in B_{1/2}, \\ v^*(x) & \text{if } x \in B_1(0) \setminus B_{1/2}(0). \end{cases}$$

Therefore, $v \in H^1(B_1(0), S^{k-1})$ and $v \equiv u$ on $\partial B_1(0)$. Since $E_\varepsilon(v^*) \leq E_\varepsilon(u)$ and u is a minimizing harmonic map, we conclude that

$$\int_{B_1(0)} |Dv|^2 dx = \int_{B_1(0)} |Du|^2 dx.$$

Hence v is a minimizing harmonic map.

As v_j minimizes E_{ε_j}, $E_{\varepsilon_j}(v_j) \leq \int_T |Du|^2 dx \leq \delta_0$. By the monotonicity identity, we have

$$r^{2-n} \int_{B_r(y)} e_{\varepsilon_j}(v_j)\, dx \leq \delta_0$$

for all $|y| = 3/4$ and all $0 < r < 1/8$. If δ_0 is sufficiently small, we then obtain by the gradient estimate that

$$r \sup_{B_{r/2}(y)} |Dv_j| \leq C(\delta_0).$$

Recall $C(\delta_0) \to 0$ as $\delta_0 \to 0$. By considering a subsequence if necessary, we have that $Dv_j \to Dv^*$ pointwise almost everywhere. So $r \sup_{B_{r/2}(y)} |Dv^*| \leq C(\delta_0)$. By simple calculation, we obtain that

$$\operatorname*{osc}_{|y|=\frac{3}{4}} |v^*| \leq C(\delta_0).$$

By the theorem due to J.Jost, we get $\operatorname*{osc}_{B_{3/4}(0)} |v| \leq C(\delta_0)$ provided δ_0 is sufficiently small. Also, by the theorem of ([6]), we can conclude that v is smooth on $B_{3/4}(0)$ if δ_0 is small enough (so that the oscillation of v is small).

By Bochner type identity for harmonic maps to spheres (under the assumption that the maps are smooth), and the the corresponding gradient estimate, we can show that $r \sup_{B_{1/2}(0)} |Du| = r \sup_{B_{1/2}(0)} |Dv| \leq C(\delta_0)$ as we require. $\square$

For a minimizing harmonic map $u : \Omega \to S^{k-1}$, we define the *regular* set of u (i.e. the set of regular point) by

$$\mathrm{Reg}(u) = \{x \in \Omega : u \text{ is smooth in a nbd of } x\},$$

and the *singular* set of u, defined by $\mathrm{Sing}(u)$, to be the complement $\Omega \setminus \mathrm{Reg}(u)$.

A consequence of the small energy regularity theorem is the following criterion for regular points of harmonic maps.

Corollary 5.6. A point $x_0 \in \Omega$ is a regular point of a minimizing harmonic map $u : \Omega \to S^{k-1}$ if and only if $\lim_{r \to 0} r^{2-n} \int_{B_r(x_0)} |Du|^2 dx = 0$.

By the above corollary, we can estimate the size of the singular set of u. In fact, it can be shown that for n-dimensional Ω, the Hausdorff dimension of the singular set is at most $n-2$. For details, see ([8]).

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