ขอบเขตเบอร์รี-เอสซีนสำหรับผลบวกสุ่มของตัวแปรสุ่มที่ไม่มีการแจกแจงเดียวกัน

นายมงคล ตุ้นทัพไทย

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต
สาขาวิชาคณิตศาสตร์ ภาควิชาคณิตศาสตร์
คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย
ปีการศึกษา 2551

BERRY-ESSEEN BOUNDS FOR RANDOM SUMS OF NON-IDENTICALLY DISTRIBUTED RANDOM VARIABLES

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A Thesis Submitted in Partial Fulfillment of the Requirements
for the Degree of Master of Science Program in Mathematics

Department of Mathematics
Faculty of Science
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Academic Year 2008

Thesis Title	Berry-Esseen bounds for random sums of non-identically distributed random variables
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มงคล ตุ้นทัพไทย : ขอบเขตของเบอร์รี-เอสซีนสำหรับผลบวกสุ่มของตัวแปรสุ่มที่ไม่มีการแจกแจงเดียวกัน. (BERRY-ESSEEN BOUNDS FOR RANDOM SUMS OF NON-IDENTICALLY DISTRIBUTED RANDOM VARIABLES) อ.ที่ปรึกษาวิทยานิพนธ์หลัก : อ. ดร.ณัฐกาญจน์ ใจดี, อ. ที่ปรึกษาวิทยานิพนธ์ร่วม : ศ. ดร.กฤษณะ เนียมมณี, 42 หน้า.

ให้ $X_1, X_2, \ldots$ เป็นตัวแปรสุ่มที่เป็นอิสระต่อกันโดยมีค่าเฉลี่ยศูนย์และโมเมนต์ที่สามจำกัด และ N เป็นตัวแปรสุ่มซึ่งมีค่าเป็นจำนวนเต็มบวกโดยที่ $N, X_1, X_2, \ldots$ เป็นอิสระต่อกัน ในงานนี้เราใช้วิธีอสมการความเข้มข้นของวิธีการของสไตน์เพื่อให้ขอบเขตของเบอร์รี-เอสซีน แบบสม่ำเสมอและแบบไม่สม่ำเสมอสำหรับผลบวกสุ่ม $X_1 + X_2 + \cdots + X_N$ ในกรณีที่ตัวแปรสุ่ม $X_1, X_2, \ldots$ ไม่จำเป็นต้องมีการแจกแจงเดียวกัน

ภาควิชา...........คณิตศาสตร์.........	ลายมือชื่อนิสิต..
สาขาวิชา.........คณิตศาสตร์.........	ลายมือชื่อ อ.ที่ปรึกษาวิทยานิพนธ์หลัก..................................
ปีการศึกษา............2551..............	ลายมือชื่อ อ.ที่ปรึกษาวิทยานิพนธ์ร่วม..................................

5072419623 : MAJOR MATHEMATICS

KEYWORDS : BERRY-ESSEEN BOUND / STEIN'S METHOD / NORMAL APPROXIMATION / CONCENTRATION INEQUALITY / RANDOM SUMS

MONGKHON TUNTAPTHAI : BERRY-ESSEEN BOUNDS FOR RANDOM SUMS OF NON-IDENTICALLY DISTRIBUTED RANDOM VARABLES. ADVISOR : NATTAKARN CHAIDEE, Ph.D., CO-ADVISOR : PROF. KRITSANA NEAMMANEE, Ph.D., 42 pp.

Let $X_1, X_2, \ldots$ be independent random variables with mean zero and finite third moment and N a positive integral-valued random variable such that $N, X_1, X_2, \ldots$ are independent. In this work, we use a concentration inequality approach of Stein's method to establish uniform and non-uniform Berry-Esseen bounds for random sums $X_1 + X_2 + \cdots + X_N$ in the case that $X_1, X_2, \ldots$ are not necessarily identically distributed random variables.

Department	: ....Mathematics....	Student's Signature :	....
Field of Study	: ....Mathematics....	Advisor's Signature :	....
Academic Year	: ....2008..........	Co-advisor's Signature :	....

ACKNOWLEDGEMENTS

I would like to express my gratitude to Dr. Nattakarn Chaidee, my thesis advisor, for his kindness and suggestions. Thanks are also due for generous comments provided by Professor Dr. Kritsana Neammanee, my thesis co-advisor, who has helped improving my work. In addition, I would like to thank Associate Professor Dr. Wicharn Lewkeeratiyutkul, Dr. Kittipat Wong, and Dr. Kittipong Laipaporn as well as my thesis committee, for constructive criticism and otherwise.

Finally, I am very thankful to my family for their encouragement throughout my graduate study.

CONTENTS

CHAPTER I
INTRODUCTION

In view of the celebrated theorem in probability theory, the central limit theorem (CLT) says that the distribution function of the sum of a large number of independent identically distributed random variables, suitably normalized to have zero mean and unit variance, converges to the standard normal distribution function.

In 1941 and 1945, Berry [2] and Esseen [5] applied CLT in particular, further assuming the finite third moment to investigate bounds of the difference between the distribution function above and its limit. Since then, many authors, e.g. [11], [13], [7], improve the bound in the Berry-Esseen theorem and the method that they used is Fourier method. In 2001, Chen and Shao [4] used a new method, Stein's method, to establish Berry-Esseen bounds for the sum of independent and non-identically distributed random variables without assuming the existence of third moment.

In many situations, we would like to apply CLT to randomly indexed sums. The CLT for random sums was obtained by many mathematicians, e.g. [8], [10], [9], [14], and they still used Fourier method. In this work, we give both uniform and non-uniform Berry-Esseen bounds for random sums of independent and not necessarily identically distributed random variables with finite third moment by using the concentration inequality approach of Stein's method. A uniform bound is introduced in Chapter III and also, a non-uniform bound is in Chapter IV.

1.1 Uniform Berry-Esseen Bounds

In 1977, Batirov, Manevich and Nagaev [1] obtained a uniform Berry-Esseen bound for random sums by using Fourier method. The following is their result.

Theorem 1.1. [1] *Let $X_1, X_2, \ldots$ be independent and not necessarily identically distributed random variables with $EX_i = 0$, $EX_i^2 = \sigma_i^2$ and $E|X_i|^3 = \gamma_i < \infty$ and N a positive integral-valued random variable such that $N, X_1, X_2, \ldots$ are independent. Denote $s_N^2 = \sum_{i=1}^{N} \sigma_i^2$, $\beta_N = \sum_{i=1}^{N} \gamma_i$ and $W = \dfrac{X_1 + X_2 + \cdots + X_N}{\sqrt{Es_N^2}}$. Then there exists a constant C such that*

$$\sup_{x\in\mathbb{R}} |P(W \le x) - \Phi(x)| \le C\Big(\frac{E\beta_N}{(Es_N^2)^{3/2}} + \frac{\sqrt{E|s_N^2 - Es_N^2|^2}}{Es_N^2}\Big)$$

where Φ is the standard normal distribution function.

In 2008, N. Chaidee and K. Neammanee [3] applied Stein's method to obtain the constant C in Theorem 1.1 in the case that $X_1, X_2, \ldots$ are identically distributed random variables.

Theorem 1.2. [3] *Let $X_1, X_2, \ldots$ be independent identically distributed random variables with $EX_1 = 0$, $EX_1^2 = \sigma^2$ and $E|X_1|^3 = \gamma < \infty$ and N a positive integral-valued random variable with $EN^{3/2} < \infty$ such that $N, X_1, X_2, \ldots$ are independent. Denote $W = \dfrac{X_1 + X_2 + \cdots + X_N}{\sqrt{EN}\sigma}$. Then*

$$\sup_{x\in\mathbb{R}} |P(W \le x) - \Phi(x)| \le 10.32\delta + \frac{0.125}{\sqrt{EN}} + \frac{1.5\delta EN^{3/2}}{(EN)^{3/2}} + \frac{E|N - EN|}{EN}$$

where $\delta = \dfrac{\gamma}{\sqrt{EN}\sigma^3}$.

In this work, we generalize Theorem 1.2 in the case that $X_1, X_2, \ldots$ are not necessarily identically distributed random variables. We use the concentration inequality approach in Chen and Shao [4] to approximate a uniform bound for W satisfying condition in Theorem 1.1. The following are our main results.

Theorem 1.3. *Let $X_1, X_2, \ldots$ be independent and not necessarily identically distributed random variables with $EX_i = 0$, $EX_i^2 = \sigma_i^2$ and $E|X_i|^3 = \gamma_i < \infty$ and N a positive integral-valued random variable such that $N, X_1, X_2, \ldots$ are independent. Denote $s_N^2 = \sum_{i=1}^{N} \sigma_i^2$, $\beta_N = \sum_{i=1}^{N} \gamma_i$ and $W = \frac{X_1 + X_2 + \cdots + X_N}{\sqrt{Es_N^2}}$. Then*

$$\sup_{x \in \mathbb{R}} |P(W \le x) - \Phi(x)| \le \frac{6.875E[\delta_N \sqrt{s_N^2}]}{\sqrt{Es_N^2}} + \frac{1.5E[\delta_N (s_N^2)^{3/2}]}{(Es_N^2)^{3/2}} + \frac{0.945E[\delta_N s_N^2]}{Es_N^2} + 100E\delta_N^2 + \frac{E|s_N^2 - Es_N^2|}{Es_N^2}$$

where $\delta_N = \frac{\beta_N}{s_N^2 \sqrt{Es_N^2}}$.

Corollary 1.4. *Under the conditions in Theorem 1.2, we have*

$$\sup_{x \in \mathbb{R}} |P(W \le x) - \Phi(x)| \le 7.82\delta + (\frac{100\gamma^2}{\sqrt{EN}\sigma^6})\frac{1}{\sqrt{EN}} + \frac{1.5\delta EN^{3/2}}{(EN)^{3/2}} + \frac{E|N - EN|}{EN}$$

where $\sigma^2 = EX_1^2$, $\gamma = E|X_1|^3$ and $\delta = \frac{\gamma}{\sqrt{EN}\sigma^3}$.

Observe that in the case of EN is large, the bound in Corollary 1.4 is better than the bound in Theorem 1.2.

1.2 Non-Uniform Berry-Esseen Bounds

In 2008, N. Chaidee and K. Neammanee [3] gave a non-uniform Berry-Esseen bound for random sums. The following is their result.

Theorem 1.5. [3] *Assume the conditions in Theorem 1.2 and that $EN^5 < \infty$. Then there exists a constant C such that for every real number x,*

$$|P(W \le x) - \Phi(x)| \le \frac{C\delta}{(1+|x|)^3}\Big(1 + \frac{EN^5}{(EN)^5}\Big) + \frac{CE|N^2 - EN^2|}{(1+|x|)^2 (EN)^2}$$

where $\sigma^2 = EX_1^2$, $\gamma = E|X_1|^3$ and $\delta = \frac{\gamma}{\sqrt{EN}\sigma^3}$.

In this work, we generalize Theorem 1.5 in the case that $X_1, X_2, \ldots$ are not necessarily identically distributed random variables. The followings are our main results.

Theorem 1.6. *Assume the conditions in Theorem 1.3 and N is a positive integral-valued random variable such that $N \geq 5$. If $s_n^2 \geq 4 \max_{1\leq i\leq n} \sigma_i^2$ for all positive integer $n \geq 5$, then there exists a constant C such that for every real number x,*

$$\begin{aligned}
&|P(W \leq x) - \Phi(x)| \\
&\leq \frac{C}{(1+|x|)^3}\big(E[\frac{\delta_N}{(s_N^2)^2}](Es_N^2)^2 + \frac{E[\delta_N(s_N^2)^5]}{(Es_N^2)^5}\big) \\
&\quad + \frac{C}{(1+|x|)^2}\big(E[\frac{\delta_N^2}{(s_N^2)^2}](Es_N^2)^2 + \frac{E[\delta_N^2(s_N^2)^5]}{(Es_N^2)^5} + \frac{E|(s_N^2)^2 - (Es_N^2)^2|}{(Es_N^2)^2}\big)
\end{aligned}$$

where $\delta_N = \dfrac{\beta_N}{s_N^2\sqrt{Es_N^2}}$.

Corollary 1.7. *Assume the conditions in Theorem 1.5 and N is a positive integral-valued random variable such that $N \geq 5$. Then there exists a constant C such that for every real number x,*

$$\begin{aligned}
&|P(W \leq x) - \Phi(x)| \\
&\leq \frac{C\delta}{(1+|x|)^3}\big(1 + \frac{EN^5}{(EN)^5}\big) + \frac{C\delta^2}{(1+|x|)^2}\big(1 + \frac{EN^5}{(EN)^5}\big) + \frac{CE|N^2 - (EN)^2|}{(1+|x|)^2(EN)^2}
\end{aligned}$$

where $\sigma^2 = EX_1^2$, $\gamma = E|X_1|^3$ *and* $\delta = \dfrac{\gamma}{\sqrt{EN}\sigma^3}$.

CHAPTER II

PRELIMINARIES

This chapter contains fundamental concept in probability and the idea of Stein's method for normal approximation.

2.1 Fundamental Concept in Probability

Probability theory is a part of mathematics concerned with the analysis of random experiments. The result of an experiment is called its **outcome**. An experiment is called **random** if its outcome cannot be predicated with certainty.

A **sample space** is the set of description of all the possible outcomes or **elementary events** of an experiment and is denoted by Ω. The subset A of Ω that some elementary event in its occurs is said to be **event**. The collection of events is a subcollection $\mathcal{F}$ of the collection of all subsets of Ω.

Given a measurable space $(\Omega, \mathcal{F})$, a measure P on $(\Omega, \mathcal{F})$ is called a **probability measure** if $P(\Omega) = 1$. In such a case, the measure space $(\Omega, \mathcal{F}, P)$ is said to be a **probability space**. Now, we can associate a probability space $(\Omega, \mathcal{F}, P)$ with any experiment, and all questions associated with the experiment can be formulated in terms of $P(A)$ the value of the **probability of event** A in this space.

There are circumstances in which the experimenter is only interested in some numerical aspects of the elementary event. After the experiment is done and the outcome $\omega \in \Omega$ is known, it is convenient to set up special numerical functions defined on the sample space and describing these aspects. In general, this numerical value is more likely to lie in certain subsets of the real line $\mathbb{R}$ than in certain others.

Let $(\Omega, \mathcal{F}, P)$ be a probability space. A function $X : \Omega \to \mathbb{R}$ is said to be a **random variable** if X is $\mathcal{F}$-measurable i.e. $X^{-1}(B) = \{\omega \in \Omega \mid X(\omega) \in B\} \in \mathcal{F}$ for all Borel set B of $\mathbb{R}$.

We shall use the notation $P(X \in B)$ in place of $P(\{\omega \in \Omega \mid X(\omega) \in B\})$ will be written as. In the case where $B = (-\infty, a]$ or $[b, \infty)$ or $[a, b]$, $P(X \in B)$ is denoted by $P(X \leq a)$ or $P(X \geq b)$ or $P(a \leq X \leq b)$, respectively.

Lex X be a random variable. A function $F : \mathbb{R} \to [0, 1]$ which is defined by

$$F(x) = P(X \leq x)$$

is called the **distribution function** of X.

A random variable X is said to be a **discrete random variable** if it takes values in some countable subset of $\mathbb{R}$. The discrete random variable X has **mass function** $f : \mathbb{R} \to [0, 1]$ given by $f(x) = P(X \leq x)$.

A random variable X with the distribution function F is called a **continuous random variable** if F can be written in the form

$$F(x) = \int_{-\infty}^{x} f(t)dt$$

for some integrable function $f : \mathbb{R} \to [0, \infty)$ called the **density function** of X.

We shall now list some examples of random variables. A type of example of discrete random variables is the **indicator function** of an event A in $\mathcal{F}$ given by

$$I_A(w) = \begin{cases} 1 & \text{if } w \in A, \\ 0 & \text{if } w \notin A. \end{cases}$$

A random variable X is said to be a **normal** random variable with parameter μ and $\sigma > 0$, denoted by $X \sim N(\mu, \sigma)$, if its density function f satisfies that

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}.$$

In the special case, if $Z \sim N(0, 1)$ then Z is said to be a **standard normal** random variable. We also denote the distribution function by the letter Φ. Thus,

$$P(Z \leq z) = \Phi(z) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{z} e^{-\frac{y^2}{2}} dy.$$

The followings are useful fact, for $z > 0$

$$1 - \Phi(z) = \frac{1}{\sqrt{2\pi}} \int_z^\infty e^{-\frac{y^2}{2}} dt \leq \frac{1}{\sqrt{2\pi}} \int_z^\infty \frac{y}{z} e^{-\frac{y^2}{2}} dt = \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi} z}, \tag{2.1}$$

and for $z < 0$, we have

$$\Phi(z) = 1 - \Phi(-z) \leq \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}(-z)} = \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}|z|}. \tag{2.2}$$

Let $(\Omega, \mathcal{F}, P)$ be a probability space, and for each i from the index set $\mathcal{I}$, let $\mathcal{F}_i$ be a sub σ-algebra of $\mathcal{F}$. A collection of σ-algebras $\{\mathcal{F}_i : i \in \mathcal{I}\}$ is said to be **independent** if every finite subset $\{i_1, i_2, \ldots, i_k\}$ of $\mathcal{I}$, we have

$$P(\bigcap_{m=1}^{k} A_{i_m}) = \prod_{m=1}^{k} P(A_{i_m})$$

where $A_{i_m} \in \mathcal{F}_{i_m}$ for all m.

For each $i \in \mathcal{I}$, let $\mathcal{E}_i$ be any collection of events in $\mathcal{F}$ i.e. $\mathcal{E}_i \subseteq \mathcal{F}$. We will say that $\{\mathcal{E}_i : i \in \mathcal{I}\}$ is **independent** if the collection of σ-algebras $\{\sigma(\mathcal{E}_i) : i \in \mathcal{I}\}$ is independent, where $\sigma(\mathcal{E}_i)$ is the smallest σ-algebra which contains $\mathcal{E}_i$.

A collection of random variables $\{X_i : i \in \mathcal{I}\}$ is said to be **independent** if the collection of σ-algebras $\{\sigma(X_i) : i \in \mathcal{I}\}$ is independent, where

$$\sigma(X) = \{X^{-1}(B) \mid B \text{ is a Borel set of } \mathbb{R}\}.$$

Remark 2.1. *Random variables $X_1, X_2, \ldots, X_n$ are independent if for any Borel sets $B_1, B_2, \ldots, B_n$, we have*

$$P\Big(\bigcap_{i=1}^{n} \{X_i \in B_i\}\Big) = \prod_{i=1}^{n} P(X_i \in B_i).$$

Let X be a random variable on a probability space $(\Omega, \mathcal{F}, P)$. If $\displaystyle\int_\Omega |X| dP < \infty$, we define its **expected value** to be

$$E(X) = \int_\Omega X dP.$$

Proposition 2.2. *Let X and Y be random variables such that $E(|X|) < \infty$ and $E(|Y|) < \infty$. Then*

1. *$E(aX + bY) = aE(X) + bE(Y)$ for all $a, b \in \mathbb{R}$.*

2. *If $X \leq Y$ then $E(X) \leq E(Y)$.*

3. *If X and Y are independent, then $E(XY) = E(X)E(Y)$.*

Proposition 2.3. *Let $f_1, f_2, \ldots, f_n$ be bounded measurable real-value functions. If random variables $X_1, X_2, \ldots, X_n$ are independent, then we have*

$$E(f_1 \circ X_1 \cdots f_n \circ X_n) = E(f_1 \circ X_1) \cdots E(f_n \circ X_n).$$

Let X be a random variable with $E(|X|^k) < \infty$. Then $E(|X|^k)$ is called the **k-th moment** of X about the origin and $E[(X - E(X))^k]$ is said to be the **k-th moment** of X about the mean. We call the second moment of X about the mean, the **variance** of X, and we write

$$\text{Var}(X) = E[(X - E(X))^2].$$

We note that the following expression of the variance in term of two moments.

$$\text{Var}(X) = E(X^2) - E^2(X).$$

Remark 2.4. *If $X \sim N(\mu, \sigma)$, then $E(X) = \mu$ and $\text{Var}(X) = \sigma^2$.*

Proposition 2.5. *Let X and Y be random variables such that $E(|X|^2) < \infty$ and $E(|Y|^2) < \infty$. Then we have for any real numbers a and b,*

$$\text{Var}(aX + bY) = a^2 \text{Var}(X) + b^2 \text{Var}(Y).$$

Let X and Y be random variables. We will note famous inequalities which are written as follows.

Theorem 2.6 (Hölder's inequality). *If $p, q \geq 1$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$, then*

$$E(|XY|) \leq E^{\frac{1}{p}}(|X|^p) E^{\frac{1}{q}}(|Y|^q).$$

Theorem 2.7 (Chebyshev's inequality).

$$P(|X - E(X)| \geq \varepsilon) \leq \frac{\mathrm{Var}(X)}{\varepsilon^2} \quad \textit{for all } \varepsilon > 0.$$

Theorem 2.8. *If* $p \geq 1$*, then for all* $\varepsilon > 0$

$$P(|X - E(X)| \geq \varepsilon) \leq \frac{E(|X - EX|^p)}{(EX - c)^p}.$$

Theorem 2.9 (Rosenthal's inequality). *If* $X_1, X_2, \ldots, X_n$ *are independent random variables such that* $E(X_i) = 0$ *for all* i*, then for* $p \geq 2$*,*

$$E\Big|\sum_{i=1}^{n} X_i\Big|^p \leq C(p)\Big\{\sum_{i=1}^{n} E|X_i|^p + \Big(\sum_{i=1}^{n} EX_i^2\Big)^{p/2}\Big\}$$

where $C(p)$ *is a positive constant depending only on* p*.*

Let $(\Omega, \mathcal{F}, P)$ be a probability space and X a random variable with $E(|X|) < \infty$. Let $\mathcal{D}$ be a sub σ-algebra of $\mathcal{F}$. Define a probability measure $P_{\mathcal{D}} : \mathcal{D} \to [0, 1]$ by

$$P_{\mathcal{D}}(E) = P(E) \text{ for any } E \in \mathcal{D},$$

and a sign-measure $\mathcal{Q}_X : \mathcal{D} \to \mathbb{R}$ by

$$\mathcal{Q}_X(E) = \int_E X dP \text{ for any } E \in \mathcal{D}.$$

Then, the definition of the integral implies that $\mathcal{Q}_X$ is absolutely continuous with respect to $P_{\mathcal{D}}$. Applying Radon-Nikodym theorem, give the existence of a unique measurable function $E^{\mathcal{D}}(X)$ on $(\Omega, \mathcal{D})$ such that for any $E \in \mathcal{D}$,

$$\int_E E^{\mathcal{D}}(X) dP_{\mathcal{D}} = \mathcal{Q}_X(E) = \int_E X dP.$$

The $\mathcal{D}$-measurable function $E^{\mathcal{D}}(X)$ is called the **conditional expectation** of X with respect to $\mathcal{D}$. Moreover, for any random variables X and Y on the same probability space $(\Omega, \mathcal{F}, P)$ such that $E(|X|) < \infty$, we will denote $E^{\sigma(Y)}(X)$ by $E^Y(X)$.

Theorem 2.10. *Let X be a random variable on a probability space $(\Omega, \mathcal{F}, P)$ such that $E(|X|) < \infty$. Then the followings hold for any sub σ-algebra $\mathcal{D}$ of $\mathcal{F}$.*

1. *If X is random variable on $(\Omega, \mathcal{D}, P_{\mathcal{D}})$, then $E^{\mathcal{D}}(X) = X$ a.s.$[P_{\mathcal{D}}]$.*

2. *$E^{\mathcal{F}}(X) = X$ a.s.$[P]$.*

3. *If $\sigma(X)$ and $\mathcal{D}$ are independent, then $E^{\mathcal{D}}(X) = E(X)$ a.s.$[P_{\mathcal{D}}]$.*

Theorem 2.11. *Let X and Y be random variables on the same probability space $(\Omega, \mathcal{F}, P)$ such that $E(|X|)$ and $E(|Y|)$ are finite. Then for any sub σ-algebra $\mathcal{D}$ of $\mathcal{F}$ the followings hold.*

1. *If $X \leq Y$, then $E^{\mathcal{D}}(X) \leq E^{\mathcal{D}}(Y)$ a.s. $[P_{\mathcal{D}}]$.*

2. *$E^{\mathcal{D}}(aX + bY) = aE^{\mathcal{D}}(X) + bE^{\mathcal{D}}(X)$ a.s. $[P_{\mathcal{D}}]$ for any $a, b \in \mathbb{R}$.*

Theorem 2.12. *Let X and Y be random variables on the same probability space $(\Omega, \mathcal{F}, P)$ such that $E(|XY|)$ and $E(|Y|)$ are finite and $\mathcal{D}_1, \mathcal{D}_2$ be sub σ-algebras of $\mathcal{F}$. If X is a random variable with respect to $\mathcal{D}_1$, then*

1. *$E^{\mathcal{D}_1}(XY) = XE^{\mathcal{D}_1}(Y)$ a.s. $[P_{\mathcal{D}_1}]$.*

2. *$E^{\mathcal{D}_2}(XY) = E^{\mathcal{D}_2}(XE^{\mathcal{D}_1}(Y))$ a.s. $[P_{\mathcal{D}_2}]$.*

For any event A on $\mathcal{F}$, we define the **conditional probability of A given $\mathcal{D}$** by

$$P(A \mid \mathcal{D}) = E^{\mathcal{D}}(I_A).$$

2.2 Stein's Method for Normal Approximation

In 1972, Stein introduced a powerful and general method for obtaining the Berry-Esseen bounds for a sum of dependent random variables. In previous work, tools such as Fourier methods were used to derive such estimates in Berry [2] and Esseen [5]. Stein's technique is differed because he used a characterization of the standard normal distribution to construct the bounds.

Let Z be a standard normally distributed random variable, and let $\mathcal{C}_{bd}$ be the set of continuous and piecewise continuously differentiable functions $f : \mathbb{R} \to \mathbb{R}$ with $E|f'(Z)| < \infty$. Stein's method rests on the followings characterization.

Lemma 2.13. *Let W be a real valued random variables. Then W has a standard normal distribution if and only if for all $f \in \mathcal{C}_{bd}$,*

$$Ef'(W) = EWf(W).$$

For a real valued measurable function h with $E|N(h)| < \infty$,

$$f'(w) - wf(w) = h(w) - N(h),\ w \in \mathbb{R} \tag{2.3}$$

is **Stein's equation for normal distribution** where

$$N(h) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} h(x) e^{-\frac{x^2}{2}} dx.$$

If $h_x = I_{(-\infty,x]}$, then the solution $f = f_x$ of Stein's equation (2.3) is given by

$$f_x(w) = \begin{cases} \sqrt{2\pi} e^{\frac{w^2}{2}} \Phi(w)\big(1 - \Phi(x)\big) & \text{if } \ w \le x \\ \sqrt{2\pi} e^{\frac{w^2}{2}} \Phi(x)\big(1 - \Phi(w)\big) & \text{if } \ w > x. \end{cases} \tag{2.4}$$

The lemmas in this section are basic properties of f_x which is defined by (2.4) and have been introduced by Chen and Shao [4] as the followings.

Lemma 2.14. *For all real numbers w and v, we have*

1. $0 \le f_x(w) \le \min\left\{\frac{\sqrt{2\pi}}{4}, \frac{1}{|x|}\right\}$;
2. $|wf_x(w)| \le 1$;
3. $|f_x'(w)| \le 1$;
4. $|f_x'(w) - f_x'(v)| \le 1$.

Lemma 2.15. *For each $s, t, x \in \mathbb{R}$, we have*

1.

$$f_x'(w+s) - f_x'(w+t) \le \begin{cases} 1 & \text{if } w+s \le x, w+t > x \\ (|w| + 0.63)(|s| + |t|) & \text{if } s \ge t \\ 0 & \text{otherwise.} \end{cases}$$

2.

$$f_x'(w+s) - f_x'(w+t) \ge \begin{cases} -1 & \text{if } w+s > x, w+t \le x \\ -(|w| + 0.63)(|s| + |t|) & \text{if } s < t \\ 0 & \text{otherwise.} \end{cases}$$

For given $x > 0$, let

$$g(w) = (wf_x(w))'.$$

Then

$$g(w) = \begin{cases} \left\{\sqrt{2\pi}(1+w^2)e^{\frac{w^2}{2}}\Phi(w) + w\right\}\big(1 - \Phi(x)\big) & \text{if } w \le x \\ \left\{\sqrt{2\pi}(1+w^2)e^{\frac{w^2}{2}}\big(1 - \Phi(w)\big) - w\right\}\Phi(x) & \text{if } w > x. \end{cases} \tag{2.5}$$

Lemma 2.16. *We have*

1. $g(w) \ge 0$ *for all* $w \in \mathbb{R}$.
2. $g(w) \le 2\big(1 - \Phi(x)\big)$ *for all* $w \le 0$.
3. $g(w) \le \dfrac{2}{1+w^3}$ *for all* $w > x$.
4. g *is increasing on* $[0, x)$.

Lemma 2.17. *For $s, t, u \in \mathbb{R}$, we have*

$$\left|f_x'(w+s) - f_x'(w+t) - \int_t^s g(w+u)du\right| \le I(x - \max\{s,t\} < w \le x - \min\{s,t\}).$$

CHAPTER III

A UNIFORM BOUND FOR RANDOM SUMS OF NON-IDENTICALLY DISTRIBUTED RANDOM VARIABLES

In this chapter, we apply Stein's method for the standard normal distribution function and concentration inequality to approximate a uniform bound for random sums. The followings are our main results.

Theorem 3.1. *Let $X_1, X_2, \dots$ be independent and not necessarily identically distributed random variables with $EX_i = 0$, $EX_i^2 = \sigma_i^2$ and $E|X_i|^3 = \gamma_i < \infty$ and N a positive integral-valued random variable such that $N, X_1, X_2, \dots$ are independent. Denote $s_N^2 = \sum_{i=1}^{N} \sigma_i^2$, $\beta_N = \sum_{i=1}^{N} \gamma_i$ and $W = \frac{X_1 + X_2 + \cdots + X_N}{\sqrt{Es_N^2}}$. Then*

$$\sup_{x\in\mathbb{R}} |P(W \le x) - \Phi(x)| \le 6.875 \frac{E[\delta_N \sqrt{s_N^2}]}{\sqrt{Es_N^2}} + 1.5 \frac{E[\delta_N (s_N^2)^{3/2}]}{(Es_N^2)^{3/2}} + 0.945 \frac{E[\delta_N s_N^2]}{Es_N^2} + 100 E\delta_N^2 + \frac{E|s_N^2 - Es_N^2|}{Es_N^2}$$

where $\delta_N = \frac{\beta_N}{s_N^2 \sqrt{Es_N^2}}$.

Corollary 3.2. *Let $X_1, X_2, \dots$ be independent identically distributed random variables with zero mean and N a positive integral-valued random variable such that $N, X_1, X_2, \dots$ are independent. Denote $W = \frac{X_1 + X_2 + \cdots + X_N}{\sqrt{EN}\sigma}$. Then*

$$\sup_{x\in\mathbb{R}} |P(W \le x) - \Phi(x)| \le 7.82\delta + (\frac{100\gamma^2}{\sqrt{EN}\sigma^6}) \frac{1}{\sqrt{EN}} + \frac{1.5\delta EN^{3/2}}{(EN)^{3/2}} + \frac{E|N - EN|}{EN}$$

where $\sigma^2 = EX_1^2$, $\gamma = E|X_1|^3$ and $\delta = \frac{\gamma}{\sqrt{EN}\sigma^3}$.

3.1 Uniform Concentration Inequality

Throughout this work, we construct some notations for a positive integer n by

$$\delta_n = \frac{\beta_n}{s_n^2\sqrt{Es_N^2}}$$

where

$$s_n^2 = \sum_{i=1}^{n} \sigma_i^2 \quad \text{and} \quad \beta_n = \sum_{i=1}^{n} \gamma_i.$$

For all $i \in \{1, 2, \ldots, n\}$, we denote

$$Y_i = \frac{X_i}{\sqrt{Es_N^2}}, \quad W_n = \sum_{i=1}^{n} Y_i, \quad W_n^{(i)} = W_n - Y_i.$$

Then $EY_i = 0$,

$$EW_n^2 = \sum_{i=1}^{n} EY_i^2 = \frac{s_n^2}{Es_N^2} \quad \text{and} \quad \sum_{i=1}^{n} E|Y_i|^3 = \frac{\beta_n}{(Es_N^2)^{3/2}} = \frac{\delta_n s_n^2}{Es_N^2}. \tag{3.1}$$

In this section, we use the concepts from Chen and Shao [4] to approximate a bound for $P(a \leq W_n^{(i)} \leq b)$ where $a < b$, which will play the key role in the proof of a concentration inequality as follows.

Proposition 3.3. *For all positive integer n and $a < b$, we have*

$$P(a \leq W_n^{(i)} \leq b) \leq \frac{2.5Es_N^2}{s_n^2}\Big\{\Big(\frac{1}{2}(b-a) + \delta_n\Big)\sqrt{\frac{s_n^2}{Es_N^2}} + 40\delta_n^2 + \delta_n\sqrt{\frac{\sigma_i^2}{Es_N^2}}\Big\}$$

for all $i = 1, 2, \ldots, n$.

Proof. Define $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(t) = \begin{cases} -\frac{1}{2}(b-a) - \delta_n & \text{for } t < a - \delta_n \\ t - \frac{1}{2}(b+a) & \text{for } a - \delta_n \leq t \leq b + \delta_n \\ \frac{1}{2}(b-a) + \delta_n & \text{for } t \geq b + \delta_n \end{cases}$$

and $M : \mathbb{R}^2 \to \mathbb{R}$ by

$$M(w, t) = w[I(-w \leq t \leq 0) - I(0 < t \leq -w)].$$

For given w, by integrating of $M(w,\cdot)$ over $|t|<y$ we get

$$\int_{|t|<y} M(w,t)dt = |w|\min\{y,|w|\}. \tag{3.2}$$

Next we note that f_n is a non-decreasing function satisfying

$$f_n'(t) = \begin{cases} 1 & \text{if } a-\delta_n < t < b+\delta_n \\ 0 & \text{if } t < a-\delta_n \text{ or } b+\delta_n > t, \end{cases} \tag{3.3}$$

and

$$|EW_n^{(i)} f_n(W_n^{(i)})| \le \big(\frac{1}{2}(b-a)+\delta_n\big)E|W_n^{(i)}|. \tag{3.4}$$

Since Y_j and $W_n^{(i)} - Y_j$ are independent for $j \neq i$, $EY_j = 0$ and $M(w,t) \ge 0$ for all $w, t \in \mathbb{R}$,

$$\begin{aligned}
EW_n^{(i)} f_n(W_n^{(i)}) &= \sum_{\substack{j=1\\ j\neq i}}^n EY_j f_n(W_n^{(i)}) \\
&= \sum_{\substack{j=1\\ j\neq i}}^n E[Y_j\{f_n(W_n^{(i)}) - f_n(W_n^{(i)} - Y_j)\}] \\
&= \sum_{\substack{j=1\\ j\neq i}}^n E\Big(Y_j \int_{-Y_j}^0 f_n'(W_n^{(i)}+t)dt\Big) \\
&= \sum_{\substack{j=1\\ j\neq i}}^n E\int_{-Y_j}^0 f_n'(W_n^{(i)}+t)M(Y_j,t)dt \\
&= \sum_{\substack{j=1\\ j\neq i}}^n E\int_{-\infty}^{\infty} f_n'(W_n^{(i)}+t)M(Y_j,t)dt \\
&= \sum_{\substack{j=1\\ j\neq i}}^n E\int_{-\infty}^{\infty} I(a-\delta_n < W_n^{(i)}+t < b+\delta_n)M(Y_j,t)dt \quad \text{(by (3.3))} \\
&\ge \sum_{\substack{j=1\\ j\neq i}}^n E\int_{|t|\le\delta_n} I(a-\delta_n < W_n^{(i)}+t < b+\delta_n)M(Y_j,t)dt \\
&\ge \sum_{\substack{j=1\\ j\neq i}}^n E\Big(I(a \le W_n^{(i)} \le b)\int_{|t|<\delta_n} M(Y_j,t)dt\Big) \\
&= E\Big[I(a \le W_n^{(i)} \le b)\sum_{\substack{j=1\\ j\neq i}}^n |Y_j|\min\{\delta_n,|Y_j|\}\Big]. \quad \text{(by (3.2))}
\end{aligned}$$

Therefore,

$$EW_n^{(i)} f_n(W_n^{(i)}) \geq E[I(a \leq W_n^{(i)} \leq b)S] - P(a \leq W_n^{(i)} \leq b)E[|Y_i| \min\{\delta_n, |Y_i|\}], \tag{3.5}$$

where

$$S = \sum_{j=1}^{n} |Y_j| \min\{\delta_n, |Y_j|\}.$$

From the fact that $\min\{a, b\} \geq b - \dfrac{b^2}{4a}$ for all $a, b > 0$ ([4], p.238) and (3.1), we obtain

$$ES \geq \sum_{j=1}^{n} E(Y_j^2 - \frac{|Y_j|^3}{4\delta_n}) \geq \sum_{j=1}^{n} EY_j^2 - \frac{1}{2\delta_n} \sum_{j=1}^{n} E|Y_j|^3 = \frac{s_n^2}{2Es_N^2}. \tag{3.6}$$

Moreover, we have

$$\begin{aligned}
ES^2 &= \sum_{j=1}^{n} E[|Y_j| \min\{\delta_n, |Y_j|\}]^2 + \sum_{\substack{j=1 \\ j\neq k}}^{n} \sum_{k=1}^{n} E[|Y_k| \min\{\delta_n, |Y_k|\}|Y_j| \min\{\delta_n, |Y_j|\}] \\
&\leq \sum_{j=1}^{n} E[|Y_j| \min\{\delta_n, |Y_j|\}]^2 + \Big[\sum_{j=1}^{n} E|Y_j| \min\{\delta_n, |Y_j|\}\Big]^2 \\
&\leq \delta_n^2 \sum_{j=1}^{n} EY_j^2 + E^2 S \\
&= \frac{\delta_n^2 s_n^2}{Es_N^2} + E^2 S. \qquad (3.7)
\end{aligned}$$

By Chebyshev's inequality (Theorem 2.7) implies that for $0 < c < ES$,

$$E(1 - S/c)I(S \leq c) \leq P(S \leq c) \leq P(|S - ES| \geq ES - c) \leq \frac{\mathrm{Var}(S)}{(ES - c)^2}. \tag{3.8}$$

From (3.6) we can choose a constant $c = \dfrac{0.4s_n^2}{Es_N^2}$ in (3.8), which implies that

$$\begin{aligned}
E(1 - \frac{2.5SEs_N^2}{s_n^2})I(S \leq \frac{0.4s_n^2}{Es_N^2}) &\leq \frac{100(Es_N^2)^2}{(s_n^2)^2} \mathrm{Var}(S) \\
&= \frac{100(Es_N^2)^2}{(s_n^2)^2}(ES^2 - E^2 S) \\
&\leq \frac{100\delta_n^2 Es_N^2}{s_n^2} \qquad (3.9)
\end{aligned}$$

where we have used (3.7) in the last inequality.

From the fact that ([4], p.238)

$$I(a \le w \le b)y \ge c\big\{I(a \le w \le b) - (1 - y/c)I(y \le c)\big\}$$

for $a < b$, $y \ge 0$ and $c > 0$, we have

$$\begin{aligned} & E\big(I(a \le W_n^{(i)} \le b)S\big) \\ & \ge \frac{0.4s_n^2}{Es_N^2}\big\{P(a \le W_n^{(i)} \le b) - E(1 - \frac{2.5SEs_N^2}{s_n^2})I(S \le \frac{0.4s_n^2}{Es_N^2})\big\}. \end{aligned} \tag{3.10}$$

By (3.5) and (3.10), we obtain

$$\begin{aligned} EW_n^{(i)}f_n(W_n^{(i)}) &\ge E\big(I(a \le W_n^{(i)} \le b)S\big) - P(a \le W_n^{(i)} \le b)E[|Y_i|\min\{\delta_n, |Y_i|\}] \\ &\ge \frac{0.4s_n^2}{Es_N^2}\big\{P(a \le W_n^{(i)} \le b) - E(1 - \frac{2.5SEs_N^2}{s_n^2})I(S \le \frac{0.4s_n^2}{Es_N^2})\big\} \\ &\quad - P(a \le W_n^{(i)} \le b)E[|Y_i|\min\{\delta_n, |Y_i|\}] \\ &\ge \frac{0.4s_n^2}{Es_N^2}\big\{P(a \le W_n^{(i)} \le b) - \frac{100\delta_n^2 Es_N^2}{s_n^2}\big\} \qquad \text{(by (3.9))} \\ &\quad - P(a \le W_n^{(i)} \le b)E[|Y_i|\min\{\delta_n, |Y_i|\}] \\ &\ge \frac{0.4s_n^2}{Es_N^2}\big\{P(a \le W_n^{(i)} \le b) - \frac{100\delta_n^2 Es_N^2}{s_n^2}\big\} - E[|Y_i|\min\{\delta_n, |Y_i|\}] \\ &\ge \frac{0.4s_n^2}{Es_N^2}P(a \le W_n^{(i)} \le b) - 40\delta_n^2 - \delta_n\sqrt{EY_i^2} \\ &= \frac{0.4s_n^2}{Es_N^2}P(a \le W_n^{(i)} \le b) - 40\delta_n^2 - \delta_n\sqrt{\frac{\sigma_i^2}{Es_N^2}}. \end{aligned}$$

Hence, by (3.4), yields

$$\begin{aligned} P(a \le W_n^{(i)} \le b) &\le \frac{2.5Es_N^2}{s_n^2}\big\{EW_n^{(i)}f_n(W_n^{(i)}) + 40\delta_n^2 + \delta_n\sqrt{\frac{\sigma_i^2}{Es_N^2}}\big\} \\ &\le \frac{2.5Es_N^2}{s_n^2}\big\{\big(\frac{1}{2}(b-a) + \delta_n\big)E|W_n^{(i)}| + 40\delta_n^2 + \delta_n\sqrt{\frac{\sigma_i^2}{Es_N^2}}\big\} \\ &\le \frac{2.5Es_N^2}{s_n^2}\big\{\big(\frac{1}{2}(b-a) + \delta_n\big)\sqrt{E|W_n^{(i)}|^2} + 40\delta_n^2 + \delta_n\sqrt{\frac{\sigma_i^2}{Es_N^2}}\big\} \\ &\le \frac{2.5Es_N^2}{s_n^2}\big\{\big(\frac{1}{2}(b-a) + \delta_n\big)\sqrt{EW_n^2} + 40\delta_n^2 + \delta_n\sqrt{\frac{\sigma_i^2}{Es_N^2}}\big\} \\ &= \frac{2.5Es_N^2}{s_n^2}\big\{\big(\frac{1}{2}(b-a) + \delta_n\big)\sqrt{\frac{s_n^2}{Es_N^2}} + 40\delta_n^2 + \delta_n\sqrt{\frac{\sigma_i^2}{Es_N^2}}\big\} \end{aligned}$$

where we have used (3.1) in the last equation. □

3.2 Proof of Main Theorem

To bound $P(W \leq x) - \Phi(x)$ in Theorem 3.1, it suffices to consider $x \geq 0$ as we can apply the result to $-W$ when $x < 0$. Let f be the unique bounded solution f_x of the Stein's equation

$$f'(w) - wf(w) = I(w \leq x) - \Phi(x). \tag{3.11}$$

For a positive integer n, if we replace w in (3.11) by W_n, then

$$Ef'(W_n) - EW_n f(W_n) = P(W_n \leq x) - \Phi(x). \tag{3.12}$$

Define $K_i : \mathbb{R} \to \mathbb{R}$ for all $i = 1, 2, \ldots, n$, by

$$K_i(t) = E\big\{Y_i\big(I(0 \leq t \leq Y_i) - I(Y_i \leq t < 0)\big)\big\}.$$

We note that $K_i(t)$ is non-negative for all $t \in \mathbb{R}$, and satisfies the followings

$$\int_{-\infty}^{\infty} K_i(t)dt = EY_i^2 \tag{3.13}$$

and

$$\int_{-\infty}^{\infty} |t|K_i(t)dt = \frac{E|Y_i|^3}{2}. \tag{3.14}$$

Also, since Y_i and $W_n^{(i)}$ are independent and $EY_i = 0$,

$$\begin{aligned} EW_n f(W_n) &= \sum_{i=1}^{n} EY_i f(W_n) \\ &= \sum_{i=1}^{n} E\big\{Y_i\big(f(W_n) - f(W_n^{(i)})\big)\big\} \\ &= \sum_{i=1}^{n} E\Big\{Y_i \int_0^{Y_i} f'(W_n^{(i)} + t)dt\Big\} \\ &= \sum_{i=1}^{n} E \int_0^{Y_i} f'(W_n^{(i)} + t)K_i(t)dt \\ &= \sum_{i=1}^{n} E \int_{-\infty}^{\infty} f'(W_n^{(i)} + t)K_i(t)dt. \end{aligned} \tag{3.15}$$

Let $p_n = P(N = n)$. By Stein's equation (3.12) and (3.15), we obtain

$$
\begin{aligned}
&P(W \le x) - \Phi(x) \\
=&\sum_{n=1}^{\infty} p_n P(W_n \le x) - \Phi(x)\sum_{n=1}^{\infty} p_n \\
=&\sum_{n=1}^{\infty} p_n \{P(W_n \le x) - \Phi(x)\} \\
=&\sum_{n=1}^{\infty} p_n \{Ef'(W_n) - EW_n f(W_n)\} && \text{(by (3.12))} \\
=&\sum_{n=1}^{\infty} p_n \Big\{Ef'(W_n) - \sum_{i=1}^{n} E\int_{-\infty}^{\infty} f'(W_n^{(i)} + t)K_i(t)dt\Big\} && \text{(by (3.15))} \\
=&\sum_{n=1}^{\infty} p_n \Big\{\sum_{i=1}^{n} E\int_{-\infty}^{\infty} f'(W_n)K_i(t)dt - \sum_{i=1}^{n} E\int_{-\infty}^{\infty} f'(W_n^{(i)} + t)K_i(t)dt\Big\} \\
&+\sum_{n=1}^{\infty} p_n \Big\{Ef'(W_n) - \sum_{i=1}^{n} E\int_{-\infty}^{\infty} f'(W_n)K_i(t)dt\Big\} \\
=&A_1 + A_2, && (3.16)
\end{aligned}
$$

where

$$
\begin{aligned}
A_1 =&\sum_{n=1}^{\infty} p_n \Big\{\sum_{i=1}^{n} E\int_{-\infty}^{\infty} f'(W_n)K_i(t)dt - \sum_{i=1}^{n} E\int_{-\infty}^{\infty} f'(W_n^{(i)} + t)K_i(t)dt\Big\}, \\
A_2 =&\sum_{n=1}^{\infty} p_n \Big\{Ef'(W_n) - \sum_{i=1}^{n} E\int_{-\infty}^{\infty} f'(W_n)K_i(t)dt\Big\}.
\end{aligned}
$$

By Lemma 2.15 (1), we have

$$
\begin{aligned}
A_1 =&\sum_{n=1}^{\infty} p_n \sum_{i=1}^{n} E\Big\{\int_{-\infty}^{\infty} \{f'(W_n^{(i)} + Y_i) - f'(W_n^{(i)} + t)\}K_i(t)dt\Big\} \\
\le&\sum_{n=1}^{\infty} p_n \sum_{i=1}^{n} E\Big\{\int_{\substack{W_n^{(i)}+t>x \\ W_n^{(i)}+Y_i\le x}} K_i(t)dt + \int_{Y_i\ge t} (|W_n^{(i)}| + 0.63)(|Y_i| + |t|)K_i(t)dt\Big\} \\
=&\sum_{n=1}^{\infty} p_n \sum_{i=1}^{n} E\int_{t>Y_i} I(x - t < W_n^{(i)} \le x - Y_i)K_i(t)dt \\
&+\sum_{n=1}^{\infty} p_n \sum_{i=1}^{n} E\int_{Y_i\ge t} (|W_n^{(i)}| + 0.63)(|Y_i| + |t|)K_i(t)dt \\
=&A_{11} + A_{12}, && (3.17)
\end{aligned}
$$

where

$$A_{11} = \sum_{n=1}^{\infty} p_n \sum_{i=1}^{n} E \int_{t>Y_i} I(x-t < W_n^{(i)} \le x - Y_i) K_i(t) dt,$$
$$A_{12} = \sum_{n=1}^{\infty} p_n \sum_{i=1}^{n} E \int_{Y_i \ge t} (|W_n^{(i)}| + 0.63)(|Y_i| + |t|) K_i(t) dt.$$

By concentration inequality (Proposition 3.3), we have

$$\begin{aligned}
A_{11} =& \sum_{n=1}^{\infty} p_n \sum_{i=1}^{n} E \int_{t>Y_i} I(x-t < W_n^{(i)} \le x - Y_i) K_i(t) dt \\
=& \sum_{n=1}^{\infty} p_n \sum_{i=1}^{n} E E^{Y_i} \int_{t>Y_i} I(x-t < W_n^{(i)} \le x - Y_i) K_i(t) dt \\
=& \sum_{n=1}^{\infty} p_n \sum_{i=1}^{n} E \int_{t>Y_i} E^{Y_i} I(x-t < W_n^{(i)} \le x - Y_i) K_i(t) dt \\
=& \sum_{n=1}^{\infty} p_n \sum_{i=1}^{n} E \int_{t>Y_i} P(x-t < W_n^{(i)} \le x - Y_i \mid Y_i) K_i(t) dt \\
\le& \sum_{n=1}^{\infty} p_n \sum_{i=1}^{n} E \int_{t>Y_i} \frac{2.5Es_N^2}{s_n^2} \{ \big(\frac{1}{2}(|t| + |Y_i|) + \delta_n \big) \sqrt{\frac{s_n^2}{Es_N^2}} \\
& \qquad + 40\delta_n^2 + \delta_n \sqrt{\frac{\sigma_i^2}{Es_N^2}} \} K_i(t) dt \qquad \text{(by Proposition 3.3)} \\
\le& \sum_{n=1}^{\infty} p_n \frac{2.5Es_N^2}{s_n^2} \sum_{i=1}^{n} E \int_{t>Y_i} \{ \big(\frac{1}{2}(|t| + |Y_i|) + 2\delta_n \big) \sqrt{\frac{s_n^2}{Es_N^2}} + 40\delta_n^2 \} K_i(t) dt \\
\le& \sum_{n=1}^{\infty} p_n \frac{2.5Es_N^2}{s_n^2} \sum_{i=1}^{n} E \int_{-\infty}^{\infty} \{ \big(\frac{1}{2}(|t| + |Y_i|) + 2\delta_n \big) \sqrt{\frac{s_n^2}{Es_N^2}} + 40\delta_n^2 \} K_i(t) dt \\
=& \sum_{n=1}^{\infty} p_n \frac{2.5Es_N^2}{s_n^2} \sum_{i=1}^{n} \{ \big(\frac{1}{2}(\frac{E|Y_i|^3}{2} + E|Y_i| E|Y_i|^2) + 2\delta_n EY_i^2 \big) \sqrt{\frac{s_n^2}{Es_N^2}} + 40\delta_n^2 EY_i^2 \} \\
& \qquad \text{(by (3.13), (3.14))} \\
\le& \sum_{n=1}^{\infty} p_n \frac{2.5Es_N^2}{s_n^2} \Big\{ \Big(\frac{3}{4} \sum_{i=1}^{n} E|Y_i|^3 + 2\delta_n \sum_{i=1}^{n} EY_i^2 \Big) \sqrt{\frac{s_n^2}{Es_N^2}} + 40\delta_n^2 \sum_{i=1}^{n} EY_i^2 \Big\} \\
=& \sum_{n=1}^{\infty} p_n \frac{2.5Es_N^2}{s_n^2} \{ (\frac{3\delta_n s_n^2}{4Es_N^2} + \frac{2\delta_n s_n^2}{Es_N^2}) \sqrt{\frac{s_n^2}{Es_N^2}} + \frac{40\delta_n^2 s_n^2}{Es_N^2} \} \\
=& \sum_{n=1}^{\infty} p_n \{ (1.857\delta_n + 5\delta_n) \sqrt{\frac{s_n^2}{Es_N^2}} + 100\delta_n^2 \} \qquad \text{(by (3.1))} \\
=& \frac{6.875E[\delta_N \sqrt{s_N^2}]}{\sqrt{Es_N^2}} + 100E\delta_N^2. \qquad (3.18)
\end{aligned}$$

Also, we have

$$
\begin{aligned}
A_{12} =& \sum_{n=1}^{\infty} p_n \sum_{i=1}^{n} E \int_{Y_i \geq t} (|W_n^{(i)}| + 0.63)(|Y_i| + |t|)K_i(t)dt \\
\leq & \sum_{n=1}^{\infty} p_n \sum_{i=1}^{n} (E|W_n^{(i)}| + 0.63) E \int_{-\infty}^{\infty} (|Y_i| + |t|)K_i(t)dt \\
=& \sum_{n=1}^{\infty} p_n \sum_{i=1}^{n} (E|W_n^{(i)}| + 0.63)(E|Y_i| E|Y_i|^2 + \frac{1}{2} E|Y_i|^3) \qquad \text{(by (3.13), (3.14))} \\
\leq & 1.5 \sum_{n=1}^{\infty} p_n \sum_{i=1}^{n} (E|W_n| + 0.63) E|Y_i|^3 \qquad \text{(by Hölder's inequality)} \\
\leq & 1.5 \sum_{n=1}^{\infty} p_n (\sqrt{E|W_n|^2} + 0.63) \sum_{i=1}^{n} E|Y_i|^3 \\
=& 1.5 \sum_{n=1}^{\infty} p_n (\sqrt{\frac{s_n^2}{Es_N^2}} + 0.63) \frac{\delta_n s_n^2}{Es_N^2} \qquad \text{(by (3.1))} \\
=& \frac{1.5 E\big[\delta_N (s_N^2)^{3/2}\big]}{(Es_N^2)^{3/2}} + \frac{0.945 E[\delta_N s_N^2]}{Es_N^2}. \qquad (3.19)
\end{aligned}
$$

Combining (3.17), (3.18) and (3.19), we obtain

$$
A_1 \leq \frac{6.875 E[\delta_N \sqrt{s_N^2}]}{\sqrt{Es_N^2}} + 100 E\delta_N^2 + \frac{1.5 E\big[\delta_N (s_N^2)^{3/2}\big]}{(Es_N^2)^{3/2}} + \frac{0.945 E[\delta_N s_N^2]}{Es_N^2}. \qquad (3.20)
$$

Similar to (3.17), we can use the fact in Lemma 2.15(2) to prove that

$$
A_1 \geq -\frac{6.875 E[\delta_N \sqrt{s_N^2}]}{\sqrt{Es_N^2}} - 100 E\delta_N^2 - \frac{1.5 E\big[\delta_N (s_N^2)^{3/2}\big]}{(Es_N^2)^{3/2}} - \frac{0.945 E[\delta_N s_N^2]}{Es_N^2}.
$$

Hence,

$$
|A_1| \leq \frac{6.875 E[\delta_N \sqrt{s_N^2}]}{\sqrt{Es_N^2}} + 100 E\delta_N^2 + \frac{1.5 E\big[\delta_N (s_N^2)^{3/2}\big]}{(Es_N^2)^{3/2}} + \frac{0.945 E[\delta_N s_N^2]}{Es_N^2}. \qquad (3.21)
$$

By Lemma 2.14(3), we have

$$
\begin{aligned}
|A_2| =& \Big| \sum_{n=1}^{\infty} p_n \Big\{ Ef'(W_n) - \sum_{i=1}^{n} E \int_{-\infty}^{\infty} f'(W_n) K_i(t)dt \Big\} \Big| \\
\leq & \sum_{n=1}^{\infty} p_n |Ef'(W_n)| \Big| 1 - \sum_{i=1}^{n} E \int_{-\infty}^{\infty} K_i(t)dt \Big| \\
\leq & \sum_{n=1}^{\infty} p_n \Big| 1 - \sum_{i=1}^{n} E \int_{-\infty}^{\infty} K_i(t)dt \Big| \qquad \text{(by Lemma 2.14 (3))}
\end{aligned}
$$

$$= \sum_{n=1}^{\infty} p_n \Big| 1 - \sum_{i=1}^{n} EY_i^2 \Big| \qquad \text{(by (3.13))}$$

$$= \sum_{n=1}^{\infty} p_n \Big| 1 - \frac{s_n^2}{Es_N^2} \Big| \qquad \text{(by (3.1))}$$

$$= \frac{E|s_N^2 - Es_N^2|}{Es_N^2}. \qquad (3.22)$$

Combining (3.16), (3.21) and (3.22), we have the theorem as desired. □

3.3 Proof of Corollary 3.2

Since $X_1, X_2, \dots$ are identically distributed random variables,

$$s_N^2 = \sum_{i=1}^{N} EX_i^2 = N\sigma^2,\ \beta_N = \sum_{i=1}^{N} E|X_i|^3 = N\gamma \ \text{ and}$$

$$\delta_N = \frac{\beta_N}{s_N^2\sqrt{Es_N^2}} = \frac{N\gamma}{N\sigma^2\sqrt{EN\sigma^2}} = \frac{\gamma}{\sqrt{EN}\sigma^3}.$$

It is immediate from Theorem 3.1 that

$$\begin{aligned}
&|P(W \le x) - \Phi(x)| \\
\le& \frac{6.875E[\delta_N\sqrt{s_N^2}]}{\sqrt{Es_N^2}} + \frac{1.5E[\delta_N(s_N^2)^{3/2}]}{(Es_N^2)^{3/2}} + \frac{0.945E[\delta_N s_N^2]}{Es_N^2} + 100E\delta_N^2 \\
&+ \frac{E|s_N^2 - Es_N^2|}{Es_N^2} \\
=& \frac{6.875E[\delta\sqrt{N\sigma^2}]}{\sqrt{EN\sigma^2}} + \frac{1.5E[\delta(N\sigma^2)^{3/2}]}{(EN\sigma^2)^{3/2}} + \frac{0.945E[\delta N\sigma^2]}{EN\sigma^2} + 100E\delta^2 \\
&+ \frac{E|N\sigma^2 - EN\sigma^2|}{EN\sigma^2} \\
=& 6.875\delta\frac{E\sqrt{N}}{\sqrt{EN}} + \frac{1.5\delta EN^{3/2}}{(EN)^{3/2}} + 0.945\delta + 100\delta^2 + \frac{E|N - EN|}{EN} \\
\le& 6.875\delta + \frac{1.5\delta EN^{3/2}}{(EN)^{3/2}} + 0.945\delta + 100\delta^2 + \frac{E|N - EN|}{EN} \\
=& 7.82\delta + \frac{1.5\delta EN^{3/2}}{(EN)^{3/2}} + \Big(\frac{100\gamma^2}{\sqrt{EN}\sigma^6}\Big)\frac{1}{\sqrt{EN}} + \frac{E|N - EN|}{EN}
\end{aligned}$$

where we have used Hölder's inequality in the second inequality. □

CHAPTER IV

A NON-UNIFORM BOUND FOR RANDOM SUMS OF NON-IDENTICALLY DISTRIBUTED RANDOM VARIABLES

In this chapter, we give a non-uniform bound for random sums W. We also use the same notations as in Chapter III, and write C instead of a positive value with possibly different values in different places. The followings are our results.

Theorem 4.1. *Let $X_1, X_2, \ldots$ be independent and not necessarily identically distributed random variables with $EX_i = 0$, $EX_i^2 = \sigma_i^2$ and $E|X_i|^3 = \gamma_i < \infty$ and N a positive integral-valued random variable such that $N \geq 5$ and $N, X_1, X_2, \ldots$ are independent. If $s_n^2 \geq 4 \max_{1\leq i\leq n} \sigma_i^2$ for all positive integer $n \geq 5$, then there exists a constant C such that for every real number x,*

$$\begin{aligned} &|P(W \leq x) - \Phi(x)| \\ &\leq \frac{C}{(1+|x|)^3}\big(E[\frac{\delta_N}{(s_N^2)^2}](Es_N^2)^2 + \frac{E[\delta_N (s_N^2)^5]}{(Es_N^2)^5}\big) \\ &\quad + \frac{C}{(1+|x|)^2}\big(E[\frac{\delta_N^2}{(s_N^2)^2}](Es_N^2)^2 + \frac{E[\delta_N^2 (s_N^2)^5]}{(Es_N^2)^5} + \frac{E|(s_N^2)^2 - (Es_N^2)^2|}{(Es_N^2)^2}\big) \end{aligned}$$

where $\delta_N = \dfrac{\beta_N}{s_N^2\sqrt{Es_N^2}}$.

Corollary 4.2. *Let $X_1, X_2, \ldots$ be independent identically distributed random variables with mean zero and N a positive integral-valued random variable such that $N \geq 5$ and $N, X_1, X_2, \ldots$ are independent. Denote $W = \dfrac{X_1 + X_2 + \cdots + X_N}{\sqrt{EN}\sigma}$. Then there exists a constant C such that for every real number x,*

$$\begin{aligned} &|P(W \leq x) - \Phi(x)| \\ &\leq \frac{C\delta}{(1+|x|)^3}\big(1 + \frac{EN^5}{(EN)^5}\big) + \frac{C\delta^2}{(1+|x|)^2}\big(1 + \frac{EN^5}{(EN)^5}\big) + \frac{CE|N^2 - (EN)^2|}{(1+|x|)^2(EN)^2} \end{aligned}$$

where $\gamma = E|X_1|^3$, $\sigma^2 = EX_1^2$ and $\delta = \dfrac{\gamma}{\sqrt{EN}\sigma^3}$.

Remark 4.3. *Let $X_1, X_2, \ldots$ be independent random variables such that*

$$P(X_i = \sqrt{i+5}) = P(X_i = -\sqrt{i+5}) = \frac{1}{2} \text{ for all } i = 1, 2, \ldots.$$

Then $EX_i = 0$, $EX_i^2 = i + 5$ and

$$\sum_{i=1}^{n} EX_i^2 = \frac{n(n+11)}{2} \geq 4n + 20 = 4 \max_{1 \leq i \leq n} EX_i^2 \text{ for all } n \geq 5.$$

Therefore, $X_1, X_2, \ldots$ satisfy the condition in Theorem 4.1 that

$$s_n^2 \geq 4 \max_{1 \leq i \leq n} \sigma_i^2 \text{ for all } n \geq 5.$$

4.1 Auxiliary Results

In this section, we begin with some notations and their conclusions. Then we provide a concentration inequality in Proposition 4.5. For a positive integer n and for all $i = 1, 2, \ldots, n$, let us define

$$T_n^{(i)} = \sum_{\substack{j=1 \\ j \neq i}}^{n} Z_j \text{ where } Z_j = Y_j I(|Y_j| \leq 1 + a) - EY_j I(|Y_j| \leq 1 + a) \text{ and}$$

$$r := r_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} EY_j I(|Y_j| > 1 + a) \text{ for } a \geq 0.$$

Because $EY_i = 0$, we first observe that

$$W_n^{(i)} = T_n^{(i)} - r, \tag{4.1}$$

whenever $\max\limits_{\substack{1 \leq j \leq n \\ j \neq i}} |Y_j| \leq 1 + a$. Also, we note that

$$r \leq \sum_{\substack{j=1 \\ j \neq i}}^{n} EY_j^2 I(|Y_j| > 1 + a) \leq \sum_{\substack{j=1 \\ j \neq i}}^{n} EY_j^2 \leq \frac{s_n^2}{Es_N^2}. \tag{4.2}$$

Lemma 4.4. *We have*

1. $E|T_n^{(i)}|^2 \leq \dfrac{s_n^2}{Es_N^2}$ *for all $n \in \mathbb{N}$ and $i = 1, 2, \ldots, n$.*

2. *There exists a constant C such that*

$$E|T_n^{(i)}|^4 \leq C\Big\{\frac{(1+a)\delta_n s_n^2}{Es_N^2} + \frac{(s_n^2)^2}{(Es_N^2)^2}\Big\}$$

   for all $n \in \mathbb{N}$ and $i = 1, 2, \ldots, n$.

Proof. (1) Since $EZ_j = 0$,

$$\begin{aligned}
E|T_n^{(i)}|^2 =& E\Big(\sum_{\substack{j=1\\ j\neq i}}^n Z_j\Big)^2 \\
=& \sum_{\substack{j=1\\ j\neq i}}^n EZ_j^2 \\
=& \sum_{\substack{j=1\\ j\neq i}}^n \{EY_j^2 I(|Y_j| \leq 1+a) - E^2 Y_j I(|Y_j| \leq 1+a)\} \\
\leq& \sum_{\substack{j=1\\ j\neq i}}^n EY_j^2 I(|Y_j| \leq 1+a) \\
\leq& \sum_{\substack{j=1\\ j\neq i}}^n EY_j^2 \\
\leq& \frac{s_n^2}{Es_N^2}.
\end{aligned}$$

(2) Now we remark some useful fact ([6], p.320) that for all $a, b \in \mathbb{R}$ and $p > 0$,

$$|a+b|^p \leq 2^{p-1}(|a|^p + |b|^p). \tag{4.3}$$

By Rosenthal's inequality (Theorem 2.9) and $EZ_j = 0$, we have

$$\begin{aligned}
E|T_n^{(i)}|^4 \leq& C\Big\{\sum_{\substack{j=1\\ j\neq i}}^n E|Z_j|^4 + \Big(\sum_{\substack{j=1\\ j\neq i}}^n EZ_j^2\Big)^2\Big\} \\
=& C\Big\{\sum_{\substack{j=1\\ j\neq i}}^n E|Z_j|^4 + \Big(E\Big|\sum_{\substack{j=1\\ j\neq i}}^n Z_j\Big|^2\Big)^2\Big\} \\
=& C\Big\{\sum_{\substack{j=1\\ j\neq i}}^n E|Y_j I(|Y_j| \leq 1+a) - EY_j I(|Y_j| \leq 1+a)|^4 + (E|T_n^{(i)}|^2)^2\Big\} \\
\leq& C\Big\{\sum_{\substack{j=1\\ j\neq i}}^n \big\{E|Y_j|^4 I(|Y_j| \leq 1+a) + E^4 Y_j I(|Y_j| \leq 1+a)\big\} + (E|T_n^{(i)}|^2)^2\Big\} \\
& \hspace{20em} \text{(by (4.3))} \\
\leq& C\Big\{\sum_{\substack{j=1\\ j\neq i}}^n E|Y_j|^4 I(|Y_j| \leq 1+a) + (E|T_n^{(i)}|^2)^2\Big\}
\end{aligned}$$

$$\begin{aligned}
&\leq C\Big\{(1+a)\sum_{j=1}^{n} E|Y_j|^3 + \frac{(s_n^2)^2}{(Es_N^2)^2}\Big\} && \text{(by (1))}\\
&= C\{\frac{(1+a)\delta_n s_n^2}{Es_N^2} + \frac{(s_n^2)^2}{(Es_N^2)^2}\}
\end{aligned}$$

where we have used Hölder's inequality in the third inequality. □

We also use the idea from Chen and Shao [4] to prove the following non-uniform concentration inequality.

Proposition 4.5. *There exists a constant C such that*

$$P(a \leq W_n^{(i)} \leq b) \leq \frac{C(b-a+\delta_n)}{(1+a)^3}\{\frac{(Es_N^2)^3}{(s_n^2)^3} + \frac{(s_n^2)^4}{(Es_N^2)^4}\}$$

where $0 \leq a < b$, $n \in \mathbb{N}$ such that $s_n^2 \geq 4 \max_{1\leq i\leq n} \sigma_i^2$ and $i = 1, 2, \ldots, n$.

Proof. We begin by noting that

$$\begin{aligned}
&P(a \leq W_n^{(i)} \leq b)\\
&= P(a \leq W_n^{(i)} \leq b, \max_{\substack{1\leq j\leq n\\ j\neq i}} |Y_j| \leq 1+a) + P(a \leq W_n^{(i)} \leq b, \max_{\substack{1\leq j\leq n\\ j\neq i}} |Y_j| > 1+a)\\
&\leq P(a+r \leq T_n^{(i)} \leq b+r) + P(\max_{\substack{1\leq j\leq n\\ j\neq i}} |Y_j| > 1+a) && \text{(by (4.1))}\\
&\leq P(a+r \leq T_n^{(i)} \leq b+r) + \sum_{j=1}^{n} P(|Y_j| > 1+a)\\
&\leq P(a+r \leq T_n^{(i)} \leq b+r) + \frac{1}{(1+a)^3}\sum_{j=1}^{n} E|Y_j|^3\\
&= P(a+r \leq T_n^{(i)} \leq b+r) + \frac{\delta_n s_n^2}{(1+a)^3 Es_N^2} && (4.4)
\end{aligned}$$

where we have used Chebyshev's inequality in the last inequality.

To complete the proof, it remains to bound $P(a+r \leq T_n^{(i)} \leq b+r)$.

We divide it into two cases.

Case 1. $(1+a)\delta_n \geq \dfrac{1}{4}$.

By the fact that $\dfrac{1}{(1+a)} \leq 4\delta_n$ and Lemma 4.4, we have

$$\begin{aligned}
&P(a+r \leq T_n^{(i)} \leq b+r) \\
\leq &P(1+a \leq 1-r+T_n^{(i)}) \\
\leq &\frac{E|1-r+T_n^{(i)}|^4}{(1+a)^4} \\
\leq &\frac{8(1-r)^4+8E|T_n^{(i)}|^4}{(1+a)^4} && \text{(by (4.3))} \\
\leq &\frac{C(1+r^4+E|T_n^{(i)}|^4)}{(1+a)^4} && \text{(by (4.3))} \\
\leq &\frac{C}{(1+a)^4}\Big\{1+\frac{(s_n^2)^4}{(Es_N^2)^4}\Big\}+\frac{C}{(1+a)^4}\Big\{\frac{(1+a)\delta_n s_n^2}{Es_N^2}+\frac{(s_n^2)^2}{(Es_N^2)^2}\Big\} && \text{(by (4.2))} \\
\leq &\frac{C\delta_n}{(1+a)^3}\Big\{1+\frac{(s_n^2)^4}{(Es_N^2)^4}+\frac{s_n^2}{Es_N^2}+\frac{(s_n^2)^2}{(Es_N^2)^2}\Big\} \\
\leq &\frac{C\delta_n}{(1+a)^3}\Big\{1+\frac{(s_n^2)^4}{(Es_N^2)^4}\Big\}. && (4.5)
\end{aligned}$$

Case 2. $(1+a)\delta_n < \dfrac{1}{4}$.

Let

$$U = \sum_{\substack{j=1 \\ j\neq i}}^{n} \eta_j,$$

where

$$\eta_j = |Z_j| \min\{\kappa_n, |Z_j|\} \quad \text{and} \quad \kappa_n = 4\delta_n.$$

Define $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(t) = \begin{cases} 0 & \text{for } t < a+r-\kappa_n \\ (1+t-r+\kappa_n)^3(t-a-r+\kappa_n) & \text{for } a+r-\kappa_n \leq t \leq b+r+\kappa_n \\ (1+t-r+\kappa_n)^3(b-a+2\kappa_n) & \text{for } t > b+r+\kappa_n, \end{cases}$$

and $M : \mathbb{R}^2 \to \mathbb{R}$ by

$$M(w,t) = w[I(-w \leq t \leq 0) - I(0 < t \leq -w)].$$

Next we note that f_n is a non-decreasing continuous function that satisfies

$$f_n'(t) \geq (1+a)^3 \text{ for } a+r-\kappa_n < t < b+r+\kappa_n \tag{4.6}$$

and

$$ET_n^{(i)} f_n(T_n^{(i)}) \leq (b-a+2\kappa_n) E\big(T_n^{(i)}(1+T_n^{(i)}-r+\kappa_n)^3\big). \tag{4.7}$$

Because Z_j and $T_n^{(i)} - Z_j$ are independent $EZ_j = 0$ and $M(w,t) \geq 0$ for all $w, t \in \mathbb{R}$, we have

$$\begin{aligned}
& ET_n^{(i)} f_n(T_n^{(i)}) \\
&= \sum_{\substack{j=1\\ j\neq i}}^n EZ_j f_n(T_n^{(i)}) \\
&= \sum_{\substack{j=1\\ j\neq i}}^n E[Z_j\{f_n(T_n^{(i)}) - f_n(T_n^{(i)} - Z_j)\}] \\
&= \sum_{\substack{j=1\\ j\neq i}}^n E\Big\{Z_j \int_{-Z_j}^0 f_n'(T_n^{(i)}+t)dt\Big\} \\
&= \sum_{\substack{j=1\\ j\neq i}}^n E\Big\{\int_{-Z_j}^0 f_n'(T_n^{(i)}+t)M(Z_j,t)dt\Big\} \\
&= \sum_{\substack{j=1\\ j\neq i}}^n E\Big\{\int_{-\infty}^\infty f_n'(T_n^{(i)}+t)M(Z_j,t)dt\Big\} \\
&\geq (1+a)^3 \sum_{\substack{j=1\\ j\neq i}}^n E\int_{-\infty}^\infty I(a+r-\kappa_n \leq T_n^{(i)}+t \leq b+r+\kappa_n)M(Z_j,t)dt \\
& \hspace{25em} \text{(by (4.6))} \\
&\geq (1+a)^3 \sum_{\substack{j=1\\ j\neq i}}^n E\Big\{I(a+r \leq T_n^{(i)} \leq b+r)\int_{|t|<\kappa_n} M(Z_j,t)dt\Big\} \\
&= (1+a)^3 E\Big[I(a+r \leq T_n^{(i)} \leq b+r)\sum_{\substack{j=1\\ j\neq i}}^n |Z_j| \min\{\kappa_n, Z_j\}\Big] \\
&= (1+a)^3 E\big(I(a+r \leq T_n^{(i)} \leq b+r)U\big).
\end{aligned} \tag{4.8}$$

From the fact that $\min\{a,b\} \geq b - \dfrac{b^2}{4a}$ for all $a, b > 0$ ([4], p.238), we obtain

$$
\begin{aligned}
EU &= \sum_{\substack{j=1\\ j\neq i}}^{n} E|Z_j| \min\{\kappa_n, |Z_j|\} \\
&\geq \sum_{\substack{j=1\\ j\neq i}}^{n} E(Z_j^2 - \frac{|Z_j|^3}{4\kappa_n}) \\
&= \sum_{\substack{j=1\\ j\neq i}}^{n} \{EY_j^2 I(|Y_j| \leq 1+a) - E^2 Y_j I(|Y_j| \leq 1+a)\} \\
&\quad - \frac{1}{4\kappa_n} \sum_{\substack{j=1\\ j\neq i}}^{n} E\big|Y_j I(|Y_j| \leq 1+a) - EY_j I(|Y_j| \leq 1+a)\big|^3 \\
&\geq \sum_{\substack{j=1\\ j\neq i}}^{n} \{EY_j^2 I(|Y_j| \leq 1+a) - E^2 Y_j I(|Y_j| > 1+a)\} \\
&\quad - \frac{1}{\kappa_n} \sum_{\substack{j=1\\ j\neq i}}^{n} E\big(|Y_j|^3 I(|Y_j| \leq 1+a) + |EY_j I(|Y_j| \leq 1+a)|^3\big) \qquad \text{(by (4.3))} \\
&\geq \sum_{\substack{j=1\\ j\neq i}}^{n} \big\{EY_j^2 - 2EY_j^2 I(|Y_j| > 1+a)\big\} - \frac{2}{\kappa_n} \sum_{\substack{j=1\\ j\neq i}}^{n} E|Y_j|^3 I(|Y_j| \leq 1+a) \\
&\qquad \text{(by Hölder's inequality)} \\
&\geq \sum_{\substack{j=1\\ j\neq i}}^{n} \big\{EY_j^2 - \frac{2}{\kappa_n} E|Y_j|^3 I(|Y_j| > 1+a)\big\} - \frac{2}{\kappa_n} \sum_{\substack{j=1\\ j\neq i}}^{n} E|Y_j|^3 I(|Y_j| \leq 1+a) \\
&\qquad \text{(by the fact that } \frac{1}{\kappa_n} < 1) \\
&= \sum_{\substack{j=1\\ j\neq i}}^{n} EY_j^2 - \frac{2}{\kappa_n} \sum_{\substack{j=1\\ j\neq i}}^{n} E|Y_j|^3 \\
&\geq \frac{s_n^2}{Es_N^2} - \frac{\sigma_i^2}{Es_N^2} - \frac{s_n^2}{2Es_N^2} \\
&= \frac{s_n^2}{4Es_N^2} + \frac{(s_n^2 - 4\sigma_i^2)}{4Es_N^2} \\
&\geq \frac{s_n^2}{4Es_N^2} \qquad (4.9)
\end{aligned}
$$

where we have used the fact $s_n^2 \geq 4 \max\limits_{1\leq i\leq n} \sigma_i^2$ in the last inequality.

By Chebyshev's inequality (Theorem 2.8) implies that for $0 < c < EU$,

$$E(1-U/c)I(U \le c) \le P(U \le c) \le P(|U-EU| \ge EU-c) \le \frac{E|U-EU|^4}{(EU-c)^4}. \tag{4.10}$$

From (4.9), we can choose a constant $c = \frac{s_n^2}{5Es_N^2}$ in (4.10), which implies that

$$\begin{aligned}
& E[(1-\frac{5UEs_N^2}{s_n^2})I(U \le \frac{s_n^2}{5Es_N^2})] \\
& \le \frac{C(Es_N^2)^4}{(s_n^2)^4} E|U-EU|^4 \\
& \le \frac{C(Es_N^2)^4}{(s_n^2)^4} \Big\{ \Big(\sum_{\substack{j=1 \\ j \ne i}}^n E(\eta_j - E\eta_j)^2 \Big)^2 + \sum_{\substack{j=1 \\ j \ne i}}^n E(\eta_j - E\eta_j)^4 \Big\} \\
& \qquad \text{(by Rosenthal's inequality)} \\
& \le \frac{C(Es_N^2)^4}{(s_n^2)^4} \Big\{ \Big(\sum_{\substack{j=1 \\ j \ne i}}^n E\eta_j^2 \Big)^2 + \sum_{\substack{j=1 \\ j \ne i}}^n E\eta_j^4 \Big\} \\
& \le \frac{C(Es_N^2)^4}{(s_n^2)^4} \Big\{ \Big(\delta_n \sum_{\substack{j=1 \\ j \ne i}}^n E|Z_j|^3 \Big)^2 + \delta_n^4 \sum_{\substack{j=1 \\ j \ne i}}^n E|Z_j|^4 \Big\} \\
& \le \frac{C(Es_N^2)^4}{(s_n^2)^4} \Big\{ \Big(\delta_n \sum_{\substack{j=1 \\ j \ne i}}^n \big\{ E|Y_j|^3 I(|Y_j| \le 1+a) + |EY_j I(|Y_j| \le 1+a)|^3 \big\} \Big)^2 \\
& \qquad + \delta_n^4 \sum_{\substack{j=1 \\ j \ne i}}^n \big\{ E|Y_j|^4 I(|Y_j| \le 1+a) + |EY_j I(|Y_j| \le 1+a)|^4 \big\} \Big\} \quad \text{(by (4.3))} \\
& \le \frac{C(Es_N^2)^4}{(s_n^2)^4} \Big\{ \Big(\delta_n \sum_{\substack{j=1 \\ j \ne i}}^n E|Y_j|^3 I(|Y_j| \le 1+a) \Big)^2 + \delta_n^4 \sum_{\substack{j=1 \\ j \ne i}}^n E|Y_j|^4 I(|Y_j| \le 1+a) \Big\} \\
& \le \frac{C(Es_N^2)^4}{(s_n^2)^4} \Big\{ \Big(\delta_n \sum_{\substack{j=1 \\ j \ne i}}^n E|Y_j|^3 \Big)^2 + (1+a)\delta_n^4 \sum_{\substack{j=1 \\ j \ne i}}^n E|Y_j|^3 I(|Y_j| \le 1+a) \Big\} \\
& \le \frac{C(Es_N^2)^4}{(s_n^2)^4} \Big\{ \Big(\delta_n \sum_{j=1}^n E|Y_j|^3 \Big)^2 + (1+a)\delta_n^4 \sum_{j=1}^n E|Y_j|^3 \Big\} \\
& = \frac{C(Es_N^2)^4}{(s_n^2)^4} \{ (\frac{\delta_n^2 s_n^2}{Es_N^2})^2 + \frac{(1+a)\delta_n^5 s_n^2}{Es_N^2} \} \\
& \le \frac{C\delta_n}{(1+a)^3} \{ \frac{(Es_N^2)^2}{(s_n^2)^2} + \frac{(Es_N^2)^3}{(s_n^2)^3} \}
\end{aligned} \tag{4.11}$$

where we have used the fact that $(1+a)\delta_n < \dfrac{1}{4}$ in the last inequality.

From the fact that ([4], p.238)

$$I(a \le w \le b)y \ge c\big(I(a \le w \le b) - (1 - y/c)I(y \le c)\big)$$

for $a < b$, $y \ge 0$ and $c > 0$, we have

$$\begin{aligned} & E\big(I(a+r \le T_n^{(i)} \le b+r)U\big) \\ & \ge \frac{s_n^2}{5Es_N^2}\{P(a+r \le T_n^{(i)} \le b+r) - E(1 - \frac{5UEs_N^2}{s_n^2})I(U \le \frac{s_n^2}{5Es_N^2})\}. \end{aligned} \tag{4.12}$$

By (4.8) and (4.12), we obtain

$$\begin{aligned} & E(T_n^{(i)} f_n(T_n^{(i)})) \\ & \ge (1+a)^3 E\big(I(a+r \le T_n^{(i)} \le b+r)U\big) \\ & \ge \frac{(1+a)^3 s_n^2}{5Es_N^2}\{P(a+r \le T_n^{(i)} \le b+r) - E(1 - \frac{5UEs_N^2}{s_n^2})I(U \le \frac{s_n^2}{5Es_N^2})\} \\ & \ge \frac{(1+a)^3 s_n^2}{5Es_N^2}\{P(a+r \le T_n^{(i)} \le b+r) - \frac{C\delta_n}{(1+a)^3}(\frac{(Es_N^2)^2}{(s_n^2)^2} + \frac{(Es_N^2)^3}{(s_n^2)^3})\} \end{aligned}$$

where we have used (4.11) in the last inequality.

Hence, by (4.7), yields

$$\begin{aligned} & P(a+r \le T_n^{(i)} \le b+r) \\ & \le \frac{C\delta_n}{(1+a)^3}\{\frac{(Es_N^2)^2}{(s_n^2)^2} + \frac{(Es_N^2)^3}{(s_n^2)^3}\} + \frac{5Es_N^2}{s_n^2(1+a)^3} ET_n^{(i)} f(T_n^{(i)}) \\ & \le \frac{C\delta_n}{(1+a)^3}\{\frac{(Es_N^2)^2}{(s_n^2)^2} + \frac{(Es_N^2)^3}{(s_n^2)^3}\} \\ & \quad + \frac{5Es_N^2}{s_n^2(1+a)^3}\{(b-a+2\kappa_n)E|T_n^{(i)}(1+T_n^{(i)} - r + \kappa_n)^3|\} \\ & \le \frac{C\delta_n}{(1+a)^3}\{\frac{(Es_N^2)^2}{(s_n^2)^2} + \frac{(Es_N^2)^3}{(s_n^2)^3}\} \\ & \quad + \frac{CEs_N^2}{s_n^2(1+a)^3}\{(b-a+2\kappa_n)\{(1+r^3)E|T_n^{(i)}| + E|T_n^{(i)}|^4\}\} \qquad \text{(by (4.3))} \\ & \le \frac{C\delta_n}{(1+a)^3}\{\frac{(Es_N^2)^2}{(s_n^2)^2} + \frac{(Es_N^2)^3}{(s_n^2)^3}\} \\ & \quad + \frac{CEs_N^2}{s_n^2(1+a)^3}\Big\{(b-a+2\kappa_n)\big\{\big(1 + \frac{(s_n^2)^3}{(Es_N^2)^3}\big)\sqrt{E|T_n^{(i)}|^2} + \frac{(s_n^2)^2}{(Es_N^2)^2} + \frac{(1+a)\delta_n s_n^2}{Es_N^2}\big\}\Big\} \\ & \qquad \text{(by (4.2) and Lemma 4.4)} \end{aligned}$$

$$
\begin{aligned}
&\le \frac{C\delta_n}{(1+a)^3}\Big\{\frac{(Es_N^2)^2}{(s_n^2)^2} + \frac{(Es_N^2)^3}{(s_n^2)^3}\Big\} \\
&\quad + \frac{CEs_N^2}{s_n^2(1+a)^3}\Big\{(b-a+2\kappa_n)\Big\{\frac{\sqrt{s_n^2}}{\sqrt{Es_N^2}} + \frac{s_n^2}{Es_N^2} + \frac{(s_n^2)^2}{(Es_N^2)^2} + \frac{(s_n^2)^{7/2}}{(Es_N^2)^{7/2}}\Big\}\Big\} \\
&\le \frac{C\delta_n}{(1+a)^3}\Big\{\frac{(Es_N^2)^2}{(s_n^2)^2} + \frac{(Es_N^2)^3}{(s_n^2)^3}\Big\} + \frac{CEs_N^2(b-a+\delta_n)}{s_n^2(1+a)^3}\Big\{\frac{\sqrt{s_n^2}}{\sqrt{Es_N^2}} + \frac{(s_n^2)^{7/2}}{(Es_N^2)^{7/2}}\Big\} \\
&\le \frac{C(b-a+\delta_n)}{(1+a)^3}\Big\{\frac{(Es_N^2)^2}{(s_n^2)^2} + \frac{(Es_N^2)^3}{(s_n^2)^3} + \frac{\sqrt{Es_N^2}}{\sqrt{s_n^2}} + \frac{(s_n^2)^{5/2}}{(Es_N^2)^{5/2}}\Big\} \\
&\le \frac{C(b-a+\delta_n)}{(1+a)^3}\Big\{\frac{(Es_N^2)^3}{(s_n^2)^3} + \frac{(s_n^2)^{5/2}}{(Es_N^2)^{5/2}}\Big\}. \qquad (4.13)
\end{aligned}
$$

Combining (4.4), (4.5) and (4.13), we have

$$
\begin{aligned}
P(a \le W_n^{(i)} \le b) \le& \frac{C\delta_n}{(1+a)^3}\Big\{1 + \frac{(s_n^2)^4}{(Es_N^2)^4}\Big\} \\
&+ \frac{C(b-a+\delta_n)}{(1+a)^3}\Big\{\frac{(Es_N^2)^3}{(s_n^2)^3} + \frac{(s_n^2)^{5/2}}{(Es_N^2)^{5/2}}\Big\} + \frac{\delta_n s_n^2}{(1+a)^3 Es_N^2} \\
\le& \frac{C(b-a+\delta_n)}{(1+a)^3}\Big\{1 + \frac{(s_n^2)^4}{(Es_N^2)^4} + \frac{(Es_N^2)^3}{(s_n^2)^3} + \frac{(s_n^2)^{5/2}}{(Es_N^2)^{5/2}} + \frac{s_n^2}{Es_N^2}\Big\} \\
\le& \frac{C(b-a+\delta_n)}{(1+a)^3}\Big\{\frac{(Es_N^2)^3}{(s_n^2)^3} + \frac{(s_n^2)^4}{(Es_N^2)^4}\Big\}.
\end{aligned}
$$

This is complete the proof. □

Let f_x be the solution of Stein's equation (2.4) and define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(w) = (wf_x(w))'.$$

Theorem 4.1 will be obtained from the followings two lemmas.

Lemma 4.6. *There exists a constant C such that*

$$E|f_x'(W_n)| \le \frac{C}{(1+x)^2}\Big(1 + \frac{s_n^2}{Es_N^2}\Big),$$

for any positive integer n and $x > 1$.

Proof. By (2.4), we note that for $w \le x$,

$$f_x'(w) = (1-\Phi(x))\big(1 + \sqrt{2\pi}we^{\frac{w^2}{2}}\Phi(w)\big).$$

From above and (2.2), it follows that for $w \le 0$,

$$0 < f_x'(w) < 1 - \Phi(w),$$

and by Lemma 2.14 (3), we have

$$|f_x'(w)| \leq 1 \quad \text{for all } w.$$

Therefore,

$$\begin{aligned}
&E|f_x'(W_n)| \\
=&E|f_x'(W_n)|I(W_n \leq 0) + E|f_x'(W_n)|I(W_n > x) + E|f_x'(W_n)|I(0 < W_n \leq x) \\
\leq&(1-\Phi(x))P(W_n \leq 0) + P(W_n > x) \\
&+ (1-\Phi(x))E[(1+\sqrt{2\pi}W_n e^{\frac{W_n^2}{2}})I(0 < W_n \leq x)] \\
\leq&(1-\Phi(x)) + \frac{E(1+W_n)^2}{(1+x)^2} + (1-\Phi(x))(1+\sqrt{2\pi}xe^{\frac{x^2}{2}}) \\
\leq&\frac{C}{(1+x)^2} + \frac{(1+EW_n^2)}{(1+x)^2} + \frac{C}{(1+x)^2} \\
\leq&\frac{C}{(1+x)^2}(1+\frac{s_n^2}{Es_N^2})
\end{aligned}$$

where we have used Chebyshev's inequality in the second inequality. □

Lemma 4.7. *There exists a constant C such that*

$$Eg(W_n^{(i)}+u) \leq \frac{C}{(1+x)^3}\big\{(1+\delta_n x)\big(\frac{(Es_N^2)^3}{(s_n^2)^3} + \frac{(s_n^2)^4}{(Es_N^2)^4}\big)\big\}$$

where $x \geq 4$, $|u| \leq 1+\frac{x}{4}$, $n \in \mathbb{N}$ such that $s_n^2 \geq 4 \max_{1\leq i\leq n} \sigma_i^2$ and $i = 1, 2, \ldots, n$.

Proof. By Lemma 2.16, we have

$$\begin{aligned}
&Eg(W_n^{(i)}+u) \\
=&Eg(W_n^{(i)}+u)I(W_n^{(i)}+u \leq 0) \\
&+ Eg(W_n^{(i)}+u)I(0 < W_n^{(i)}+u \leq x-1) \\
&+ Eg(W_n^{(i)}+u)I(x-1 < W_n^{(i)}+u < x) \\
&+ Eg(W_n^{(i)}+u)I(W_n^{(i)}+u \geq x) \\
\leq&2(1-\Phi(x)) + g(x-1) + Eg(W_n^{(i)}+u)I(x-1 < W_n^{(i)}+u \leq x) + \frac{2}{1+x^3} \\
\leq&\frac{C}{(1+x)^3} + g(x-1) + Eg(W_n^{(i)}+u)I(x-1 < W_n^{(i)}+u < x).
\end{aligned}$$

By (2.5) and the fact that

$$h(x) = \{(x - 2 + \frac{2}{x})e^{-x+\frac{1}{2}} + \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\}(1+x)^3$$

is decreasing on $[4, \infty)$, we obtain

$$\begin{aligned} g(x-1) =& \big\{\sqrt{2\pi}(1+(x-1)^2)e^{\frac{(x-1)^2}{2}}\Phi(x-1) + (x-1)\big\}(1-\Phi(x)) \\ \le& \sqrt{2\pi}(1+(x-1)^2)e^{\frac{(x-1)^2}{2}}(1-\Phi(x)) + x(1-\Phi(x)) \\ =& (x-2+\frac{2}{x})e^{-x+\frac{1}{2}} + \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} \\ \le& \frac{C}{(1+x)^3}. \end{aligned}$$

Also, by Proposition 4.5, we have

$$\begin{aligned} & Eg(W_n^{(i)} + u)I(x-1 < W_n^{(i)} + u < x) \\ =& \int_{x-1}^{x} -g(w)dP(w < W_n^{(i)} + u < x) \\ =& g(x-1)P(w < W_n^{(i)} + u < x) + \int_{x-1}^{x} g'(w)P(w < W_n^{(i)} + u < x)dw \\ \le& g(x-1) + \frac{C}{(1+x)^3}\Big(\frac{(Es_N^2)^3}{(s_n^2)^3} + \frac{(s_n^2)^4}{(Es_N^2)^4}\Big)\int_{x-1}^{x} g'(w)(x-w+\delta_n)dw \\ \le& \frac{C}{(1+x)^3} + \frac{C}{(1+x)^3}\Big(\frac{(Es_N^2)^3}{(s_n^2)^3} + \frac{(s_n^2)^4}{(Es_N^2)^4}\Big)\Big\{\delta_n g(x) + \int_{x-1}^{x} g'(w)(x-w)dw\Big\} \\ \le& \frac{C}{(1+x)^3}\Big(\frac{(Es_N^2)^3}{(s_n^2)^3} + \frac{(s_n^2)^4}{(Es_N^2)^4}\Big)\Big\{1 + \delta_n g(x) + \int_{x-1}^{x} g'(w)(x-w)dw\Big\} \\ \le& \frac{C}{(1+x)^3}\Big(\frac{(Es_N^2)^3}{(s_n^2)^3} + \frac{(s_n^2)^4}{(Es_N^2)^4}\Big)\Big\{1 + \delta_n x + \int_{x-1}^{x} (x-w)dg(w)\Big\} \\ =& \frac{C}{(1+x)^3}\Big(\frac{(Es_N^2)^3}{(s_n^2)^3} + \frac{(s_n^2)^4}{(Es_N^2)^4}\Big)\Big\{1 + \delta_n x - g(x-1) + \int_{x-1}^{x} g(w)dw\Big\} \\ \le& \frac{C}{(1+x)^3}\Big(\frac{(Es_N^2)^3}{(s_n^2)^3} + \frac{(s_n^2)^4}{(Es_N^2)^4}\Big)\{1 + \delta_n x + xf_x(x)\} \qquad \text{(by Lemma 2.16 (1))} \\ \le& \frac{C}{(1+x)^3}\Big\{(1+\delta_n x)\Big(\frac{(Es_N^2)^3}{(s_n^2)^3} + \frac{(s_n^2)^4}{(Es_N^2)^4}\Big)\Big\}. \end{aligned}$$

This completes the proof. □

4.2 Proof of Theorem 4.1

To bound $P(W \leq x) - \Phi(x)$ in Theorem 4.1, it suffices to consider $x \geq 0$. We divide the proof into two cases.

Case 1. $0 \leq x < 4$.

By Theorem 3.1, we have

$$
\begin{aligned}
&|P(W \leq x) - \Phi(x)| \\
\leq & 6.875\frac{E[\delta_N\sqrt{s_N^2}]}{\sqrt{Es_N^2}} + 1.5\frac{E\delta_N(s_N^2)^{3/2}}{(Es_N^2)^{3/2}} + 0.945\frac{E[\delta_N s_N^2]}{Es_N^2} + 100E\delta_N^2 + \frac{E|s_N^2 - Es_N^2|}{Es_N^2} \\
\leq & \frac{C}{(1+|x|)^3}\left(\frac{E[\delta_N\sqrt{s_N^2}]}{\sqrt{Es_N^2}} + \frac{E[\delta_N(s_N^2)^{3/2}]}{(Es_N^2)^{3/2}} + \frac{E[\delta_N s_N^2]}{Es_N^2}\right) \\
& + \frac{C}{(1+|x|)^2}\left(E\delta_N^2 + \frac{E|s_N^2 - Es_N^2||s_N^2 + Es_N^2|}{(Es_N^2)^2}\right) \\
\leq & \frac{C}{(1+|x|)^3}\left(E[\frac{\delta_N}{(s_N^2)^2}](Es_N^2)^2 + \frac{E[\delta_N(s_N^2)^5]}{(Es_N^2)^5}\right) \\
& + \frac{C}{(1+|x|)^2}\left(E[\frac{\delta_N^2}{(s_N^2)^2}](Es_N^2)^2 + \frac{E[\delta_N^2(s_N^2)^5]}{(Es_N^2)^5} + \frac{E|(s_N^2)^2 - (Es_N^2)^2|}{(Es_N^2)^2}\right)
\end{aligned}
$$

where we have used (4.14) in the last inequality.

Now, we remark the above line of the proof, one can be obtained it by a technique in analysis. Considering an event $A = \{w \in \Omega \mid 0 < \frac{s_N^2(w)}{Es_N^2} < 1\}$, for $i = 1, 2$ and $-2 \leq p \leq 5$, we have

$$
\begin{aligned}
\frac{E[\delta_N^i(s_N^2)^p]}{(Es_N^2)^p} &= \int_A \frac{\delta_N^i(s_N^2)^p}{(Es_N^2)^p}dP + \int_{A^C} \frac{\delta_N^i(s_N^2)^p}{(Es_N^2)^p}dP \\
&\leq \int_A \frac{\delta_N^i(s_N^2)^{-2}}{(Es_N^2)^{-2}}dP + \int_{A^C} \frac{\delta_N^i(s_N^2)^5}{(Es_N^2)^5}dP \\
&\leq \int_\Omega \frac{\delta_N^i(Es_N^2)^2}{(s_N^2)^2}dP + \int_\Omega \frac{\delta_N^i(s_N^2)^5}{(Es_N^2)^5}dP \\
&= E[\frac{\delta_N^i}{(s_N^2)^2}](Es_N^2)^2 + \frac{E[\delta_N^i(s_N^2)^5]}{(Es_N^2)^5}.
\end{aligned}
\tag{4.14}
$$

Case 2. $x \geq 4$.

By the same argument as (3.16), we have

$$\begin{aligned}
&P(W \leq x) - \Phi(x) \\
&= \sum_{n=1}^{\infty} p_n \{Ef'(W_n) - EW_n f(W_n)\} \\
&= \sum_{n=1}^{\infty} p_n \sum_{i=1}^{n} E\Big\{I(|Y_i| \leq 1 + \frac{x}{4}) \int_{-\infty}^{\infty} \{f'(W_n^{(i)} + Y_i) - f'(W_n^{(i)} + t)\} K_i(t)dt\Big\} \\
&\quad + \sum_{n=1}^{\infty} p_n \sum_{i=1}^{n} E\Big\{I(|Y_i| > 1 + \frac{x}{4}) \int_{-\infty}^{\infty} \{f'(W_n^{(i)} + Y_i) - f'(W_n^{(i)} + t)\} K_i(t)dt\Big\} \\
&\quad + \sum_{n=1}^{\infty} p_n \Big\{Ef'(W_n) - \sum_{i=1}^{n} E\int_{-\infty}^{\infty} f'(W_n) K_i(t)dt\Big\} \\
&= A_1 + A_2 + A_3, \qquad (4.15)
\end{aligned}$$

where

$$\begin{aligned}
A_1 &= \sum_{n=1}^{\infty} p_n \sum_{i=1}^{n} E\Big\{I(|Y_i| \leq 1 + \frac{x}{4}) \int_{-\infty}^{\infty} \{f'(W_n^{(i)} + Y_i) - f'(W_n^{(i)} + t)\} K_i(t)dt\Big\}, \\
A_2 &= \sum_{n=1}^{\infty} p_n \sum_{i=1}^{n} E\Big\{I(|Y_i| > 1 + \frac{x}{4}) \int_{-\infty}^{\infty} \{f'(W_n^{(i)} + Y_i) - f'(W_n^{(i)} + t)\} K_i(t)dt\Big\}, \\
A_3 &= \sum_{n=1}^{\infty} p_n \Big\{Ef'(W_n) - \sum_{i=1}^{n} E\int_{-\infty}^{\infty} f'(W_n) K_i(t)dt\Big\}.
\end{aligned}$$

By Lemma 2.17, we have

$$\begin{aligned}
|A_1| &\leq \sum_{n=1}^{\infty} p_n \sum_{i=1}^{n} \Big|E\Big\{I(|Y_i| \leq 1 + \frac{x}{4}) \int_{-\infty}^{\infty} \{f'(W_n^{(i)} + Y_i) - f'(W_n^{(i)} + t)\} K_i(t)dt\Big\}\Big| \\
&\leq \sum_{n=1}^{\infty} p_n \sum_{i=1}^{n} \Big|E\Big\{I(|Y_i| \leq 1 + \frac{x}{4}) \int_{-\infty}^{\infty} K_i(t) \int_{t}^{Y_i} Eg(W_n^{(i)} + u)du\, dt\Big\}\Big| \\
&\quad + \sum_{n=1}^{\infty} p_n \sum_{i=1}^{n} E\Big\{I(|Y_i| \leq 1 + \frac{x}{4}) \\
&\quad \times \int_{-\infty}^{\infty} I(x - \max(Y_i, t) < W_n^{(i)} \leq x - \min(Y_i, t)) K_i(t)dt\Big\} \\
&= |A_{11}| + |A_{12}|, \qquad (4.16)
\end{aligned}$$

where

$$|A_{11}| = \sum_{n=1}^{\infty} p_n \sum_{i=1}^{n} \Big| E\Big\{ I(|Y_i| \le 1 + \frac{x}{4}) \int_{-\infty}^{\infty} K_i(t) \int_t^{Y_i} Eg(W_n^{(i)} + u) du\, dt \Big\} \Big|,$$

$$|A_{12}| = \sum_{n=1}^{\infty} p_n \sum_{i=1}^{n} E\Big\{ I(|Y_i| \le 1 + \frac{x}{4}) \times \int_{-\infty}^{\infty} I(x - \max(Y_i, t) < W_n^{(i)} \le x - \min(Y_i, t)) K_i(t) dt \Big\}.$$

By Lemma 4.7, we have

$$\begin{aligned}
|A_{11}| &= \sum_{n=1}^{\infty} p_n \sum_{i=1}^{n} \Big| E\Big\{ I(|Y_i| \le 1 + \frac{x}{4}) \int_{-\infty}^{\infty} K_i(t) \int_t^{Y_i} Eg(W_n^{(i)} + u) du\, dt \Big\} \Big| \\
&\le \frac{C}{(1+x)^3} \sum_{n=1}^{\infty} p_n \sum_{i=1}^{n} \Big| E\Big\{ I(|Y_i| \le 1 + \frac{x}{4}) \\
&\quad \times \int_{-\infty}^{\infty} K_i(t) \int_t^{Y_i} (1 + \delta_n x)\big(\frac{(Es_N^2)^3}{(s_n^2)^3} + \frac{(s_n^2)^4}{(Es_N^2)^4} \big) du\, dt \Big\} \Big| \\
&\le \frac{C}{(1+x)^3} \sum_{n=1}^{\infty} p_n (1 + \delta_n x)\big(\frac{(Es_N^2)^3}{(s_n^2)^3} + \frac{(s_n^2)^4}{(Es_N^2)^4} \big) \sum_{i=1}^{n} E \int_{-\infty}^{\infty} (|Y_i| + |t|) K_i(t) dt \\
&= \frac{C}{(1+x)^3} \sum_{n=1}^{\infty} p_n (1 + \delta_n x)\big(\frac{(Es_N^2)^3}{(s_n^2)^3} + \frac{(s_n^2)^4}{(Es_N^2)^4} \big) \sum_{i=1}^{n} (E|Y_i| E|Y_i|^2 + \frac{1}{2} E|Y_i|^3) \\
&\le \frac{C}{(1+x)^3} \sum_{n=1}^{\infty} p_n (1 + \delta_n x)\big(\frac{(Es_N^2)^3}{(s_n^2)^3} + \frac{(s_n^2)^4}{(Es_N^2)^4} \big) \sum_{i=1}^{n} E|Y_i|^3 \\
&\qquad\qquad \text{(by Hölder's inequality)} \\
&= \frac{C}{(1+x)^3} \sum_{n=1}^{\infty} p_n (1 + \delta_n x)\big(\frac{(Es_N^2)^3}{(s_n^2)^3} + \frac{(s_n^2)^4}{(Es_N^2)^4} \big) \frac{\delta_n s_n^2}{Es_N^2} \\
&= \frac{C}{(1+x)^3} \sum_{n=1}^{\infty} p_n \Big(\frac{\delta_n (Es_N^2)^2}{(s_n^2)^2} + \frac{\delta_n (s_n^2)^5}{(Es_N^2)^5} \Big) \\
&\quad + \frac{Cx}{(1+x)^3} \sum_{n=1}^{\infty} p_n \Big(\frac{\delta_n^2 (Es_N^2)^2}{(s_n^2)^2} + \frac{\delta_n^2 (s_n^2)^5}{(Es_N^2)^5} \Big) \\
&\le \frac{C}{(1+x)^3} \big(E[\frac{\delta_N}{(s_N^2)^2}](Es_N^2)^2 + \frac{E[\delta_N (s_N^2)^5]}{(Es_N^2)^5} \big) \\
&\quad + \frac{C}{(1+x)^2} \big(E[\frac{\delta_N^2}{(s_N^2)^2}](Es_N^2)^2 + \frac{E[\delta_N^2 (s_N^2)^5]}{(Es_N^2)^5} \big). \qquad (4.17)
\end{aligned}$$

Concentration inequality (Proposition 4.5) yields

$$
\begin{aligned}
&|A_{12}| \\
&= \sum_{n=1}^{\infty} p_n \sum_{i=1}^{n} E\Big\{ I(|Y_i| \le 1 + \frac{x}{4}) \\
&\quad \times \int_{-\infty}^{\infty} I(x - \max(Y_i, t) < W_n^{(i)} \le x - \min(Y_i, t)) K_i(t) dt \Big\} \\
&= \sum_{n=1}^{\infty} p_n \sum_{i=1}^{n} E\Big\{ I(|Y_i| \le 1 + \frac{x}{4}) \\
&\quad \times \int_{-\infty}^{\infty} P(x - \max(Y_i, t) < W_n^{(i)} \le x - \min(Y_i, t) | Y_i) K_i(t) dt \Big\} \\
&\le \frac{C}{(1+x)^3} \sum_{n=1}^{\infty} p_n \big(\frac{(Es_N^2)^3}{(s_n^2)^3} + \frac{(s_n^2)^4}{(Es_N^2)^4} \big) \\
&\quad \times \sum_{i=1}^{n} E\Big\{ I(|Y_i| \le 1 + \frac{x}{4}) \int_{-\infty}^{\infty} \big(\max\{Y_i, t\} - \min\{Y_i, t\} + \delta_n \big) K_i(t) dt \Big\} \\
&\le \frac{C}{(1+x)^3} \sum_{n=1}^{\infty} p_n \big(\frac{(Es_N^2)^3}{(s_n^2)^3} + \frac{(s_n^2)^4}{(Es_N^2)^4} \big) \\
&\quad \times \sum_{i=1}^{n} E\Big\{ I(|Y_i| \le 1 + \frac{x}{4}) \int_{|t| \le 1 + \frac{x}{4}} (|t| + |Y_i| + \delta_n) K_i(t) dt \Big\} \\
&\le \frac{C}{(1+x)^3} \sum_{n=1}^{\infty} p_n \big(\frac{Es_N^2}{s_n^2} + \frac{(s_n^2)^4}{(Es_N^2)^4} \big) \sum_{i=1}^{n} E \int_{-\infty}^{\infty} (|t| + |Y_i| + \delta_n) K_i(t) dt \\
&= \frac{C}{(1+x)^3} \sum_{n=1}^{\infty} p_n \big(\frac{(Es_N^2)^3}{(s_n^2)^3} + \frac{(s_n^2)^4}{(Es_N^2)^4} \big) \sum_{i=1}^{n} \big(\frac{1}{2} E|Y_i|^3 + E|Y_i| E|Y_i|^2 + \delta_n E|Y_i|^2 \big) \\
&\le \frac{C}{(1+x)^3} \sum_{n=1}^{\infty} p_n \big(\frac{(Es_N^2)^3}{(s_n^2)^3} + \frac{(s_n^2)^4}{(Es_N^2)^4} \big) \sum_{i=1}^{n} (E|Y_i|^3 + \delta_n E|Y_i|^2) \\
&= \frac{C}{(1+x)^3} \sum_{n=1}^{\infty} p_n \big(\frac{(Es_N^2)^3}{(s_n^2)^3} + \frac{(s_n^2)^4}{(Es_N^2)^4} \big) \big(\frac{\delta_n s_n^2}{Es_N^2} + \frac{\delta_n s_n^2}{Es_N^2} \big) \\
&= \frac{C}{(1+x)^3} \sum_{n=1}^{\infty} p_n \big(\frac{\delta_n (Es_N^2)^2}{(s_n^2)^2} + \frac{\delta_n (s_n^2)^5}{(Es_N^2)^5} \big) \\
&= \frac{C}{(1+x)^3} \Big(E[\frac{\delta_N}{(s_N^2)^2}] (Es_N^2)^2 + \frac{E[\delta_N (s_N^2)^5]}{(Es_N^2)^5} \Big).
\end{aligned}
\tag{4.18}
$$

Combining (4.16), (4.17) and (4.18), we have

$$
\begin{aligned}
|A_1| \le & \frac{C}{(1+x)^3} \Big(E[\frac{\delta_N}{(s_N^2)^2}] (Es_N^2)^2 + \frac{E[\delta_N (s_N^2)^5]}{(Es_N^2)^5} \Big) \\
& + \frac{C}{(1+x)^2} \Big(E[\frac{\delta_N^2}{(s_N^2)^2}] (Es_N^2)^2 + \frac{E[\delta_N^2 (s_N^2)^5]}{(Es_N^2)^5} \Big).
\end{aligned}
\tag{4.19}
$$

By Lemma 2.14(4), we have

$$
\begin{aligned}
|A_2| &\le \sum_{n=1}^{\infty} p_n \sum_{i=1}^{n} E\Big\{I(|Y_i| > 1 + \frac{x}{4}) \int_{-\infty}^{\infty} \big|f'(W_n^{(i)} + Y_i) - f'(W_n^{(i)} + t)\big| K_i(t)dt\Big\} \\
&\le \sum_{n=1}^{\infty} p_n \sum_{i=1}^{n} E\Big\{I(|Y_i| > 1 + \frac{x}{4}) \int_{-\infty}^{\infty} K_i(t)dt\Big\} \\
&= \sum_{n=1}^{\infty} p_n \sum_{i=1}^{n} P(|Y_i| > 1 + \frac{x}{4}) \frac{\sigma_i^2}{Es_N^2} \\
&\le \sum_{n=1}^{\infty} p_n \sum_{i=1}^{n} P(|Y_i| > 1 + \frac{x}{4}) \frac{s_n^2}{Es_N^2} \\
&\le \frac{C}{(1+x)^3} \sum_{n=1}^{\infty} p_n \frac{s_n^2}{Es_N^2} \sum_{i=1}^{n} E|Y_i|^3 \qquad \text{(by Chebysev's inequality)} \\
&= \frac{C}{(1+x)^3} \sum_{n=1}^{\infty} p_n \frac{\delta_n (s_n^2)^2}{(Es_N^2)^2} \\
&= \frac{CE[\delta_N (s_N^2)^2]}{(1+x)^3 (Es_N^2)^2}. \qquad (4.20)
\end{aligned}
$$

By Lemma 4.6, we have

$$
\begin{aligned}
|A_3| &\le \sum_{n=1}^{\infty} p_n \Big| Ef'(W_n) - \sum_{i=1}^{n} E \int_{-\infty}^{\infty} f'(W_n) K_i(t)dt \Big| \\
&\le \sum_{n=1}^{\infty} p_n |Ef'(W_n)| \Big| 1 - \sum_{i=1}^{n} \int_{-\infty}^{\infty} K_i(t)dt \Big| \\
&= \sum_{n=1}^{\infty} p_n |Ef'(W_n)| \Big| 1 - \frac{s_n^2}{Es_N^2} \Big| \\
&\le \frac{C}{(1+x)^2} \sum_{i=1}^{n} p_n (1 + \frac{s_n^2}{Es_N^2}) \Big| 1 - \frac{s_n^2}{Es_N^2} \Big| \\
&= \frac{CE|(s_N^2)^2 - (Es_N^2)^2|}{(1+x)^2 (Es_N^2)^2}. \qquad (4.21)
\end{aligned}
$$

Combining (4.15), (4.19), (4.20) and (4.21) yields,

$$
\begin{aligned}
&|P(W \le x) - \Phi(x)| \\
&\le \frac{C}{(1+x)^3} \big(E[\frac{\delta_N}{(s_N^2)^2}](Es_N^2)^2 + \frac{E[\delta_N (s_N^2)^5]}{(Es_N^2)^5} + \frac{E[\delta_N (s_N^2)^2]}{(Es_N^2)^2} \big) \\
&\quad + \frac{C}{(1+x)^2} \big(E[\frac{\delta_N^2}{(s_N^2)^2}](Es_N^2)^2 + \frac{E[\delta_N^2 (s_N^2)^5]}{(Es_N^2)^5} + \frac{E|(s_N^2)^2 - (Es_N^2)^2|}{(Es_N^2)^2} \big).
\end{aligned}
$$

Hence, the theorem is proved by applying (4.14) again. □

4.3 Proof of Corollary 4.2

Similar to the proof of Corollary 3.2 and by Theorem 4.1, it is trivial that

$$
\begin{aligned}
&|P(W \le x) - \Phi(x)| \\
&\le \frac{C}{(1+|x|)^3}\big(E[\frac{\delta_N}{(s_N^2)^2}](Es_N^2)^2 + \frac{E[\delta_N (s_N^2)^5]}{(Es_N^2)^5}\big) \\
&\quad + \frac{C}{(1+|x|)^2}\big(E[\frac{\delta_N^2}{(s_N^2)^2}](Es_N^2)^2 + \frac{E[\delta_N^2 (s_N^2)^5]}{(Es_N^2)^5} + \frac{E|(s_N^2)^2 - (Es_N^2)^2|}{(Es_N^2)^2}\big) \\
&= \frac{C\delta}{(1+|x|)^3}(\frac{(EN\sigma^2)^2}{E(N\sigma^2)^2} + \frac{E(N\sigma^2)^5}{(EN\sigma^2)^5}) \\
&\quad + \frac{C\delta^2}{(1+|x|)^2}(\frac{(EN\sigma^2)^2}{E(N\sigma^2)^2} + \frac{E(N\sigma^2)^5}{(EN\sigma^2)^5}) + \frac{CE|(N\sigma^2)^2 - (EN\sigma^2)^2|}{(1+|x|)^2(EN\sigma^2)^2} \\
&= \frac{C\delta}{(1+|x|)^3}(\frac{(EN)^2}{EN^2} + \frac{EN^5}{(EN)^5}) \\
&\quad + \frac{C\delta^2}{(1+|x|)^2}(\frac{(EN)^2}{EN^2} + \frac{EN^5}{(EN)^5}) + \frac{CE|N^2 - (EN)^2|}{(1+|x|)^2(EN)^2} \\
&\le \frac{C\delta}{(1+|x|)^3}(1 + \frac{EN^5}{(EN)^5}) + \frac{C\delta^2}{(1+|x|)^2}(1 + \frac{EN^5}{(EN)^5}) + \frac{CE|N^2 - (EN)^2|}{(1+|x|)^2(EN)^2}
\end{aligned}
$$

where we have used Hölder's inequality in the last inequality. $\square$

REFERENCES

[1] Batirov, Kh., Manevich, D.V., Nagaev, S.V.: The Esseen inequality for sum of a random number of differently distributed random variables, *Matematicheskie Zametki* 22, 143-146 (1980).

[2] Berry, A.C.: The accuracy of Gaussian approximation to the sum of independent variables, *Trans. Amer. Math Soc.* 49, 122-136 (1941).

[3] Chaidee, N., Neammanee, K.: Berry-Esseen bound for independent random sum via Stein's method, *International Mathematical Forum*, 15, 721-738 (2008).

[4] Chen, L.H.Y., Shao, Q.M.: A non-uniform Berry-Esseen bound via Stein's method, *Probab. Theory Related Fields* 120, 236-254 (2001).

[5] Esseen, C.G.: Fourier analysis of distribution functions: a mathematical study of the Laplace-Gaussian law, *Acta Math* 77, 11-25 (1945).

[6] Grimmett, G., Stirzaker, D.: *Probability and random Processes*, 3rd ed., Oxford university press Inc., New York, 2001.

[7] Feller, W.: On the Berry-Esseen theorem, *Z. Wahrsch. Verw. Gebite* 10, 261-268 (1986).

[8] Krajka, A., Rychilk, Z.: Necessary and sufficient conditions for weak convergence of random sums of independent random variables, *Comment. Math. Univ. Carolin.* 34, 465-482 (1993).

[9] Landers, D., Rogge, L.: The exact approximation order in the central limit theorem for random summation, *Z. Wahrsch. Vers. Gebiete* 36, 269-283 (1976).

[10] Nakas, A.: Estimate of the remainder term in the central limit theorem with a random number of random summands, *Liet. Mat. Rink.* 12, 157-163 (1972).

[11] Shiganov, I.S.: A refinement of the upper bound of the constant in the remainder term of the central limit theorem, *J. Soviet Math.* 3, 171-182 (1986).

[12] Stein, C.: *Approximation computation of expectation*, Hayword, California, 1986.

[13] Van Beeck, P.: An application of Fourier methods to the the problem of sharpening the Berry-Esseen inequality, *Z. Wahrsdh. Verw. Gebiete* 23, 187-196 (1972).

[14] Vellaisamy, P., Chaudhuri, B.: Poisson and compound Poisson approximations for random sums of random variables, *J. Applied Prob.* 33, 127-137 (1996).

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