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NORMAL APPROXIMATION FOR CALL FUNCTION VIA STEIN'S METHOD

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A call function is an important tool in finance. One uses this function to compute a payoff from investment. In this work, we are attentive to approximate the call function by dividing this work into three parts. First, we give uniform and non-uniform bounds on normal approximation for the call function without correction term. Second, we improve and find the explicit constants of a bound on normal approximation for the call function with a correction term proposed by Karoui and Jiao in 2009. Third, we propose uniform and non-uniform bounds of such approximation. In the part of non-uniform bounds we present both polynomial and exponential non-uniform bounds. Our techniques are Stein's method and the zero bias transformation.

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CONTENTS

CHAPTER I

INTRODUCTION

For a fixed positive real number k, the call function is given by

$$C_k(x) = (x - k)^+,$$

where $x^+ = \max\{x, 0\}$. It is an effective function for computing a payoff in finance. An interesting example of using the call function is appeared in collateralized debt obligation (CDO) tranche pricing.

CDO, which played a leading role in the financial crisis in the United State of America, is a type of financial instrument. The CDO is split into different risk classes known as tranche. Each tranche is assigned a different payment priority and interest rate. Investors can choose to invest on each tranche according to their risk aversion. In the standard CDO tranche pricing with underlying n assets, the i^{th} asset is assumed to have a recovery rate R_i. We can obtain the percentage loss at the time T by the total loss on the portfolio,

$$L(T) = \sum_{i=1}^{n} \frac{1}{n}(1 - R_i)\mathbb{I}_{\{\tau_i \leq T\}},$$

where τ_i is the default time of the i^{th} asset and $\mathbb{I}_A$ is an indicator function of A. The key of investing in CDO is to compute

$$E(L(T) - k)^+,$$

which is a call function on the percentage loss $L(T)$, where k is the attachment or the detachment point of the tranche. As a result, many methods are proposed to approximate this call function such as the analytic method by Hull and White

([7]) in 2004 and the Monte Carlo method by Cao et al. ([1]) in 2008. Our method used in this approximation problem is the Stein's method.

Let Z be the standard normal random variable. The main tool of the Stein's method for normal approximation is the Stein's equation, for a given function h, namely

$$xf(x) - f'(x) = h(x) - Eh(Z), \tag{1.1}$$

where f is a continuous and piecewise continuously differentiable function. Observe that the Stein's equation is a differential equation, so we are going to solve this equation and use the properties of its solution. If we apply the Stein's equation with $h(x) = (x-k)^+$, then (1.1) becomes

$$xf(x) - f'(x) = (x-k)^+ - E(Z-k)^+. \tag{1.2}$$

Denoted the solution of (1.2) by f_k.

Let $X_1, X_2, X_3, \ldots, X_n$ be independent random variables with zero means and finite variances $\sigma_1^2, \sigma_2^2, \sigma_3^2, \ldots, \sigma_n^2$ such that $\sum_{i=1}^{n} \sigma_i^2 = 1$. Define

$$W = \sum_{i=1}^{n} X_i.$$

In 2011, Chen et al. ([5], pp. 46) gave a bound on normal approximation for the Lipschitz function. The following is their result.

Theorem 1.1. *([5]) Let h be a Lipschitz function. If $E|X_i|^3 < \infty$ for $i = 1, 2, 3, \ldots, n$, then*

$$|Eh(W) - Eh(Z)| \leq 3 \sum_{i=1}^{n} E|X_i|^3.$$

It is obvious that the bound in Theorem 1.1 is a uniform bound. In this work, we are attentive to the case $h(x) = (x-k)^+$, which is a Lipschitz function. We give a non-uniform bound on normal approximation for the call function. Theorem 1.2 is our result.

Theorem 1.2. *Assume that EX_i^4 exists for $i = 1, 2, 3, \ldots, n$. Then*

$$\begin{aligned}&\left|E(W-k)^+ - E(Z-k)^+\right| \\ &\leq \frac{2.86}{e^{\frac{3k^2}{8}}} \sum_{i=1}^{n} E\left|X_i\right|^3 + \frac{16}{\sqrt{6}k}\left[\frac{1}{\sqrt{3}}\left(\sum_{i=1}^{n} EX_i^4 + 1\right)^{\frac{1}{2}} + 1\right]\left[\sum_{i=1}^{n} EX_i^4\right]^{\frac{1}{2}}.\end{aligned}$$

Notice that the rate of convergence of Theorem 1.1 and Theorem 1.2 is of order $\frac{1}{\sqrt{n}}$. However, Karoui and Jiao ([8]) improved the rate of convergence of the bound by adding a correction term which made the bound to be of order $\frac{1}{n}$ in 2009. Theorem 1.3 is their result.

Theorem 1.3. *([8]) Assume that EX_i^4 exists for $i = 1, 2, 3, \ldots, n$. Then*

$$\begin{aligned}\delta_n \leq& \left(60.68 + 3k + 4.8E(Z-k)^+\right)\left(\sum_{i=1}^{n} E|X_i|^3\right)^2 + \left(Var\left(f_k''(W)\right)\right)\left(\sum_{i=1}^{n} \sigma_i^6\right)^{\frac{1}{2}} \\ &+ \left(16.65 + 1.34k + 3.2E(Z-k)^+\right)\sum_{i=1}^{n} EX_i^4 \\ &+ \sum_{i=1}^{n} E\left|X_i\right|^3\left[4\sum_{j=1}^{n} \sigma_j^3 + 2.5\left(\sum_{j=1}^{n} \sigma_j^4\right)^{\frac{1}{2}} + 16\sigma_i\right],\end{aligned}$$

where $\delta_n := \left|E(W-k)^+ - E(Z-k)^+ - \frac{ke^{-\frac{k^2}{2}}}{6\sqrt{2\pi}}\sum_{i=1}^{n} EX_i^3\right|.$

There are unknown constants in Theorem 1.3, i.e., $E(Z-k)^+$ and $Var\left(f_k''(W)\right).$ In this work, we give bounds of these two terms and improve the constants in Theorem 1.3. Theorem 1.4 is the result.

Theorem 1.4. *Assume that EX_i^4 exists for $i = 1, 2, 3, \ldots, n$. Then*

$$\delta_n \leq (18.46 + 2k)\left(\sum_{i=1}^{n} E|X_i|^3\right)^2 + 2\left(\sum_{i=1}^{n} \sigma_i^6\right)^{\frac{1}{2}} + (9.38 + 1.34k)\sum_{i=1}^{n} EX_i^4.$$

The bound in Theorem 1.4 is not satisfactory in a case of large k. We improve this weak point by giving a uniform bound in Theorem 1.5. The bound in Theorem 1.5 does not depend on k. We also provide non-uniform bounds which is great for large k in Theorem 1.6 and Theorem 1.8.

Theorem 1.5 (Uniform bound)**.** *Assume that EX_i^6 exists for $i = 1, 2, 3, \ldots, n$. Then for $k < 1$, we have*

$$\delta_n \leq 20.46 \left(\sum_{i=1}^{n} E|X_i|^3 \right)^2 + 2 \left(\sum_{i=1}^{n} \sigma_i^6 \right)^{\frac{1}{2}} + 10.72 \sum_{i=1}^{n} EX_i^4,$$

and for $k \geq 1$, we have

$$\begin{aligned}\delta_n \leq\ & 13.76 \left(\sum_{i=1}^{n} E\left|X_i\right|^3 \right)^2 + 2 \left(\sum_{i=1}^{n} \sigma_i^6 \right)^{\frac{1}{2}} + \left[\frac{1}{\sqrt{3}} \left(\sum_{j=1}^{n} EX_j^4 + 1 \right)^{\frac{1}{2}} + 1 \right] \\ & \times \left[11.97 \left(\sum_{i=1}^{n} EX_i^6 \right)^{\frac{1}{2}} + 15.45 \sum_{i=1}^{n} E\left|X_i\right|^3 \left(\sum_{j=1}^{n} EX_j^4 \right)^{\frac{1}{2}} \right] + 6.25 \sum_{i=1}^{n} EX_i^4.\end{aligned}$$

Theorem 1.6 (Polynomial Non-uniform)**.** *Assume that EX_i^6 exists for $i = 1, 2, 3, \ldots, n$. Then for $k \geq 4$, we have*

$$\begin{aligned}\delta_n \leq\ & \left[\frac{4.74}{e^{\frac{k^2}{4}}} + \frac{285.94}{k} \right] \left(\sum_{i=1}^{n} E\left|X_i\right|^3 \right)^2 + \left[\frac{3.16}{e^{\frac{k^2}{4}}} + \frac{1.36}{k} \right] \sum_{i=1}^{n} EX_i^4 \\ & + \frac{1}{k} \left\{ \frac{1}{\sqrt{5}} \left[\sum_{j=1}^{n} EX_j^6 + \sum_{j=1}^{n} EX_j^4 + \left(\sum_{j=1}^{n} E\left|X_j\right|^3 \right)^2 + 1 \right]^{\frac{1}{2}} + \left[\sum_{j=1}^{n} EX_j^4 + 1 \right]^{\frac{1}{2}} \right\} \\ & \times \left[30.90 \left[\sum_{i=1}^{n} E\left|X_i\right|^3 \right] \left[\sum_{i=1}^{n} EX_i^4 \right]^{\frac{1}{2}} + 23.94 \left[\sum_{i=1}^{n} EX_i^6 \right]^{\frac{1}{2}} \right] \\ & + \frac{1}{k} \sum_{i=1}^{n} \sigma_i \left[14.32 \left[EX_i^6 \right]^{\frac{1}{2}} + 31.56 \sum_{j=1}^{n} E\left|X_j\right|^3 \left[EX_i^4 \right]^{\frac{1}{2}} \right] \left[\frac{EX_i^4}{3\sigma_i^2} + \sigma_i^2 \right]^{\frac{1}{2}} \\ & + \left(\frac{0.64}{e^{\frac{3k^2}{8}}} + \frac{16}{k^2} \right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} \sigma_i^6 \right)^{\frac{1}{2}}.\end{aligned}$$

Corollary 1.7. *If X_i's are independent and identically distributed random variable for $i = 1, 2, 3, \ldots, n$, then for $k \geq 4$, we have*

$$\delta_n \leq \left[\frac{4.74}{e^{\frac{k^2}{4}}} + \frac{285.94}{k}\right] \frac{\left(E|X_1|^3\right)^2}{\sigma_1^6 n} + \left[\frac{3.16}{e^{\frac{k^2}{4}}} + \frac{1.36}{k}\right] \frac{EX_1^4}{\sigma_1^4 n} + \frac{1}{n}\left(\frac{0.64}{e^{\frac{3k^2}{8}}} + \frac{16}{k^2}\right)^{\frac{1}{2}}$$
$$+ \frac{1}{kn}\left\{\frac{1}{\sqrt{5}}\left[\frac{EX_1^6}{\sigma_1^6 n^2} + \frac{EX_1^4}{\sigma_1^4 n} + \frac{\left(E|X_1|^3\right)^2}{\sigma_1^6 n} + 1\right]^{\frac{1}{2}} + \left[\frac{EX_i^4}{\sigma_1^4 n} + 1\right]^{\frac{1}{2}}\right\}$$
$$\times \left[\frac{30.90E|X_1|^3 (EX_1^4)^{\frac{1}{2}}}{\sigma_1^5} + \frac{23.94\,(EX_1^6)^{\frac{1}{2}}}{\sigma_1^3}\right]$$
$$+ \frac{1}{kn\sqrt{n}}\left[\frac{14.32(EX_1^6)^{\frac{1}{2}}}{\sigma_1^3} + \frac{31.56E|X_1|^3 (EX_1^4)^{\frac{1}{2}}}{\sigma_1^5}\right]\left[\frac{EX_1^4}{3\sigma_1^4} + 1\right]^{\frac{1}{2}}.$$

Theorem 1.8 (Exponential Non-uniform)**.** *Assume that $|X_i| < \beta$ for $i = 1, 2, 3, \ldots, n$ and for some $\beta > 0$. Then*

$$\delta_n \leq \frac{7.90}{e^{\frac{k^2}{4}}}\beta^2 + \frac{109.67}{e^{\frac{k}{8}}} e^{\frac{\theta}{2}}\left(e^{\beta} + 1\right)\beta^2 + \frac{C_\beta}{2e^{\frac{k}{2}}}\beta^2 + \left(\frac{2.86}{e^{\frac{3k^2}{8}}} + \frac{4e^{\theta}}{e^{\frac{k}{2}}}\right)^{\frac{1}{2}}\beta^2$$
$$+ \frac{1}{e^{\frac{3k}{8}}}\left(17.34e^{\frac{\theta}{2}+\frac{3\beta}{4}} + 2e^{\frac{\theta}{2}+\frac{\beta}{2}}\right)\beta^2 + \frac{16.23}{e^{\frac{k}{8}}} e^{\frac{\beta}{2}}\beta^2,$$

where $C_\beta = 4.45 + 2.21e^{2\beta+\beta^{-2}\left(e^{2\beta}-1-2\beta\right)}$ *and* $\theta = \beta^{-2}\left(e^{\beta} - 1 - \beta\right)$.

Observe that if $\beta \to 0$ we have $e^{\theta} \to \sqrt{e}$ and $C_\beta \to 20.78$. Hence $\delta_n \to 0$.

In this thesis, we organize as follows. Chapter I is used to describe Introduction and main results. A background of the Stein's method, some useful properties of the Stein's solution and the zero bias transformation are presented in Chapter II. Uniform and non-uniform bounds on normal approximation for the call function without correction term are obtained in Chapter III. Improvement of the constant in Theorem 1.3, uniform bound, polynomial and exponential non-uniform bounds on normal approximation for the call function with a correction term are presented in Chapter IV.

CHAPTER II

STEIN'S METHOD FOR NORMAL APPROXIMATION

In this chapter, the idea of the Stein's method for normal approximation is presented in Section 2.1. We also give some properties of the Stein's solution which are useful facts for obtaining a bound in this section. The definition of zero bias transformation is shown in Section 2.2.

2.1 Stein's Method for Normal Approximation

In this section, we introduce a brilliant method provided by Stein ([12]) in 1972. Let Z be the standard normal random variable with distribution function Φ, $\mathcal{C}_{bd}$ the set of continuous and piecewise continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$ with $E|f'(Z)| < \infty$. The Stein's method begins with the Stein's equation for normal approximation

$$xf(x) - f'(x) = h(x) - Eh(Z), \tag{2.1}$$

for a given function h and $f \in \mathcal{C}_{bd}$.

In this work, we apply the Stein's equation (2.1) with $h(x) = (x-k)^+$ and $k > 0$. By (2.1), we have

$$xf(x) - f'(x) = (x-k)^+ - E(Z-k)^+. \tag{2.2}$$

Denoted the solution of (2.2) by f_k. From (2.2), we have

$$EWf_k(W) - Ef_k'(W) = E(W-k)^+ - E(Z-k)^+$$

for any random variable W. Thus, we can bound $|EWf_k(W) - Ef_k'(W)|$ instead of $|E(W-k)^+ - E(Z-k)^+|$. This technique is called the Stein's method.

The next proposition shows an explicit form of $E(Z-k)^+$ and its bound.

Proposition 2.1.

(1) $E\left(Z-k\right)^+ = \dfrac{e^{-\frac{k^2}{2}}}{\sqrt{2\pi}} - k\Phi(-k).$

(2) $E\left(Z-k\right)^+ \leq \dfrac{e^{-\frac{k^2}{2}}}{\sqrt{2\pi}} \min\left\{1, \dfrac{1}{k^2}\right\}.$

Proof.

(1) By the definition of an expectation and the call function, we obtain

$$\begin{aligned} E(Z-k)^+ &= \int_{-\infty}^{\infty} (x-k)^+ \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\, dx \\ &= \int_{k}^{\infty} (x-k) \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\, dx \\ &= \int_{k}^{\infty} \frac{xe^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\, dx - k\int_{k}^{\infty} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\, dx \\ &= \frac{e^{-\frac{k^2}{2}}}{\sqrt{2\pi}} - k\Phi(-k). \end{aligned}$$

(2) By the fact that

$$\int_x^{\infty} e^{-\frac{t^2}{2}} dt \geq \frac{x}{1+x^2} e^{-\frac{x^2}{2}} \qquad \text{for } x > 0,$$

([3], pp. 252) and

$$\frac{x}{1+x^2} \geq \frac{1}{x} - \frac{1}{x^3} \qquad \text{for } x > 0,$$

we have

$$\begin{aligned}\Phi(x) &= 1-\Phi(-x)\\ &= \frac{1}{\sqrt{2\pi}}\int_{-x}^{\infty} e^{-\frac{t^2}{2}}\,dt\\ &\geq \frac{-x}{1+x^2}\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\\ &\geq \left(-\frac{1}{x}+\frac{1}{x^3}\right)\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \qquad \text{for } x<0. \end{aligned} \tag{2.3}$$

From (2.3) and (1), we obtain

$$\begin{aligned}E(Z-k)^+ &\leq \frac{e^{-\frac{k^2}{2}}}{\sqrt{2\pi}} - k\left(\frac{1}{k}-\frac{1}{k^3}\right)\frac{e^{-\frac{k^2}{2}}}{\sqrt{2\pi}}\\ &= \frac{e^{-\frac{k^2}{2}}}{\sqrt{2\pi}k^2}.\end{aligned}$$

This implies that

$$E\left(Z-k\right)^+ \leq \frac{e^{-\frac{k^2}{2}}}{\sqrt{2\pi}}\min\left\{1,\frac{1}{k^2}\right\}.$$

□

In order to bound $|EWf_k(W)-Ef_k'(W)|$ for any random variable W, we need some properties of f_k''. In 1986, Stein ([11], pp. 25) showed that for an absolutely continuous function h, if h' exists and bounded on $\mathbb{R}$, then f_h'' exists and

$$\|f_h''\| \leq 2\|h'\|, \tag{2.4}$$

where $\|f\| = \sup_{x\in\mathbb{R}}|f(x)|$ and f_h is the solution of (2.1). In fact, we can apply his proof to show that for an absolutely continuous function h, if h' exists and bounded on a subset B of $\mathbb{R}$, then

$$\|f_h''\|_B \leq 2\|h'\|_B, \tag{2.5}$$

where $\|f\|_B = \sup_{x\in B}|f(x)|$. In the case of the call function, which is an absolutely continuous function, we have $h'(x) = C_k'(x) = \mathbb{I}\left(x>k\right)$ ([8], pp. 161), where

$\mathbb{I}(A) = \mathbb{I}_A$ is an indicator function of a set A. This implies that $\|h'\| \le 1$. By (2.4), we obtain

$$\|f_k''\| \le 2. \tag{2.6}$$

In our work, we need other properties of f_k'', which are shown in the following propositions.

Proposition 2.2.

$$|f_k''(x)| \le \frac{1.43}{e^{\frac{3k^2}{8}}} \qquad \text{for } x \le \frac{k}{2}.$$

Proof. To bound f_k'', we first give an explicit form of f_k. Stein ([11], pp. 15) showed that the unique solution f_k is given by

$$f_k(x) = e^{\frac{x^2}{2}} \int_x^{\infty} e^{-\frac{t^2}{2}} [(t-k)^+ - E\,(Z-k)^+]dt. \tag{2.7}$$

From (2.7) and the fact that $\int_{-\infty}^{\infty} e^{-\frac{t^2}{2}} \left[(t-k)^+ - E\,(Z-k)^+\right] dt = 0$, we have

$$\begin{aligned}
f_k(x) &= -e^{\frac{x^2}{2}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} [(t-k)^+ - E(Z-k)^+]dt \\
&= e^{\frac{x^2}{2}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} E(Z-k)^+ dt \\
&= e^{\frac{x^2}{2}} \sqrt{2\pi} E(Z-k)^+ \int_{-\infty}^{x} \frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}} dt \\
&= e^{\frac{x^2}{2}} \sqrt{2\pi} E(Z-k)^+ \Phi(x)
\end{aligned}$$

for $x \le k$, and

$$\begin{aligned}
f_k(x) &= e^{\frac{x^2}{2}} \int_x^{\infty} e^{-\frac{t^2}{2}} [(t-k) - E(Z-k)^+]dt \\
&= e^{\frac{x^2}{2}} \int_x^{\infty} t e^{-\frac{t^2}{2}} dt - e^{\frac{x^2}{2}} [k + E(Z-k)^+] \int_x^{\infty} e^{-\frac{t^2}{2}} dt \\
&= e^{\frac{x^2}{2}} e^{-\frac{x^2}{2}} - \sqrt{2\pi} e^{\frac{x^2}{2}} [k + E(Z-k)^+] \Phi(-x) \\
&= 1 - \sqrt{2\pi} e^{\frac{x^2}{2}} [k + E(Z-k)^+] \Phi(-x)
\end{aligned}$$

for $x > k$. Thus

$$f_k'(x) = \begin{cases} E(Z-k)^+ \left(1+\sqrt{2\pi}x\Phi(x)e^{\frac{x^2}{2}}\right) & \text{if } x < k; \\ [k+E(Z-k)^+] \left(1-\sqrt{2\pi}x\Phi(-x)e^{\frac{x^2}{2}}\right) & \text{if } x > k. \end{cases}$$

We can see that the derivative of f_k does not exist only at the point k. Then, for making the Stein's equation (2.2) holds, we let

$$f_k'(k) = E(Z-k)^+ \left(1+\sqrt{2\pi}\Phi(k)e^{\frac{k^2}{2}}\right).$$

Hence,

$$f_k'(x) = \begin{cases} E(Z-k)^+ \left(1+\sqrt{2\pi}x\Phi(x)e^{\frac{x^2}{2}}\right) & \text{if } x \leq k; \\ [k+E(Z-k)^+] \left(1-\sqrt{2\pi}x\Phi(-x)e^{\frac{x^2}{2}}\right) & \text{if } x > k. \end{cases}$$

It follows that

$$f_k''(x) = \begin{cases} E(Z-k)^+ \left(x+\sqrt{2\pi}\Phi(x)e^{\frac{x^2}{2}}(x^2+1)\right) & \text{if } x < k; \\ (k+E(Z-k)^+) \left(x-\sqrt{2\pi}\Phi(-x)e^{\frac{x^2}{2}}(x^2+1)\right) & \text{if } x > k. \end{cases}$$

Again, $f_k'(x)$ is not differentiable at $x = k$. For convenience, we assume that

$$f_k''(k) = E(Z-k)^+ \left(k+\sqrt{2\pi}\Phi(k)e^{\frac{k^2}{2}}(k^2+1)\right).$$

Hence,

$$f_k''(x) = \begin{cases} E(Z-k)^+ \left(x+\sqrt{2\pi}\Phi(x)e^{\frac{x^2}{2}}(x^2+1)\right) & \text{if } x \leq k; \\ (k+E(Z-k)^+) \left(x-\sqrt{2\pi}\Phi(-x)e^{\frac{x^2}{2}}(x^2+1)\right) & \text{if } x > k. \end{cases}$$

Now, we find a bound of $f_k''(x)$ for $x \leq \frac{k}{2}$.

If $-\frac{1}{2} < x < 0$, then $\Phi(x) \leq \frac{1}{2}$. This implies that

$$-\frac{1}{2} < x+\sqrt{2\pi}\Phi(x)e^{\frac{x^2}{2}}(x^2+1) < \frac{\sqrt{2\pi}}{2}e^{\frac{1}{8}}\left(\frac{1}{4}+1\right) \leq 1.78.$$

Hence,

$$|f_k''(x)| \leq 1.78E(Z-k)^+ \qquad \text{for } -\frac{1}{2} < x < 0. \tag{2.8}$$

In the case of $x \le -\frac{1}{2}$, from (2.3) and the fact that

$$\Phi(x) \le -\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}x} \qquad \text{for } x < 0 \tag{2.9}$$

([11], pp. 23), we have

$$x + \sqrt{2\pi}\Phi(x)e^{\frac{x^2}{2}}(x^2+1) \le -\frac{1}{x} \le 2,$$

and

$$x + \sqrt{2\pi}\Phi(x)e^{\frac{x^2}{2}}(x^2+1) \ge x + \sqrt{2\pi}\Phi(x)e^{\frac{x^2}{2}}x^2 \ge \frac{1}{x} \ge -2.$$

Hence, $\left|x + \sqrt{2\pi}\Phi(x)e^{\frac{x^2}{2}}(x^2+1)\right| \le 2$. Consequently,

$$|f_k''(x)| \le 2E(Z-k)^+ \qquad \text{for } x \le -\frac{1}{2}. \tag{2.10}$$

From (2.8), (2.10) and the fact that

$$0 \le f_k''(x) \le E\,(Z-k)^+\left[\frac{k}{2} + \sqrt{2\pi}e^{\frac{k^2}{8}}\left(\frac{k^2}{4}+1\right)\right] \qquad \text{for } 0 \le x \le \frac{k}{2},$$

we have

$$|f_k''(x)| \le \begin{cases} 2E(Z-k)^+ & \text{for } x < 0; \\ E(Z-k)^+\left[\frac{k}{2} + \sqrt{2\pi}e^{\frac{k^2}{8}}\left(\frac{k^2}{4}+1\right)\right] & \text{for } 0 \le x \le \frac{k}{2}. \end{cases}$$

Since $\sqrt{2\pi}e^{\frac{k^2}{8}}\left(\frac{k^2}{4}+1\right) \ge \sqrt{2\pi} \ge 2$,

$$\begin{aligned} |f_k''(x)| &\le E\,(Z-k)^+\left[\frac{k}{2} + \sqrt{2\pi}e^{\frac{k^2}{8}}\left(\frac{k^2}{4}+1\right)\right] \\ &\le \frac{e^{-\frac{k^2}{2}}}{\sqrt{2\pi}}\left[\frac{k}{2} + \sqrt{2\pi}e^{\frac{k^2}{8}}\left(\frac{k^2}{4}+1\right)\right]\min\left\{1, \frac{1}{k^2}\right\} \qquad \text{for } x \le \frac{k}{2}. \end{aligned} \tag{2.11}$$

If $k \ge 1$, then

$$\begin{aligned} \frac{e^{-\frac{k^2}{2}}}{\sqrt{2\pi}}\left[\frac{k}{2} + \sqrt{2\pi}e^{\frac{k^2}{8}}\left(\frac{k^2}{4}+1\right)\right]\min\left\{1, \frac{1}{k^2}\right\} &= \frac{e^{-\frac{k^2}{2}}}{\sqrt{2\pi}}\left[\frac{k}{2} + \sqrt{2\pi}e^{\frac{k^2}{8}}\left(\frac{k^2}{4}+1\right)\right]\frac{1}{k^2} \\ &= \frac{e^{-\frac{k^2}{2}}}{2\sqrt{2\pi}k} + \frac{e^{-\frac{3k^2}{2}}}{4} + \frac{e^{-\frac{3k^2}{8}}}{k^2} \\ &= e^{-\frac{3k^2}{8}}\left(\frac{e^{-\frac{k^2}{8}}}{2\sqrt{2\pi}k} + \frac{1}{4} + \frac{1}{k^2}\right) \\ &\le \frac{1.43}{e^{\frac{3k^2}{8}}}, \end{aligned} \tag{2.12}$$

where we use the fact that $\frac{e^{-\frac{k^2}{8}}}{2\sqrt{2\pi}k} + \frac{1}{4} + \frac{1}{k^2}$ has the maximum value at $k = 1$ in the last inequality.

For $k < 1$, by the fact that $\frac{ke^{-\frac{k^2}{8}}}{2\sqrt{2\pi}} + \frac{k^2}{4} + 1$ has the maximum value at $k = 1$, we have

$$\begin{aligned}
\frac{e^{-\frac{k^2}{2}}}{\sqrt{2\pi}}\left[\frac{k}{2} + \sqrt{2\pi}e^{\frac{k^2}{8}}\left(\frac{k^2}{4}+1\right)\right]\min\left\{1, \frac{1}{k^2}\right\} &= \frac{e^{-\frac{k^2}{2}}}{\sqrt{2\pi}}\left[\frac{k}{2} + \sqrt{2\pi}e^{\frac{k^2}{8}}\left(\frac{k^2}{4}+1\right)\right] \\
&= \frac{ke^{-\frac{k^2}{2}}}{2\sqrt{2\pi}} + \frac{k^2e^{-\frac{3k^2}{2}}}{4} + e^{-\frac{3k^2}{8}} \\
&= e^{-\frac{3k^2}{8}}\left(\frac{ke^{-\frac{k^2}{8}}}{2\sqrt{2\pi}} + \frac{k^2}{4} + 1\right) \\
&\leq \frac{1.43}{e^{\frac{3k^2}{8}}}. \qquad (2.13)
\end{aligned}$$

By (2.11), (2.12) and (2.13), we have Proposition 2.2. □

Proposition 2.3. *For $k \geq 4$, let W be a random variable with $EW^2 = 1$. Then*

$$E\left(f_k''(W)\right)^2 \leq \frac{0.64}{e^{\frac{3k^2}{8}}} + \frac{16}{k^2}.$$

Proof. From (2.6), (2.11) and $k \geq 4$, we have

$$\begin{aligned}
E\left|f_k''(W)\right| &= E\left[\left|f_k''(W)\right|\mathbb{I}\left(W \leq \frac{k}{2}\right)\right] + E\left[\left|f_k''(W)\right|\mathbb{I}\left(W > \frac{k}{2}\right)\right] \qquad (2.14) \\
&\leq \frac{e^{-\frac{k^2}{2}}}{\sqrt{2\pi}}\left[\frac{k}{2} + \sqrt{2\pi}e^{\frac{k^2}{8}}\left(\frac{k^2}{4}+1\right)\right]\min\left\{1, \frac{1}{k^2}\right\} + 2P\left(W > \frac{k}{2}\right) \\
&\leq \frac{e^{-\frac{k^2}{2}}}{\sqrt{2\pi}k^2}\left[\frac{k}{2} + \sqrt{2\pi}e^{\frac{k^2}{8}}\left(\frac{k^2}{4}+1\right)\right] + \frac{8EW^2}{k^2} \\
&= e^{-\frac{3k^2}{8}}\left[\frac{e^{-\frac{k^2}{8}}}{2\sqrt{2\pi}k} + \frac{1}{4} + \frac{1}{k^2}\right] + \frac{8}{k^2}.
\end{aligned}$$

From this fact and (2.6), we have

$$\begin{aligned}
E\left(f_k''(W)\right)^2 &\leq 2E\left|f_k''(W)\right| \\
&\leq e^{-\frac{3k^2}{8}}\left[\frac{e^{-\frac{k^2}{8}}}{\sqrt{2\pi}k} + \frac{1}{2} + \frac{2}{k^2}\right] + \frac{16}{k^2} \\
&\leq \frac{0.64}{e^{\frac{3k^2}{8}}} + \frac{16}{k^2},
\end{aligned}$$

where we use the fact that $f(k) = \frac{e^{-\frac{k^2}{8}}}{\sqrt{2\pi}k} + \frac{1}{2} + \frac{2}{k^2}$ has the maximum value $f(4) \leq 0.64$ at $k = 4$. $\square$

In the next proposition, we give an exponential bound of $E\left(f_k''(W)\right)^2$. To do this, we use the following Bennett-Hoeffding inequality.

Lemma 2.4 ([4], pp. 40)**.** *Let $X_1, X_2, X_3, \ldots, X_n$ be independent random variables with $EX_i \leq 0, X_i \leq \beta$ for $i = 1, 2, 3, \ldots, n$, for some real number β and $\sum_{i=1}^{n} EX_i^2 \leq B_n^2$. Put $W = \sum_{i=1}^{n} X_i$. Then*

$$Ee^{tW} \leq e^{\beta^{-2}\left(e^{t\beta}-1-t\beta\right)B_n^2} \qquad \text{for } t > 0.$$

Proposition 2.5. *Let $X_1, X_2, X_3, \ldots, X_n$ be independent random variables with zero means and finite variances $\sigma_1^2, \sigma_2^2, \sigma_3^2, \ldots, \sigma_n^2$ such that $\sum_{i=1}^{n} \sigma_i^2 = 1$. Assume that $|X_i| \leq \beta$ for $i = 1, 2, 3, \ldots, n$ and for some $\beta > 0$. Let $W = \sum_{i=1}^{n} X_i$. Then for $k \geq 1$, we have*

$$E(f_k''(W))^2 \leq \frac{2.86}{e^{\frac{3k^2}{8}}} + \frac{4e^{\theta}}{e^{\frac{k}{2}}},$$

where $\theta = \beta^{-2}\left(e^{\beta} - 1 - \beta\right)$.

Proof. By Lemma 2.4 with $B_n^2 = 1$ and $t = 1$, we have

$$P\left(W > \frac{k}{2}\right) \leq \frac{Ee^{W}}{e^{\frac{k}{2}}} \leq \frac{1}{e^{\frac{k}{2}}} e^{\beta^{-2}\left(e^{\beta}-1-\beta\right)} = \frac{e^{\theta}}{e^{\frac{k}{2}}}, \tag{2.15}$$

where $\theta = \beta^{-2}\left(e^{\beta} - 1 - \beta\right)$. From (2.6), (2.14) and (2.15), we obtain

$$\begin{aligned} E\left(f_k''(W)\right)^2 &\leq 2E\left|f_k''(W)\right| \\ &\leq \frac{2.86}{e^{\frac{3k^2}{8}}} + 4P\left(W > \frac{k}{2}\right) \\ &\leq \frac{2.86}{e^{\frac{3k^2}{8}}} + \frac{4e^{\theta}}{e^{\frac{k}{2}}}. \end{aligned}$$

$\square$

In the next proposition, we give bounds of G'' where $G(x) = xf_k(x)$. Note that

$$G'(x) = xf_k'(x) + f_k(x) \tag{2.16}$$

and

$$\begin{aligned} G''(x) &= xf_k''(x) + 2f_k'(x) \\ &= \begin{cases} E(Z-k)^+ \left(x^2 + 2 + \sqrt{2\pi}\Phi(x)e^{\frac{x^2}{2}}\left(x^3+3x\right)\right) & \text{if } x \le k; \\ (k + E(Z-k)^+) \left(x^2 + 2 - \sqrt{2\pi}\Phi(-x)e^{\frac{x^2}{2}}\left(x^3+3x\right)\right) & \text{if } x > k. \end{cases} \end{aligned}$$

Proposition 2.6.

(1) $\|G''\|_{(-\infty, \frac{k}{2}]} \le \dfrac{2.37}{e^{\frac{k^2}{4}}}$ *for* $k \ge 1$.

(2) $\|G''\| \le \dfrac{3e^{-\frac{k^2}{2}}}{\sqrt{2\pi}} + k + 3$ *for* $k > 0$.

Proof.

(1) Let $x \le \frac{k}{2}$ and $k \ge 1$.

If $x \ge 0$, then by Proposition 2.1 (2), we obtain

$$\begin{aligned} 0 \le G''(x) &\le \frac{e^{-\frac{k^2}{2}}}{\sqrt{2\pi}k^2}\left(\frac{k^2}{4} + 2 + \sqrt{2\pi}e^{\frac{k^2}{8}}\left(\frac{k^3}{8} + \frac{3k}{2}\right)\right) \\ &= \frac{e^{-\frac{k^2}{2}}}{4\sqrt{2\pi}} + \frac{2e^{-\frac{k^2}{2}}}{\sqrt{2\pi}k^2} + \frac{e^{-\frac{3k^2}{8}}k}{8} + \frac{3e^{-\frac{3k^2}{8}}}{2k}. \end{aligned} \tag{2.17}$$

Suppose that $x < 0$. By (2.9), we have

$$\begin{aligned} \Phi(x)e^{\frac{x^2}{2}}\left(x^3+3x\right) &\ge \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}(-x)}e^{\frac{x^2}{2}}(x^3+3x) \\ &= -\frac{1}{\sqrt{2\pi}}\left(x^2+3\right) \end{aligned}$$

and by (2.3), we have

$$\begin{aligned} \Phi(x)e^{\frac{x^2}{2}}\left(x^3+3x\right) &\le \left(-\frac{1}{x} + \frac{1}{x^3}\right)\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}e^{\frac{x^2}{2}}(x^3+3x) \\ &= \frac{1}{\sqrt{2\pi}}\left(-x^2 - 2 + \frac{3}{x^2}\right). \end{aligned}$$

This implies that

$$-1 \leq x^2 + 2 + \sqrt{2\pi}\Phi(x)e^{\frac{x^2}{2}}\left(x^3 + 3x\right) \leq \frac{3}{x^2} \qquad \text{for } x < 0. \tag{2.18}$$

Hence,

$$\left| x^2 + 2 + \sqrt{2\pi}\Phi(x)e^{\frac{x^2}{2}}\left(x^3 + 3x\right) \right| \leq \max\left\{1, \frac{3}{k^2}\right\} \leq 1 + \frac{2}{k^2} \qquad \text{for } x < -k.$$

From this fact and Proposition 2.1 (2), we have

$$|G''(x)| \leq \frac{e^{-\frac{k^2}{2}}}{\sqrt{2\pi}}\left(1 + \frac{2}{k^2}\right) = \frac{e^{-\frac{k^2}{2}}}{\sqrt{2\pi}} + \frac{2e^{-\frac{k^2}{2}}}{\sqrt{2\pi}k^2} \qquad \text{for } x < -k. \tag{2.19}$$

Let $-k \leq x < 0$. By (2.18) and the fact that

$$x^2 + 2 + \sqrt{2\pi}\Phi(x)e^{\frac{x^2}{2}}\left(x^3 + 3x\right) \leq x^2 + 2 \qquad \text{for } x < 0, \tag{2.20}$$

we obtain $\left| x^2 + 2 + \sqrt{2\pi}\Phi(x)e^{\frac{x^2}{2}}\left(x^3 + 3x\right) \right| \leq k^2 + 2$. Thus, by Proposition 2.1 (2), we obtain

$$|G''(x)| \leq \frac{e^{-\frac{k^2}{2}}}{\sqrt{2\pi}k^2}(k^2 + 2) = \frac{e^{-\frac{k^2}{2}}}{\sqrt{2\pi}} + \frac{2e^{-\frac{k^2}{2}}}{\sqrt{2\pi}k^2} \qquad \text{for } -k \leq x < 0. \tag{2.21}$$

It follows from (2.17), (2.19) and (2.21) that

$$|G''(x)| \leq \frac{e^{-\frac{k^2}{2}}}{\sqrt{2\pi}} + \frac{2e^{-\frac{k^2}{2}}}{\sqrt{2\pi}k^2} + \frac{e^{-\frac{3k^2}{8}}k}{8} + \frac{3e^{-\frac{3k^2}{8}}}{2k} \qquad \text{for } x \leq \frac{k}{2} \text{ and } k \geq 1. \tag{2.22}$$

Hence,

$$\begin{aligned} |G''(x)| &\leq \frac{1}{e^{\frac{k^2}{4}}}\left(\frac{e^{-\frac{k^2}{4}}}{\sqrt{2\pi}} + \frac{2e^{-\frac{k^2}{4}}}{\sqrt{2\pi}k^2} + \frac{e^{-\frac{k^2}{8}}k}{8} + \frac{3e^{-\frac{k^2}{8}}}{2k}\right) \\ &\leq \frac{2.37}{e^{\frac{k^2}{4}}} \qquad \text{for } x \leq \frac{k}{2} \text{ and } k \geq 1, \end{aligned}$$

where we use the fact that $\frac{e^{-\frac{k^2}{4}}}{\sqrt{2\pi}} + \frac{2e^{-\frac{k^2}{4}}}{\sqrt{2\pi}k^2} + \frac{e^{-\frac{k^2}{8}}k}{8} + \frac{3e^{-\frac{k^2}{8}}}{2k}$ has the maximum value at $k = 1$ in the second inequality.

(2) We divide the proof into 2 cases.

Case 1. $k \geq 1$.

Let $k \geq 1$. If $\frac{k}{2} < x \leq k$, then by Proposition 2.1 (2), we have

$$\begin{aligned} 0 \leq G''(x) &\leq \frac{e^{-\frac{k^2}{2}}}{\sqrt{2\pi}k^2}\left(k^2+2+\sqrt{2\pi}e^{\frac{k^2}{2}}\left(k^3+3k\right)\right) \\ &= \frac{e^{-\frac{k^2}{2}}}{\sqrt{2\pi}} + \frac{2e^{-\frac{k^2}{2}}}{\sqrt{2\pi}k^2} + k + \frac{3}{k}. \end{aligned} \tag{2.23}$$

Suppose that $x > k$. By (2.9), we have

$$\begin{aligned} \Phi(-x)e^{\frac{x^2}{2}}\left(x^3+3x\right) &\leq \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}x}e^{\frac{x^2}{2}}(x^3+3x) \\ &= \frac{1}{\sqrt{2\pi}}(x^2+3) \end{aligned}$$

and by (2.3), we have

$$\begin{aligned} \Phi(-x)e^{\frac{x^2}{2}}\left(x^3+3x\right) &\geq \left(\frac{1}{x}-\frac{1}{x^3}\right)\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}e^{\frac{x^2}{2}}(x^3+3x) \\ &= \frac{1}{\sqrt{2\pi}}\left(x^2+2-\frac{3}{x^2}\right). \end{aligned}$$

Hence

$$-1 \leq x^2+2-\sqrt{2\pi}\Phi(-x)e^{\frac{x^2}{2}}\left(x^3+3x\right) \leq \frac{3}{x^2} \leq \frac{3}{k^2} \leq 1+\frac{2}{k^2} \qquad \text{for } x > k. \tag{2.24}$$

From this fact and Proposition 2.1 (2), we have

$$\begin{aligned} |G''(x)| &\leq \left(k+E(Z-k)^+\right)\left(1+\frac{2}{k^2}\right) \\ &\leq \left(k+\frac{e^{-\frac{k^2}{2}}}{\sqrt{2\pi}}\right)\left(1+\frac{2}{k^2}\right) \\ &= \frac{2}{k}+k+\frac{e^{-\frac{k^2}{2}}}{\sqrt{2\pi}}+\frac{2e^{-\frac{k^2}{2}}}{\sqrt{2\pi}k^2} \qquad \text{for } x > k. \end{aligned} \tag{2.25}$$

Hence, by (2.23) and (2.25), we obtain

$$|G''(x)| \leq \frac{3}{k}+k+\frac{e^{-\frac{k^2}{2}}}{\sqrt{2\pi}}+\frac{2e^{-\frac{k^2}{2}}}{\sqrt{2\pi}k^2} \qquad \text{for } x > \frac{k}{2}. \tag{2.26}$$

Thus, by (2.22), (2.26) and the fact that

$$\frac{e^{-\frac{3k^2}{8}}k}{8}+\frac{3e^{-\frac{3k^2}{8}}}{2k}\le k+\frac{3}{k},$$

we have

$$\begin{aligned}\|G''\| &\le \frac{e^{-\frac{k^2}{2}}}{\sqrt{2\pi}}+\frac{2e^{-\frac{k^2}{2}}}{\sqrt{2\pi}k^2}+k+\frac{3}{k}\\ &\le \frac{3e^{-\frac{k^2}{2}}}{\sqrt{2\pi}}+k+3 \qquad \text{for } k\ge 1.\end{aligned}$$

Case 2. $0<k<1$.

If $x<0<k<1$, then by (2.18) and (2.20), we have

$$\left|x^2+2+\sqrt{2\pi}\Phi(x)e^{\frac{x^2}{2}}(x^3+3x)\right|\le \min\left\{x^2+2,\frac{3}{x^2}\right\}\le 3.$$

From this fact and Proposition 2.1 (2), we have

$$|G''(x)|\le 3E(Z-k)^+\le \frac{3e^{-\frac{k^2}{2}}}{\sqrt{2\pi}} \qquad \text{for } x<0<k<1. \tag{2.27}$$

In the case of $0\le x\le k<1$, by Proposition 2.1 (2), we have

$$\begin{aligned}0\le G''(x) &\le E\,(Z-k)^+\left(k^2+2+\sqrt{2\pi}\Phi(k)e^{\frac{k^2}{2}}\left(k^3+3k\right)\right)\\ &\le \frac{e^{-\frac{k^2}{2}}}{\sqrt{2\pi}k^2}\left(k^2+\sqrt{2\pi}e^{\frac{k^2}{2}}k^3\right)+\frac{e^{-\frac{k^2}{2}}}{\sqrt{2\pi}}\left(2+3\sqrt{2\pi}e^{\frac{k^2}{2}}k\right)\\ &= \frac{3e^{-\frac{k^2}{2}}}{\sqrt{2\pi}}+4k\\ &\le \frac{3e^{-\frac{k^2}{2}}}{\sqrt{2\pi}}+k+3 \qquad \text{for } 0\le x\le k<1.\end{aligned} \tag{2.28}$$

Let $x>k$. By (2.24) and the fact that

$$x^2+2-\sqrt{2\pi}\Phi(-x)e^{\frac{x^2}{2}}(x^3+3x)\le x^2+2,$$

we obtain

$$-1\le x^2+2-\sqrt{2\pi}\Phi(-x)e^{\frac{x^2}{2}}(x^3+3x)\le \min\left\{x^2+2,\frac{3}{x^2}\right\}\le 3.$$

This implies that

$$|G''(x)| \leq 3[1 + E(Z-k)^+] \leq 3 + \frac{3e^{-\frac{k^2}{2}}}{\sqrt{2\pi}} \qquad \text{for } x > k. \tag{2.29}$$

Hence, by (2.27), (2.28) and (2.29), we obtain

$$\|G''\| \leq \frac{3e^{-\frac{k^2}{2}}}{\sqrt{2\pi}} + k + 3 \qquad \text{for } k < 1.$$

□

2.2 Zero Bias Transformation

In this section, we give the definition of the zero bias transformation and an example. This identity is useful for rewriting $EWf_k(W)$ in a suitable form, where W is a random variable.

Let X be a zero mean and finite variance σ^2 random variable. Goldstein and Reinert ([6]) introduced the zero bias transformation in 1997 as follow. We say that X^* is the X-zero biased distribution if

$$\sigma^2 E[f'(X^*)] = E[Xf(X)] \tag{2.30}$$

for all absolutely continuous function f which these expectations exist. Note that, the zero bias distribution exists for every random variables having zero mean and finite variance ([5], pp. 27).

By the definition of the zero bias transformation, for any $m \in \mathbb{N}$, if X has an $(m+2)^{\text{th}}$ moment, then X^* has an m^{th} moment and

$$\sigma^2 E\,|X^*|^m = \frac{E\,|X|^{m+2}}{m+1} \qquad ([6], \text{ pp. } 939). \tag{2.31}$$

Note that, if X is bounded by some constants then X^* is also bounded by the same constant, that is

$$|X| \leq C \quad \text{implies} \quad |X^*| \leq C \qquad ([5], \text{ pp. } 29). \tag{2.32}$$

Here is an example of the zero bias transformation.

Example 2.7. ([5], pp. 27) Let ξ be a Bernoulli random variable with success probability $p \in (0,1)$. Set $X = \xi - p$ having variance $\sigma^2 = p(1-p)$. Then

$$EXf(X) = \sigma^2 Ef'(U),$$

where U is a uniformly distributed over $[-p, 1-p]$. Thus the zero bias distribution of $X = \xi - p$ is a uniform random variable on $[-p, 1-p]$.

CHAPTER III

BOUNDS ON NORMAL APPROXIMATION FOR CALL FUNCTION WITHOUT CORRECTION TERM

In this chapter, uniform and non-uniform bounds on normal approximation for the call function is presented. The rate of convergence of this bound is of order $\frac{1}{\sqrt{n}}$.

Let $X_1, X_2, X_3, \ldots, X_n$ be independent random variables with zero means and finite variances $\sigma_1^2, \sigma_2^2, \sigma_3^2, \ldots, \sigma_n^2$ such that $\sum_{i=1}^{n} \sigma_i^2 = 1$. Define

$$W = \sum_{i=1}^{n} X_i.$$

Let X_i^* be the X_i-zero biased distribution, I a random variable on $\{1, 2, 3, \ldots, n\}$ with $P(I = i) = \sigma_i^2$ for $i = 1, 2, 3, \ldots, n$. Assume that X_i, X_i^* and I are independent. The W-zero biased distribution W^* is given by

$$W^* = W - X_I + X_I^* \tag{3.1}$$

([5], pp. 29).

First, we review a uniform bound on normal approximation for a Lipschitz function h given by Chen et al. ([5], pp. 46).

Theorem 3.1. *([5]) Let h be a Lipschitz function, i.e., $|h(x) - h(y)| \leq C\,|x - y|$ for all real numbers x, y and for some positive constants C. If $E\,|X_i|^3$ exists for $i = 1, 2, 3, \ldots, n$, then*

$$|Eh(W) - Eh(Z)| \leq 3 \sum_{i=1}^{n} E\,|X_i|^3\,.$$

In this work, we pay attention to the case $h(x) = (x-k)^+$, which is a Lipschitz function. Thus, by Theorem 3.1, we obtain a uniform bound on normal approximation for the call function as follow:

Theorem 3.2. *Assume that $E|X_i|^3$ exists for $i = 1, 2, 3, \ldots, n$. Then*

$$\left|E(W-k)^+ - E(Z-k)^+\right| \leq 3\sum_{i=1}^{n} E|X_i|^3 .$$

Proof. By Theorem 3.1, it suffices to show that the call function is a Lipschitz function.

Let $x, y \in \mathbb{R}$. Without loss of generality, we assume that $x \leq y$.

If $x \leq y \leq k$, then

$$\left|(y-k)^+ - (x-k)^+\right| = 0 \leq |y-x| .$$

If $k \leq x \leq y$, then

$$\left|(y-k)^+ - (x-k)^+\right| = |y-k-x+k| = |y-x| .$$

Suppose that $x \leq k \leq y$. Then

$$\left|(y-k)^+ - (x-k)^+\right| = y-k \leq y-x = |y-x| .$$

These imply that the call function is a Lipschitz function. □

Next, we give a non-uniform bound on normal approximation for the call function. Theorem 3.3 is our result.

Theorem 3.3. *Assume that EX_i^4 exists for $i = 1, 2, 3, \ldots, n$. Then*

$$\begin{aligned}&\left|E(W-k)^+ - E(Z-k)^+\right| \\ &\leq \frac{2.86}{e^{\frac{3k^2}{8}}}\sum_{i=1}^{n} E|X_i|^3 + \frac{16}{\sqrt{6}k}\left[\frac{1}{\sqrt{3}}\left(\sum_{i=1}^{n} EX_i^4 + 1\right)^{\frac{1}{2}} + 1\right]\left[\sum_{i=1}^{n} EX_i^4\right]^{\frac{1}{2}} .\end{aligned}$$

Proof. To prove this theorem, let $\tilde{X}_i$ be an independent copy of X_i and $X_i^s = X_i - \tilde{X}_i$ for $i = 1, 2, 3, \ldots, n$. We note the useful fact obtained by Karoui and Jiao that

$$E\left|W^* - W\right|^m = \frac{1}{2(m+1)} \sum_{i=1}^{n} E\left|X_i^s\right|^{m+2} \qquad ([8], \text{ pp. } 158),$$

where W^* is the W-zero biased distribution given by (3.1) and $m \in \mathbb{N}$. By the fact that

$$(a+b)^n \leq 2^{n-1}\left(a^n + b^n\right)$$

for $a, b > 0$ and $n \in \mathbb{N}$, we have

$$E\left|X_i^s\right|^m = E\left|X_i - \tilde{X}_i\right|^m \leq 2^m E\left|X_i\right|^m. \tag{3.2}$$

Hence,

$$E\left|W^* - W\right|^m \leq \frac{2^{m+1}}{(m+1)} \sum_{i=1}^{n} E\left|X_i\right|^{m+2}. \tag{3.3}$$

By (2.2) and (2.30), we obtain

$$\begin{aligned} E(W-k)^+ - E(Z-k)^+ &= EWf_k(W) - Ef_k'(W) \\ &= Ef_k'(W^*) - Ef_k'(W) \\ &= E\left[\int_0^{W^*-W} f_k''(W+t)dt\right] \\ &=: A_1 + A_2, \end{aligned}$$

where

$$A_1 = E\left[\int_0^{W^*-W} f_k''(W+t)\mathbb{I}\left(W + t \leq \frac{k}{2}\right)dt\right]$$

and

$$A_2 = E\left[\int_0^{W^*-W} f_k''(W+t)\mathbb{I}\left(W + t > \frac{k}{2}\right)dt\right].$$

By (3.3), we have

$$E\left[\int_{0\wedge W^*-W}^{0\vee W^*-W} dt\right] \leq E\left[\int_0^{W^*-W} \mathbb{I}\left(W^*-W>0\right)dt + \int_{W^*-W}^{0} \mathbb{I}\left(W^*-W\leq 0\right)dt\right]$$
$$= E\left[(W^*-W)\,\mathbb{I}\left(W^*-W>0\right) - (W^*-W)\,\mathbb{I}\left(W^*-W\leq 0\right)\right]$$
$$= E\left|W^*-W\right|$$
$$\leq 2\sum_{i=1}^{n} E\left|X_i\right|^3$$

where $a\wedge b = \min\{a,b\}$ and $a\vee b = \max\{a,b\}$ for some real numbers a, b. From this fact, we obtain

$$\begin{aligned}|A_1| &\leq E\left[\int_{0\wedge W^*-W}^{0\vee W^*-W} \left|f_k''\left(W+t\right)\right| \mathbb{I}\left(W+t\leq \frac{k}{2}\right)dt\right]\\ &\leq \|f_k''\|_{\left(-\infty,\frac{k}{2}\right]} E\left[\int_{0\wedge W^*-W}^{0\vee W^*-W} dt\right]\\ &\leq 2\|f_k''\|_{\left(-\infty,\frac{k}{2}\right]}\sum_{i=1}^{n} E\left|X_i\right|^3. \end{aligned} \tag{3.4}$$

By Proposition 2.2,

$$|A_1| \leq \frac{2.86}{e^{\frac{3k^2}{8}}}\sum_{i=1}^{n} E\left|X_i\right|^3. \tag{3.5}$$

The fact (3.3) implies

$$\begin{aligned}&E\left[\int_{0\wedge W^*-W}^{0\vee W^*-W} \mathbb{I}\left(W+t>\frac{k}{2}\right)dt\right]\\ &\leq E\left[\int_0^{W^*-W} \mathbb{I}\left(W+t>\frac{k}{2}\right)\mathbb{I}\left(W^*-W>0\right)dt\right]\\ &\qquad + E\left[\int_{W^*-W}^{0} \mathbb{I}\left(W+t>\frac{k}{2}\right)\mathbb{I}\left(W^*-W\leq 0\right)dt\right]\\ &\leq E\left[\mathbb{I}\left(W^*>\frac{k}{2}\right)\mathbb{I}\left(W^*-W>0\right)\int_0^{W^*-W} dt\right]\\ &\qquad + E\left[\mathbb{I}\left(W>\frac{k}{2}\right)\mathbb{I}\left(W^*-W\leq 0\right)\int_{W^*-W}^{0} dt\right]\\ &= E\left[\mathbb{I}\left(W^*>\frac{k}{2}\right)\mathbb{I}\left(W^*-W>0\right)\left(W^*-W\right)\right]\\ &\qquad - E\left[\mathbb{I}\left(W>\frac{k}{2}\right)\mathbb{I}\left(W^*-W\leq 0\right)\left(W^*-W\right)\right]\end{aligned}$$

$$\leq E\left[\mathbb{I}\left(W^* > \frac{k}{2}\right)|W^* - W|\right] + E\left[\mathbb{I}\left(W > \frac{k}{2}\right)|W^* - W|\right]$$

$$\leq \left[P\left(W^* > \frac{k}{2}\right)\right]^{\frac{1}{2}}\left[E\left(W^* - W\right)^2\right]^{\frac{1}{2}} + \left[P\left(W > \frac{k}{2}\right)\right]^{\frac{1}{2}}\left[E\left(W^* - W\right)^2\right]^{\frac{1}{2}}$$

$$\leq \left\{\left[P\left(W^* > \frac{k}{2}\right)\right]^{\frac{1}{2}} + \left[P\left(W > \frac{k}{2}\right)\right]^{\frac{1}{2}}\right\}\left[\frac{8}{3}\sum_{i=1}^{n} EX_i^4\right]^{\frac{1}{2}}.$$

Hence,

$$|A_2| \leq E\left[\int_{0\wedge W^*-W}^{0\vee W^*-W} |f_k''(W+t)|\,\mathbb{I}\left(W+t > \frac{k}{2}\right)dt\right]$$

$$\leq \|f_k''\| E\left[\int_{0\wedge W^*-W}^{0\vee W^*-W} \mathbb{I}\left(W+t > \frac{k}{2}\right)dt\right]$$

$$\leq \|f_k''\|\left\{\left[P\left(W^* > \frac{k}{2}\right)\right]^{\frac{1}{2}} + \left[P\left(W > \frac{k}{2}\right)\right]^{\frac{1}{2}}\right\}\left[\frac{8}{3}\sum_{i=1}^{n} EX_i^4\right]^{\frac{1}{2}}. \quad (3.6)$$

Since $EX_i = 0$ for $i = 1, 2, 3, \ldots, n$,

$$EW^4 = E\left(\sum_{i=1}^{n} X_i^4 + \sum_{i=1}^{n} X_i^2 \sum_{\substack{j=1 \\ j\neq i}}^{n} X_j^2\right) \leq \sum_{i=1}^{n} EX_i^4 + 1. \quad (3.7)$$

From (2.31) and (3.7), we have

$$E(W^*)^2 = \frac{EW^4}{3} \leq \frac{1}{3}\left(\sum_{i=1}^{n} EX_i^4 + 1\right),$$

which implies that

$$P\left(W^* > \frac{k}{2}\right) \leq \frac{4E\left(W^*\right)^2}{k^2} \leq \frac{4}{3k^2}\left(\sum_{i=1}^{n} EX_i^4 + 1\right). \quad (3.8)$$

By the fact that

$$P\left(W > \frac{k}{2}\right) \leq \frac{4EW^2}{k^2} = \frac{4}{k^2}, \quad (3.9)$$

(2.6), (3.6) and (3.8), we obtain

$$|A_2| \leq 2\left\{\frac{2}{\sqrt{3}k}\left[\sum_{i=1}^{n} EX_i^4 + 1\right]^{\frac{1}{2}} + \frac{2}{k}\right\}\left[\frac{8}{3}\sum_{i=1}^{n} EX_i^4\right]^{\frac{1}{2}}$$

$$= \frac{16}{\sqrt{6}k}\left\{\frac{1}{\sqrt{3}}\left[\sum_{i=1}^{n} EX_i^4 + 1\right]^{\frac{1}{2}} + 1\right\}\left[\sum_{i=1}^{n} EX_i^4\right]^{\frac{1}{2}}. \quad (3.10)$$

Combine (3.5) and (3.10), we have Theorem 3.3. □

CHAPTER IV

BOUNDS ON NORMAL APPROXIMATION FOR CALL FUNCTION WITH A CORRECTION TERM

In this chapter, we are interested in normal approximation for the call function with a correction term. The Stein's method and the zero bias transformation are used to find bounds. We divide this chapter into 4 sections. First, we improve and find explicit constants of the bound in Theorem 4.1 which is obtained by Karoui and Jiao. Next, we propose a uniform bound on normal approximation for the call function. To sharpen the uniform bound, we give polynomial and exponential non-uniform bounds in the third and forth section, respectively. Notice that, we use the notation in Chapter III throughout the proof of this Chapter.

In 2009, Karoui and Jiao ([8]) proposed a correction term on normal approximation for the call function and obtained a bound. Their correction term is

$$\frac{ke^{-\frac{k^2}{2}}}{6\sqrt{2\pi}}\sum_{i=1}^{n}EX_i^3,$$

so, the purpose of this chapter is finding bounds of

$$\left|E(W-k)^+ - E(Z-k)^+ - \frac{ke^{-\frac{k^2}{2}}}{6\sqrt{2\pi}}\sum_{i=1}^{n}EX_i^3\right| := \delta_n.$$

The bound given by Karoui and Jiao is shown in Theorem 4.1.

Theorem 4.1. *([8]) Assume that EX_i^4 exists for $i = 1, 2, 3, \dots, n$. Then*

$$\delta_n \leq \left(60.68 + 3k + 4.8E(Z-k)^+\right)\left(\sum_{i=1}^{n} E|X_i|^3\right)^2 + \left(Var\left(f_k''(W)\right)\right)\left(\sum_{i=1}^{n} \sigma_i^6\right)^{\frac{1}{2}}$$
$$+ \left(16.65 + 1.34k + 3.2E(Z-k)^+\right)\sum_{i=1}^{n} EX_i^4$$
$$+ \sum_{i=1}^{n} E\left|X_i\right|^3 \left(4\sum_{j=1}^{n}\sigma_j^3 + 2.5\left(\sum_{j=1}^{n}\sigma_j^4\right)^{\frac{1}{2}} + 16\sigma_i\right).$$

4.1 Improvement of the Bound Obtained by Karoui and Jiao

In this section, we improve the constants in Theorem 4.1. The following is our result.

Theorem 4.2. *Assume that EX_i^4 exists for $i = 1, 2, 3, \dots, n$. Then*

$$\delta_n \leq (18.46 + 2k)\left(\sum_{i=1}^{n} E|X_i|^3\right)^2 + 2\left(\sum_{i=1}^{n}\sigma_i^6\right)^{\frac{1}{2}} + (9.38 + 1.34k)\sum_{i=1}^{n} EX_i^4.$$

Proof. First, note that Karoui and Jiao ([8], pp. 158, 165-166) showed that

$$\delta_n \leq A_n + B_n + C_n + D_n,$$

where

$$A_n = \frac{1}{2}\sum_{i=1}^{n} E\left|X_i\right|^3 \left|E\left(G'(W) - G'(Z)\right)\right|$$
$$+ \left|E\left(G(W^*) - G(W) - G'(W)(X_I^* - X_I)\right)\right|,$$
$$B_n = \frac{1}{2}\sum_{i=1}^{n} E\left|X_i\right|^3 \left|P\left(W \leq k\right) - P\left(Z \leq k\right)\right|,$$
$$C_n \leq \left[Var\left(f_k''(W)\right)\right]^{\frac{1}{2}}\left(\sum_{i=1}^{n}\sigma_i^6\right)^{\frac{1}{2}},$$
$$D_n = \sum_{i=1}^{n}\sigma_i^2 E\Big[\left|X_i^* - X_i\right| D_n'\Big]$$

and $D'_n = E\left[\mathbb{I}\left(k - \max\left\{X_i^*, X_i\right\} \leq W - X_i \leq k - \min\left\{X_i^*, X_i\right\}\right) \mid X_i^*, X_i\right]$. We use this notation in our proof.

By Proposition 2.6 (2), we have $\|G''\| \leq 4.2 + k$. From this fact, $P(I = i) = \sigma_i^2$ and the fact that

$$A_n \leq \|G''\| \left[\frac{1}{4}\sum_{i=1}^{n} E|X_i|^3 \sum_{j=1}^{n} E\left|X_j^s\right|^3 + \frac{1}{2}E\left(X_I^* - X_I\right)^2\right]$$

([8], pp. 165-166), we have

$$A_n \leq (4.2 + k)\left[\frac{1}{4}\sum_{i=1}^{n} E|X_i|^3 \sum_{j=1}^{n} E\left|X_j^s\right|^3 + \frac{1}{2}\sum_{i=1}^{n} \sigma_i^2 E\left(X_i^* - X_i\right)^2\right]. \tag{4.1}$$

Note that

$$E|X_i^* - X_i|^m = \frac{1}{2(m+1)\sigma_i^2} E|X_i^s|^{m+2} \tag{4.2}$$

for any random variable X_i's with zero means and finite variances σ_i^2's ([8], pp. 155). By (3.2) and (4.2), we have

$$E|X_i^* - X_i|^m = \frac{E|X_i^s|^{m+2}}{2(m+1)\sigma_i^2} \leq \frac{2^{m+1}E|X_i|^{m+2}}{(m+1)\sigma_i^2}. \tag{4.3}$$

From (3.2), (4.1) and (4.3), we obtain

$$A_n \leq (8.4 + 2k)\left[\left(\sum_{i=1}^{n} E|X_i|^3\right)^2 + \frac{2}{3}\sum_{i=1}^{n} EX_i^4\right]. \tag{4.4}$$

Siganov ([10], pp. 2546) showed in 1986 that

$$|P(W \leq k) - P(Z \leq k)| \leq 0.7915 \sum_{i=1}^{n} E|X_i|^3,$$

which implies that

$$B_n \leq 0.4\left(\sum_{i=1}^{n} E|X_i|^3\right)^2. \tag{4.5}$$

By (2.6), we obtain

$$
\begin{aligned}
C_n &\leq [Var\,(f_k''(W))]^{\frac{1}{2}} \left(\sum_{i=1}^{n} \sigma_i^6\right)^{\frac{1}{2}} \\
&\leq \left[E\,(f_k''(W))^2\right]^{\frac{1}{2}} \left(\sum_{i=1}^{n} \sigma_i^6\right)^{\frac{1}{2}} \\
&\leq 2\left(\sum_{i=1}^{n} \sigma_i^6\right)^{\frac{1}{2}}. \qquad (4.6)
\end{aligned}
$$

To find a bound of D_n, by the concentration inequality

$$P(a \leq W - X_i \leq b) \leq \sqrt{2}(b-a) + 2(\sqrt{2}+1)\sum_{i=1}^{n} E\,|X_i|^3 \qquad (4.7)$$

for all real number $a < b$ and $i = 1, 2, 3, \ldots, n$ ([5], pp. 54), we have

$$
\begin{aligned}
D_n' &= E\left[\mathbb{I}\left(k - \max\{X_i^*, X_i\} \leq W - X_i \leq k - \min\{X_i^*, X_i\}\right) \mid X_i^*, X_i\right] \\
&= P\left(k - \max\{X_i^*, X_i\} \leq W - X_i \leq k - \min\{X_i^*, X_i\} \mid X_i^*, X_i\right) \qquad (4.8) \\
&\leq \sqrt{2}\,|X_i^* - X_i| + 2(\sqrt{2}+1)\sum_{j=1}^{n} E\,|X_j|^3. \qquad (4.9)
\end{aligned}
$$

By (4.3) and (4.9), we have

$$
\begin{aligned}
D_n &\leq \sum_{i=1}^{n} \sigma_i^2 E\left[|X_i^* - X_i|\left(\sqrt{2}\,|X_i^* - X_i| + 2(\sqrt{2}+1)\sum_{j=1}^{n} E\,|X_j|^3\right)\right] \\
&= \sqrt{2}\sum_{i=1}^{n} \sigma_i^2 E\,(X_i^* - X_i)^2 + 2\left(\sqrt{2}+1\right)\sum_{j=1}^{n} E\,|X_j|^3 \sum_{i=1}^{n} \sigma_i^2 E\,|X_i^* - X_i| \\
&= \frac{8\sqrt{2}}{3}\sum_{i=1}^{n} EX_i^4 + 4(\sqrt{2}+1)\left(\sum_{i=1}^{n} E\,|X_i|^3\right)^2 \\
&\leq 3.78\sum_{i=1}^{n} EX_i^4 + 9.66\left(\sum_{i=1}^{n} E\,|X_i|^3\right)^2. \qquad (4.10)
\end{aligned}
$$

Combine (4.4), (4.5), (4.6) and (4.10), we have Theorem 4.2. □

4.2 Uniform Bound

Observe that, the bound in Theorem 4.2 is not satisfactory in a case of large k. In this section, we give a uniform bound on normal approximation for the call function. This bound does not depend on k. Theorem 4.3 is our result.

Theorem 4.3 (Uniform bound)**.** *Assume that EX_i^6 exists for $i = 1, 2, 3, \ldots, n$. Then for $k < 1$, we have*

$$\delta_n \leq 20.46 \left(\sum_{i=1}^{n} E|X_i|^3 \right)^2 + 2 \left(\sum_{i=1}^{n} \sigma_i^6 \right)^{\frac{1}{2}} + 10.72 \sum_{i=1}^{n} EX_i^4,$$

and for $k \geq 1$, we have

$$\delta_n \leq 13.76 \left(\sum_{i=1}^{n} E|X_i|^3 \right)^2 + 2 \left(\sum_{i=1}^{n} \sigma_i^6 \right)^{\frac{1}{2}} + \left[\frac{1}{\sqrt{3}} \left(\sum_{j=1}^{n} EX_j^4 + 1 \right)^{\frac{1}{2}} + 1 \right]$$
$$\times \left[11.97 \left(\sum_{i=1}^{n} EX_i^6 \right)^{\frac{1}{2}} + 15.45 \sum_{i=1}^{n} E|X_i|^3 \left(\sum_{j=1}^{n} EX_j^4 \right)^{\frac{1}{2}} \right] + 6.25 \sum_{i=1}^{n} EX_i^4.$$

Proof. For $k < 1$, by Theorem 4.2, we have

$$\delta_n \leq 20.46 \left(\sum_{i=1}^{n} E|X_i|^3 \right)^2 + 2 \left(\sum_{i=1}^{n} \sigma_i^6 \right)^{\frac{1}{2}} + 10.72 \sum_{i=1}^{n} EX_i^4.$$

Suppose that $k \geq 1$. Let A_n, B_n, C_n and D_n be defined as in the proof of Theorem 4.2. From (4.5), (4.6) and (4.10), we have

$$B_n + C_n + D_n \leq 10.06 \left(\sum_{i=1}^{n} E|X_i|^3 \right)^2 + 2 \left(\sum_{i=1}^{n} \sigma_i^6 \right)^{\frac{1}{2}} + 3.78 \sum_{i=1}^{n} EX_i^4. \quad (4.11)$$

Now, it suffices to obtain a bound for A_n without k. Let

$$A_n = \frac{1}{2} \sum_{i=1}^{n} E|X_i|^3 |A_{n1}| + |A_{n2}|, \quad (4.12)$$

where

$$A_{n1} := E\left(G'(W) - G'(Z)\right),$$

$$A_{n2} := E\left(G(W^*) - G(W) - G'(W)(X_I^* - X_I)\right).$$

To find a bound of A_{n1}, let g be the solution of the Stein's equation (2.1) for $h(x) = G'(x)$, which is defined in (2.16). By (2.1) and (2.30), we obtain

$$\begin{aligned} A_{n1} &= EWg(W) - Eg'(W) \\ &= Eg'(W^*) - Eg'(W) \\ &= E\left[\int_0^{W^*-W} g''(W+t)\,dt\right] \\ &=: T_1 + T_2, \end{aligned} \tag{4.13}$$

where

$$T_1 = E\left[\int_0^{W^*-W} g''(W+t)\mathbb{I}\left(W+t \le \frac{k}{2}\right)\,dt\right]$$

and

$$T_2 = E\left[\int_0^{W^*-W} g''(W+t)\mathbb{I}\left(W+t > \frac{k}{2}\right)\,dt\right].$$

Replacing f_k'' in (3.4) with g'' and using (2.5), we have

$$\begin{aligned} |T_1| &\le 2\|g''\|_{\left(-\infty,\frac{k}{2}\right]} \sum_{i=1}^{n} E\left|X_i\right|^3 \\ &\le 4\|G''\|_{\left(-\infty,\frac{k}{2}\right]} \sum_{i=1}^{n} E\left|X_i\right|^3. \end{aligned} \tag{4.14}$$

It follows from Proposition 2.6 (1) and (4.14) that

$$|T_1| \le \frac{9.48}{e^{\frac{k^2}{4}}} \sum_{i=1}^{n} E\left|X_i\right|^3 \tag{4.15}$$

$$\le 7.39 \sum_{i=1}^{n} E\left|X_i\right|^3. \tag{4.16}$$

Replacing f_k'' in (3.6) with g'' and using (2.4), we obtain

$$\begin{aligned} |T_2| &\le \|g''\| \left\{\left[P\left(W^* > \frac{k}{2}\right)\right]^{\frac{1}{2}} + \left[P\left(W > \frac{k}{2}\right)\right]^{\frac{1}{2}}\right\} \left[\frac{8}{3}\sum_{i=1}^{n} EX_i^4\right]^{\frac{1}{2}} \\ &\le 2\|G''\| \left\{\left[P\left(W^* > \frac{k}{2}\right)\right]^{\frac{1}{2}} + \left[P\left(W > \frac{k}{2}\right)\right]^{\frac{1}{2}}\right\} \left[\frac{8}{3}\sum_{i=1}^{n} EX_i^4\right]^{\frac{1}{2}}. \end{aligned} \tag{4.17}$$

By Proposition 2.6 (2), we have

$$\begin{aligned}\frac{\|G''\|}{k} &\le \frac{3e^{-\frac{k^2}{2}}}{\sqrt{2\pi}k} + 1 + \frac{3}{k} \\ &\le 4.73, \end{aligned} \tag{4.18}$$

where we use the fact that $\frac{3e^{-\frac{k^2}{2}}}{\sqrt{2\pi}k} + 1 + \frac{3}{k}$ has the maximum value at $k = 1$.

By (3.8), (3.9), (4.17) and (4.18), we have

$$\begin{aligned}|T_2| &\le 2\|G''\| \left[\frac{2}{\sqrt{3}k} \left(\sum_{j=1}^{n} EX_j^4 + 1 \right)^{\frac{1}{2}} + \frac{2}{k} \right] \left[\frac{8}{3} \sum_{i=1}^{n} EX_i^4 \right]^{\frac{1}{2}} \\ &= \frac{16\|G''\|}{\sqrt{6}k} \left[\frac{1}{\sqrt{3}} \left(\sum_{j=1}^{n} EX_j^4 + 1 \right)^{\frac{1}{2}} + 1 \right] \left[\sum_{i=1}^{n} EX_i^4 \right]^{\frac{1}{2}} \\ &\le 30.90 \left[\frac{1}{\sqrt{3}} \left(\sum_{j=1}^{n} EX_j^4 + 1 \right)^{\frac{1}{2}} + 1 \right] \left[\sum_{i=1}^{n} EX_i^4 \right]^{\frac{1}{2}}. \end{aligned} \tag{4.19}$$

Combine (4.16) and (4.19), we obtain

$$|A_{n1}| \le 30.90 \left[\frac{1}{\sqrt{3}} \left(\sum_{j=1}^{n} EX_j^4 + 1 \right)^{\frac{1}{2}} + 1 \right] \left[\sum_{i=1}^{n} EX_i^4 \right]^{\frac{1}{2}} + 7.39 \sum_{i=1}^{n} E\left|X_i\right|^3. \tag{4.20}$$

Next, consider

$$\begin{aligned}A_{n2} &= E\left[\int_0^{W^*-W} G'(W+t)\,dt - G'(W) \int_0^{W^*-W} dt \right] \\ &= E\left[\int_0^{W^*-W} \int_0^t G''(W+u)du\,dt \right] \\ &=: T_3 + T_4, \end{aligned} \tag{4.21}$$

where

$$T_3 = E \int_0^{W^*-W} \int_0^t G''(W+u)\mathbb{I}\left(W + u \le \frac{k}{2} \right) du\,dt$$

and

$$T_4 = E \int_0^{W^*-W} \int_0^t G''(W+u)\mathbb{I}\left(W + u > \frac{k}{2} \right) du\,dt.$$

Note that

$$
\begin{aligned}
&E\left[\int_{0\wedge W^*-W}^{0\vee W^*-W}\int_{0\wedge t}^{0\vee t} du\ dt\right] \\
&\leq E\left[\int_0^{W^*-W}\int_0^t \mathbb{I}\left(W^*-W>0\right) du\ dt + \int_{W^*-W}^0\int_t^0 \mathbb{I}\left(W^*-W\leq 0\right) du\ dt\right] \\
&= E\left[\mathbb{I}\left(W^*-W>0\right)\int_0^{W^*-W} t\ dt - \mathbb{I}\left(W^*-W\leq 0\right)\int_{W^*-W}^0 t\ dt\right] \\
&= \frac{1}{2}E\left[\mathbb{I}\left(W^*-W>0\right)\left(W^*-W\right)^2 + \mathbb{I}\left(W^*-W\leq 0\right)\left(W^*-W\right)^2\right] \\
&= \frac{1}{2}E\left(W^*-W\right)^2 \\
&\leq \frac{4}{3}\sum_{i=1}^n EX_i^4,
\end{aligned}
$$

where we have used (3.3) in the last inequality. This implies that

$$
\begin{aligned}
|T_3| &\leq E\left[\int_{0\wedge W^*-W}^{0\vee W^*-W}\int_{0\wedge t}^{0\vee t} |G''(W+u)|\,\mathbb{I}\left(W+u\leq\frac{k}{2}\right) du\ dt\right] \\
&\leq \|G''\|_{(-\infty,\frac{k}{2}]} E\left[\int_{0\wedge W^*-W}^{0\vee W^*-W}\int_{0\wedge t}^{0\vee t} du\ dt\right] \\
&\leq \frac{4}{3}\|G''\|_{(-\infty,\frac{k}{2}]}\sum_{i=1}^n EX_i^4. \qquad (4.22)
\end{aligned}
$$

By Proposition 2.6 (1) and (4.22), we have

$$
\begin{aligned}
|T_3| &\leq \frac{3.16}{e^{\frac{k^2}{4}}}\sum_{i=1}^n EX_i^4 \qquad (4.23) \\
&\leq 2.47\sum_{i=1}^n EX_i^4. \qquad (4.24)
\end{aligned}
$$

The fact (3.3) implies

$$
\begin{aligned}
&E\left[\int_{0\wedge W^*-W}^{0\vee W^*-W}\int_{0\wedge t}^{0\vee t} \mathbb{I}\left(W+u>\frac{k}{2}\right) du\ dt\right] \\
&\leq E\left[\int_0^{W^*-W}\int_0^t \mathbb{I}\left(W+u>\frac{k}{2}\right)\mathbb{I}\left(W^*-W>0\right) du\ dt\right] \\
&\quad + E\left[\int_{W^*-W}^0\int_t^0 \mathbb{I}\left(W+u>\frac{k}{2}\right)\mathbb{I}\left(W^*-W\leq 0\right) du\ dt\right] \\
&\leq E\left[\mathbb{I}\left(W^*>\frac{k}{2}\right)\mathbb{I}\left(W^*-W>0\right)\int_0^{W^*-W}\int_0^t du\ dt\right] \\
&\quad + E\left[\mathbb{I}\left(W>\frac{k}{2}\right)\mathbb{I}\left(W^*-W\leq 0\right)\int_{W^*-W}^0\int_t^0 du\ dt\right]
\end{aligned}
$$

$$\leq E\left[\mathbb{I}\left(W^* > \frac{k}{2}\right)\mathbb{I}\left(W^* - W > 0\right)\int_0^{W^*-W} t\ dt\right]$$
$$- E\left[\mathbb{I}\left(W > \frac{k}{2}\right)\mathbb{I}\left(W^* - W \leq 0\right)\int_{W^*-W}^{0} t\ dt\right]$$
$$\leq \frac{1}{2}E\left[\mathbb{I}\left(W^* > \frac{k}{2}\right)(W^* - W)^2\right] + \frac{1}{2}E\left[\mathbb{I}\left(W > \frac{k}{2}\right)(W^* - W)^2\right]$$
$$\leq \frac{1}{2}\left\{\left[P\left(W^* > \frac{k}{2}\right)\right]^{\frac{1}{2}}\left[E\left(W^* - W\right)^4\right]^{\frac{1}{2}} + \left[P\left(W > \frac{k}{2}\right)\right]^{\frac{1}{2}}\left[E\left(W^* - W\right)^4\right]^{\frac{1}{2}}\right\}$$
$$\leq \frac{1}{2}\left\{\left[P\left(W^* > \frac{k}{2}\right)\right]^{\frac{1}{2}} + \left[P\left(W > \frac{k}{2}\right)\right]^{\frac{1}{2}}\right\}\left[\frac{32}{5}\sum_{i=1}^{n} EX_i^6\right]^{\frac{1}{2}}. \tag{4.25}$$

Hence, by (3.8), (3.9), (4.18) and (4.25), we obtain

$$|T_4| \leq E\left[\int_{0\wedge W^*-W}^{0\vee W^*-W}\int_{0\wedge t}^{0\vee t} |G''(W+u)|\,\mathbb{I}\left(W + u > \frac{k}{2}\right) du\ dt\right]$$
$$\leq \|G''\|E\left[\int_{0\wedge W^*-W}^{0\vee W^*-W}\int_{0\wedge t}^{0\vee t} \mathbb{I}\left(W + u > \frac{k}{2}\right) du\ dt\right]$$
$$\leq \frac{\|G''\|}{2}\left\{\left[P\left(W^* > \frac{k}{2}\right)\right]^{\frac{1}{2}} + \left[P\left(W > \frac{k}{2}\right)\right]^{\frac{1}{2}}\right\}\left[\frac{32}{5}\sum_{i=1}^{n} EX_i^6\right]^{\frac{1}{2}} \tag{4.26}$$
$$\leq 11.97\left[\frac{1}{\sqrt{3}}\left(\sum_{j=1}^{n} EX_j^4 + 1\right)^{\frac{1}{2}} + 1\right]\left[\sum_{i=1}^{n} EX_i^6\right]^{\frac{1}{2}}. \tag{4.27}$$

By (4.21), (4.24) and (4.27), we have

$$\left|A_{n2}\right| \leq 2.47\sum_{i=1}^{n} EX_i^4 + 11.97\left[\frac{1}{\sqrt{3}}\left(\sum_{j=1}^{n} EX_j^4 + 1\right)^{\frac{1}{2}} + 1\right]\left[\sum_{i=1}^{n} EX_i^6\right]^{\frac{1}{2}}. \tag{4.28}$$

Combine (4.12), (4.20) and (4.28), we obtain

$$A_n \leq 3.70\left(\sum_{i=1}^{n} E\left|X_i\right|^3\right)^2 + 11.97\left[\frac{1}{\sqrt{3}}\left(\sum_{j=1}^{n} EX_j^4 + 1\right)^{\frac{1}{2}} + 1\right]\left[\sum_{i=1}^{n} EX_i^6\right]^{\frac{1}{2}}$$
$$+ 15.45\sum_{l=1}^{n} E\left|X_l\right|^3\left[\frac{1}{\sqrt{3}}\left(\sum_{j=1}^{n} EX_j^4 + 1\right)^{\frac{1}{2}} + 1\right]\left[\sum_{i=1}^{n} EX_i^4\right]^{\frac{1}{2}}$$
$$+ 2.47\sum_{i=1}^{n} EX_i^4. \tag{4.29}$$

From (4.11) and (4.29), we have Theorem 4.3. □

4.3 Polynomial Non-uniform Bound

In this section, we improve the bound in Theorem 4.3 by proposing a polynomial non-uniform bound on normal approximation for the call function. To do this, we give a polynomial non-uniform concentration inequality as shown in the next proposition.

Proposition 4.4 (Polynomial Non-uniform Concentration Inequality)**.** *For* $i = 1, 2, 3, \ldots, n$, *we have*

$$P\left(a \leq W - X_i \leq b\right) \leq \frac{5.264}{(1+a)^3}(b-a) + \left[\frac{7.018}{(1+a)^3} + \frac{80(1+a)}{a^2}\right] \sum_{i=1}^{n} E\left|X_i\right|^3,$$

where $3 \leq a < b < \infty$.

Proof. Case 1. $(1+a)\sum_{i=1}^{n} E\left|X_i\right|^3 \leq \dfrac{1}{80}$. Let

$$\delta_a = \frac{\alpha_a}{(1+a)^2} + \frac{\beta_a}{(1+a)^3},$$

where

$$\alpha_a = \sum_{i=1}^{n} EX_i^2 \mathbb{I}\left(|X_i| > 1+a\right) \qquad \text{and} \qquad \beta_a = \sum_{i=1}^{n} E\left|X_i\right|^3 \mathbb{I}\left(|X_i| \leq 1+a\right).$$

Thongtha and Neammanee ([13], pp. 5) showed in 2007 that

$$P\left(a \leq W - X_i \leq b\right) \leq \frac{5.264}{(1+a)^3}(b-a) + 7.018\delta_a$$

for $a \geq 3$ and $(1+a)^2\alpha_a + (1+a)\beta_a \leq \frac{1}{80}$. Note that

$$\begin{aligned}
&(1+a)^2\alpha_a + (1+a)\beta_a \\
&= (1+a)^2 \sum_{i=1}^{n} EX_i^2 \mathbb{I}\left(|X_i| > 1+a\right) + (1+a)\sum_{i=1}^{n} E\left|X_i\right|^3 \mathbb{I}\left(|X_i| \leq 1+a\right) \\
&\leq (1+a)\sum_{i=1}^{n} E\left|X_i\right|^3 \mathbb{I}\left(|X_i| > 1+a\right) + (1+a)\sum_{i=1}^{n} E\left|X_i\right|^3 \mathbb{I}\left(|X_i| \leq 1+a\right) \\
&= (1+a)\sum_{i=1}^{n} E\left|X_i\right|^3 \\
&\leq \frac{1}{80}
\end{aligned}$$

and

$$\delta_a \leq \frac{1}{(1+a)^3} \sum_{i=1}^{n} E\left|X_i\right|^3 \mathbb{I}\left(\left|X_i\right| > 1+a\right) + \frac{1}{(1+a)^3} \sum_{i=1}^{n} E\left|X_i\right|^3 \mathbb{I}\left(\left|X_i\right| \leq 1+a\right)$$
$$= \frac{1}{(1+a)^3} \sum_{i=1}^{n} E\left|X_i\right|^3.$$

Hence,

$$P\left(a \leq W - X_i \leq b\right) \leq \frac{5.264}{(1+a)^3}(b-a) + \frac{7.018}{(1+a)^3} \sum_{i=1}^{n} E\left|X_i\right|^3 \qquad \text{for } a \geq 3. \tag{4.30}$$

Case 2. $(1+a) \sum_{i=1}^{n} E\left|X_i\right|^3 > \frac{1}{80}$.

Since $EX_i = 0$ for all i,

$$\begin{aligned} E\left(W - X_i\right)^2 &= EW^2 - 2E\left(WX_i\right) + EX_i^2 \\ &= 1 - 2\left(\sum_{\substack{j=1 \\ j \neq i}}^{n} E\left(X_j X_i\right) + EX_i^2\right) + \sigma_i^2 \\ &= 1 - \sigma_i^2 \\ &\leq 1. \end{aligned}$$

This implies that

$$\begin{aligned} P\left(a \leq W - X_i \leq b\right) &\leq P\left(W - X_i \geq a\right) \\ &\leq \frac{1}{a^2} E\left(W - X_i\right)^2 \\ &\leq \frac{1}{a^2} \\ &\leq \frac{80(1+a)}{a^2} \sum_{i=1}^{n} E\left|X_i\right|^3. \end{aligned} \tag{4.31}$$

Combine (4.30) and (4.31), we have Proposition 4.4. □

The following theorem is the polynomial non-uniform bound on normal approximation for the call function.

Theorem 4.5 (Polynomial Non-uniform)**.** *Assume that* EX_i^6 *exists for* $i = 1, 2, 3, \ldots, n$. *Then for* $k \geq 4$, *we have*

$$\delta_n \leq \left[\frac{4.74}{e^{\frac{k^2}{4}}} + \frac{285.94}{k}\right]\left(\sum_{i=1}^{n} E|X_i|^3\right)^2 + \left[\frac{3.16}{e^{\frac{k^2}{4}}} + \frac{1.36}{k}\right]\sum_{i=1}^{n} EX_i^4$$
$$+ \frac{1}{k}\left\{\frac{1}{\sqrt{5}}\left[\sum_{j=1}^{n} EX_j^6 + \sum_{j=1}^{n} EX_j^4 + \left(\sum_{j=1}^{n} E|X_j|^3\right)^2 + 1\right]^{\frac{1}{2}} + \left[\sum_{j=1}^{n} EX_j^4 + 1\right]^{\frac{1}{2}}\right\}$$
$$\times \left[30.90\left[\sum_{i=1}^{n} E|X_i|^3\right]\left[\sum_{i=1}^{n} EX_i^4\right]^{\frac{1}{2}} + 23.94\left[\sum_{i=1}^{n} EX_i^6\right]^{\frac{1}{2}}\right]$$
$$+ \frac{1}{k}\sum_{i=1}^{n}\sigma_i\left[14.32\left[EX_i^6\right]^{\frac{1}{2}} + 31.56\sum_{j=1}^{n} E|X_j|^3\left[EX_i^4\right]^{\frac{1}{2}}\right]\left[\frac{EX_i^4}{3\sigma_i^2} + \sigma_i^2\right]^{\frac{1}{2}}$$
$$+ \left(\frac{0.64}{e^{\frac{3k^2}{8}}} + \frac{16}{k^2}\right)^{\frac{1}{2}}\left(\sum_{i=1}^{n}\sigma_i^6\right)^{\frac{1}{2}}.$$

Proof. Asuume that $k \geq 4$. Let A_n, B_n, C_n and D_n be definded as in the proof of Theorem 4.3. First, we find a bound of A_n. From (4.12), (4.13) and (4.21), we note that

$$A_n = \frac{1}{2}\sum_{i=1}^{n} E|X_i|^3 |T_1 + T_2| + |T_3 + T_4|, \tag{4.32}$$

where T_1, T_2, T_3 and T_4 are definded as in the proof of Theorem 4.3. So, it suffices to bound T_1, T_2, T_3 and T_4.

By (4.15), we have

$$|T_1| \leq \frac{9.48}{e^{\frac{k^2}{4}}}\sum_{i=1}^{n} E|X_i|^3 \tag{4.33}$$

and by (4.23), we have

$$|T_3| \leq \frac{3.16}{e^{\frac{k^2}{4}}}\sum_{i=1}^{n} EX_i^4. \tag{4.34}$$

To bound T_2 and T_4, from (2.31) and (3.7), we note that

$$\begin{aligned}
&P\left(W^* > \frac{k}{2}\right)\\
&\leq \frac{16E(W^*)^4}{k^4}\\
&= \frac{16}{5k^4}EW^6\\
&= \frac{16}{5k^4}E\left(\sum_{i=1}^{n}X_i^6 + \sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}X_i^4X_j^2 + \sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}|X_i|^3|X_j|^3 + \sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\sum_{\substack{l=1\\ l\neq i\\ l\neq j}}^{n}X_i^2X_j^2X_l^2\right)\\
&\leq \frac{16}{5k^4}\left(\sum_{i=1}^{n}EX_i^6 + \sum_{i=1}^{n}EX_i^4 + \left(\sum_{i=1}^{n}E|X_i|^3\right)^2 + 1\right) \qquad (4.35)
\end{aligned}$$

and

$$P\left(W > \frac{k}{2}\right) \leq \frac{16EW^4}{k^4} \leq \frac{16}{k^4}\left(\sum_{i=1}^{n}EX_i^4 + 1\right). \qquad (4.36)$$

From these facts and (4.17), we have

$$\begin{aligned}
|T_2| &\leq 2\|G''\|\left\{\frac{4}{\sqrt{5}k^2}\left[\sum_{j=1}^{n}EX_j^6 + \sum_{j=1}^{n}EX_j^4 + \left(\sum_{j=1}^{n}E|X_j|^3\right)^2 + 1\right]^{\frac{1}{2}}\right.\\
&\qquad\left. + \frac{4}{k^2}\left[\sum_{j=1}^{n}EX_j^4 + 1\right]^{\frac{1}{2}}\right\}\left[\frac{8}{3}\sum_{i=1}^{n}EX_i^4\right]^{\frac{1}{2}}\\
&\leq \frac{61.80}{k}\left\{\frac{1}{\sqrt{5}}\left[\sum_{j=1}^{n}EX_j^6 + \sum_{j=1}^{n}EX_j^4 + \left(\sum_{j=1}^{n}E|X_j|^3\right)^2 + 1\right]^{\frac{1}{2}}\right.\\
&\qquad\left. + \left[\sum_{j=1}^{n}EX_j^4 + 1\right]^{\frac{1}{2}}\right\}\left[\sum_{i=1}^{n}EX_i^4\right]^{\frac{1}{2}}, \qquad (4.37)
\end{aligned}$$

where we use (4.18) in the last inequality. Similarly to (4.37), by (4.18), (4.26), (4.35) and (4.36), we obtain

$$\begin{aligned}
|T_4| &\leq \frac{23.94}{k}\left\{\frac{1}{\sqrt{5}}\left(\sum_{i=1}^{n}EX_i^6 + \sum_{i=1}^{n}EX_i^4 + \left(\sum_{i=1}^{n}E|X_i|^3\right)^2 + 1\right)^{\frac{1}{2}}\right.\\
&\qquad\left. + \left(\sum_{i=1}^{n}EX_i^4 + 1\right)^{\frac{1}{2}}\right\}\left[\sum_{i=1}^{n}EX_i^6\right]^{\frac{1}{2}}. \qquad (4.38)
\end{aligned}$$

Combine (4.32), (4.33), (4.34), (4.37) and (4.38), we obtain

$$\begin{aligned} A_n \le& \frac{4.74}{e^{\frac{k^2}{4}}}\left[\sum_{i=1}^n E\left|X_i\right|^3\right]^2 + \frac{3.16}{e^{\frac{k^2}{4}}}\sum_{i=1}^n EX_i^4 \\ &+ \frac{1}{k}\left\{\frac{1}{\sqrt{5}}\left[\sum_{j=1}^n EX_j^6 + \sum_{j=1}^n EX_j^4 + \left(\sum_{j=1}^n E\left|X_j\right|^3\right)^2 + 1\right]^{\frac{1}{2}} + \left[\sum_{j=1}^n EX_j^4 + 1\right]^{\frac{1}{2}}\right\} \\ &\times \left[30.90\left[\sum_{i=1}^n E\left|X_i\right|^3\right]\left[\sum_{i=1}^n EX_i^4\right]^{\frac{1}{2}} + 23.94\left[\sum_{i=1}^n EX_i^6\right]^{\frac{1}{2}}\right]. \end{aligned} \tag{4.39}$$

Next, we will bound B_n. To do this, we note that

$$\left|P(W \le k) - P(Z \le k)\right| \le \frac{31.935}{1+k^3}\sum_{i=1}^n E\left|X_i\right|^3 \qquad \text{([9], pp. 455).}$$

Then, by the fact that $1 + k^3 \ge 16k$, we have

$$B_n \le \frac{15.97}{16k}\left(\sum_{i=1}^n E\left|X_i\right|^3\right)^2 \le \frac{1}{k}\left(\sum_{i=1}^n E\left|X_i\right|^3\right)^2. \tag{4.40}$$

From Proposition 2.3, we have

$$C_n \le \left(\frac{0.64}{e^{\frac{3k^2}{8}}} + \frac{16}{k^2}\right)^{\frac{1}{2}}\left(\sum_{i=1}^n \sigma_i^6\right)^{\frac{1}{2}}. \tag{4.41}$$

To find a bound of D_n, by (4.8), note that

$$\begin{aligned} D_n &= \sum_{i=1}^n \sigma_i^2 E\Big[\left|X_i^* - X_i\right| D_n'\Big] \\ &=: \sum_{i=1}^n \sigma_i^2 E\Big[\left|X_i^* - X_i\right| D_{n1}\Big] + \sum_{i=1}^n \sigma_i^2 E\Big[\left|X_i^* - X_i\right| D_{n2}\Big], \end{aligned}$$

where

$$D_{n1} = \mathbb{I}_B\Big[P\big(k - \max\{X_i^*, X_i\} \le W - X_i \le k - \min\{X_i^*, X_i\} \mid X_i^*, X_i\big)\Big],$$

$$D_{n2} = \mathbb{I}_{B^c}\Big[P\big(k - \max\{X_i^*, X_i\} \le W - X_i \le k - \min\{X_i^*, X_i\} \mid X_i^*, X_i\big)\Big],$$

$$\mathbb{I}_B = \mathbb{I}\left(\left|X_i^*\right| < \frac{k}{4} \wedge \left|X_i\right| < \frac{k}{4}\right) \qquad \text{and} \qquad \mathbb{I}_{B^c} = \mathbb{I}\left(\left|X_i^*\right| \ge \frac{k}{4} \vee \left|X_i\right| \ge \frac{k}{4}\right).$$

Observe that, in the case of $|X_i^*| < \frac{k}{4}$ and $|X_i| < \frac{k}{4}$, we have

$$k - \max\{X_i^*, X_i\} \geq \frac{3k}{4} \geq 3, \tag{4.42}$$

thus,

$$1 + k - \max\{X_i^*, X_i\} \geq \frac{4+3k}{4} \qquad \text{and} \qquad 2 + k - \max\{X_i^*, X_i\} \geq \frac{8+3k}{4}.$$

Proposition 4.4 implies that

$$\begin{aligned} D_{n1} &\leq \mathbb{I}_B \left[\frac{336.90}{(4+3k)^3} |X_i^* - X_i| + \left[\frac{449.16}{(8+3k)^3} + \frac{1280}{9k^2} + \frac{320}{3k} \right] \sum_{i=1}^n E|X_i|^3 \right] \\ &\leq \left[\frac{336.90}{(4+3k)^3} |X_i^* - X_i| + \left[\frac{449.16}{(8+3k)^3} + \frac{1280}{9k^2} + \frac{320}{3k} \right] \sum_{i=1}^n E|X_i|^3 \right] \\ &\leq \frac{0.34}{k} |X_i^* - X_i| + \frac{142.47}{k} \sum_{i=1}^n EX_i^3, \end{aligned}$$

where we use the fact that $(4+3k)^3 \geq 1008k$, $(8+3k)^3 \geq 1872k$ and $9k^2 \geq 36k$ in the last inequality. Hence, by (4.3), we obtain

$$\begin{aligned} &\sum_{i=1}^n \sigma_i^2 E(|X_i^* - X_i| D_{n1}) \\ &\leq \frac{0.34}{k} \sum_{i=1}^n \sigma_i^2 E(X_i^* - X_i)^2 + \frac{142.47}{k} \sum_{j=1}^n EX_j^3 \sum_{i=1}^n \sigma_i^2 E|X_i^* - X_i| \\ &\leq \frac{1.36}{k} \sum_{i=1}^n EX_i^4 + \frac{284.94}{k} \left(\sum_{i=1}^n E|X_i|^3 \right)^2. \end{aligned} \tag{4.43}$$

From (4.7), we have

$$D_{n2} \leq \mathbb{I}_{B^c} \left(\sqrt{2} |X_i^* - X_i| + 2(\sqrt{2}+1) \sum_{j=1}^n E|X_j|^3 \right). \tag{4.44}$$

By (4.3), we have

$$\begin{aligned} E(X_i^* - X_i)^2 \mathbb{I}_{B^c} &\leq \left(E(X_i^* - X_i)^4 \right)^{\frac{1}{2}} (P(B^c))^{\frac{1}{2}} \\ &\leq \left[\frac{32}{5\sigma_i^2} EX_i^6 \right]^{\frac{1}{2}} (P(B^c))^{\frac{1}{2}} \end{aligned} \tag{4.45}$$

and

$$\begin{aligned} E\left|X_i^* - X_i\right| \mathbb{I}_{B^c} &\le \left(E\left(X_i^* - X_i\right)^2\right)^{\frac{1}{2}} \left(P(B^c)\right)^{\frac{1}{2}} \\ &\le \left[\frac{8}{3\sigma_i^2} EX_i^4\right]^{\frac{1}{2}} \left(P(B^c)\right)^{\frac{1}{2}}. \end{aligned} \tag{4.46}$$

Note by (2.31) that

$$\begin{aligned} P\left(B^c\right) &\le P\left(\left|X_i^*\right| \ge \frac{k}{4}\right) + P\left(\left|X_i\right| \ge \frac{k}{4}\right) \\ &\le \frac{16E\left(X_i^*\right)^2}{k^2} + \frac{16EX_i^2}{k^2} \\ &\le \frac{16}{k^2}\left[\frac{EX_i^4}{3\sigma_i^2} + \sigma_i^2\right]. \end{aligned} \tag{4.47}$$

Hence, (4.44), (4.45), (4.46) and (4.47) imply

$$\begin{aligned} &\sum_{i=1}^n \sigma_i^2 E\left(\left|X_i^* - X_i\right| D_{n2}\right) \\ &\le \sqrt{2}\sum_{i=1}^n \sigma_i^2 E\left(X_i^* - X_i\right)^2 \mathbb{I}_{B^c} + 2(\sqrt{2}+1)\sum_{j=1}^n E\left|X_j\right|^3 \sum_{i=1}^n \sigma_i^2 E\left|X_i^* - X_i\right| \mathbb{I}_{B^c} \\ &\le \sum_{i=1}^n \sigma_i^2 \left[3.58\left(\frac{EX_i^6}{\sigma_i^2}\right)^{\frac{1}{2}} + 7.89\sum_{j=1}^n E\left|X_j\right|^3 \left(\frac{EX_i^4}{\sigma_i^2}\right)^{\frac{1}{2}}\right] \left(P(B^c)\right)^{\frac{1}{2}} \qquad (4.48) \\ &\le \frac{1}{k}\sum_{i=1}^n \sigma_i \left[14.32\left[EX_i^6\right]^{\frac{1}{2}} + 31.56\sum_{j=1}^n E\left|X_j\right|^3 \left[EX_i^4\right]^{\frac{1}{2}}\right] \left[\frac{EX_i^4}{3\sigma_i^2} + \sigma_i^2\right]^{\frac{1}{2}}. \qquad (4.49) \end{aligned}$$

By (4.43) and (4.49), we can conclude that

$$\begin{aligned} D_n \le{}& \frac{1}{k}\sum_{i=1}^n \sigma_i \left[14.32\left[EX_i^6\right]^{\frac{1}{2}} + 31.56\sum_{j=1}^n E\left|X_j\right|^3 \left[EX_i^4\right]^{\frac{1}{2}}\right] \left[\frac{EX_i^4}{3\sigma_i^2} + \sigma_i^2\right]^{\frac{1}{2}} \\ &+ \frac{1.36}{k}\sum_{i=1}^n EX_i^4 + \frac{284.94}{k}\left(\sum_{i=1}^n E\left|X_i\right|^3\right)^2. \end{aligned} \tag{4.50}$$

Combine (4.39)-(4.41) and (4.50), we have Theorem 4.5. □

4.4 Exponential Non-uniform Bound

In this section, we consider bounded random variables. We give an exponential non-uniform bound on normal approximation for the call function. Before proving the result, we give an exponential non-uniform concentration inequality. We follow the proof in [4] to prove this proposition.

Proposition 4.6 (Exponential Non-uniform Concentration Inequality)**.** *Assume that* $|X_i| \leq \beta$ *for* $i = 1, 2, 3, \ldots, n$ *and for some real number* $\beta > 0$. *Then*

$$P\left(a \leq W \leq b\right) \leq e^{-\frac{a}{2}}\left[2(b-a)e^{\frac{\theta}{2}+\frac{\beta}{4}} + \left(e^{\frac{\theta}{2}} + 2e^{\frac{\theta}{2}+\frac{\beta}{4}}\right)\beta\right]$$

where $\theta = \beta^{-2}\left(e^{\beta} - 1 - \beta\right)$.

Proof. By following the proof of Proposition 6.1 in [4] (pp. $33, 41-43$), we have

$$EWf(W) \geq e^{-\frac{\delta}{2}}\left(H_{2,1} - H_{2,2}\right), \quad H_{2,2} \leq \delta e^{\frac{\theta}{2}} \qquad \text{and} \quad t \geq 0.5, \tag{4.51}$$

where

$$f(w) = \begin{cases} 0 & \text{if } w < a - \delta; \\ e^{\frac{w}{2}}\left(w - a + \delta\right) & \text{if } a - \delta \leq w \leq b + \delta; \\ e^{\frac{w}{2}}\left(b - a + 2\delta\right) & \text{if } w > b + \delta, \end{cases}$$

$$\begin{aligned} H_{2,1} &= E\left[\mathbb{I}\left(a \leq W \leq b\right)e^{\frac{W}{2}}t\right], \quad & H_{2,2} &= E\left[e^{\frac{W}{2}}\left|Y - t\right|\right], \\ Y &= \sum_{j=1}^{n}\left|X_j\right|\min\left\{\delta, \left|X_j\right|\right\}, & t &= \sum_{j=1}^{n}E\left|X_j\right|\min\left\{\delta, \left|X_j\right|\right\} \quad \text{and} \\ \delta &= \frac{1}{2}\sum_{i=1}^{n}E\left|X_i\right|^3. \end{aligned}$$

Hence,

$$H_{2,1} \geq 0.5e^{\frac{a}{2}}P\left(a \leq W \leq b\right). \tag{4.52}$$

By Hölder's inequality and the fact that f is increasing, we have

$$\begin{aligned} EWf(W) &\leq (b-a+2\delta)\,E\left(We^{\frac{W}{2}}\right) \\ &\leq (b-a+2\delta)\left(EW^2\right)^{\frac{1}{2}}\left(Ee^{W}\right)^{\frac{1}{2}} \\ &\leq (b-a+2\delta)\,e^{\frac{\theta}{2}}. \end{aligned} \tag{4.53}$$

By (4.51)-(4.53), we obtain

$$\begin{aligned} P\left(a \leq W \leq b\right) &\leq 2e^{-\frac{a}{2}}H_{2,1} \\ &\leq 2e^{-\frac{a}{2}}\left(e^{\frac{\delta}{2}}EWf(W) + H_{2,2}\right) \\ &\leq 2e^{-\frac{a}{2}}\left[(b-a+2\delta)\,e^{\frac{\theta}{2}+\frac{\delta}{2}} + \delta e^{\frac{\theta}{2}}\right] \\ &= e^{-\frac{a}{2}}\left[2(b-a)e^{\frac{\theta}{2}+\frac{\gamma}{4}} + \left(e^{\frac{\theta}{2}} + 2e^{\frac{\theta}{2}+\frac{\gamma}{4}}\right)\gamma\right] \\ &\leq e^{-\frac{a}{2}}\left[2(b-a)e^{\frac{\theta}{2}+\frac{\beta}{4}} + \left(e^{\frac{\theta}{2}} + 2e^{\frac{\theta}{2}+\frac{\beta}{4}}\right)\beta\right], \end{aligned}$$

where $\gamma = \sum_{i=1}^{n} E\left|X_i\right|^3$. □

The next theorem is an exponential non-uniform bound on normal approximation for the call function.

Theorem 4.7 (Exponential Non-uniform)**.** *Assume that* $|X_i| < \beta$ *for* $i = 1, 2, 3, \ldots, n$ *and* $k \geq 1$. *Then*

$$\begin{aligned} \delta_n \leq{}& \frac{7.90}{e^{\frac{k^2}{4}}}\beta^2 + \frac{109.67}{e^{\frac{k}{8}}}e^{\frac{\theta}{2}}\left(e^{\beta}+1\right)\beta^2 + \frac{C_\beta}{2e^{\frac{k}{2}}}\beta^2 + \left(\frac{2.86}{e^{\frac{3k^2}{8}}} + \frac{4e^{\theta}}{e^{\frac{k}{2}}}\right)^{\frac{1}{2}}\beta^2 \\ &+ \frac{1}{e^{\frac{3k}{8}}}\left(17.34e^{\frac{\theta}{2}+\frac{3\beta}{4}} + 2e^{\frac{\theta}{2}+\frac{\beta}{2}}\right)\beta^2 + \frac{16.23}{e^{\frac{k}{8}}}e^{\frac{\beta}{2}}\beta^2, \end{aligned}$$

where $C_\beta = 4.45 + 2.21e^{2\beta+\beta^{-2}\left(e^{2\beta}-1-2\beta\right)}$ *and* $\theta = \beta^{-2}\left(e^{\beta}-1-\beta\right)$.

Observe that if $\beta \to 0$ we have $e^{\theta} \to \sqrt{e}$ and $C_\beta \to 20.78$. Hence $\delta_n \to 0$.

Proof. Assume that $k \geq 1$. To prove the theorem, we use the same notation with the proof of Theorem 4.5. First, we will bound A_n. By (4.33), we have

$$|T_1| \leq \frac{9.48}{e^{\frac{k^2}{4}}}\sum_{i=1}^{n} E\left|X_i\right|^3 \leq \frac{9.48}{e^{\frac{k^2}{4}}}\beta. \tag{4.54}$$

By Lemma 2.4 and (2.32), we have

$$Ee^{W^*} = Ee^{W-X_I+X_{I^*}} \leq Ee^{W+|X_I|+\left|X_I^*\right|} \leq Ee^{W+2\beta} \leq e^{\theta+2\beta}.$$

Then

$$P\left(W^* > \frac{k}{2}\right) \leq \frac{Ee^{W^*}}{e^{\frac{k}{2}}} \leq \frac{e^{\theta+2\beta}}{e^{\frac{k}{2}}} \tag{4.55}$$

and

$$P\left(W > \frac{k}{2}\right) \leq \frac{Ee^{W}}{e^{\frac{k}{2}}} \leq \frac{e^{\theta}}{e^{\frac{k}{2}}}. \tag{4.56}$$

Note that, by (4.18) and the fact that $\frac{1}{e^{\frac{k}{8}}} \leq \frac{8}{k}$, we obtain

$$\frac{\|G''\|}{e^{\frac{k}{8}}} \leq \frac{8\|G''\|}{k} \leq 37.84. \tag{4.57}$$

Thus, (4.17), (4.55), (4.56) and (4.57) imply

$$\begin{aligned}
|T_2| &\leq 2\|G''\| \left\{ \left[\frac{e^{\theta+2\beta}}{e^{\frac{k}{2}}}\right]^{\frac{1}{2}} + \left[\frac{e^{\theta}}{e^{\frac{k}{2}}}\right]^{\frac{1}{2}} \right\} \left[\frac{8}{3}\sum_{i=1}^{n} EX_i^4\right]^{\frac{1}{2}} \\
&\leq \frac{8\|G''\|e^{\frac{\theta}{2}}}{\sqrt{6}e^{\frac{k}{4}}}(e^{\beta}+1)\beta \\
&= \frac{8e^{\frac{\theta}{2}}}{\sqrt{6}e^{\frac{k}{8}}}\frac{\|G''\|}{e^{\frac{k}{8}}}\left(e^{\beta}+1\right)\beta \\
&\leq \frac{123.59}{e^{\frac{k}{8}}}e^{\frac{\theta}{2}}\left(e^{\beta}+1\right)\beta. \qquad (4.58)
\end{aligned}$$

By (4.23), we have

$$|T_3| \leq \frac{3.16}{e^{\frac{k^2}{4}}}\sum_{i=1}^{n} EX_i^4 \leq \frac{3.16}{e^{\frac{k^2}{4}}}\beta^2. \tag{4.59}$$

From (4.26), (4.55), (4.56) and (4.57), we have

$$\begin{aligned}
|T_4| &\leq \frac{\|G''\|}{2}\left\{\frac{e^{\frac{\theta}{2}+\beta}}{e^{\frac{k}{4}}} + \frac{e^{\frac{\theta}{2}}}{e^{\frac{k}{4}}}\right\}\left[\frac{32}{5}\sum_{i=1}^{n} EX_i^6\right]^{\frac{1}{2}} \\
&\leq \frac{4\|G''\|e^{\frac{\theta}{2}}}{\sqrt{10}e^{\frac{k}{4}}}\left(e^{\beta}+1\right)\left[\sum_{i=1}^{n} EX_i^6\right]^{\frac{1}{2}} \\
&\leq \frac{4\|G''\|e^{\frac{\theta}{2}}}{\sqrt{10}e^{\frac{k}{4}}}\left(e^{\beta}+1\right)\beta^2 \\
&\leq \frac{47.87}{e^{\frac{k}{8}}}e^{\frac{\theta}{2}}\left(e^{\beta}+1\right)\beta^2. \qquad (4.60)
\end{aligned}$$

Combine (4.54), (4.58), (4.59) and (4.60), we obtain

$$A_n \leq \frac{7.90}{e^{\frac{k^2}{4}}}\beta^2 + \frac{109.67}{e^{\frac{k}{8}}}e^{\frac{\theta}{2}}\left(e^{\beta}+1\right)\beta^2. \tag{4.61}$$

By [2] (pp. 75), we have

$$|P(W \leq k) - P(Z \leq k)| \leq C_\beta e^{-\frac{k}{2}}\beta,$$

where $C_\beta = 4.45 + 2.21e^{2\beta+\beta^{-2}\left(e^{2\beta-1-2\beta}\right)}$. Then

$$B_n \leq \frac{C_\beta \beta}{2e^{\frac{k}{2}}}\sum_{i=1}^{n} E|X_i|^3 \leq \frac{C_\beta \beta^2}{2e^{\frac{k}{2}}}. \tag{4.62}$$

To bound C_n, by Proposition 2.5, we obtain

$$C_n \leq \left(\frac{2.86}{e^{\frac{3k^2}{8}}} + \frac{4e^{\theta}}{e^{\frac{k}{2}}}\right)^{\frac{1}{2}}\beta^2. \tag{4.63}$$

By Proposition 4.6, we have

$$\begin{aligned} P(a \leq W - X_i \leq b) &\leq P(a - \beta \leq W \leq b + \beta) \\ &\leq e^{-\frac{a-\beta}{2}}\left[2(b-a+2\beta)e^{\frac{\theta}{2}+\frac{\beta}{4}} + \left(e^{\frac{\theta}{2}} + 2e^{\frac{\theta}{2}+\frac{\beta}{4}}\right)\beta\right] \\ &\leq e^{-\frac{a}{2}}\left[2(b-a)e^{\frac{\theta}{2}+\frac{3\beta}{4}} + \left(e^{\frac{\theta}{2}+\frac{\beta}{2}} + 6e^{\frac{\theta}{2}+\frac{3\beta}{4}}\right)\beta\right]. \end{aligned}$$

Hence, by (4.42), we have

$$D_{n1} \leq e^{-\frac{3k}{8}}\left[2|X_i^* - X_i|e^{\frac{\theta}{2}+\frac{3\beta}{4}} + \left(e^{\frac{\theta}{2}+\frac{\beta}{2}} + 6e^{\frac{\theta}{2}+\frac{3\beta}{4}}\right)\beta\right].$$

From this fact and (4.3), we have

$$\begin{aligned} &\sum_{i=1}^{n}\sigma_i^2 E\left(|X_i^* - X_i| D_{n1}\right) \\ &\leq \frac{2}{e^{\frac{3k}{8}}}e^{\frac{\theta}{2}+\frac{3\beta}{4}}\sum_{i=1}^{n}\sigma_i^2 E\left(X_i^* - X_i\right)^2 + \frac{1}{e^{\frac{3k}{8}}}\left(e^{\frac{\theta}{2}+\frac{\beta}{2}} + 6e^{\frac{\theta}{2}+\frac{3\beta}{4}}\right)\beta\sum_{i=1}^{n}\sigma_i^2 E|X_i^* - X_i| \\ &\leq \frac{16}{3e^{\frac{3k}{8}}}e^{\frac{\theta}{2}+\frac{3\beta}{4}}\sum_{i=1}^{n}EX_i^4 + \frac{2}{e^{\frac{3k}{8}}}\left(e^{\frac{\theta}{2}+\frac{\beta}{2}} + 6e^{\frac{\theta}{2}+\frac{3\beta}{4}}\right)\beta\sum_{i=1}^{n}E|X_i|^3 \\ &\leq \frac{16}{3e^{\frac{3k}{8}}}e^{\frac{\theta}{2}+\frac{3\beta}{4}}\beta^2 + \frac{2}{e^{\frac{3k}{8}}}\left(e^{\frac{\theta}{2}+\frac{\beta}{2}} + 6e^{\frac{\theta}{2}+\frac{3\beta}{4}}\right)\beta^2 \\ &= \frac{1}{e^{\frac{3k}{8}}}\left(17.34e^{\frac{\theta}{2}+\frac{3\beta}{4}} + 2e^{\frac{\theta}{2}+\frac{\beta}{2}}\right)\beta^2. \end{aligned} \tag{4.64}$$

Note that, by (2.32), we have

$$\begin{aligned} P\left(B^c\right) &\leq P\left(|X_i^*| \geq \frac{k}{4}\right) + P\left(|X_i| \geq \frac{k}{4}\right) \\ &\leq \frac{e^{|X_i^*|}}{e^{\frac{k}{4}}} + \frac{e^{|X_i|}}{e^{\frac{k}{4}}} \\ &\leq \frac{2e^{\beta}}{e^{\frac{k}{4}}}. \end{aligned}$$

From this fact and (4.48), we have

$$\begin{aligned} &\sum_{i=1}^{n} \sigma_i^2 E\left(|X_i^* - X_i| D_{n2}\right) \\ &\leq \sum_{i=1}^{n} \sigma_i^2 \left[3.58\left(\frac{EX_i^6}{\sigma_i^2}\right)^{\frac{1}{2}} + 7.89\sum_{j=1}^{n} E\left|X_j\right|^3 \left(\frac{EX_i^4}{\sigma_i^2}\right)^{\frac{1}{2}}\right]\left(\frac{2e^{\beta}}{e^{\frac{k}{4}}}\right)^{\frac{1}{2}} \\ &\leq \left(3.58\beta^2 + 7.89\beta^2\right)\sum_{i=1}^{n}\sigma_i^2\left(\frac{2e^{\beta}}{e^{\frac{k}{4}}}\right)^{\frac{1}{2}} \\ &\leq \frac{16.23}{e^{\frac{k}{8}}} e^{\frac{\beta}{2}}\beta^2. \end{aligned} \tag{4.65}$$

By (4.64) and (4.65), we have

$$D_n \leq \frac{1}{e^{\frac{3k}{8}}}\left(17.34e^{\frac{\theta}{2}+\frac{3\beta}{4}} + 2e^{\frac{\theta}{2}+\frac{\beta}{2}}\right)\beta^2 + \frac{16.23}{e^{\frac{k}{8}}}e^{\frac{\beta}{2}}\beta^2. \tag{4.66}$$

Combine (4.61), (4.62), (4.63) and (4.66), we have Theorem 4.7.

□

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