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FIRST DIGIT DISTRIBUTIONS OF PARTIAL DERIVATIVES OF COPULAS

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เราเสนอตัวบ่งชี้การขึ้นต่อกันของสองตัวแปรสุ่มที่มีฟังก์ชันการแจกแจงต่อเนื่อง การศึกษาของเรามีจุดเริ่มต้นจากข้อเท็จจริงที่ทราบกันดีว่าค่าคาดหมายของตัวแปรสุ่มหนึ่งภายใต้เงื่อนไขของอีกตัวแปรสุ่มหนึ่งถูกกำหนดอย่างสมบูรณ์โดยอนุพันธ์ย่อยของคอปูลาของสองตัวแปรสุ่มนั้น เราจึงศึกษาว่าฟังก์ชันการแจกแจงเลขโดดหลักแรกของอนุพันธ์ย่อยของคอปูลาก่อให้เกิดตัววัดการขึ้นต่อกันหรือไม่ สิ่งที่เราสนใจคือค่าสูงสุดแบบบรรทัดฐานของฟังก์ชันการแจกแจงเลขโดดหลักแรกของอนุพันธ์ย่อยของคอปูลา เรายังแสดงว่าค่าสูงสุดแบบบรรทัดฐานนี้บรรลุค่าต่ำสุดถ้าตัวแปรสุ่มอิสระต่อกัน และบรรลุค่าสูงสุดถ้าตัวแปรสุ่มขึ้นต่อกันอย่างสมบูรณ์ อย่างไรก็ตามเราพบตัวอย่างค้านของบทกลับของทั้งคู่ ท้ายที่สุดเราเสนอวิธีการคำนวณเชิงตัวเลขของตัวบ่งชี้การขึ้นต่อกัน ถึงแม้ว่าทั้งความอิสระต่อกันและความขึ้นต่อกันอย่างสมบูรณ์ไม่สามารถถูกแสดงลักษณะโดยตัวบ่งชี้การขึ้นต่อกัน แต่เราแสดงผ่านคอปูลาบางคลาสที่มีพารามิเตอร์ว่าค่าเชิงตัวเลขของตัวบ่งชี้การขึ้นต่อกันแปรผันตรงกับค่าของตัววัดการขึ้นต่อกันของซีเบิร์ก-สโทเมนอฟและของทรัตช์นิก

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We propose an indicator of dependence between two random variables X and Y whose distribution functions are continuous. The starting point of our investigation is the well-known fact that if C is the copula of X and Y, then the conditional expectations of Y given X and of X given Y are completely determined by the first partial derivatives of C. We then investigate whether the distributions of the first digits of $\partial_1 C$ and $\partial_2 C$ can give rise to a measure of dependence. Our candidates are normalized maxima of the first digit distributions of $\partial_i C$. We also show that these normalized maxima attain minimum value if X and Y are independent and attain maximum value when they are completely dependent. However, both converses are not true as we are able to produce counterexamples. At last, we introduce a numerical computation of the dependence indicators. Although neither independence nor complete dependence can be characterized by the dependence indicators, we illustrate via some parametric classes of copulas that their numerical values are directly proportional to those of the dependence measures introduced by Siburg-Stoimenov and of Trutschnig.

Department :	Mathematics and Computer Science	Student's Signature ....
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CONTENTS

CHAPTER I

INTRODUCTION

In measuring the level of dependence between two random variables, since it largely depends on a given circumstance, measures of dependence are needed for consistency. Unsurprisingly, measures of dependence have been extensively studied. Their simple properties are ability to detect independence and complete dependence and invariance under certain types of transformations. Many dependence measures have been proposed and investigated via bivariate copulas owing to Sklar's theorem [10]. It says that copulas are functions describing the dependence structure of two random variables regardless of their marginal distributions. Such dependence measures are called copula-based measures of dependence. Via many copula-based measures of dependence, e.g. Siburg and Stoimenov's ω_1 (see [15]) and Trutschnig's ζ_1 (see [16]), it has become more transparent that dependence information is encoded in the first order partial derivatives of copulas. Also Darsow, Nguyen and Olsen [2] have shown that the first order partial derivatives of copulas coincide with the conditional distributions of one random variable given the other. From experiments, the first digit distribution yields some information about the function and is computationally robust for most copulas. This motivates us to propose a new dependence indicator based on the first digit distributions of the partial derivatives of copulas.

For $i = 1, 2$, we define functions $\mathrm{FD}_d\left(\partial_i C\right)$ and $\mathrm{MF}_i\left(C\right)$ to introduce a dependence indicator $\mu_i(C)$ of copula C as follows:

$$\mathrm{FD}_d(\partial_i C) = \frac{\lambda_2\left(\left\{(u,v)\in[0,1]^2\colon \text{ the first digit of } \partial_i C(u,v) \text{ is } d\right\}\right)}{\lambda_2\left(\{(u,v)\in[0,1]^2\colon \partial_i C(u,v)>0\}\right)} \in [0,1],$$

$$\mathrm{MF}_i(C) = \max_{d=1,\ldots,9} \mathrm{FD}_d(\partial_i C) \in \left[\frac{1}{9}, 1\right] \quad \text{and} \quad \mu_i(C) = \sqrt{\frac{9\,\mathrm{MF}_i(C)-1}{8}} \in [0,1],$$

where λ_2 denotes Lebesgue measure on $[0,1]^2$. We shall only investigate μ_1 as μ_2 can be studied in a completely similar manner. We show that although μ_1 lacks the ability to totally detect independence and complete dependence, it yields levels of dependence as well as ω_1 and ζ_1 for certain parametric classes of copulas.

In Chapter II, we give a brief introduction to copulas and selected tools for proving the main results. Chapter III covers two sections. In the first section, theoretical facts of the upper level sets are proved. After all the prerequisites for $\mathrm{FD}_d, \mathrm{MF}_1$ and μ_1 are ready, we prove the main results in the second section. The last chapter is devoted to numerical computation of μ_1, examples and comparisons.

CHAPTER II

PRELIMINARIES

First we introduce some notations used throughout this thesis. Let $\mathbb{I}$ denote the closed unit interval $[0,1]$, $\mathcal{B}(\mathbb{I})$ the Borel σ-algebra on $\mathbb{I}$, λ and λ_2 the Lebesgue measures on $\mathbb{I}$ and $\mathbb{I}^2$, respectively. We may simply write $\int_A \mathrm{d}x$ for the Lebesgue integral on a subset A of $\mathbb{I}$. For any set S, $\mathbb{1}_S$ denotes a characteristic function of S. As usual, $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ denote the sets of natural, integer and real numbers, respectively. Let us start with a definition of copulas.

Definition 2.1. A *bivariate copula* is a function C from $\mathbb{I}^2$ to $\mathbb{I}$ with the following properties.

I. (Boundary conditions) For every u, v in $\mathbb{I}$, $C(u,0)=0=C(0,v)$, $C(u,1)=u$ and $C(1,v)=v$.

II. (2-increasing property) For each rectangle $R:=(u_1,u_2]\times(v_1,v_2]\subseteq\mathbb{I}^2$,

$$V_C(R):=C(u_2,v_2)-C(u_2,v_1)-C(u_1,v_2)+C(u_1,v_1)\geq 0.$$

Each copula C induces a unique measure μ_C on $\mathbb{I}^2$ via $\mu_C(R):=V_C(R)$ for any rectangle $R\subseteq\mathbb{I}^2$ and a standard extension to all Borel subsets of $\mathbb{I}^2$ using Carathéodory-Hahn extension theorem. The *support of* C is defined as the support of its induced measure μ_C, i.e. it is the complement of the union of all open subsets of $\mathbb{I}^2$ that are μ_C-null sets. Moreover, the induced measure μ_C is a *doubly stochastic measure*, as for every Borel subset B of $\mathbb{I}$,

$$\mu_C(\mathbb{I}\times B)=\lambda(B)=\mu_C(B\times\mathbb{I}).$$

Conversely, for each doubly stochastic measure μ, the copula corresponding to that measure μ is obtained by $C(u,v)=\mu([0,u]\times[0,v])$. In fact, this is a one-to-one correspondence between the set of copulas and the set of doubly stochastic measures.

The joint continuity of copulas can be obtained from a simple fact that every copula C is Lipschitz, i.e. for every $u_1, u_2, v_1, v_2 \in \mathbb{I}$,

$$|C(u_2, v_2) - C(u_1, v_1)| \leq |u_2 - u_1| + |v_2 - v_1|.$$

As a consequence, C is absolutely continuous in each argument, or, for all $u, v \in \mathbb{I}$, $t \mapsto C(u,t)$ and $t \mapsto C(t,v)$ are absolutely continuous on $\mathbb{I}$. Another important property of copulas is the non-decreasing property in each argument being a virtue of the 2-increasing property. By the Lebesgue Differentiation Theorem, this implies that for any $v \in \mathbb{I}$, the first order partial derivative $\partial_1 C(u,v) := \frac{\partial}{\partial u} C(u,v)$ exists for almost all $u \in \mathbb{I}$ and for any $u \in \mathbb{I}$, the first order partial derivative $\partial_2 C(u,v) := \frac{\partial}{\partial v} C(u,v)$ exists for almost all $v \in \mathbb{I}$ and the derivatives $\partial_i C$ $(i = 1, 2)$ are measurable. For $u, v \in \mathbb{I}$ such that $\partial_i C(u,v)$ exists, $0 \leq \partial_i C(u,v) \leq 1$ $(i = 1, 2)$. This is again a result of the Lipschitz condition. Moreover, using the 2-increasing property, it is straightforward to show that the partial derivatives of copulas are non-decreasing almost everywhere in the sense that for every $u, v \in \mathbb{I}$, the functions $t \mapsto \partial_1 C(u,t)$ and $t \mapsto \partial_2 C(t,v)$ are non-decreasing on a subset of $\mathbb{I}$ of full measure.

Owing to Sklar's theorem, see [10], copulas become very useful.

Theorem 2.2 (Sklar's theorem)**.** *Let X and Y be random variables with distribution functions F and G, respectively, and joint distribution function H. Then there exists a* ($unique$ *on* $\operatorname{Ran} F \times \operatorname{Ran} G$) *copula C such that for all $x, y \in \mathbb{R}$,*

$$H(x,y) = C(F(x), G(y)). \tag{2.1}$$

Conversely, if C is a copula, F and G are distribution functions, then the function H defined by (2.1) *is a joint distribution function with the margins F and G, respectively.*

Additionally, the uniqueness of the copula of random variables immediately follows when the marginal distribution functions are continuous. Because of this, we will assume that every random variable in this thesis has continuous distribution function. Here, under this assumption, we denote by $C_{X,Y}$ (called the copula of X and Y) the unique copula of random variables X and Y satisfying (2.1). When the random variables X and Y are understood, we may simply write C.

Surprisingly, the first order partial derivatives of copulas of random variables are related to the conditional probabilities of those random variables. We recall some basic facts on conditional probabilities.

For a random variable X, we denote by $\sigma(X)$ the σ-algebra generated by all inverse images of Borel sets under X, that is $\sigma(X) := \left\{X^{-1}(B) : B \in \mathcal{B}(\mathbb{R})\right\}$, where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra on $\mathbb{R}$.

Definition 2.3. Let X and Y be random variables on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$ with $\mathrm{E}[|X|] < \infty$. Then the *conditional expectation of X with respect to Y* is a random variable Z on a measurable space $(\Omega, \sigma(Y))$ satisfying

$$\int_A X \, \mathrm{dP} = \int_A Z \, \mathrm{dP} \quad \text{for all } A \in \sigma(Y). \tag{2.2}$$

It is straightforward to show that set function on $\sigma(Y)$ defined by the left hand side of identity (2.2) is absolutely continuous with respect to P. Hence, by Radon-Nikodym Theorem, see [11], the conditional expectation is in fact the Radon-Nikodym derivative and it is unique up to a set of measure zero and we then denote by $\mathrm{E}[X|Y]$ the conditional expectation of X with respect to Y. Naturally, we define the *conditional probability of $B \in \mathcal{F}$ given Y* by $\mathrm{P}(B|Y) := \mathrm{E}[\mathbb{1}_B|Y]$. One can see that $\mathrm{P}(B|Y)$ is the unique (a.e.) $\sigma(Y)$-measurable function such that

$$\mathrm{P}(A \cap B) = \int_A \mathbb{1}_B \, \mathrm{dP} = \int_A \mathrm{P}(B|Y) \, \mathrm{dP} \quad \text{for all } A \in \sigma(Y).$$

Let us summarize essential properties of the conditional expectation. For more details, see [4, 11, 13, 18].

Theorem 2.4. *Let X and Y be random variables on a common probability space. Suppose that* $\mathrm{E}\left[|X|\right] < \infty$.

I. $\mathrm{E}\left[\mathrm{E}\left[X|Y\right]\right] = \mathrm{E}\left[X\right]$

II. If $Y = c$ *a.s.* $(c \in \mathbb{R})$, *then* $\mathrm{E}\left[X|Y\right] = \mathrm{E}\left[X\right]$ *a.s.*

III. If X and Y are independent, then $\mathrm{E}\left[X|Y\right] = \mathrm{E}\left[X\right]$ *a.s.*

IV. If Z is $\sigma(Y)$-measurable and bounded, then $\mathrm{E}\left[ZX|Y\right] = Z\,\mathrm{E}\left[X|Y\right]$ *a.s.*

We are now ready to prove a surprising result.

Theorem 2.5. *Let C be the copula of random variables X and Y whose continuous marginal distributions are F and G, respectively. Then for all $x, y \in \mathbb{R}$,*

I. $\mathrm{P}\left(Y \leq y|X\right)(\omega) = \mathrm{E}\left[\mathbb{1}_{\{Y \leq y\}}|X\right](\omega) = \partial_1 C\left(F(X(\omega)), G(y)\right)$ *a.e.* $\omega \in \Omega$*; and*

II. $\mathrm{P}\left(X \leq x|Y\right)(\omega) = \mathrm{E}\left[\mathbb{1}_{\{X \leq x\}}|Y\right](\omega) = \partial_2 C\left(F(x), G(Y(\omega))\right)$ *a.e.* $\omega \in \Omega$.

Proof. Since $\mathrm{P}\left(X \leq x|Y\right) = \mathrm{E}\left[\mathbb{1}_{\{X \leq x\}}|Y\right]$ is the unique a.e. $\sigma(Y)$-measurable function such that

$$\int_A \mathbb{1}_{\{X \leq x\}}(\omega)\,\mathrm{dP}(\omega) = \int_A \mathrm{E}\left[\mathbb{1}_{\{X \leq x\}}|Y\right](\omega)\,\mathrm{dP}(\omega)$$

for all $A \in \sigma(Y)$. Note that $\partial_2 C\left(F(x), G(Y(\cdot))\right)$ is $\sigma(Y)$-measurable. It suffices to show that

$$\int_A \mathbb{1}_{\{X \leq x\}}(\omega)\,\mathrm{dP}(\omega) = \int_A \partial_2 C\left(F(x), G(Y(\omega))\right)\mathrm{dP}(\omega)$$

for $A = Y^{-1}\left((-\infty, a]\right), a \in \mathbb{R}$. By making changes of variables,

$$\begin{aligned}\int_{Y^{-1}((-\infty,a])} \partial_2 C\left(F(x), G(Y(\omega))\right)\mathrm{dP}(\omega) &= \int_{-\infty}^{a} \partial_2 C\left(F(x), G(y)\right)\mathrm{d}G(y) \\ &= \int_0^{G(a)} \partial_2 C\left(F(x), v\right)\mathrm{d}v\end{aligned}$$

$$
\begin{aligned}
&= C(F(x), G(a)) - C(F(x), 0) \\
&= C(F(x), G(a)) = \mathrm{P}(X \leq x, Y \leq a) \\
&= \int_{\Omega} \mathbb{1}_{\{X \leq x\}}(\omega) \cdot \mathbb{1}_{\{Y \leq a\}}(\omega) \, \mathrm{dP}(\omega) \\
&= \int_{\{Y \leq a\}} \mathbb{1}_{\{X \leq x\}}(\omega) \, \mathrm{dP}(\omega) \\
&= \int_{Y^{-1}((-\infty, a])} \mathbb{1}_{\{X \leq x\}}(\omega) \, \mathrm{dP}(\omega),
\end{aligned}
$$

where we have applied the fundamental theorem of calculus and the Sklar's theorem. Using the uniqueness of the conditional expectation, II. immediately follows. I. can be proved in a similar way. □

Next, we give important examples of copulas of random variables and their probabilistic meanings. First, the product copula $\Pi(u, v) := uv$ is sometimes called the independence copula Π because it is the copula of independent random variables.

Theorem 2.6. *For random variables X and Y with continuous distribution functions, $C_{X,Y} = \Pi$ if and only if X and Y are independent.*

Its proof directly follows from Sklar's theorem and will be omitted. We then give the copulas of completely dependent random variables in the sense defined as follows.

Definition 2.7. Let X and Y be random variables. Y is said to be *completely dependent* on X if there exists a Borel measurable function f such that $\mathrm{P}(Y = f(X)) = 1$. The copula $C_{X,Y}$ of X and Y is called a *complete dependence copula*. In the case that the function f is strictly monotonic, Y is said to be *monotonically dependent* on X and $C_{X,Y}$ is said to be a *monotonic dependence copula*. Moreover, X and Y are said to be *mutually completely dependent* if Y is completely dependent on X and X is completely dependent on Y or if there is an invertible Borel measurable function f with $\mathrm{P}(Y = f(X)) = 1$. Surely, such $C_{X,Y}$ is said to be a *mutual complete dependence copula*.

The copulas $W(u, v) := \max(u + v - 1, 0)$ and $M(u, v) := \min(u, v)$ are the most important examples of complete dependence copulas, as they are the two monotonic dependence copulas. The following theorem was given in Schweizer and Wolff (see [14]).

Theorem 2.8. *Let X and Y be random variables with continuous distribution functions.*

I. $C_{X,Y} = W$ *if and only if Y is strictly decreasingly dependent on X.*

II. $C_{X,Y} = M$ *if and only if Y is strictly increasingly dependent on X.*

So, by Theorem 2.8, W and M are often called the countermonotonic and comonotonic copulas, respectively. Furthermore, W and M are pointwise lower and upper bounds, respectively, of copulas, i.e. $W \leq C \leq M$ for all copulas C.

The following theorem is given by Darsow, Nguyen and Olsen (see [2]). They gave equivalent forms of complete dependence for random variables.

Theorem 2.9. *Let C be the copula of random variables X and Y having continuous marginal distributions. Then the following statements are equivalent.*

I. Y is completely dependent on X. (X is completely dependent on Y.)

II. For $y \in \mathbb{R}$, $\mathrm{P}(Y \leq y|X) \in \{0,1\}$ a.s. (For $x \in \mathbb{R}$, $\mathrm{P}(X \leq x|Y) \in \{0,1\}$ a.s.)

III. For $v \in \mathbb{I}$, $\partial_1 C(u,v) \in \{0,1\}$ almost every $u \in \mathbb{I}$. (For $u \in \mathbb{I}$, $\partial_2 C(u,v) \in \{0,1\}$ almost every $v \in \mathbb{I}$.)

Moreover, by virtue of Theorem 2.9, one obtains the following corollary.

Corollary 2.10. *Let C be the copula of random variables X and Y having continuous marginal distributions. Then the following statements are equivalent.*

I. X and Y are mutually completely dependent.

II. For $u, v \in \mathbb{I}$, $\partial_1 C(\cdot, v), \partial_2 C(u, \cdot) \in \{0,1\}$ almost everywhere.

Next, we present methods for constructing copulas used in this thesis. The first one is a method for constructing patched copulas introduced by Zheng, Yang and Huang (see [19]). Recall from the identity (2.1) that a copula is the joint distribution function of two uniform random variables on $\mathbb{I}$. So, given a sample space Ω with probability measure P, let C be a copula, it is indeed a joint distribution function, of random variables U and V that are uniformly distributed on $\mathbb{I}$. Fix $m, n \in \mathbb{N}$, let $\{c_i\}_{i=1}^m$ and $\{d_j\}_{j=1}^n$ be such that

$$0 = c_0 < c_1 < \cdots < c_m = 1 \quad \textit{and} \quad 0 = d_0 < d_1 < \cdots < d_n = 1.$$

Let $I_0 = [c_0, c_1]$ and $I_i = (c_i, c_{i+1}]$ for $i = 1, 2, \ldots, m-1$, $J_0 = [d_0, d_1]$ and $J_j = (d_j, d_{j+1}]$ for $j = 1, 2, \ldots, n-1$. Then $\{I_i \times J_j : i \in N_{m-1}, j \in N_{n-1}\}$ is a partition of $\mathbb{I}^2$, where $N_k = \{0, 1, \ldots, k\}$. For $i \in N_{m-1}$, $j \in N_{n-1}$, set $A_{ij} = \{\omega \in \Omega : (U(\omega), V(\omega)) \in I_i \times J_j\}$ then $\{A_{ij} : i \in N_{m-1}, j \in N_{n-1}\}$ is a partition of Ω. Using the law of total probability, one can derive that

$$C(u, v) = \mathrm{P}(U \leq u, V \leq v) = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \mathrm{P}(A_{ij})\, \mathrm{P}(U \leq u, V \leq v | A_{ij}). \tag{2.3}$$

One can show that each $\mathrm{P}(U \leq u, V \leq v | A_{ij})$ is a joint distribution function on $I_i \times J_j$ whose continuous marginal distributions are $F_{ij}(u) := \mathrm{P}(U \leq u | A_{ij})$ and $G_{ij}(v) := \mathrm{P}(V \leq v | A_{ij})$. Here, the continuities of F_{ij} and G_{ij} follow from the continuity of C. According to Sklar's theorem, there is a unique corresponding copula C_{ij} such that

$$\mathrm{P}(U \leq u, V \leq v | A_{ij}) = C_{ij}(F_{ij}(u), G_{ij}(v)). \tag{2.4}$$

Plugging (2.4) into (2.3), we have

$$C(u, v) = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \mathrm{P}(A_{ij}) C_{ij}(F_{ij}(u), G_{ij}(v)). \tag{2.5}$$

Note that $[\mathrm{P}(A_{ij})]_{m \times n}$ is a doubly stochastic matrix. The decomposition in (2.5) is called the $(\{c_i\}_{i=1}^m, \{d_j\}_{j=1}^n)$*-patched decomposition of* C or just *patched decomposition of* C if $\{c_i\}_{i=1}^m$ and $\{d_j\}_{j=1}^n$ are understood. This is in fact a generalization of patched copulas, which was introduced by Zheng, Yang and Huang [19], in the sense that the unit square is partitioned into unequal-sized sub-rectangles. Any given copula C can be decomposed in the form of (2.5). On the whole, the converse is not true. However, we obtain the following characterization on some assumptions of the coefficients and marginal distributions. Indeed, the following theorem is a generalization of the case that $n = m$ and $c_i = d_i = \frac{i}{n}$ for all $i = 0, 1, \ldots, n$, which was given by Chaidee, Santiwipanont and Sumetkijakan (see [1]).

Theorem 2.11. *For each $i = 0, 1, \ldots, m-1$ and $j = 0, 1, \ldots, n-1$, let H_{ij} and L_{ij} denote distribution functions over I_i and J_j, respectively, let D_{ij} be a copula and $a_{ij} \geq 0$. The function D defined by*

$$D(u, v) = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} a_{ij} D_{ij}(H_{ij}(u), L_{ij}(v)), \quad u, v \in \mathbb{I},$$

is a copula if and only if

I. *for each $h = 0, 1, \ldots, m-1$, $\sum_{j=0}^{n-1} a_{hj} H_{hj}(u) = u - c_h$ for all $u \in I_h$; and*

II. *for each $k = 0, 1, \ldots, n-1$, $\sum_{i=0}^{m-1} a_{ik} L_{ik}(v) = v - d_k$ for all $v \in J_k$.*

Proof. ($\Rightarrow$) We prove I., the proof of II. is similar. Fix $h = 0, 1, \ldots, m-1$. Since for each $j \geq 0$, $H_{ij}(c_{h+1}) = 1$ for all $i \leq h$ and $H_{ij}(c_{h+1}) = 0$ for all $i > h$ and D is a copula,

$$c_{h+1} = D(c_{h+1}, 1) = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} a_{ij} D_{ij}(H_{ij}(c_{h+1}), L_{ij}(1)) = \sum_{i=0}^{h} \sum_{j=0}^{n-1} a_{ij}$$

which implies in particular that $\sum_{j=0}^{n-1} a_{ij} = c_{i+1} - c_i$ for all $i = 0, 1, \ldots, m-1$. We are now ready to show I. Let $u \in I_h$. So for each $j \geq 0$, $H_{ij}(u) = 1$ for all $i < h$ and $H_{ij}(u) = 0$ for all $i > h$. Then

$$D(u, 1) = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} a_{ij} D_{ij}(H_{ij}(u), L_{ij}(1))$$

$$\begin{aligned}
&= \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} a_{ij} D_{ij}(H_{ij}(u),1) = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} a_{ij} H_{ij}(u) \\
&= \sum_{i=0}^{h-1}\sum_{j=0}^{n-1} a_{ij} + \sum_{j=0}^{n-1} a_{hj} H_{hj}(u) \qquad (2.6)\\
&= \sum_{i=0}^{h-1}(c_{i+1}-c_i) + \sum_{j=0}^{n-1} a_{hj} H_{hj}(u) \\
&= c_h + \sum_{j=0}^{n-1} a_{hj} H_{hj}(u).
\end{aligned}$$

Thus I. follows because D is a copula.

($\Leftarrow$) The statement $D(u,0) = 0 = D(0,v)$ for $u, v \in \mathbb{I}$ follows directly from the condition $H_{ij}(0) = 0 = L_{ij}(0)$ and D_{ij} is a copula for all i, j. Moreover, it is also straightforward to prove that D is 2-increasing by using the assumption that H_{ij} and L_{ij} are distribution functions, that D_{ij} is 2-increasing and that $a_{ij} \geq 0$. It remains to show that $D(u,1) = u$ and $D(1,v) = v$. For each $h = 0, 1, \ldots, m-1$, let $u \in I_h$. Then

$$D(u,1) = \sum_{i=0}^{h-1}\sum_{j=0}^{n-1} a_{ij} + \sum_{j=0}^{n-1} a_{hj} H_{hj}(u)$$

follows from equation (2.6), which is not a consequence of condition that D is a copula. Then the condition I. particularly gives $\sum_{j=0}^{n-1} a_{ij} = c_{i+1} - c_i$ for $i = 0, 1, \ldots, m-1$ and yields

$$D(u,1) = \sum_{i=0}^{h-1}(c_{i+1}-c_i) + \sum_{j=0}^{n-1} a_{hj} H_{hj}(u) = c_h + u - c_h = u.$$

The last property required for definition of copulas is $D(1,v) = v$ for all $v \in \mathbb{I}$. It can be proved in a similar manner using II. □

One can obtain a new copula from old by the following binary operation on the space of all copulas, denoted by $\mathcal{C}$. A binary operation $*$ on $\mathcal{C}$ is defined as

$$C * D(u,v) = \int_{\mathbb{I}} \partial_2 C(u,t)\partial_1 D(t,v)\,\mathrm{d}t \quad \text{for } C, D \in \mathcal{C} \text{ and } u, v \in \mathbb{I}.$$

This was introduced by Darsow, Nguyen and Olsen [2], they proved that $*$ is a binary operation on $\mathcal{C}$ and $(\mathcal{C}, *)$ is a monoid with null element Π and identity M.

In the following theorem, they also proved that for any random variables X, Y and Z, the copula $C_{X,Z}$ can be decomposed into $*$-product of the copulas $C_{X,Y}$ and $C_{Y,Z}$ where some conditions on three random variables are assumed.

Theorem 2.12**.** *Let X, Y and Z be random variables having continuous marginal distributions on a common probability space such that*

$$\mathrm{E}\left[\mathbb{1}_{\{X\in B_1\}} \cdot \mathbb{1}_{\{Z\in B_2\}}|Y\right] = \mathrm{E}\left[\mathbb{1}_{\{X\in B_1\}}|Y\right] \cdot \mathrm{E}\left[\mathbb{1}_{\{Z\in B_2\}}|Y\right] \; a.s.$$

for any Borel sets B_1, B_2, one indeed says that X and Z are conditionally independent given Y. Then $C_{X,Z} = C_{X,Y} * C_{Y,Z}$.

Proof. Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a probability space and F_X, F_Y and F_Z continuous distribution functions of X, Y and Z, respectively. Let $u, v \in \mathbb{I}$. If u or v are in $\{0, 1\}$, it is obvious that $C_{X,Z}(u, v) = C_{X,Y} * C_{Y,Z}(u, v)$. Suppose that $u, v \notin \{0, 1\}$. Since F_X and F_Z are continuous, by applying Intermediate Value Theorem, there exist real numbers x and z such that $F_X(x) = u$ and $F_Z(z) = v$. Then

$$\begin{aligned}
&C_{X,Y} * C_{Y,Z}(u, v)\\
&= \int_{\mathbb{I}} \partial_2 C_{X,Y}(u, t)\partial_1 C_{Y,Z}(t, v)\,\mathrm{d}t\\
&= \int_{-\infty}^{\infty} \partial_2 C_{X,Y}(F_X(x), F_Y(y))\partial_1 C_{Y,Z}(F_Y(y), F_Z(z))\,\mathrm{d}F_Y(y)\\
&= \int_{\Omega} \partial_2 C_{X,Y}(F_X(x), F_Y(Y(\omega)))\partial_1 C_{Y,Z}(F_Y(Y(\omega)), F_Z(z))\,\mathrm{dP}(\omega)\\
&= \int_{\Omega} \mathrm{E}\left[\mathbb{1}_{\{X\le x\}}|Y\right](\omega)\,\mathrm{E}\left[\mathbb{1}_{\{Z\le z\}}|Y\right](\omega)\,\mathrm{dP}(\omega)\\
&= \int_{\Omega} \mathrm{E}\left[\mathbb{1}_{\{X\le x\}}\mathbb{1}_{\{Z\le z\}}|Y\right](\omega)\,\mathrm{dP}(\omega) = \int_{\Omega} \mathrm{E}\left[\mathbb{1}_{\{X\le x, Z\le z\}}|Y\right](\omega)\,\mathrm{dP}(\omega)\\
&= \int_{\Omega} \mathbb{1}_{\{X\le x, Z\le z\}}\,\mathrm{dP} = \mathrm{P}\left(X \le x, Z \le z\right)\\
&= C_{X,Z}\left(F_X(x), F_Z(z)\right) = C_{X,Z}(u, v),
\end{aligned}$$

where we have used changes of variables theorem in the second and third equalities, Theorem 2.5 in the fourth equality, the assumption that X and Z are conditionally independent given Y in the fifth equality, the definition of the conditional expectation in the seventh equality and Sklar's theorem in the next to last equality. □

Moreover, Ruankong, Santiwipanont and Sumetkijakan (see [12]) gave the following special case frequently seen.

Corollary 2.13. *Let X and Y be random variables having continuous marginal distributions and f a Borel measurable function. Then $C_{f(X),Y} = C_{f(X),X} * C_{X,Y}$.*

They proved that $f(X)$ and Y are conditionally independent given X and then the corollary above immediately follows from Theorem 2.12.

Moreover, Darsow and Olsen [3] showed that the $*$-product on $\mathcal{C}$ is jointly continuous with respect to the Sobolev norm.

Let $p \in \mathbb{N}$. The *Sobolev norm* $\|\cdot\|_{(p)}$ of a copula C is defined by

$$\|C\|_{(p)} = \left(\int_{\mathbb{I}}\int_{\mathbb{I}} (|\partial_1 C(u,v)|^p + |\partial_2 C(u,v)|^p)\,\mathrm{d}u\,\mathrm{d}v\right)^{1/p}.$$

Theorem 2.14. *The $*$-product on $\mathcal{C}$ is jointly continuous with respect to the Sobolev norm. Formally, let $p \in \mathbb{N}$ and for any $n \in \mathbb{N}$, let C_n, D_n, C and D be copulas such that $\lim_{n\to\infty} \|C_n - C\|_{(p)} = 0$ and $\lim_{n\to\infty} \|D_n - D\|_{(p)} = 0$. Then $\lim_{n\to\infty} \|C_n * D_n - C * D\|_{(p)} = 0$.*

Since the Sobolev norm of a copula is defined in term of its derivatives, the following results are easily obtained.

Theorem 2.15. *Let $(C_n)_{n\in\mathbb{N}}$ be a sequence of copulas and C be a copula such that $(\partial_i C_n)_{n\in\mathbb{N}}$ converges to $\partial_i C$ almost everywhere for $i = 1, 2$, and let $p \in \mathbb{N}$. Then $\lim_{n\to\infty} \|C_n - C\|_{(p)} = 0$.*

Proof. By the assumption, we have, for almost every $(u,v) \in \mathbb{I}^2$,

$$\lim_{n\to\infty} |\partial_i C_n(u,v) - \partial_i C(u,v)| = 0, \qquad i = 1,2.$$

Therefore,

$$\begin{aligned}
&\lim_{n\to\infty} \|C_n - C\|_{(p)} \\
&= \lim_{n\to\infty} \left(\int_{\mathbb{I}} \int_{\mathbb{I}} (|\partial_1 (C_n - C)|^p + |\partial_2 (C_n - C)|^p) \,\mathrm{d}u \,\mathrm{d}v \right)^{1/p} \\
&= \lim_{n\to\infty} \left(\int_{\mathbb{I}} \int_{\mathbb{I}} (|\partial_1 C_n - \partial_1 C)|^p + |\partial_2 C_n - \partial_2 C)|^p) \,\mathrm{d}u \,\mathrm{d}v \right)^{1/p} \\
&= \left(\lim_{n\to\infty} \left(\int_{\mathbb{I}} \int_{\mathbb{I}} (|\partial_1 C_n - \partial_1 C)|^p + |\partial_2 C_n - \partial_2 C)|^p) \,\mathrm{d}u \,\mathrm{d}v \right) \right)^{1/p} \\
&= \left(\int_{\mathbb{I}} \int_{\mathbb{I}} \lim_{n\to\infty} ((|\partial_1 C_n - \partial_1 C)|^p + |\partial_2 C_n - \partial_2 C)|^p)) \,\mathrm{d}u \,\mathrm{d}v \right)^{1/p} \\
&= 0,
\end{aligned}$$

where we have used the Dominated Convergence Theorem in the next to last equality. $\square$

Moreover, we will show that convergence of copulas in Sobolev norm $\|\cdot\|_{(p)}$ implies the existence of a subsequence whose first order partial derivatives converge pointwise almost everywhere.

Theorem 2.16**.** *Let $p \in \mathbb{N}$ and $(C_n)_{n\in\mathbb{N}}$ a sequence of copulas and C a copula such that $\lim_{n\to\infty} \|C_n - C\|_{(p)} = 0$. Then there exists a subsequence $(C_{n_k})_{k\in\mathbb{N}}$ such that $(\partial_i C_{n_k})_{k\in\mathbb{N}}$ converges to $\partial_i C$ almost everywhere for $i = 1,2$.*

Proof. Since $\lim_{n\to\infty} \|C_n - C\|_{(p)} = 0$, we can find a strictly increasing sequence of positive integers $(n_k)_{k\in\mathbb{N}}$ such that $\int_{\mathbb{I}} \int_{\mathbb{I}} (|\partial_1 C_{n_k} - \partial_1 C|^p + |\partial_2 C_{n_k} - \partial_2 C|^p) \,\mathrm{d}u \,\mathrm{d}v \le \frac{1}{2^k}$. And then

$$\begin{aligned}
&\int_{\mathbb{I}} \int_{\mathbb{I}} \sum_{k=1}^{\infty} (|\partial_1 C_{n_k} - \partial_1 C|^p + |\partial_2 C_{n_k} - \partial_2 C|^p) \,\mathrm{d}u \,\mathrm{d}v \\
&= \sum_{k=1}^{\infty} \int_{\mathbb{I}} \int_{\mathbb{I}} (|\partial_1 C_{n_k} - \partial_1 C|^p + |\partial_2 C_{n_k} - \partial_2 C|^p) \,\mathrm{d}u \,\mathrm{d}v \le \sum_{k=1}^{\infty} \frac{1}{2^k} < \infty.
\end{aligned}$$

Consequently,

$$\sum_{k=1}^{\infty} (|\partial_1 C_{n_k} - \partial_1 C|^p + |\partial_2 C_{n_k} - \partial_2 C|^p) < \infty \quad \text{a.e.}$$

In particular,

$$\sum_{k=1}^{\infty} |\partial_1 C_{n_k} - \partial_1 C|^p < \infty \quad \text{and} \quad \sum_{k=1}^{\infty} |\partial_2 C_{n_k} - \partial_2 C|^p < \infty \quad \text{a.e.}$$

Thus, $\lim_{k\to\infty} |\partial_i C_{n_k} - \partial_1 C| = 0$ a.e. for all $i = 1, 2$. □

Another interesting method for constructing copulas is according to the following procedure.

I. Place the support for M on $\mathbb{I}^2$.

II. Cut $\mathbb{I}^2$ into a finite number of strips vertically.

III. Shuffle the strips (some of them can be flipped around their vertical axes of symmetry).

IV. Put them together again to form the square.

A copula whose support is obtained by the above method is called a *shuffle of Min*. The formal definition is given by Mikusiński, Sherwood and Taylor [9].

Definition 2.17. Let $n \in \mathbb{N}$, $0 = s_0 < s_1 < \cdots < s_n = 1$ and $0 = t_0 < t_1 < \cdots < t_n = 1$ two partitions of $\mathbb{I}$ and σ a permutation on $\{1, 2, \ldots, n\}$. A copula C is called the *shuffle of Min generated by* $(n, \{s_i\}, \{t_i\}, \sigma)$ if each $[s_{i-1}, s_i] \times [t_{\sigma(i)-1}, t_{\sigma(i)}]$ is a square in which C puts a probability mass $s_i - s_{i-1}$ spread uniformly on one of the diagonals. For each $i \in \{1, 2, \ldots, n\}$, let $m(i)$ denote the slope of the diagonal of $[s_{i-1}, s_i] \times [t_{\sigma(i)-1}, t_{\sigma(i)}]$ on which the probability mass in that square is distributed. We also say that C is the *shuffle of Min generated by* $(n, \{s_i\}, \{t_i\}, \sigma, m)$.

Mikusiński, Sherwood and Taylor (see [9]) also gave a probabilistic meaning of shuffles of Min.

Theorem 2.18. *Let X and Y be random variables with continuous distribution functions. Then $C_{X,Y}$ is a shuffle of Min if and only if there exists an invertible Borel measurable function with finitely many discontinuity points f such that $Y = f(X)$ a.s.*

Durante, Sarkoci and Sempi [5] characterize shuffles of Min in terms of push-forward measures, let us recall the definition of push-forwards. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $(\Omega_1, \mathcal{F}_1)$ a measurable space. For each measurable function $f : \Omega \to \Omega_1$, a *push-forward of μ under f* is a set function $f \star \mu$ on $(\Omega_1, \mathcal{F}_1)$ defined by $f \star \mu(A) = \mu\left(f^{-1}(A)\right)$ for every $A \in \mathcal{F}_1$. One can obtain that the push-forward $f \star \mu$ is a measure on $\mathcal{F}_1$. Moreover, if g is a measurable function from Ω_1 to another measurable space $(\Omega_2, \mathcal{F}_2)$, then

$$(g \circ f) \star \mu = g \star (f \star \mu). \tag{2.7}$$

Before characterizing shuffles of Min, the notions of shufflings and measure-preserving functions are needed. A *shuffling* $S_T : \mathbb{I}^2 \to \mathbb{I}^2$ is defined by

$$S_T(u, v) = (T(u), v), \tag{2.8}$$

where $T : \mathbb{I} \to \mathbb{I}$. Note that ${S_T}^{-1} = S_{T^{-1}}$. A Borel measurable function $f : \mathbb{I} \to \mathbb{I}$ is *measure-preserving* if for any $B \in \mathcal{B}(\mathbb{I})$, $\lambda(f^{-1}(B)) = \lambda(B)$. Borel measurability of the inverse of a Borel measurable injection yields the measure-preserving of the inverse of a measure-preserving bijection. Moreover, the following result is also needed.

Lemma 2.19. *Let $T : \mathbb{I} \to \mathbb{I}$ be a Borel measurable, S_T a shuffling and μ a doubly stochastic measure. Then the following statements are equivalent.*

I. $S_T \star \mu$ is doubly stochastic measure.

II. T is a measure-preserving transformation.

Proof. Using doubly stochastic property of μ, one can directly show that for every Borel set $B \subseteq \mathbb{I}$,

$$
\begin{aligned}
S_T \star \mu\left(\mathbb{I} \times B\right) &= \mu\left(S_T^{-1}(\mathbb{I} \times B)\right) = \mu\left(\mathbb{I} \times B\right) = \lambda(B), \\
S_T \star \mu\left(B \times \mathbb{I}\right) &= \mu\left(S_T^{-1}(B \times \mathbb{I})\right) = \mu\left(T^{-1}(B) \times \mathbb{I}\right) = \lambda\left(T^{-1}(B)\right).
\end{aligned}
$$

Then the lemma immediately follows. □

Theorem 2.20 (The characterization of shuffles of Min)**.** *A copula C is a shuffle of Min if and only if there exists a piecewise-continuous measure-preserving bijection $T : \mathbb{I} \to \mathbb{I}$ such that $\mu_C = S_T \star \mu_M$.*

Proof. ($\Leftarrow$) Let V be a uniform $(0,1)$ random variable. Then (V,V) is distributed according to M, i.e. the copula of V and V is M, and so $(V,V) \star \mathrm{P} = \mu_M$. Equations (2.7) and (2.8) yield, for all measurable transformation T of $\mathbb{I}$,

$$
(T \circ V, V) \star \mathrm{P} = (S_T \circ (V,V)) \star \mathrm{P} = S_T \star ((V,V) \star \mathrm{P}) = S_T \star \mu_M. \tag{2.9}
$$

Note that T is piecewise-continuous measure-preserving bijection by assumption and μ_M is doubly stochastic. Then by Lemma 2.19, $S_T \star \mu_M$ is doubly stochastic and so corresponds to a copula C. By equation (2.9), joint distribution function of $T \circ V$ and V is the connecting copula C. Thus, according to Theorem 2.18, C is a shuffle of Min.
($\Rightarrow$) Let C be a shuffle of Min. Then there exist uniform $(0,1)$ random variables U and V whose connecting copula is C. So $(U,V) \star \mathrm{P} = \mu_C$. Furthermore, by Theorem 2.18, there exists a piecewise-continuous bijection T on $\mathbb{I}$ for which $U = T(V)$ a.s. Then

$$
(T \circ V, V) \star \mathrm{P} = (U,V) \star \mathrm{P} = \mu_C.
$$

Using (2.9), which is not a consequence of condition that T is measure-preserving, we obtain $\mu_C = S_T \star \mu_M$. Finally, Lemma 2.19 yields the measure-preserving of T. □

The characterization of shuffles of Min leads Durante, Sarkoci and Sempi [5] to generalize the concept of shuffle of Min by dropping piecewise-continuous of T.

Definition 2.21. A *generalized shuffle of Min* is a copula C whose induced measure is

$$\mu_C := S_T \star \mu_M \tag{2.10}$$

for some measure-preserving bijection $T : \mathbb{I} \to \mathbb{I}$ (not necessary to be piecewise-continuous). Replacing M by a given copula D in (2.10), C is called a *generalized shuffle of* D. If the measure-preserving bijection T is piecewise-continuous, then C is called a *shuffle of* D.

The most useful properties used in the next chapter are listed in the following theorem. They are given by Ruankong, Santiwipanont and Sumetkijakan (see [12]).

Theorem 2.22. *Let C and D be copulas. Then*

1. *C is a T-shuffle of D if and only if there exists a shuffle of Min A, defined by $\mu_A = S_T \star \mu_M$, such that $C = A * D$.*
2. *C is a generalized T-shuffle of D if and only if there exists a generalized shuffle of Min A, defined by $\mu_A = S_T \star \mu_M$, such that $C = A * D$.*

CHAPTER III

DEPENDENCE INDICATORS FROM FIRST DIGIT DISTRIBUTIONS

We divide this chapter into two sections. The first section covers theoretical facts of the upper level sets needed to prove main properties of the first digit distributions of partial derivatives of copulas. As a result, we obtain an indicator of dependence μ_1 in the second section.

3.1 Upper level functions

Let $\mathcal{L}(\mathbb{I}^2)$ be the space of Lebesgue measurable subsets of $\mathbb{I}^2$. The function

$$\rho(A, B) := \lambda_2(A \triangle B), \quad \text{for } A, B \in \mathcal{L}(\mathbb{I}^2),$$

is a pseudometric on $\mathcal{L}(\mathbb{I}^2)$ where $A \triangle B := (A \setminus B) \cup (B \setminus A)$. ρ becomes a metric if $\mathcal{L}(\mathbb{I}^2)$ is considered modulo the equivalence relation $\sim$: $A \sim B$ if and only if $\lambda_2(A \triangle B) = 0$.

Definition 3.1. For a copula C, let $\Phi_{\partial_1 C}$ be a function on $\mathbb{I}$ defined by

$$\Phi_{\partial_1 C}(\alpha) = [\partial_1 C]_\alpha := \left\{(u, v) \in \mathbb{I}^2 : \partial_1 C(u, v) \text{ exists and } \partial_1 C(u, v) \geq \alpha\right\}.$$

$[\partial_1 C]_\alpha$ are called *upper α-level-sets*, introduced by Fernández-Sánchez and Trutschnig [6]. We may sometimes write $\left\{(u, v) \in \mathbb{I}^2 : \partial_1 C(u, v) \geq \alpha\right\}$ or just $\{\partial_1 C \geq \alpha\}$ instead of $[\partial_1 C]_\alpha$.

With respect to the metric ρ, we obtain the following theorems.

Theorem 3.2. *For every copula C the upper level fuction $\Phi_{\partial_1 C}$ has at most countably many discontinuities.*

Proof. Let C be a copula and J_C a function on $\mathbb{I}$ defined by

$$J_C(\alpha) = \lambda_2([\partial_1 C]_\alpha)$$

for all α in $\mathbb{I}$. Let D_{J_C} denote the set of all discontinuities of J_C. Since J_C is decreasing, D_{J_C} is countable. To show that every continuous point of J_C is a continuous point of $\Phi_{\partial_1 C}$, let $\alpha \in \mathbb{I}$. Suppose that J_C is continuous at α. Let $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence of real numbers in $\mathbb{I}$ such that $(\alpha_n)_{n\in\mathbb{N}}$ converges to α and $\epsilon > 0$. Then there exists an $N \in \mathbb{N}$ such that for all $n \geq N$,

$$|J_C(\alpha_n) - J_C(\alpha)| < \epsilon.$$

Fix $n \geq N$, then

$$\begin{aligned}\rho(\Phi_{\partial_1 C}(\alpha_n), \Phi_{\partial_1 C}(\alpha)) &= \rho([\partial_1 C]_{\alpha_n}, [\partial_1 C]_\alpha) \\ &= \lambda_2(([\partial_1 C]_{\alpha_n} \setminus [\partial_1 C]_\alpha) \cup ([\partial_1 C]_\alpha \setminus [\partial_1 C]_{\alpha_n})).\end{aligned}$$

Case 1: $\alpha_n \leq \alpha$

Then $[\partial_1 C]_\alpha \subseteq [\partial_1 C]_{\alpha_n}$ and so

$$\begin{aligned}\rho(\Phi_{\partial_1 C}(\alpha_n), \Phi_{\partial_1 C}(\alpha)) &= \lambda_2([\partial_1 C]_{\alpha_n} \setminus [\partial_1 C]_\alpha) = \lambda_2([\partial_1 C]_{\alpha_n}) - \lambda_2([\partial_1 C]_\alpha) \\ &= J_C(\alpha_n) - J_C(\alpha) = |J_C(\alpha_n) - J_C(\alpha)| < \epsilon.\end{aligned}$$

Case 2: $\alpha_n > \alpha$

Then $[\partial_1 C]_{\alpha_n} \subseteq [\partial_1 C]_\alpha$ and so

$$\begin{aligned}\rho(\Phi_{\partial_1 C}(\alpha_n), \Phi_{\partial_1 C}(\alpha)) &= \lambda_2([\partial_1 C]_\alpha \setminus [\partial_1 C]_{\alpha_n}) = \lambda_2([\partial_1 C]_\alpha) - \lambda_2([\partial_1 C]_{\alpha_n}) \\ &= J_C(\alpha) - J_C(\alpha_n) = |J_C(\alpha_n) - J_C(\alpha)| < \epsilon.\end{aligned}$$

We conclude that $\rho(\Phi_{\partial_1 C}(\alpha_n), \Phi_{\partial_1 C}(\alpha)) < \epsilon$ for all $n \geq N$. This shows that $\Phi_{\partial_1 C}$ is continuous at α. Hence, the set of all discontinuities of $\Phi_{\partial_1 C}$ is a subset of D_{J_C}, which is countable. Therefore, $\Phi_{\partial_1 C}$ has at most countably many discontinuities. $\square$

Theorem 3.3. *Let $(C_n)_{n\in\mathbb{N}}$ be a sequence of copulas and C a copula such that $(\partial_1 C_n)_{n\in\mathbb{N}}$ converges to $\partial_1 C$ almost everywhere. If $\alpha \in \mathbb{I}$ is a continuity point of $\Phi_{\partial_1 C}$, then*

$$\lim_{n\to\infty} \rho([\partial_1 C_n]_\alpha, [\partial_1 C]_\alpha) = 0.$$

Proof. To prove this, Egorov's Theorem is needed. Let us state here for convenience.

Egorov's Theorem. *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, $E \in \mathcal{F}$ with $\mu(E) < \infty$ and let $(f_n)_{n\in\mathbb{N}}$ be a sequence of real-valued measurable functions on E that converges pointwise to a real-valued measurable function f almost everywhere on E. Then for any $\epsilon > 0$, there exists a measurable set A contained in E such that $\mu(E \setminus A) < \epsilon$ and $(f_n)_{n\in\mathbb{N}}$ converges to f uniformly on A.*

Let $\epsilon > 0$. Since $\Phi_{\partial_1 C}$ is continuous at α, there exists $\delta \in (0, \epsilon)$ such that for all $\beta \in \mathbb{I}$,

$$\text{if } |\alpha - \beta| \le \delta \text{ then } \lambda_2([\partial_1 C]_\alpha \triangle [\partial_1 C]_\beta) \le \frac{\epsilon}{4}. \tag{3.1}$$

Since $(\partial_1 C_n)_{n\in\mathbb{N}}$ converges to $\partial_1 C$ almost everywhere, by Egorov's Theorem, there exists a measurable set $Z \subseteq \mathbb{I}^2$ such that $\lambda_2(Z) < \frac{\epsilon}{4}$ and $(\partial_1 C_n)_{n\in\mathbb{N}}$ converges uniformly to $\partial_1 C$ on $\mathbb{I}^2 \setminus Z$. Thus there exists an $N \in \mathbb{N}$ such that for every $n \ge N$,

$$\sup_{(u,v)\in\mathbb{I}^2\setminus Z} |\partial_1 C_n(u,v) - \partial_1 C(u,v)| \le \delta. \tag{3.2}$$

We want to show that for every $n \ge N$,

$$\rho([\partial_1 C_n]_\alpha, [\partial_1 C]_\alpha) < \epsilon.$$

Fix $n \ge N$, then

$$\begin{aligned} \rho([\partial_1 C_n]_\alpha, [\partial_1 C]_\alpha) &= \lambda_2([\partial_1 C_n]_\alpha \triangle [\partial_1 C]_\alpha) = \lambda_2(([\partial_1 C_n]_\alpha \setminus [\partial_1 C]_\alpha) \cup ([\partial_1 C]_\alpha \setminus [\partial_1 C_n]_\alpha)) \\ &= \lambda_2([\partial_1 C_n]_\alpha \setminus [\partial_1 C]_\alpha) + \lambda_2([\partial_1 C]_\alpha \setminus [\partial_1 C_n]_\alpha). \end{aligned} \tag{3.3}$$

It suffices to show that

$$(3.3.1)\quad \lambda_2([\partial_1 C_n]_\alpha \setminus [\partial_1 C]_\alpha) < \frac{\epsilon}{2} \text{ and } \qquad (3.3.2)\quad \lambda_2([\partial_1 C]_\alpha \setminus [\partial_1 C_n]_\alpha) < \frac{\epsilon}{2}.$$

If we obtain (3.3.1) and (3.3.2) then $\rho([\partial_1 C_n]_\alpha, [\partial_1 C]_\alpha) < \epsilon$ by (3.3).

Next, we prove the above two inequalities.

(3.3.1): Note that $[\partial_1 C_n]_\alpha \setminus [\partial_1 C]_{\alpha-\delta} \subseteq Z$. Indeed, let $(u, v) \in [\partial_1 C_n]_\alpha \setminus [\partial_1 C]_{\alpha-\delta}$. Then $\partial_1 C_n(u, v) \geq \alpha$ and $(u, v) \notin [\partial_1 C]_{\alpha-\delta}$. If $(u, v) \notin Z$, then $\partial_1 C(u, v) < \alpha - \delta$ and so $\partial_1 C_n(u, v) - \partial_1 C(u, v) > \delta$, this contradicts (3.2). Thus $(u, v) \in Z$. Hence it follows that

$$\begin{aligned}
\lambda_2([\partial_1 C_n]_\alpha \setminus [\partial_1 C]_\alpha) &\leq \lambda_2([\partial_1 C_n]_\alpha \setminus [\partial_1 C]_{\alpha-\delta}) + \lambda_2([\partial_1 C]_{\alpha-\delta} \setminus [\partial_1 C]_\alpha) \\
&\leq \lambda_2(Z) + \lambda_2([\partial_1 C]_\alpha \triangle [\partial_1 C]_{\alpha-\delta}) \\
&< \frac{\epsilon}{4} + \frac{\epsilon}{4} \quad \text{by (3.1)} \\
&= \frac{\epsilon}{2}.
\end{aligned}$$

(3.3.2): Note that $[\partial_1 C]_{\alpha+\delta} \setminus [\partial_1 C_n]_\alpha \subseteq Z$. Indeed, let $(u, v) \in [\partial_1 C]_{\alpha+\delta} \setminus [\partial_1 C_n]_\alpha$. Then $\partial_1 C(u, v) \geq \alpha + \delta$ and $(u, v) \notin [\partial_1 C_n]_\alpha$. If $(u, v) \notin Z$, then $\partial_1 C_n(u, v) < \alpha$ and so $\partial_1 C(u, v) - \partial_1 C_n(u, v) > \delta$, this contradicts (3.2). Thus $(u, v) \in Z$. Hence it follows that

$$\begin{aligned}
\lambda_2([\partial_1 C]_\alpha \setminus [\partial_1 C_n]_\alpha) &\leq \lambda_2([\partial_1 C]_\alpha \setminus [\partial_1 C]_{\alpha+\delta}) + \lambda_2([\partial_1 C]_{\alpha+\delta} \setminus [\partial_1 C_n]_\alpha) \\
&\leq \lambda_2([\partial_1 C]_\alpha \triangle [\partial_1 C]_{\alpha+\delta}) + \lambda_2(Z) \\
&< \frac{\epsilon}{4} + \frac{\epsilon}{4} \quad \text{by (3.1)} \\
&= \frac{\epsilon}{2}.
\end{aligned}$$

This completes the proof. □

According to Theorems 3.2 and 3.3, point-wise a.e. convergence of a sequence of first partial derivative of copulas implies the convergence of the corresponding upper α-level-sets for all but at most countably many $\alpha \in \mathbb{I}$. Furthermore, Theorem 3.2 is very useful in our proof of tranformation invariance $\lambda_2([\partial_1 C_{X,Y}]_\alpha) = \lambda_2([\partial_1 C_{f(X),Y}]_\alpha)$ where

f is invertible Borel measurable function and $\alpha \in \mathbb{I}$ is a continuity point of $\Phi_{\partial_1 C_{f(X),Y}}$. To prove this, the following lemma is required.

Lemma 3.4. *Let C be a copula and S a generalized shuffle of Min. Then*

$$\lambda_2([\partial_1(S * C)]_\alpha) = \lambda_2([\partial_1 C]_\alpha)$$

for any $\alpha \in \mathbb{I}$.

Proof. According to [17], for each $v \in \mathbb{I}$, $\partial_1(S * C)(u, v) = \partial_1 C(T^{-1}(u), v)$ for almost every $u \in \mathbb{I}$ for some measure-preserving bijection $T : \mathbb{I} \to \mathbb{I}$. We prove this again. For a copula A, $K_A : \mathbb{I} \times \mathcal{B}(\mathbb{I}) \to \mathbb{I}$ is defined, for every $B \in \mathcal{B}(\mathbb{I})$, by

$$K_A(U(\omega), B) = \mathrm{E}\left(\mathbb{1}_B \circ V | U\right)(\omega) \quad \text{almost all } \omega \in \Omega,$$

where U and V are uniform $(0, 1)$ random variables whose connecting copula is $A_{U,V} = A$. Then for every Borel subset G of $\mathbb{I}^2$ we set $G_u := \{v \in \mathbb{I} : (u, v) \in G\}$ and obtain

$$\begin{aligned}
\mu_A(G) &= \mathrm{P}((U, V) \in G) \\
&= \int_\Omega \mathbb{1}_{G_{U(\omega)}}(V(\omega))\, \mathrm{dP}(\omega) \\
&= \int_\Omega \mathrm{E}\left(\mathbb{1}_{G_U} \circ V | U\right)(\omega)\, \mathrm{dP}(\omega) \\
&= \int_{\mathbb{I}} \mathrm{E}\left(\mathbb{1}_{G_u} \circ V | u\right) \mathrm{dP}_U(u) \\
&= \int_{\mathbb{I}} \mathrm{E}\left(\mathbb{1}_{G_u} \circ V | u\right) \mathrm{d}u \\
&= \int_{\mathbb{I}} K_A\left(u, G_u\right) \mathrm{d}u,
\end{aligned}$$

where $\mathrm{P}_U(B) := \mathrm{P}(U^{-1}(B))$ for any $B \in \mathcal{B}(\mathbb{I})$. Notice that we have made a change of variable in the fourth equality and the uniqueness of Lebesgue measure, that it assigns to each bounded open interval its length, in the fifth equality. In particular, $\int_{\mathbb{I}} K_A\left(u, F\right) \mathrm{d}u = \lambda(F)$ for every $F \in \mathcal{B}(\mathbb{I})$. Next, let T be a measure-preserving bijection so that $\mu_S = S_T \star \mu_M$ (by Definition 2.21) and $\mu_{S*C} = S_T \star \mu_C$ (by Theorem

2.22). This leads to

$$\begin{aligned}\int_E K_{S*C}(t,F)\,\mathrm{d}t &= \mu_{S*C}(E\times F)\\ &= S_T\star\mu_C(E\times F)\\ &= \mu_C\left((S_T)^{-1}(E\times F)\right)\\ &= \mu_C\left(T^{-1}(E)\times F\right)\\ &= \int_{T^{-1}(E)} K_C(t,F)\,\mathrm{d}t,\end{aligned}$$

for every $E,F\in\mathcal{B}(\mathbb{I})$. Set $E:=[0,u]$ so that both sides of the above equation are viewed as increasing functions. Then $K_{S*C}(u,F)=\frac{\mathrm{d}}{\mathrm{d}u}\int_{T^{-1}([0,u])}K_C(t,F)\,\mathrm{d}t$ for almost every $u\in\mathbb{I}$ where the left hand side follows from integrability of $K_{S*C}(\cdot,F)$. Moreover, it follows directly from the measure-preserving of T that

$$\begin{aligned}K_{S*C}(u,F) &= \frac{\mathrm{d}}{\mathrm{d}u}\int_{T^{-1}([0,u])}K_C(t,F)\,\mathrm{d}t\\ &= \frac{\mathrm{d}}{\mathrm{d}u}\int_{\mathbb{I}}\mathbb{1}_{T^{-1}([0,u])}(t)K_C(t,F)\,\mathrm{d}t\\ &= \frac{\mathrm{d}}{\mathrm{d}u}\int_{\mathbb{I}}\mathbb{1}_{T^{-1}([0,u])}(T^{-1}(t))K_C\left(T^{-1}(t),F\right)\mathrm{d}t\\ &= \frac{\mathrm{d}}{\mathrm{d}u}\int_{\mathbb{I}}\mathbb{1}_{[0,u]}(t)K_C\left(T^{-1}(t),F\right)\mathrm{d}t\\ &= \frac{\mathrm{d}}{\mathrm{d}u}\int_{[0,u]}K_C\left(T^{-1}(t),F\right)\mathrm{d}t\\ &= K_C\left(T^{-1}(u),F\right),\end{aligned}$$

for almost every $u\in\mathbb{I}$ where the last equality is obtained by again using integrability of $K_C\left(T^{-1}(\cdot),F\right)$. In particular, for every $v\in\mathbb{I}$, $\partial_1(S*C)(u,v)=\partial_1C(T^{-1}(u),v)$ for almost every $u\in\mathbb{I}$. Using Fubini's Theorem and measure-preserving property of T, the following equalities follow,

$$\begin{aligned}\lambda_2([\partial_1(S*C)]_\alpha) &= \int_{\mathbb{I}}\int_{\mathbb{I}}\mathbb{1}_{\{\partial_1(S*C)(u,v)\geq\alpha\}}\,\mathrm{d}u\,\mathrm{d}v\\ &= \int_{\mathbb{I}}\int_{\mathbb{I}}\mathbb{1}_{\{\partial_1C(T^{-1}(u),v)\geq\alpha\}}\,\mathrm{d}u\,\mathrm{d}v\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{I}}\int_{\mathbb{I}} \mathbb{1}_{\{\partial_1 C \geq \alpha\}} \circ T^{-1} \,\mathrm{d}u\,\mathrm{d}v \\
&= \int_{\mathbb{I}}\int_{\mathbb{I}} \mathbb{1}_{\{\partial_1 C \geq \alpha\}} \,\mathrm{d}u\,\mathrm{d}v \\
&= \lambda_2([\partial_1 C]_\alpha),
\end{aligned}
$$

for all $\alpha \in \mathbb{I}$. □

Theorem 3.5. *Let X and Y be random variables with continuous distribution functions. Then*

$$\lambda_2([\partial_1 C_{X,Y}]_\alpha) = \lambda_2([\partial_1 C_{f(X),Y}]_\alpha)$$

for every invertible Borel measurable function f and every continuity point α of $\Phi_{\partial_1 C_{f(X),Y}}$.

Proof. Since f is Borel measurable, by Corollary 2.13,

$$C_{f(X),Y} = C_{f(X),X} * C_{X,Y}.$$

Since $f(X)$ and X are mutually completely dependent, there exists a sequence of shuffles of Min $(S_n)_{n\in\mathbb{N}}$ such that $(\partial_1(S_n * C_{X,Y}))_{n\in\mathbb{N}}$ converges pointwise to $\partial_1(C_{f(X),X} * C_{X,Y}) = \partial_1 C_{f(X),Y}$ almost everwhere. To prove this, we quote here without proof Theorem 3.1 in [12].

Theorem 3.1 in Ruankong, Santiwipanont and Sumetkijakan [12]. *Let C be a mutual complete dependence copula. Then there exists a sequence of shuffles of Min $(S_n)_{n\in\mathbb{N}}$ that converges to C in the Sobolev norm $\|\cdot\|_{(2)}$.*

As a virtue of the above theorem, there exists a sequence of shuffles of Min $(S_n)_{n\in\mathbb{N}}$ that converges to $C_{f(X),X}$ in the Sobolev norm $\|\cdot\|_{(2)}$. By Theorem 2.14,

$$\lim_{n\to\infty} \|S_n * C_{X,Y} - C_{f(X),X} * C_{X,Y}\|_{(2)} = 0.$$

Using Theorem 2.16, we can find a subsequence of $(S_n * C_{X,Y})_{n\in\mathbb{N}}$, which, for convenience, will again be denoted by $(S_n * C_{X,Y})_{n\in\mathbb{N}}$, such that $(\partial_1(S_n * C_{X,Y}))_{n\in\mathbb{N}}$ converges pointwise to $\partial_1(C_{f(X),X} * C_{X,Y}) = \partial_1 C_{f(X),Y}$ almost everwhere. By virtue of Theorem 3.3, for every α in $\mathbb{I}$ which is a continuity point of $\Phi_{\partial_1 C_{f(X),Y}}$,

$$\lim_{n\to\infty} \rho([\partial_1(S_n * C_{X,Y})]_\alpha, [\partial_1 C_{f(X),Y}]_\alpha) = 0.$$

This implies that $(\lambda_2([\partial_1(S_n * C_{X,Y})]_\alpha))_{n\in\mathbb{N}}$ converges to $\lambda_2([\partial_1 C_{f(X),Y}]_\alpha)$. Indeed, let $\epsilon > 0$. Then there exists a positive integer N such that for any integer $n \geq N$

$$\rho([\partial_1(S_n * C_{X,Y})]_\alpha, [\partial_1 C_{f(X),Y}]_\alpha) < \epsilon.$$

Then

$$\begin{aligned}
\lambda_2([\partial_1(S_n * C_{X,Y})]_\alpha) &= \lambda_2([\partial_1(S_n * C_{X,Y})]_\alpha \setminus [\partial_1 C_{f(X),Y}]_\alpha) \\
&\quad + \lambda_2([\partial_1(S_n * C_{X,Y})]_\alpha \cap [\partial_1 C_{f(X),Y}]_\alpha) \\
&\leq \lambda_2([\partial_1(S_n * C_{X,Y})]_\alpha \setminus [\partial_1 C_{f(X),Y}]_\alpha) \\
&\quad + \lambda_2([\partial_1 C_{f(X),Y}]_\alpha).
\end{aligned}$$

So we obtain that

$$\begin{aligned}
&\lambda_2([\partial_1(S_n * C_{X,Y})]_\alpha) - \lambda_2([\partial_1 C_{f(X),Y}]_\alpha) \\
&\qquad \leq \lambda_2([\partial_1(S_n * C_{X,Y})]_\alpha \setminus [\partial_1 C_{f(X),Y}]_\alpha) \\
&\qquad \leq \rho([\partial_1(S_n * C_{X,Y})]_\alpha, [\partial_1 C_{f(X),Y}]_\alpha) \\
&\qquad < \epsilon.
\end{aligned}$$

Similarly, for $n \geq N$, $\lambda_2([\partial_1 C_{f(X),Y}]_\alpha) - \lambda_2([\partial_1(S_n * C_{X,Y})]_\alpha) < \epsilon$ and so we have

$$\left|\lambda_2([\partial_1(S_n * C_{X,Y})]_\alpha) - \lambda_2([\partial_1 C_{f(X),Y}]_\alpha)\right| < \epsilon.$$

Hence $(\lambda_2([\partial_1(S_n * C_{X,Y})]_\alpha))_{n\in\mathbb{N}}$ converges to $\lambda_2([\partial_1 C_{f(X),Y}]_\alpha)$. Since, by Lemma 3.4,

$\lambda_2([\partial_1(S_n * C_{X,Y})]_\alpha) = \lambda_2([\partial_1 C_{X,Y}]_\alpha)$ for all $n \in \mathbb{N}$,

$$\lambda_2([\partial_1 C_{X,Y}]_\alpha) = \lambda_2([\partial_1 C_{f(X),Y}]_\alpha). \qquad \square$$

3.2 An indicator of dependence

First of all, we give the formal definition of the i^{th} significant digit of non-zero real numbers.

Definition 3.6. Let $\mathfrak{m} : \mathbb{R} \setminus \{0\} \to [1, 10)$ be defined by

$$\mathfrak{m}(x) = r, \quad \text{for } x \in \mathbb{R} \setminus \{0\},$$

where r is the unique number in $[1, 10)$ such that $|x| = r \times 10^n$ for some necessarily unique $n \in \mathbb{Z}$. Following [7], $\mathfrak{m}$ is called the *mantissa function.* Let $D^{(i)}(x)$ denote the i^{th} *significant digit (or i^{th} digit)* of x. Formally, $D^{(i)} : \mathbb{R} \setminus \{0\} \to \{0, 1, 2, \ldots, 9\}$ is given by

$$D^{(i)}(x) = ((t^{-1} \circ \mathfrak{m})(x))_i, \quad \text{for } x \in \mathbb{R} \setminus \{0\},$$

where $(\cdot)_i$ is the coordinate projection $(c_1, c_2, \ldots)_i = c_i$, t is the function $t(t_0, t_1, t_2, \ldots) = \sum_{k=0}^{\infty} \frac{t_k}{10^k}$ for $t_0 \in \{1, 2, \ldots, 9\}$ and $t_k \in \{0, 1, 2, \ldots, 9\}$ for $k \in \mathbb{N}$, and t^{-1} is taken to be the terminating inverse when it is non-unique (e.g., $t^{-1}(0.1999\ldots) = t^{-1}(0.2) = (2, 0, 0, \ldots)$). Note that this definition leads the first digit of $0.1999\ldots$ to be 2.

To ensure the well-defined property of first digit distributions of partial derivatives of copulas, the following results are useful.

Lemma 3.7. *For every copula C and for $d = 1, 2, \ldots, 9$,*

$$\sum_{k=0}^{\infty} \lambda_2 \left([\partial_1 C]_{\frac{d}{10^k}} \setminus [\partial_1 C]_{\frac{d+1}{10^k}} \right)$$

is finite.

Proof. Since $\left\{[\partial_1 C]_{\frac{d}{10^k}} \setminus [\partial_1 C]_{\frac{d+1}{10^k}}\right\}_{k=0}^{\infty}$ is a countable collection of pairwise disjoint sets,

$$\begin{aligned}\sum_{k=0}^{\infty} \lambda_2\left([\partial_1 C]_{\frac{d}{10^k}} \setminus [\partial_1 C]_{\frac{d+1}{10^k}}\right) &= \lambda_2\left(\bigcup_{k=0}^{\infty}\left([\partial_1 C]_{\frac{d}{10^k}} \setminus [\partial_1 C]_{\frac{d+1}{10^k}}\right)\right)\\ &\leq \lambda_2\left(\mathbb{I}^2\right) = 1.\end{aligned}$$

□

Lemma 3.8. *For every copula C, $\lambda_2\left(\left\{(u,v) \in \mathbb{I}^2 : \partial_1 C(u,v) > 0\right\}\right) \neq 0$.*

Proof. Suppose that $\lambda_2\left(\left\{(u,v) \in \mathbb{I}^2 : \partial_1 C(u,v) > 0\right\}\right) = 0$. So $\partial_1 C(u,v) = 0$ for almost every v in $\mathbb{I}$ almost every u in $\mathbb{I}$. Then for almost every v in $\mathbb{I}$,

$$v = C(1,v) - C(0,v) = \int_{\mathbb{I}} \partial_1 C(u,v)\,\mathrm{d}u = 0,$$

where the next to last equality follows from the absolute continuity of C in each argument. This is a contradiction and so $\lambda_2\left(\left\{(u,v) \in \mathbb{I}^2 : \partial_1 C(u,v) > 0\right\}\right) \neq 0$. □

Definition 3.9. For a copula C, let $\mathrm{FD}_d(\partial_1 C)$ be the first digit distribution of $\partial_1 C$ defined by

$$\begin{aligned}\mathrm{FD}_d(\partial_1 C) &= \frac{\lambda_2\left(\left\{(u,v) \in \mathbb{I}^2 : \text{the first digit of } \partial_1 C(u,v) \text{ is } d\right\}\right)}{\lambda_2\left(\{(u,v) \in \mathbb{I}^2 : \partial_1 C(u,v) > 0\}\right)}\\ &= \frac{\sum_{k=0}^{\infty} \lambda_2\left([\partial_1 C]_{\frac{d}{10^k}} \setminus [\partial_1 C]_{\frac{d+1}{10^k}}\right)}{\lambda_2\left(\{(u,v) \in \mathbb{I}^2 : \partial_1 C(u,v) > 0\}\right)} \qquad (3.4)\end{aligned}$$

for $d = 1, 2, \ldots, 9$. Since $\partial_1 C$ exists a.e. and $0 \leq \partial_1 C \leq 1$ a.e., FD_d can be computed via upper α-level sets as (3.4).

Remark 3.10. By Lemmas 3.7 and 3.8, the first digit distributions of $\partial_1 C$ are well-defined. Moreover, $\sum_{d=1}^{9} \mathrm{FD}_d(\partial_1 C) = 1$.

Proof. Since $\left\{[\partial_1 C]_{\frac{d}{10^k}} \setminus [\partial_1 C]_{\frac{d+1}{10^k}}\right\}_{k \geq 0, d=1,2,\ldots,9}$ is a countable collection of pairwise disjoint sets,

$$\sum_{d=1}^{9} \mathrm{FD}_d(\partial_1 C) = \sum_{d=1}^{9} \frac{\sum_{k=0}^{\infty} \lambda_2\left([\partial_1 C]_{\frac{d}{10^k}} \setminus [\partial_1 C]_{\frac{d+1}{10^k}}\right)}{\lambda_2\left(\{(u,v) \in \mathbb{I}^2 : \partial_1 C(u,v) > 0\}\right)}$$

$$= \sum_{d=1}^{9} \frac{\lambda_2\left(\bigcup_{k=0}^{\infty}\left([\partial_1 C]_{\frac{d}{10^k}} \setminus [\partial_1 C]_{\frac{d+1}{10^k}}\right)\right)}{\lambda_2\left(\{(u,v)\in\mathbb{I}^2 : \partial_1 C(u,v) > 0\}\right)}$$

$$= \frac{\lambda_2\left(\bigcup_{d=1}^{9}\bigcup_{k=0}^{\infty}\left([\partial_1 C]_{\frac{d}{10^k}} \setminus [\partial_1 C]_{\frac{d+1}{10^k}}\right)\right)}{\lambda_2\left(\{(u,v)\in\mathbb{I}^2 : \partial_1 C(u,v) > 0\}\right)}$$

$$= \frac{\lambda_2\left(\left\{(u,v)\in\mathbb{I}^2 : \partial_1 C(u,v) > 0\right\}\right)}{\lambda_2\left(\{(u,v)\in\mathbb{I}^2 : \partial_1 C(u,v) > 0\}\right)} = 1. \qquad \square$$

Definition 3.11**.** For a copula C, denote $\mathrm{MF}_1(C) = \max_{d=1,2,\cdots,9} \mathrm{FD}_d(\partial_1 C)$. To rescale so that it takes value in $\mathbb{I}$, let us define function μ_1 on the space of all copulas $\mathcal{C}$ by

$$\mu_1(C) = \sqrt{\frac{9\,\mathrm{MF}_1(C) - 1}{8}}.$$

Remark 3.12**.** For a copula C, $\frac{1}{9} \leq \mathrm{MF}_1(C) \leq 1$ and so $0 \leq \mu_1(C) \leq 1$.

Proof. Suppose that $\mathrm{MF}_1(C) < \frac{1}{9}$. So $\sum_{d=1}^{9} \mathrm{FD}_d(\partial_1 C) < 1$, which is a contradiction. Clearly, for $d = 1, 2, \ldots, 9$, $\mathrm{FD}_d(\partial_1 C) \leq \sum_{d=1}^{9} \mathrm{FD}_d(\partial_1 C) = 1$. Then $\frac{1}{9} \leq \mathrm{MF}_1(C) \leq 1$. $\square$

Theorem 3.13**.** *Let X and Y be random variables with continuous distribution functions. If Y is completely dependent on X, then* $\mathrm{MF}_1(C_{X,Y}) = 1$ *and so* $\mu_1(C_{X,Y}) = 1$.

Proof. Since Y is completely dependent on X, $\partial_1 C \in \{0, 1\}$ almost everywhere. So for any $k \in \mathbb{N}$ and $d = 1, 2, \ldots, 9$, up to subsets of measure zero

$$[\partial_1 C]_{\frac{d}{10^k}} = \left\{(u,v)\in\mathbb{I}^2 : \partial_1 C(u,v) = 1\right\} = \left\{(u,v)\in\mathbb{I}^2 : \partial_1 C(u,v) > 0\right\}.$$

Then

$$\mathrm{FD}_1(\partial_1 C) = \frac{\lambda_2\left([\partial_1 C]_1\right) + \sum_{k=1}^{\infty} \lambda_2\left([\partial_1 C]_{\frac{1}{10^k}} \setminus [\partial_1 C]_{\frac{2}{10^k}}\right)}{\lambda_2\left(\{(u,v)\in\mathbb{I}^2 : \partial_1 C(u,v) > 0\}\right)}$$

$$= \frac{\lambda_2([\partial_1 C]_1) + \sum_{k=1}^{\infty} 0}{\lambda_2([\partial_1 C]_1)} = 1.$$

This implies that $\mathrm{MF}_1(C) = 1$. $\square$

The converse is not true, that is, if $\mathrm{MF}_1(C) = 1$ then the copula C is not necessarily a complete dependence copula. A counterexample can be constructed via patching method. Let $J_0 = [0, \frac{1}{10}]$, $J_1 = (\frac{1}{10}, 1]$, $a_0 = \frac{1}{10}$ and $a_1 = \frac{9}{10}$. And for each $j = 0, 1$, let L_j be the uniform distribution function on J_j. Then, by Theorem 2.11, the function C defined by

$$C(u, v) = \sum_{j=0}^{1} a_j M(u, L_j(v)) = \frac{1}{10} M(u, L_0(v)) + \frac{9}{10} M(u, L_1(v)), \quad \text{for } u, v \in \mathbb{I}, \quad (3.5)$$

is a copula.

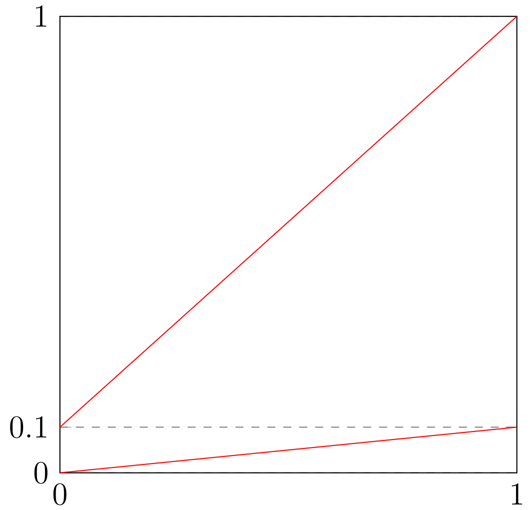

Figure 3.1: Support of the copula defined in (3.5)

To show that $\mathrm{MF}_1(C) = 1$, let $(u, v) \in \mathbb{I} \times J_0$. Then $L_0(v) = 10v$ and $L_1(v) = 0$. So $C(u, v) = \frac{1}{10} \min\{u, 10v\}$. Consequently, $\partial_1 C(u, v) = 0$ for all $(u, v) \in \mathbb{I} \times J_0$ such that $v < \frac{u}{10}$ and $\partial_1 C(u, v) = \frac{1}{10}$ for all $(u, v) \in \mathbb{I} \times J_0$ such that $v > \frac{u}{10}$.

Next, let $(u, v) \in \mathbb{I} \times J_1$. Then $L_0(v) = 1$ and $L_1(v) = \frac{1}{9}(10v - 1)$. Consequently, $C(u, v) = \frac{1}{10} u + \frac{9}{10} \min\{u, \frac{1}{9}(10v - 1)\}$. Therefore, $\partial_1 C(u, v) = \frac{1}{10}$ for all $(u, v) \in \mathbb{I} \times J_1$ such that $v < \frac{9}{10} u + \frac{1}{10}$ and $\partial_1 C(u, v) = 1$ for all $(u, v) \in \mathbb{I} \times J_1$ such that $v > \frac{9}{10} u + \frac{1}{10}$. Then

$$\mathrm{FD}_1(\partial_1 C) = \frac{\sum_{k=0}^{\infty} \lambda_2 \left([\partial_1 C]_{\frac{1}{10^k}} \setminus [\partial_1 C]_{\frac{2}{10^k}} \right)}{\lambda_2 \left(\{(u, v) \in \mathbb{I}^2 : \partial_1 C(u, v) > 0\} \right)}$$

$$= \frac{\lambda_2\left(\{(u,v)\in\mathbb{I}^2:\partial_1 C(u,v)=1\}\right)+\lambda_2\left(\{(u,v)\in\mathbb{I}^2:\partial_1 C(u,v)=\frac{1}{10}\}\right)}{\lambda_2\left(\{(u,v)\in\mathbb{I}^2:\partial_1 C(u,v)=1\}\right)+\lambda_2\left(\{(u,v)\in\mathbb{I}^2:\partial_1 C(u,v)=\frac{1}{10}\}\right)}$$

$$= 1,$$

where the second equality is obtained by using $\partial_1 C \in \{0, \frac{1}{10}, 1\}$ almost everywhere. This implies $\mathrm{MF}_1(C) = 1$.

Theorem 3.14. $\mathrm{MF}_1(\Pi) = 1/9$, *i.e.* $\mu_1(\Pi) = 0$.

Proof. Note that

$$\begin{aligned}\lambda_2\left([\partial_1\Pi]_1\right) &= \lambda_2\left(\{(u,v)\in\mathbb{I}^2:\partial_1\Pi(u,v)=1\}\right)\\ &= \lambda_2\left(\{(u,v)\in\mathbb{I}^2:v=1\}\right)\\ &= \lambda_2\left(\mathbb{I}\times\{1\}\right)=0.\end{aligned}$$

Moreover,

$$\lambda_2(\{(u,v)\in\mathbb{I}^2:\partial_1\Pi(u,v)>0\}) = \lambda_2(\{(u,v)\in\mathbb{I}^2:v>0\}) = \lambda_2\left(\mathbb{I}\times(0,1]\right)=1.$$

Then for any $d = 1, 2, \ldots, 9$,

$$\begin{aligned}\mathrm{FD}_d(\partial_1\Pi) &= \sum_{k=1}^{\infty}\lambda_2\left([\partial_1\Pi]_{\frac{d}{10^k}}\setminus[\partial_1\Pi]_{\frac{d+1}{10^k}}\right)\\ &= \sum_{k=1}^{\infty}\lambda_2\left(\left\{(u,v)\in\mathbb{I}^2:\frac{d}{10^k}\le\partial_1\Pi(u,v)<\frac{d+1}{10^k}\right\}\right)\\ &= \sum_{k=1}^{\infty}\lambda_2\left(\left\{(u,v)\in\mathbb{I}^2:\frac{d}{10^k}\le v<\frac{d+1}{10^k}\right\}\right)\\ &= \sum_{k=1}^{\infty}\lambda_2\left(\mathbb{I}\times\left[\frac{d}{10^k},\frac{d+1}{10^k}\right)\right)=\sum_{k=1}^{\infty}\frac{1}{10^k}=\frac{1}{9}.\end{aligned}$$

This implies that $\mathrm{MF}_1(\Pi) = \frac{1}{9}$ and so $\mu_1(\Pi) = 0$. □

However, we found a copula $C \neq \Pi$ whose $\mu_1(C) = 0$. Let $I_0 = [0, \frac{1}{10}]$ and $I_i = (\frac{i}{10}, \frac{i+1}{10}]$ for $i = 1, 2, \ldots, 9$. For each $i \in \{0, 1, \ldots, 9\}$ and $j \in \{1, 2, \ldots, 9\}$, let F_i

be the uniform distribution function on I_i, $C_{i0} = \Pi$ and $C_{ij} = M$. Then Theorem 2.11 implies that the function C defined by

$$C(u,v) = \sum_{i=0}^{9}\sum_{j=0}^{9}\frac{1}{100}C_{ij}(F_i(u), F_j(v)), \quad u, v \in \mathbb{I}, \tag{3.6}$$

is a copula.

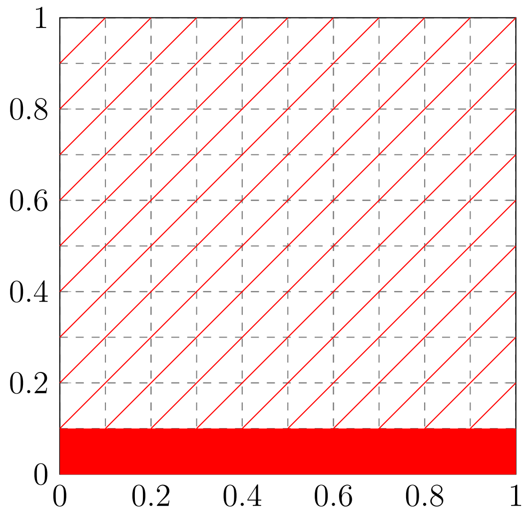

Figure 3.2**:** Support of the copula in (3.6)

To show that $\mathrm{MF}_1(C) = \frac{1}{9}$, for each $h, k \in \{0, 1, \ldots, 9\}$, let $(u, v) \in I_h \times I_k$. Then $F_i(u) = 1$ for all $i < h$, $F_i(u) = 0$ for all $i > h$, $F_j(v) = 1$ for all $j < k$ and $F_j(v) = 0$ for all $j > k$. Consequently,

$$\begin{aligned}
C(u,v) &= \sum_{i=0}^{h}\sum_{j=0}^{k}\frac{1}{100}C_{ij}(F_i(u), F_j(v)) \\
&= \sum_{i=0}^{h-1}\sum_{j=0}^{k-1}\frac{1}{100} + \sum_{i=0}^{h-1}\frac{10v-k}{100} + \sum_{j=0}^{k-1}\frac{10u-h}{100} + \frac{1}{100}C_{hk}(10u-h, 10v-k) \\
&= \frac{hk}{100} + \frac{h}{100}(10v-k) + \frac{k}{100}(10u-h) + \frac{1}{100}C_{hk}(10u-h, 10v-k).
\end{aligned}$$

Therefore, $\partial_1 C(u,v) = \dfrac{k}{10} + \dfrac{1}{10}\partial_1 C_{hk}(10u-h, 10v-k)$ for almost every $(u, v) \in I_h \times I_k$. In particular, for each $h \geq 0$, if $k = 0$ then $\partial_1 C(u,v) = \dfrac{1}{10}\partial_1\Pi(10u-h, 10v) = v$ for all $(u,v) \in I_h \times I_0$ and if $k > 0$ then $\partial_1 C(u,v) = \dfrac{k}{10} + \dfrac{1}{10}\partial_1 M(10u-h, 10v-k) = \dfrac{k+1}{10}$ for all

$(u,v) \in I_h \times I_k$ such that $v > u - \frac{h}{10} + \frac{k}{10}$ and $\partial_1 C(u,v) = \dfrac{k}{10} + \dfrac{1}{10}\partial_1 M(10u-h, 10v-k)$ $= \dfrac{k}{10}$ for all $(u,v) \in I_h \times I_k$ such that $v < u - \frac{h}{10} + \frac{k}{10}$. This formula implies that $\partial_1 C(u,v) \in [0, \frac{1}{10}]$ for all $(u,v) \in \mathbb{I} \times I_0$ and $\partial_1 C(u,v) \in \left\{\frac{1}{10}, \frac{2}{10}, \frac{3}{10}, \frac{4}{10}, \frac{5}{10}, \frac{6}{10}, \frac{7}{10}, \frac{8}{10}, \frac{9}{10}, 1\right\}$ for almost every $(u,v) \notin \mathbb{I} \times I_0$. Next, fix $d = 1, 2, \ldots, 9$. First of all, notice that $\lambda_2\left(\left\{(u,v) \in \mathbb{I}^2 : \partial_1 C(u,v) > 0\right\}\right) = \lambda_2\left(\mathbb{I} \times (0,1]\right) = 1$. If $d = 1$, then

$$\begin{aligned}
\mathrm{FD}_1(\partial_1 C) &= \sum_{k=0}^{\infty} \lambda_2\left([\partial_1 C]_{\frac{1}{10^k}} \setminus [\partial_1 C]_{\frac{2}{10^k}}\right) \\
&= \lambda_2\left([\partial_1 C]_1\right) + \lambda_2\left([\partial_1 C]_{\frac{1}{10}} \setminus [\partial_1 C]_{\frac{2}{10}}\right) \\
&\quad + \sum_{k=2}^{\infty} \lambda_2\left([\partial_1 C]_{\frac{1}{10^k}} \setminus [\partial_1 C]_{\frac{2}{10^k}}\right).
\end{aligned} \tag{3.7}$$

We consider the first, second and last terms of the above equation. Firstly,

$$\begin{aligned}
\lambda_2\left([\partial_1 C]_1\right) &= \lambda_2\left(\left\{(u,v) \in \mathbb{I}^2 : \partial_1 C(u,v) = 1\right\}\right) \\
&= \lambda_2\left(\bigcup_{h=0}^{9} \left\{(u,v) \in I_h \times I_9 : v > u - \frac{h}{10} + \frac{9}{10}\right\}\right) \\
&= \sum_{h=0}^{9} \lambda_2\left(\left\{(u,v) \in I_h \times I_9 : v > u - \frac{h}{10} + \frac{9}{10}\right\}\right) \\
&= \sum_{h=0}^{9} \int_{\frac{h}{10}}^{\frac{h+1}{10}} \int_{u-\frac{h}{10}+\frac{9}{10}}^{1} 1 \,\mathrm{d}v\,\mathrm{d}u = \sum_{h=0}^{9} \frac{1}{200} = \frac{1}{20}.
\end{aligned}$$

Secondly,

$$\begin{aligned}
\lambda_2\left([\partial_1 C]_{\frac{1}{10}} \setminus [\partial_1 C]_{\frac{2}{10}}\right) &= \lambda_2\left(\left\{(u,v) \in \mathbb{I}^2 : \frac{1}{10} \le \partial_1 C(u,v) < \frac{2}{10}\right\}\right) \\
&= \lambda_2\left(\left\{(u,v) \in \mathbb{I}^2 : \partial_1 C(u,v) = \frac{1}{10}\right\}\right) \\
&= \lambda_2\left(\bigcup_{h=0}^{9} \left\{(u,v) \in I_h \times I_1 : v < u - \frac{h}{10} + \frac{1}{10}\right\}\right) \\
&= \sum_{h=0}^{9} \lambda_2\left(\left\{(u,v) \in I_h \times I_1 : v < u - \frac{h}{10} + \frac{1}{10}\right\}\right) \\
&= \sum_{h=0}^{9} \int_{\frac{h}{10}}^{\frac{h+1}{10}} \int_{\frac{1}{10}}^{u-\frac{h}{10}+\frac{1}{10}} 1 \,\mathrm{d}v\,\mathrm{d}u = \sum_{h=0}^{9} \frac{1}{200} = \frac{1}{20}.
\end{aligned}$$

And the last term is

$$\begin{aligned}\sum_{k=2}^{\infty}\lambda_2\left([\partial_1 C]_{\frac{1}{10^k}}\setminus[\partial_1 C]_{\frac{2}{10^k}}\right)&=\sum_{k=2}^{\infty}\lambda_2\left(\left\{(u,v)\in\mathbb{I}^2:\frac{1}{10^k}\le\partial_1 C(u,v)<\frac{2}{10^k}\right\}\right)\\&=\sum_{k=2}^{\infty}\lambda_2\left(\mathbb{I}\times\left[\frac{1}{10^k},\frac{2}{10^k}\right)\right)\\&=\sum_{k=2}^{\infty}\frac{1}{10^k}=\frac{1}{90}.\end{aligned}$$

Therefore, by (3.7), $\mathrm{FD}_1(\partial_1 C)=\frac{1}{9}$.

For $d>1$

$$\begin{aligned}\mathrm{FD}_d(\partial_1 C)&=\sum_{k=0}^{\infty}\lambda_2\left([\partial_1 C]_{\frac{d}{10^k}}\setminus[\partial_1 C]_{\frac{d+1}{10^k}}\right)\\&=\lambda_2\left([\partial_1 C]_{\frac{d}{10}}\setminus[\partial_1 C]_{\frac{d+1}{10}}\right)+\sum_{k=2}^{\infty}\lambda_2\left([\partial_1 C]_{\frac{d}{10^k}}\setminus[\partial_1 C]_{\frac{d+1}{10^k}}\right).\end{aligned}\tag{3.8}$$

Similarly, we consider two terms of equation (3.8). Firstly,

$$\begin{aligned}\lambda_2\left([\partial_1 C]_{\frac{d}{10}}\setminus[\partial_1 C]_{\frac{d+1}{10}}\right)&=\lambda_2\left(\left\{(u,v)\in\mathbb{I}^2:\frac{d}{10}\le\partial_1 C(u,v)<\frac{d+1}{10}\right\}\right)\\&=\lambda_2\left(\left\{(u,v)\in\mathbb{I}^2:\partial_1 C(u,v)=\frac{d}{10}\right\}\right)\\&=\lambda_2\left(\left(\bigcup_{h=0}^{9}\left\{(u,v)\in I_h\times I_{d-1}:v>u-\frac{h}{10}+\frac{d-1}{10}\right\}\right)\right.\\&\qquad\left.\bigcup\left(\bigcup_{h=0}^{9}\left\{(u,v)\in I_h\times I_d:v<u-\frac{h}{10}+\frac{d}{10}\right\}\right)\right)\\&=\sum_{h=0}^{9}\lambda_2\left(\left\{(u,v)\in I_h\times I_{d-1}:v>u-\frac{h}{10}+\frac{d-1}{10}\right\}\right)\\&\quad+\sum_{h=0}^{9}\lambda_2\left(\left\{(u,v)\in I_h\times I_d:v<u-\frac{h}{10}+\frac{d}{10}\right\}\right)\\&=\sum_{h=0}^{9}\int_{\frac{h}{10}}^{\frac{h+1}{10}}\int_{u-\frac{h}{10}+\frac{d-1}{10}}^{\frac{d}{10}}1\,\mathrm{d}v\,\mathrm{d}u+\sum_{h=0}^{9}\int_{\frac{h}{10}}^{\frac{h+1}{10}}\int_{\frac{d}{10}}^{u-\frac{h}{10}+\frac{d}{10}}1\,\mathrm{d}v\,\mathrm{d}u\\&=\sum_{h=0}^{9}\frac{1}{200}+\sum_{h=0}^{9}\frac{1}{200}=\frac{1}{20}+\frac{1}{20}=\frac{1}{10}.\end{aligned}$$

Secondly,

$$\begin{aligned}\sum_{k=2}^{\infty}\lambda_2\left([\partial_1 C]_{\frac{d}{10^k}}\setminus[\partial_1 C]_{\frac{d+1}{10^k}}\right)&=\sum_{k=2}^{\infty}\lambda_2\left(\left\{(u,v)\in\mathbb{I}^2:\frac{d}{10^k}\le\partial_1 C(u,v)<\frac{d+1}{10^k}\right\}\right)\\&=\sum_{k=2}^{\infty}\lambda_2\left(\mathbb{I}\times\left[\frac{d}{10^k},\frac{d+1}{10^k}\right)\right)\\&=\sum_{k=2}^{\infty}\frac{1}{10^k}=\frac{1}{90}.\end{aligned}$$

Thus, by (3.8), $\mathrm{FD}_d(\partial_1 C)=\frac{1}{9}$. This shows that $\mathrm{MF}_1(C)=\frac{1}{9}$.

Naturally, under certain transformations of random variables, dependence level should not vary when random variables are transformed. In lieu of Theorems 3.2 and 3.5, we have to restrict such transformations to injective transformations with finitely many discontinuity points.

Theorem 3.15. *Let X and Y be random variables with continuous marginal distributions. If f is an invertible Borel measurable function with finitely many discontinuity points, then* $\mathrm{MF}_1(C_{X,Y})=\mathrm{MF}_1(C_{f(X),Y})$, *i.e.* $\mu_1(C_{X,Y})=\mu_1(C_{f(X),Y})$.

Proof. Its proof follows from Theorem 2.18 and Lemma 3.4. □

CHAPTER IV

NUMERICAL COMPUTATION

In this chapter, we investigate a procedure for approximating first digit distributions of partial derivatives of copulas. Moreover, comparisons between μ_1 and other measures of dependence are included.

4.1 Numerical computation

Definition 4.1. Let $a, b \in \mathbb{I}$ and the functions $f_{0,a}$ and $f_{1,b}$ be defined, for u in $\mathbb{I}$, by

$$\begin{aligned} f_{0,a}(u) &= \inf\{v \in \mathbb{I} : \partial_1 C(u,v) \geq a\} \quad \text{and} \\ f_{1,b}(u) &= \sup\{v \in \mathbb{I} : \partial_1 C(u,v) < b\}. \end{aligned}$$

Lemma 4.2. *For $a, b \in \mathbb{I}$, $f_{0,a}$ and $f_{1,b}$ are measurable.*

Proof. Let $\hat{\mathbb{I}} \subseteq \mathbb{I}$ be the set of full measure of u's such that $\partial_1 C(u,v)$ exists for almost every $v \in \mathbb{I}$. And for every $u \in \hat{\mathbb{I}}$, since we can redefine $\partial_1 C(u,v)$ on a subset of $\mathbb{I}$ of measure zero, we may assume that $\partial_1 C(u,v)$ exists and is nondecreasing for all $v \in \mathbb{I}$. We also extend C in such a way that $C(u,v) = 0$ if u or v is negative; $C(u,v) = 1$ if $u > 1$ and $v > 1$; $C(u,v) = u$ for $v > 1$; and $C(u,v) = v$ for $u > 1$. Let $c \in \mathbb{I}$. Using the nondecreasing property of $\partial_1 C(u,\cdot)$, we obtain

$$\begin{aligned} \left\{u \in \hat{\mathbb{I}} : f_{0,a}(u) < c\right\} &= \left\{u \in \hat{\mathbb{I}} : \exists v < c, \partial_1 C(u,v) \geq a\right\} \\ &= \bigcup_{n=1}^{\infty} \left\{u \in \hat{\mathbb{I}} : \partial_1 C(u, c - \frac{1}{n}) \geq a\right\}, \text{ and} \\ \left\{u \in \hat{\mathbb{I}} : f_{1,b}(u) > c\right\} &= \left\{u \in \hat{\mathbb{I}} : \exists v > c, \partial_1 C(u,v) < b\right\} \\ &= \bigcup_{n=1}^{\infty} \left\{u \in \hat{\mathbb{I}} : \partial_1 C(u, c + \frac{1}{n}) < b\right\}. \end{aligned}$$

They are measurable sets because any $\partial_1 C(\cdot, c - \frac{1}{n})$ and $\partial_1 C(\cdot, c + \frac{1}{n})$ are measurable. □

Lemma 4.3. *For $a, b \in \mathbb{I}$,*

$$\begin{aligned}&\lambda_2\left(\left\{(u,v)\in\mathbb{I}^2 : a \le \partial_1 C(u,v) < b \text{ and } \partial_1 C(u,v) \text{ exists}\right\}\right)\\&= \lambda_2\left(\left\{(u,v)\in\mathbb{I}^2 : f_{0,a}(u) \le v < f_{1,b}(u)\right\}\right).\end{aligned}$$

Proof. This follows from

$$\begin{aligned}&\left\{(u,v)\in\mathbb{I}^2 : a \le \partial_1 C(u,v) < b \text{ and } \partial_1 C(u,v)\,\text{exists}\right\}\\&\subseteq \left\{(u,v)\in\mathbb{I}^2 : f_{0,a}(u) \le v \le f_{1,b}(u)\right\},\\&\left\{(u,v)\in\mathbb{I}^2 : f_{0,a}(u) < v < f_{1,b}(u)\right\}\\&\subseteq \left\{(u,v)\in\mathbb{I}^2 : a \le \partial_1 C(u,v) < b \text{ and } \partial_1 C(u,v)\,\text{exists}\right\}\\&\quad\cup \left\{(u,v)\in\mathbb{I}^2 : \partial_1 C(u,v)\,\text{does not exist}\right\},\end{aligned}$$

and the fact that the graph of a measurable function has Lebesgue measure zero. □

Then, for $n \in \mathbb{N}$, we also define functions $F_{0,a,n}$ and $F_{1,b,n}$ on $\mathbb{I}$ for approximating $\lambda_2\left(\left\{(u,v)\in\mathbb{I}^2 : f_{0,a}(u) \le v < f_{1,b}(u)\right\}\right)$. It is indeed a way to numerically compute $\lambda_2\left(\left\{(u,v)\in\mathbb{I}^2 : a \le \partial_1 C(u,v) < b \text{ and } \partial_1 C(u,v)\,\text{exists}\right\}\right)$. As a result, we obtain numerical computation of the first digit distributions of partial derivatives of copulas.

Theorem 4.4. *Denote the set of odd numbers by $\mathcal{O}$. Let $a, b \in \mathbb{I}$. For each $n \in \mathbb{N}$, let $J_j^n = \left[\frac{j-1}{n}, \frac{j}{n}\right)$ for $j = 1, 2, \ldots, n-1$ and $J_n^n = \left[\frac{n-1}{n}, 1\right]$ and define measurable functions*

$$\begin{aligned}F_{0,a,n}(u) &= n\int_{\frac{j-1}{n}}^{\frac{j}{n}} f_{0,a}(t)\,\mathrm{d}t \quad \textit{and}\\F_{1,b,n}(u) &= n\int_{\frac{j-1}{n}}^{\frac{j}{n}} f_{1,b}(t)\,\mathrm{d}t,\end{aligned}$$

for $u \in J_j^n$ $(j = 1, 2, \ldots, n)$. Then

$$\begin{aligned}&\lim_{n\to\infty} \frac{\#\left(\left(2n\left\{(u,v)\in\mathbb{I}^2 : F_{0,a,n}(u) \le v < F_{1,b,n}(u)\right\}\right) \bigcap \mathcal{O}^2\right)}{n^2}\\&= \lambda_2\left(\left\{(u,v)\in\mathbb{I}^2 : a \le \partial_1 C(u,v) < b \text{ and } \partial_1 C(u,v) \text{ exists}\right\}\right),\end{aligned}$$

where, for any subset B of $\mathbb{I}^2$ and $c \in \mathbb{R}$, $cB := \{(cu, cv) : (u, v) \in B\}$ and, for any finite set A, $\#(A)$ is the number of elements in A.

Before proving this theorem, the following definition and theorem are needed. For more details, see [13].

Nicely Shrinking Sets. Let $x \in \mathbb{R}$. A sequence $(B_n)_{n\in\mathbb{N}}$ of Borel subsets of $\mathbb{R}$ is said to *shrink to x nicely* if there exists an $s > 0$ and there is a sequence $(B(x, r_n))_{n\in\mathbb{N}}$, with $\lim_{n\to\infty} r_n = 0$, such that $B_n \subseteq B(x, r_n)$ and $\lambda(B_n) \geq s \cdot \lambda(B(x, r_n))$ for all $n \in \mathbb{N}$, where $B(x, r) := \{y \in \mathbb{R} : |x - y| < r\}$ and λ is the Lebesgue measure on $\mathbb{R}$.

Lebesgue Point Theorem. *Let $\overline{\mathbb{R}}$ be the extended real line $[-\infty, \infty]$ and let $f : \mathbb{R} \to \overline{\mathbb{R}}$ be an integrable function. For each $x \in \mathbb{R}$, a sequence $(B_n(x))_{n\in\mathbb{N}}$ of Borel subsets of $\mathbb{R}$ shrinks to x nicely. Then for almost every $x \in \mathbb{R}$,*

$$f(x) = \lim_{n\to\infty} \frac{1}{\lambda(B_n(x))} \int_{B_n(x)} f \, \mathrm{d}\lambda.$$

Proof of Theorem 4.4. There are two technical functions which are useful in proving this theorem. Define measurable functions, for every $u \in \mathbb{I}$,

$$f_{0,a,n}(u) = \sum_{k=0}^{n} \sum_{j=1}^{n} \frac{k}{n} \mathbb{1}_{F_{0,a,n}^{-1}\left(\frac{2k-1}{2n}, \frac{2k+1}{2n}\right]}\left(\tfrac{2j-1}{2n}\right) \mathbb{1}_{J_j^n}(u)$$
$$f_{1,b,n}(u) = \sum_{k=0}^{n} \sum_{j=1}^{n} \frac{k}{n} \mathbb{1}_{F_{1,b,n}^{-1}\left(\frac{2k-1}{2n}, \frac{2k+1}{2n}\right]}\left(\tfrac{2j-1}{2n}\right) \mathbb{1}_{J_j^n}(u).$$

The functions $f_{0,a}$ and $f_{1,b}$ may not exist at $\frac{2j-1}{2n}$. To get around this technicality, we have to use their average functions over the interval J_j^n, i.e. $F_{0,a,n}$ and $F_{1,b,n}$. Note that $f_{0,a}$ and $f_{1,b}$ are integrable and for each $u \in \mathbb{I}$, there exists $j_{u,n} = 1, 2, \ldots, n$ such that $u \in J_{j_{u,n}}^n$ for every $n \in \mathbb{N}$. Moreover, for each $u \in \mathbb{I}$, the sequence $(J_{j_{u,n}}^n)_{n\in\mathbb{N}}$ shrinks to u nicely with $r_n = \frac{1}{n}$ and $s = \frac{1}{2}$. Then, by Lebesgue Point Theorem, $(F_{0,a,n})_{n\in\mathbb{N}}$ and $(F_{1,b,n})_{n\in\mathbb{N}}$ converge a.e. to $f_{0,a}$ and $f_{1,b}$, respectively. Next, we will show that $(f_{0,a,n})_{n\in\mathbb{N}}$ converges a.e. to $f_{0,a}$. Let $u \in \mathbb{I}$ be such that $(F_{0,a,n}(u))_{n\in\mathbb{N}}$ converges to $f_{0,a}(u)$ and let $\epsilon > 0$. Then there exists an $N_1 \in \mathbb{N}$ such that $\frac{1}{N_1} < \epsilon$. Since $(F_{0,a,n}(u))_{n\in\mathbb{N}}$

converges to $f_{0,a}(u)$, there exists an $N_2 \in \mathbb{N}$ such that $|F_{0,a,n}(u) - f_{0,a}(u)| < \frac{\epsilon}{2}$ for every $n \geq N_2$. Let $n \geq \max(N_1, N_2)$. Then $u \in J_j^n$ for some $j = 1, 2, \ldots, n$. Thus $F_{0,a,n}\left(\frac{2j-1}{2n}\right) = F_{0,a,n}(u)$ and so $\frac{2k-1}{2n} < F_{0,a,n}(u) \leq \frac{2k+1}{2n}$ for some $k = 0, 1, 2, \ldots, n$. This implies that $|F_{0,a,n}(u) - f_{0,a,n}(u)| \leq \frac{1}{2n}$ because $f_{0,a,n}(u) = \frac{k}{n}$. Therefore,

$$|f_{0,a,n}(u) - f_{0,a}(u)| \leq |f_{0,a,n}(u) - F_{0,a,n}(u)| + |F_{0,a,n}(u) - f_{0,a}(u)| \leq \frac{1}{2n} + \frac{\epsilon}{2} < \epsilon.$$

This shows that $(f_{0,a,n})_{n\in\mathbb{N}}$ converges a.e. to $f_{0,a}$. In a similar way, $(f_{1,b,n})_{n\in\mathbb{N}}$ converges a.e. to $f_{1,b}$. Then, because for any $n \in \mathbb{N}$, $|f_{1,b,n} - f_{0,a,n}|$ is dominated by an integrable function on $\mathbb{I}$, $\mathbb{1}_{\mathbb{I}}$, and $(f_{1,b,n} - f_{0,a,n})_{n\in\mathbb{N}}$ converges a.e. to $f_{1,b} - f_{0,a}$, we have, by the Dominated Convergence Theorem,

$$\lim_{n\to\infty} \int_{\mathbb{I}} f_{1,b,n}(u) - f_{0,a,n}(u)\, \mathrm{d}u = \int_{\mathbb{I}} f_{1,b}(u) - f_{0,a}(u)\, \mathrm{d}u.$$

But, by Lemma 4.3,

$$\begin{aligned}
&\lambda_2\left(\left\{(u,v) \in \mathbb{I}^2 : a \leq \partial_1 C(u,v) < b \text{ and } \partial_1 C(u,v) \text{ exists}\right\}\right) \\
&= \lambda_2\left(\left\{(u,v) \in \mathbb{I}^2 : f_{0,a}(u) \leq v < f_{1,b}(u)\right\}\right) \\
&= \int_{\mathbb{I}} f_{1,b}(u) - f_{0,a}(u)\, \mathrm{d}u,
\end{aligned}$$

and

$$\begin{aligned}
\int_{\mathbb{I}} f_{1,b,n}(u) - f_{0,a,n}(u)\, \mathrm{d}u &= \sum_{k=0}^{n}\sum_{j=1}^{n} \frac{k}{n^2}\left(\mathbb{1}_{F_{1,b,n}^{-1}\left(\frac{2k-1}{2n}, \frac{2k+1}{2n}\right]} - \mathbb{1}_{F_{0,a,n}^{-1}\left(\frac{2k-1}{2n}, \frac{2k+1}{2n}\right]}\right)\left(\frac{2j-1}{2n}\right) \\
&= \frac{\#\left(\left(2n\left\{(u,v) \in \mathbb{I}^2 : F_{0,a,n}(u) \leq v < F_{1,b,n}(u)\right\}\right) \bigcap \mathcal{O}^2\right)}{n^2}.
\end{aligned}$$

This shows that

$$\begin{aligned}
&\lim_{n\to\infty} \frac{\#\left(\left(2n\left\{(u,v) \in \mathbb{I}^2 : F_{0,a,n}(u) \leq v < F_{1,b,n}(u)\right\}\right) \bigcap \mathcal{O}^2\right)}{n^2} \\
&= \lambda_2\left(\left\{(u,v) \in \mathbb{I}^2 : a \leq \partial_1 C(u,v) < b \text{ and } \partial_1 C(u,v) \text{ exists}\right\}\right). \qquad \square
\end{aligned}$$

4.2 Examples and comparisons

In previous section, let us observe that we use measurable functions $F_{0,a,n}$ and $F_{1,b,n}$ as technical tools for proving Theorem 4.4. Nevertheless, we can unquestionably use smooth level curves of partial derivatives of copulas to approximate their first digit distributions. In this section, we numerically compute μ_1 of some classes of copulas.

Example 4.1**.** A pair of random variables X and Y has the standard bivariate normal distribution with correlation coefficient $\rho \in (-1,1)$ if their joint probability density function is given by

$$\phi(x,y;\rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}\right)$$

for $x, y \in \mathbb{R}$. One can show that the marginal distributions of X and Y are standard normal, denoted as Φ. Then let $N_\rho(x,y)$ denote the standard bivariate normal joint distribution function with correlation coefficient $\rho \in (-1,1)$ so that C_ρ, the *Gaussian copula*, is defined by Sklar's theorem

$$C_\rho(\Phi(x), \Phi(y)) = N_\rho(x,y) \quad (x, y \in \mathbb{R}).$$

One can show that $C_0 = \Pi$. Moreover, we can extend continuously for $\rho \in \{-1, 1\}$:

$$\begin{aligned} C_{-1}(u,v) &:= \lim_{\rho \to -1} C_\rho(u,v) = \max(u+v-1, 0), \\ C_{+1}(u,v) &:= \lim_{\rho \to +1} C_\rho(u,v) = \min(u,v), \end{aligned}$$

for every $u, v \in \mathbb{I}$, so that C_ρ approaches W and M as ρ approaches -1 and 1, respectively. Then partial derivatives of Gaussian copulas are

$$\begin{aligned} \partial_1 C_\rho(u,v) &= \Phi\left(\frac{\Phi^{-1}(v) - \rho\Phi^{-1}(u)}{\sqrt{1-\rho^2}}\right), \\ \partial_2 C_\rho(u,v) &= \Phi\left(\frac{\Phi^{-1}(u) - \rho\Phi^{-1}(v)}{\sqrt{1-\rho^2}}\right). \end{aligned}$$

We then numerically compute μ_1 of $C_{0.8}$, the Gaussian copula with $\rho = 0.8$. Let us begin by considering level curves of $\partial_1 C_{0.8}$. By Theorem 4.4, to approximate $\mathrm{FD}_1(\partial_1 C_{0.8})$ we

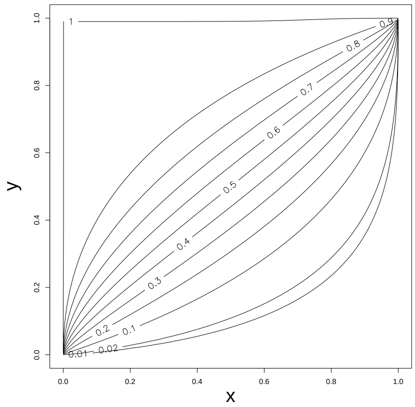

Figure 4.1**:** Level curves of Gaussian copula $\partial_1 C_{0.8}$

firstly count grid points which lie above or on level curve $\{\partial_1 C(u,v) = 1\}$. Similarly, we find the number of grid points lying above or on level curve $\{\partial_1 C(u,v) = \frac{1}{10^i}\}$ and below $\{\partial_1 C(u,v) = \frac{2}{10^i}\}$ for every $i = 1, 2, \ldots, n$. The quotient of total of such grid points and the number of grid points such that $\partial_1 C_{0.8} > 0$ is an approximation of $\mathrm{FD}_1(\partial_1 C_{0.8})$. Thus, in a similar manner, an approximation of $\mathrm{MF}_1(C_{0.8})$ is obtained and an approximate value of $\mu_1(C_{0.8})$ follows. In Figure 4.2, μ_1 of Gaussian copulas are graphed where the parameter ρ vary.

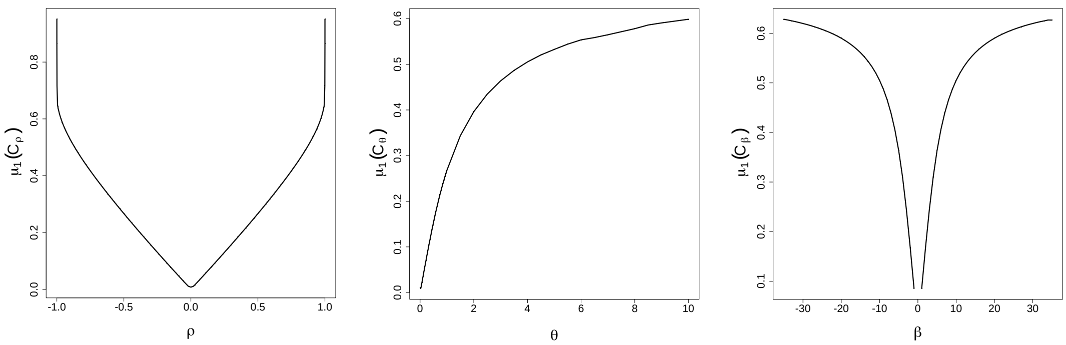

Figure 4.2**:** μ_1 of Gaussian C_ρ (Left), Clayton C_θ (Center) and Frank C_β (Right) copulas

For various classes, we also numerically compute μ_1 of Clayton and Frank copulas whose graphs are shown in Figure 4.2. Next, let us recall the definitions of some copula-based measures of dependence to compare their values with μ_1:

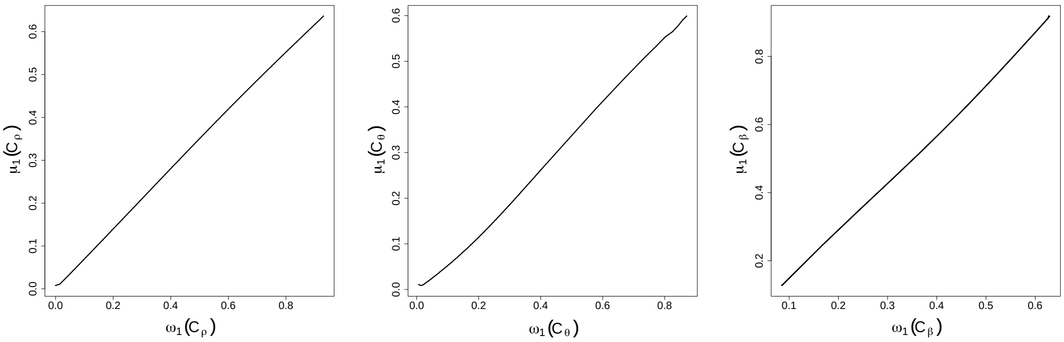

Figure 4.3**:** Graphs of $\mu_1(C)$ against $\omega_1(C)$ for Gaussian C_ρ (Left), Clayton C_θ (Center) and Frank C_β (Right) copulas

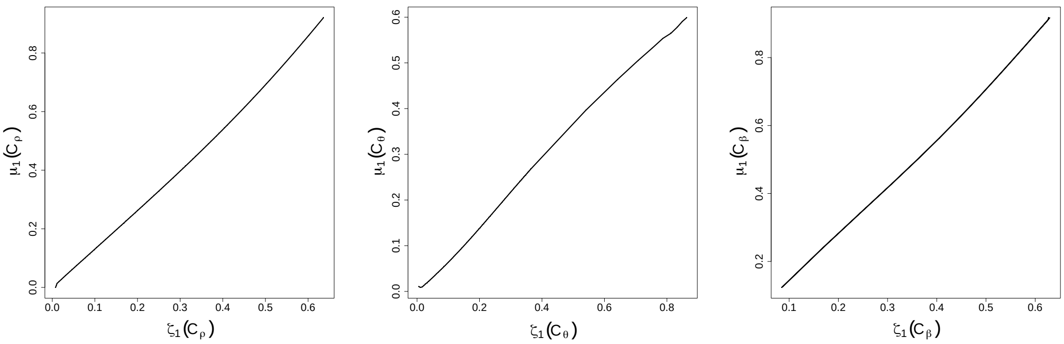

Figure 4.4**:** Graphs of $\mu_1(C)$ against $\zeta_1(C)$ for Gaussian C_ρ (Left), Clayton C_θ (Center) and Frank C_β (Right) copulas

I. (Siburg and Stoimenov [15])

$$\omega_i(C) = \left(6\int_{\mathbb{I}}\int_{\mathbb{I}} |\partial_i C(u,v) - \partial_i \Pi(u,v)|^2 \, \mathrm{d}u\, \mathrm{d}v\right)^{1/2}, \quad i = 1,2;$$

II. (Trutschnig [16])

$$\zeta_i(C) = 3\int_{\mathbb{I}}\int_{\mathbb{I}} |\partial_i C(u,v) - \partial_i \Pi(u,v)| \, \mathrm{d}u\, \mathrm{d}v, \quad i = 1,2.$$

In the Figures 4.3 and 4.4, numerical values μ_1 of Gaussian, Clayton and Frank copulas are plotted against ω_1 and ζ_1, respectively. One can see that for all three classes considered, the values of μ_1 vary quite proportional to both ω_1 and ζ_1. This means that μ_1 could be a good indicator of dependence. For finer detection, one can investigate in a future exploration dependence indicators based on the first two, three or more digit distributions of the partial derivatives of copulas.

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APPENDIX

APPENDIX : Examples of R code for calculating and plotting μ_1

In calculating, by applying Theorem 4.4, the approximate values μ_1 of each example in this thesis, we use R code to compute that robust results. Let us give the code for MF_1 and, as a result, μ_1 used in Example 4.1.

```
n = 6000
#Define grid points used in calculating
m <- matrix(rep(seq(1, 2*n-1, by = 2),n),nrow=n)
mt <- t(m)
x <- m/(2*n)
y <- mt/(2*n)

#Use parameter 0.5 in this example
r <- 0.5
#Define first derivative of Gaussian copula (parameter=r)
dG <- function(r) pnorm((qnorm(y)-(r*qnorm(x)))/sqrt(1-(r^2)))

#Calculate first digit distribution
FG = matrix(nrow=1,ncol=9)
Z <- sum(dG>0)
for(d in 1:9)
{s=0;
for(i in 0:100){s=s+sum(dG>=d/(10^i))-sum(dG>=(d+1)/(10^i))};
FG[1,d]=s/Z}

#Set approximate value of MF_1 and mu_1
MF1 <- max(FG)
mu1 <- ((MF1-(1/9))*(9/8))^0.5
```

For plotting μ_1 against ω_1 and ζ_1, the following R code is needed to calculate ω_1:

```
library(rmutil) #Load the package "rmutil"
Omega1 <- function(r){
#Define first derivative of Gaussian copula
dG<- function(x,y) pnorm((qnorm(y)-(r*qnorm(x)))/sqrt(1-(r^2)))
#Define first derivative of Product copula
dPi<-function(x,y) y
integrand <- function(x,y) (dG(x,y)-dPi(x,y))^2
int<-int2(integrand, a=c(0,0), b=c(1,1))
return(int)}
```

while ζ_1 use:

```
library(rmutil) #Load the package "rmutil"
Zeta1 <- function(r){
#Define first derivative of Gaussian copula
dG<- function(x,y) pnorm((qnorm(y)-(r*qnorm(x)))/sqrt(1-(r^2)))
#Define first derivative of Product copula
dPi<-function(x,y) y
integrand <- function(x,y) abs(dG(x,y)-dPi(x,y))
int<-int2(integrand, a=c(0,0), b=c(1,1))
return(int)}
```

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