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นายธีรพล สุคนธ์วิมลมาลย์

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STABILITY OF FRÉCHET FUNCTIONAL EQUATIONS ON CERTAIN GROUPOIDS

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เราพิจารณาปัญหาเสถียรภาพแบบ Hyers-Ulam ของสมการเชิงฟังก์ชันเฟรเช

$$\Delta^{n+1}_{h_1,h_2,\ldots,h_{n+1}} f(\mathrm{x}) = 0$$

และ

$$\Delta^{n+1}_{h} f(\mathrm{x}) = 0 \qquad (1)$$

เมื่อโดเมนของ f เป็นกรุปพอยด์ที่มีสมบัติเปลี่ยนหมู่ยกกำลังและสมบัติสมมาตรกำลัง นอกจากนี้ เราได้ศึกษาวิธีการใหม่ในการแก้ปัญหาเสถียรภาพของสมการเชิงฟังก์ชัน (1) เมื่อโดเมนของ f เป็นกึ่งกรุปสลับที่

ภาควิชา คณิตศาสตร์และวิทยาการคอมพิวเตอร์ ลายมือชื่อนิสิต
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We considered the Hyers-Ulam stability problem for Fréchet functional equations

$$\Delta^{n+1}_{h_1,h_2,\ldots,h_{n+1}} f(x) = 0$$

and

$$\Delta^{n+1}_{h} f(x) = 0 \qquad (1)$$

for the case where the domain of f is a power-associative, power-symmetric groupoid. We also gave a new approach for Hyers-Ulam stability of (1) when the domain of f is a commutative semigroup.

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CONTENTS

CHAPTER I
INTRODUCTION

The aim of this chapter is to give a general concept of functional equations, stability problems, and history concerning our proposed problem.

1.1 Functional Equations

A functional equation is an equation expressing certain property of an unknown function. Any function that satisfies the equation is called a solution of the functional equation. Solving a functional equation means to find all solutions of the equation, the collection of which is called the *general solution* of the equation.

Example. *Consider the functional equation*

$$f(x+y) = xf(y) + yf(x)$$

for all $x, y \in \mathbb{R}$*, where* $f : \mathbb{R} \to \mathbb{R}$*. Substituting* $y = 0$ *results in* $f(x) = xf(0)$ *for all* $x \in \mathbb{R}$*. In particular,* $f(0) = 0$*. So* $f(x) = 0$ *for all* $x \in \mathbb{R}$*.*

Substituting back shows that the zero function satisfies the above equation. Hence, a function f *is a solution of the functional equation* $f(x+y) = xf(y)+yf(x)$ *for all* $x, y \in \mathbb{R}$ *if and only if it is the zero function.*

Among the most studied functional equations is the *Cauchy functional equation*

$$f(x+y) = f(x) + f(y) \tag{1.1}$$

for all x, y in the domain of f. This equation is named after A.L. Cauchy, who showed that, when assigned $f : \mathbb{R} \to \mathbb{R}$, all of its continuous solutions are of the form $f(x) = cx$ for all $x \in \mathbb{R}$, where $c \in \mathbb{R}$ [3]. The existence of nonlinear solution of eq. (1.1) was discovered in 1905, when Hamel constructed the general solution through the use of a basis of $\mathbb{R}$ considered as a vector space over $\mathbb{Q}$ (see [11]). An interesting property of any nonlinear solution of eq. (1.1) is that the graph for each of them is densed in $\mathbb{R}^2$. This means these functions are nowhere continuous and are unbounded on any open interval.

A functional equation that is closely related to eq. (1.1) is the *Jensen functional equation*

$$f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2} \tag{1.2}$$

for all $x, y \in \mathbb{R}$. It is known that f is a solution of eq. (1.2) if and only if $f(x) = c + A(x)$, where A is a solution of eq. (1.1) and c is a constant (see [4]).

A further generalization of eq. (1.1) was done by M. Fréchet in 1909 when he studied a functional equation which can be written as

$$\Delta^{n+1}_{h_1,h_2,\ldots,h_{n+1}} f(0) = 0 \tag{1.3}$$

for all $h_1, h_2, \ldots, h_{n+1} \in \mathbb{R}$, where n is a positive integer ([9], also see [1]). M. Fréchet showed that the only continuous solutions of eq. (1.3) are the zero function and the polynomials of degree not exceeding n.

Observe that if a function $f : \mathbb{R} \to \mathbb{R}$ satisfies eq. (1.3) for all $h_1, h_2, \ldots, h_{n+1} \in \mathbb{R}$, then

$$\Delta^{n+1}_{h_1,h_2,\ldots,h_{n+1}} f(x) = \Delta^{n+1}_{x+h_1,h_2,\ldots,h_{n+1}} f(0) - \Delta^{n+1}_{x,h_2,\ldots,h_{n+1}} f(0) = 0$$

for all $x, h_1, h_2, \ldots, h_{n+1} \in \mathbb{R}$. Hence eq. (1.3) implies

$$\Delta^{n+1}_{h_1,h_2,\ldots,h_{n+1}} f(x) = 0 \tag{1.4}$$

for all $x, h_1, h_2, \ldots, h_{n+1} \in \mathbb{R}$. The converse is obviously true, so the two functional equations are equivalent.

Many researchers have studied eq. (1.4) and its generalizations [2, 5, 7, 14, 15, 19].

It was also shown that eq. (1.4) is equivalent to

$$\Delta_h^{n+1} f(x) = 0 \tag{1.5}$$

for all $x, h \in \mathbb{R}$ [6]. Hence, both eq. (1.4) and eq. (1.5) were named Fréchet functional equation. Their solutions are called *generalized polynomial functions of order at most n.*

1.2 Stability Problems of Functional Equations

The stability problems was initiated in 1940 by S.M. Ulam [21] during his talk at the Mathematics Club of the University of Wisconsin. His proposed problem is as follows: "Let G_1 be a group and G_2 be a metric group with the metric d. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if $f : G_1 \to G_2$ satisfies the inequality $d(f(xy), f(x)f(y)) \leq \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(f(x), H(x)) \leq \epsilon$ for all $x \in G_1$?"

A partial answer for this question came in the following year by D.H. Hyers [12]. He proved that for a function f between Banach spaces E_1 and E_2, if f satisfies the inequality

$$\|f(x + y) - f(x) - f(y)\| \leq \epsilon$$

for all $x, y \in E_1$ and for certain $\epsilon > 0$, then there exists a unique additive mapping $A : E_1 \to E_2$ satisfying the inequality

$$\|f(x) - A(x)\| \leq \epsilon$$

for all $x \in E_1$.

So any results answering a question of this sense are called *Hyers-Ulam stability.* This kind of stability became one of fundamental concepts in the study of functional equations.

There are other kinds of stability of functional equations, but in this dissertation we will only refer to Hyers-Ulam stability.

1.3 Motivation and Proposed Problem

Solutions and stability of functional equation eq. (1.4) have been studied under many different assumptions, such as one point continuity or assumption that the domain of f is an algebraic structure other than $(\mathbb{R}, +)$. Among the most general domains are commutative semigroups, on which the stability of both eqs. (1.4) and (1.5) were proved by M. Albert and J. Baker [2]. A more general stability result for eq. (1.4) has been obtained by L. Szekélyhidi [20], who confirmed an affirmative result on amenable semigroups.

Stability of other functional equations on nonassociative, noncommutative domains were also widely investigated [8, 16, 17, 18, 22]. Among them is the *monomial functional equation*,

$$\Delta_h^n f(x) = n! f(y)$$

for all x, h in the domain. This functional equation is also a generalization of eq. (1.1) and is closely related to eq. (1.5), as each of its solutions (see [7]) must also satisfy eq. (1.5). Its stability on power-associative, power-symmetric groupoids was proved by A. Gilányi in 1999 [10].

Inspired by these results, we aim to investigate the stability problem for eq. (1.4) and eq. (1.5) on power-associative, power-symmetric groupoids.

CHAPTER II
PRELIMINARIES

In this chapter, we covers some basic theorems concerning difference operator, additivity of functions, power-associativity and power-symmetry on groupoids. Throughout this dissertation, let $(S,+)$ be a commutative semigroup and B be a real Banach space.

2.1 Groupoids and Special Properties

A groupoid $(G,\circ)$ consists of a set G with a binary operation $\circ : G\times G \to G$. Since $\circ$ is not necessarily associative, order of operations must be made clear before we proceed.

Let $k \in \mathbb{N}$. For $x_1, x_2, \ldots, x_k \in G$, denote

$$x_1 \circ x_2, \circ \cdots \circ x_k = (\ldots((x_1 \circ x_2) \circ x_3) \circ \ldots) \circ x_k,$$

that is, the operation $\circ$ will be done from left to right whenever there are no written parenthesis. With this notion, define

$$x^k = \underbrace{x \circ x \circ \cdots \circ x}_{k \text{ terms}}.$$

Next, we give the definition of power-associativity and power-symmetry, and properties that they imply.

Definition 2.1. A groupoid $(G,\circ)$ is *power-associative* if

$$x^{k+l} = x^k \circ x^l$$

for all $x \in G$ and all $k, l \in \mathbb{N}$.

Theorem 2.2. [10] *Let* $(G, \circ)$ *be a power-associative groupoid. Then*

$$(x^k)^l = x^{kl}$$

for all $x \in G$ *and for all* $k, l \in \mathbb{N}$.

Remark 2.3. *Let* $(G, \circ)$ *be a power-associative groupoid and let* $x \in G$. *Define*

$$G_x = \{x^k \mid k \in \mathbb{N}\}.$$

Then it is not hard to see that G_x *is a commutative semigroup.*

Definition 2.4. Let $m > 1$ be an integer. A groupoid $(G, \circ)$ is mth-*power-symmetric* if

$$(x \circ y)^m = x^m \circ y^m$$

for all $x, y \in G$.

We say that $(G, \circ)$ is *power-symmetric* if $(G, \circ)$ is mth-power-symmetric for some $m > 1$.

Theorem 2.5. [10] *Let* $(G, \circ)$ *be a power-associative groupoid and let an integer* $m > 1$. *If* $(G, \circ)$ *is* mth-*power-symmetric, then it is also* m^kth-*power-symmetric for all* $k \in \mathbb{N}$.

2.2 Difference Operators

The difference operator is a very important tool in our study. Let $(G, \circ)$ be a groupoid and $f : G \to B$. The *difference operator* Δ_h with the span $h \in G$ is defined by

$$\Delta_h f(x) = f(x \circ h) - f(x)$$

for all $x \in G$.

The iteration of Δ is defined recursively by

$$\Delta^{k+1}_{h_1,h_2,\ldots,h_{k+1}} f = \Delta_{h_1}\left(\Delta^{k}_{h_2,h_3,\ldots,h_{k+1}} f\right)$$

for every $h_1, h_2, \ldots, h_{k+1} \in G$. When the spans are all equal, denote

$$\Delta^k_h f = \Delta^k_{\underbrace{h, h, \ldots, h}_{k \text{ terms}}} f.$$

In the following theorem and later, denote $x \circ y^0 := x$ for all $x, y \in G$. Theorems referenced from [4] have originally been stated for mappings between linear spaces over $\mathbb{Q}$. The proof presented in [4] remains valid for functions whose domain and range are as restated in here.

Theorem 2.6. [4] *Let $(G, \circ)$ be a groupoid, $f : G \to B$ and $k \in \mathbb{N}$. Then*

$$\Delta^k_{h_1,h_2\ldots,h_k} f(x) = \sum_{\epsilon_1,\epsilon_2,\ldots,\epsilon_k \in \{0,1\}} (-1)^{k-\epsilon_1-\epsilon_2-\cdots-\epsilon_k} f(x \circ h_1^{\epsilon_1} \circ h_2^{\epsilon_2} \circ \cdots \circ h_k^{\epsilon_k})$$

for all $x, h_1, h_2, \ldots, h_k \in G$. Furthermore,

$$\Delta^k_h f(x) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} f(x \circ \underbrace{h \circ h \circ \cdots \circ h}_{i \text{ terms}})$$

for all $x, h \in G$.

The difference operators commute when acting on a function whose domain is a commutative semigroup.

Theorem 2.7. [4] *Let $f : S \to B$. Then*

$$\Delta_{h_1}\Delta_{h_2} f = \Delta_{h_2}\Delta_{h_1} f$$

for all $h_1, h_2 \in S$.

2.3 Additivity of Functions

Definition 2.8. A function $A : S \to B$ is called an *additive function* if

$$A(x+y) = A(x) + A(y)$$

for all $x, y \in S$.

Example. *Let $c \in \mathbb{R}$. A linear function $f : \mathbb{R} \to \mathbb{R}$ defined by*

$$f(x) = cx$$

for all $x \in \mathbb{R}$, is an additive function.

A generalization of additivity is *multi-additivity*, which is an important tool in establishment of our results.

Definition 2.9. Let $n \in \mathbb{N}$. A function $A_n : S^n \to B$ is said to be *n-additive* if it is additive respect to each of its components, that is,

$$\begin{aligned} A_n(x_1, x_2, \ldots, x_{i-1}, x_i + x^*, x_{i+1}, \ldots, x_n) =& A_n(x_1, x_2, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) \\ &+ A_n(x_1, x_2, \ldots, x_{i-1}, x^*, x_{i+1}, \ldots, x_n) \end{aligned}$$

for all $x_1, x_2, \ldots, x_n, x^* \in S$ and for all $i \in \{1, 2, \ldots, n\}$.

Definition 2.10. Let n be a positive integer and $A_n : S^n \to B$ be an n-additive function. The *diagonalization* of A_n is defined to be the function $A^n : S \to B$ such that

$$A^n(x) = A_n(x, x, \ldots, x)$$

for all $x \in S$.

Example. *Let $f : \mathbb{R} \to \mathbb{R}$ be defined by*

$$f(x) = x^4$$

for all $x \in \mathbb{R}$. *Then* f *is the diagonalization of the* 4*-additive function* $g : \mathbb{R}^4 \to \mathbb{R}$ *defined by*

$$g(x_1, x_2, x_3, x_4) = x_1 x_2 x_3 x_4$$

for all $x_1, x_2, x_3, x_4 \in \mathbb{R}$.

The next two theorems are basic properties of diagonalization of n-additive functions.

Theorem 2.11. [4] *Let* $n \in \mathbb{N}$ *and* $A^n : S \to Y$ *be the diagonalization of an* n*-additive function. Then*

$$A^n(kx) = k^n A^n(x)$$

for all $x \in S$ *and for all* $k \in \mathbb{N}$.

Theorem 2.12. [4] *Let* $k \in \mathbb{N}$, $A_k : S^k \to B$ *be a* k*-additive function and* A^k *be the diagonalization of* A_k. *Then*

$$\Delta^k_{h_1,h_2,\ldots,h_k} A^k(x) = k! A_k(h_1, h_2, \ldots, h_k)$$

for all $x, h_1, h_2, \ldots, h_k \in S$. *Moreover, if* $n > k$, *then*

$$\Delta^n_{h_1,h_2,\ldots,h_n} A^k(x) = 0$$

for all $x, h_1, h_2, \ldots, h_n \in S$.

2.4 Generalized Polynomial Functions

In 1983, M. Albert and J. Baker gave the general solution and proved a stability result of Fréchet functional equation eq. (1.4) on commutative semigroups. The solutions are called *generalized polynomial functions of order at most* n, and are described in the following theorem.

Theorem 2.13. [2] *Let* $n \in \mathbb{N}$ *and* $f : S \to B$. *Assume that*

$$\Delta^{n+1}_{h_1,h_2,\ldots,h_{n+1}} f(x) = 0$$

for all $x, h_1, h_2, \ldots, h_{n+1} \in S$*. Then there exist* $A^1, A^2, \ldots, A^n : S \to B$ *and* $c \in B$ *such that each* A^i *is the diagonalization of an* i*-additive function and*

$$f(x) = c + A^1(x) + A^2(x) + \cdots + A^n(x)$$

for all $x \in S$*.*

The stability of eq. (1.4) proved by M. Albert and J. Baker is restated in the following theorem.

Theorem 2.14. [2] *Let* $n \in \mathbb{N}$*,* $\epsilon \in \mathbb{R}^+$*, and* $f : S \to B$*. Assume that* f *satisfies*

$$\|\Delta^{n+1}_{h_1,h_2,\ldots,h_{n+1}} f(x)\| \leq \epsilon$$

for all $x, h_1, h_2, \ldots, h_{n+1} \in S$*. Then there exist* $A^1, A^2, \ldots, A^n : S \to B$ *and* $c \in B$ *such that each* A^i *is a diagonalization of an* i*-additive function and*

$$\|f(x) - c - \sum_{i=1}^{n} A^i(x)\| \leq 2\epsilon$$

for all $x \in S$*. Furthermore, if* S *has an identity element* e *then*

$$\|f(x) - f(e) - \sum_{i=1}^{n} A^i(x)\| \leq \epsilon$$

for all $x \in S$*.*

Proposition 2.15. *We will investigate a relation between* f *and* $A^1, A^2, \ldots, A^n$ *from Theorem 2.14. Define*

$$E(x) = f(x) - c - \sum_{i=1}^{n} A^i(x)$$

for all $x \in S$*. Then* E *is a bounded function and* $\lim_{k\to\infty} \dfrac{E(kx) + c}{k^l} = 0$ *for all* $l \in \mathbb{N}$*.*

Hence

$$\begin{aligned} A^n(x) &= \lim_{k\to\infty} \frac{\sum_{i=1}^{n} k^i A^i(x)}{k^n} \\ &= \lim_{k\to\infty} \frac{\sum_{i=1}^{n} A^i(kx)}{k^n} + 0 \\ &= \lim_{k\to\infty} \frac{f(kx) - E(kx) - c}{k^n} + \lim_{k\to\infty} \frac{E(kx) + c}{k^n} \\ &= \lim_{k\to\infty} \frac{f(kx)}{k^n} \end{aligned}$$

for all $x \in S$. *In the similar manner,*

$$\begin{aligned} A^l(x) &= \lim_{k\to\infty} \frac{\sum_{i=1}^{l} k^i A^i(x)}{k^l} \\ &= \lim_{k\to\infty} \frac{\sum_{i=1}^{l} A^i(kx)}{k^l} + 0 \\ &= \lim_{k\to\infty} \frac{f(kx) - \sum_{j=l+1}^{n} A^j(kx) - E(kx) - c}{k^l} + \lim_{k\to\infty} \frac{E(kx) + c}{k^l} \\ &= \lim_{k\to\infty} \frac{f(kx) - \sum_{j=l+1}^{n} A^j(kx)}{k^l} \end{aligned}$$

for all $x \in S$ *and for all* $l \in \{1, 2, \ldots, n-1\}$. *This relation will be essential in our study.*

Proposition 2.16. *The functions* $A^1, A^2, \ldots, A^n$ *in Theorem 2.14 are unique: Assume that there exist* $B^1, B^2, \ldots, B^n : S \to B$ *where each* B^i *is the diagonalization of an* i*-additive function and there exists* $\delta > 0$ *such that*

$$\|f(x) - f(e) - \sum_{i=1}^{n} B^i(x)\| \leq \delta$$

for all $x \in G$. *Then*

$$\begin{aligned}\|\sum_{i=1}^{n}(A^i(x) - B^i(x))\| \leq & \|(f(x) - f(e) - \sum_{i=1}^{n} B^i(x)) - (f(x) - f(e) - \sum_{i=1}^{n} A^i(x))\| \\ \leq & \|f(x) - f(e) - \sum_{i=1}^{n} B^i(x)\| + \|f(x) - f(e) - \sum_{i=1}^{n} A^i(x)\| \\ \leq & \delta + \epsilon. \end{aligned} \tag{2.1}$$

Suppose that there exists the largest $k \in \{1, 2, \ldots, n\}$ *such that* $A^k \neq B^k$. *By Theorem 2.12,*

$$\Delta_h^k(A^k - B^k)(x) = k!A^k(h) - k!B^k(h)$$

for all $x, h \in S$.

From Theorem 2.11, Theorem 2.12, and inequality (2.1), we have

$$\begin{aligned}\|A^k(h) - B^k(h)\| =& \frac{1}{k!l^k}\|k!A^k(lh) - k!B^k(lh) + 0\| \\ =& \frac{1}{k!l^k}\|\Delta_{lh}^k(A^k - B^k)(x) + \Delta_{lh}^k \sum_{i=1}^{k-1}(A^i - B^i)(x)\| \\ =& \frac{1}{k!l^k}\left\|\Delta_{lh}^k \sum_{i=1}^{k}(A^i - B^i)(x)\right\| \\ =& \frac{1}{k!l^k}\left\|\sum_{j=0}^{k}\left[(-1)^{k-j}\binom{k}{j}\sum_{i=1}^{k}(A^i - B^i)(x(lh)^j)\right]\right\| \\ =& \frac{1}{k!l^k}\left\|\sum_{j=0}^{k}\left[(-1)^{k-j}\binom{k}{j}\sum_{i=1}^{k}\left(A^i(x(lh)^j) - B^i(x(lh)^j)\right)\right]\right\| \\ \leq& \frac{1}{k!l^k}\sum_{j=0}^{k}\left[\binom{k}{j}\sum_{i=1}^{k}\|A^i(x(lh)^j) - B^i(x(lh)^j)\|\right] \\ \leq& \frac{1}{k!l^k}\sum_{j=0}^{k}\binom{k}{j}(\delta + \epsilon) \\ \leq& \frac{1}{k!l^k}2^k(\delta + \epsilon)\end{aligned}$$

for all $x, h \in S$ *and for all* $l \in \mathbb{N}$. *Hence,*

$$\begin{aligned}\|A^k(h) - B^k(h)\| &= \lim_{l\to\infty} \|A^k(h) - B^k(h)\| \\ &\leq \lim_{l\to\infty} \frac{1}{k!l^k} 2^k(\delta + \epsilon) \\ &= 0\end{aligned}$$

for all $h \in S$.

This contradicts the assumption that $A^k - B^k$ *is not the zero function. Hence* $A^i = B^i$ *for all* $i \in \{1, 2, \ldots, n\}$.

This also implies that $A^1 + A^2 + \cdots + A^n$, *as a function, is unique.*

CHAPTER III
HYERS-ULAM STABILITY OF FRÉCHET FUNCTIONAL EQUATION ON COMMUTATIVE SEMIGROUPS

We recall the work of D. Ž. Djoković [6] and A. Albert and J. Baker [2] on the stability of symmetric Fréchet functional equation,

$$\Delta_h^{n+1} f(x) = 0 \tag{1.5}$$

for all $x, h \in S$. Firstly, we state Djoković's representation theorem here.

Theorem 3.1. [6] *Let Y be an abelian group and $n \in \mathbb{N}$. Then there exist $s, k \in \mathbb{N}$ and $m_1, m_2, \ldots m_k \in \mathbb{Z}$ such that the following statement holds:*

For every $h_1, h_2, \ldots, h_n \in S$ there exist $u_1, v_1, u_2, v_2, \ldots, u_k, v_k \in S$ such that

$$(n!)^{2^s} \Delta^n_{h_1,h_2,\ldots,h_n} f(x) = \sum_{i=1}^{k} m_i \Delta^n_{v_i} f(x+u_i)$$

for all $f : S \to Y$ and for all $x \in S$.

Now consider $\epsilon \in \mathbb{R}^+$, an integer $n \in \mathbb{N}$, and a function $f : S \to B$ such that

$$\|\Delta_h^{n+1} f(x)\| \le \epsilon$$

for all $x, h \in S$. Theorem 3.1 implies that there exist integers $s, k \in \mathbb{N}$ and $m_1, m_2, \ldots, m_k \in \mathbb{Z}$ such that

$$(n+1)!^{2^s} \Delta^{n+1}_{h_1,h_2,\ldots,h_{n+1}} f(x) = \sum_{i=1}^{k} m_i \Delta^{n+1}_{v_i} f(x+u_i)$$

for any $x, h_1, h_2, \ldots, h_{n+1} \in S$. Hence

$$\begin{aligned}\|\Delta^{n+1}_{h_1,h_2,\ldots,h_{n+1}} f(x)\| &\leq \sum_{i=1}^{k} \|\frac{m_i}{(n+1)!^{2^s}} \Delta^{n+1}_{v_i} f(x+u_i)\| \\ &\leq \sum_{i=1}^{k} \|\frac{m_i}{(n+1)!^{2^s}}\| \epsilon\end{aligned}$$

for all $x, h_1, h_2, \ldots, h_n \in S$. Letting $M_n = \sum_{i=1}^{k} \|\frac{m_i}{(n+1)!^{2^s}}\|$, the following theorem follows from Theorem 2.14.

Theorem 3.2. [2] *Let $n \in \mathbb{N}$, $\epsilon \in \mathbb{R}^+$, and $f : S \to B$. Assume that f satisfies*

$$\|\Delta^{n+1}_{h} f(x)\| \leq \epsilon$$

for all $x, h \in S$. Then there exist $M_n \in \mathbb{R}^+$ and $A^1, A^2, \ldots, A^n : S \to B$ and $c \in B$ such that each A^i is the diagonalization of an i-additive function and

$$\|f(x) - c - \sum_{i=1}^{n} A^i(x)\| \leq M_n \epsilon$$

for all $x \in G$.

However, it is not easy to determine M_n in Theorem 3.2 for large n. To see this, we will consider the procedure from which the combination

$$(n+1)!^{2^s} \Delta^{n+1}_{h_1,h_2,\ldots,h_{n+1}} f(x) = \sum_{i=1}^{k} m_i \Delta^{n+1}_{v_i} f(x+u_i)$$

is derived. It was shown in the proof of Theorem 3.1 that

$$\begin{aligned}(n+1)!^{2^s} \Delta^{n+1}_{h_1,\ldots,h_{n+1}} f = (-1)^{n+1} &\prod_{r=0}^{s-1} \left[(n+1)!^{2^r} + Q^{2^r}\right] \\ &\cdot \left[\sum_{\epsilon_1,\ldots,\epsilon_n \in \{0,1\}} (-1)^{\epsilon_1+\cdots+\epsilon_{n+1}} \Delta^{n+1}_{\epsilon_1 h_1+\cdots+\epsilon_n h_{n+1}}\right] f \\ &+ Q^{2^s} \Delta^{n+1}_{h_1,\ldots,h_{n+1}} f\end{aligned} \tag{3.1}$$

for all $h_1, \ldots, h_{n+1} \in S$ and $n, s \in \mathbb{N}$, where

$$Q = -\sum_{m=1}^{n+1} (-1)^{m+n+1} \binom{n+1}{m} \sum_{i_1,\ldots,i_{n+1} \in \{0,1,\ldots,m-1\}} \Delta_{i_1 h_1 + \cdots + i_{n+1} h_{n+1}}.$$

The multiplication in the above equation represents operator composition, and the number $(n+1)!$ in the right side represents the operator defined by

$$((n+1)! f)(x) = (n+1)! [f(x)]$$

for all $f : S \to B$ and all $x \in S$.

It was concluded that if s is large enough (apparently, if $2^s \geq n^2$), then there exist $s, k \in \mathbb{N}$, $m_1, m_2, \ldots, m_k \in \mathbb{Z}$, and $u_1, v_1, u_2, v_2, \ldots, u_k, v_k \in S$ such that

$$Q^{2^s} \Delta^{n+1}_{h_1,h_2,\ldots,h_{n+1}} f(x) = \sum_{i=1}^{k} m_i \Delta^{n+1}_{v_i} f(x + u_i),$$

for every $x \in S$.

In particular, when $n = 1$ and $s = 1$, this procedure yields

$$\begin{aligned}
4\Delta^2_{h_1,h_2} f(x) = & -5\Delta^2_{h_1} f(x) - 5\Delta^2_{h_2} f(x) + 5\Delta^2_{h_1+h_2} f(x) + \Delta^2_{h_1} f(x+h_1) + \Delta^2_{h_2} f(x+h_1) \\
& + \Delta^2_{h_1+h_2} f(x+h_1) + \Delta^2_{h_1} f(x+h_2) + \Delta^2_{h_2} f(x+h_2) + \Delta^2_{h_1+h_2} f(x+h_2) \\
& + \Delta^2_{h_1} f(x+h_1+h_2) + \Delta^2_{h_2} f(x+h_1+h_2) + \Delta^2_{h_1+h_2} f(x+h_1+h_2) \\
& + \Delta^3_{h_1} \Delta^3_{h_2} f(x) + 4\Delta^3_{h_1} \Delta_{h_2} f(x) + 4\Delta_{h_1} \Delta^3_{h_2} f(x) + 4\Delta^3_{h_1} \Delta^2_{h_2} f(x) \\
& + 4\Delta^2_{h_1} \Delta^3_{h_2} f(x) + 8\Delta^2_{h_1} \Delta^2_{h_2} f(x)
\end{aligned}$$

for all $x, h_1, h_2 \in S$. Note that the last six terms can be finalized in more than one way, for example,

$$\begin{aligned}\Delta_{h_1}^3\Delta_{h_2}^2 f(x) =&\Delta_{h_2}^2 f(x+3h_1) - 3\Delta_{h_2}^2 f(x+2h_1)\\ &+3\Delta_{h_2}^2 f(x+h_1) - \Delta_{h_2}^2 f(x)\\ =&\Delta_{h_1}^2 f(x+h_1+2h_2) - \Delta_{h_1}^2 f(x+2h_2) - 2\Delta_{h_1}^2 f(x+h_1+h_2)\\ &+2\Delta_{h_1}^2 f(x+h_2) + \Delta_{h_1}^2 f(x+h_1) - \Delta_{h_1}^2 f(x).\end{aligned}$$

Hence, the method of construction is not without difficulties, especially for large n. Since M_n in Theorem 3.2 is defined using eq. (3.1), it could be considerably large and hard to determine.

Our goal in this chapter is to give an alternative result on the stability of eq. (1.5), which will have a simpler bound than M_n.

3.1 Auxiliary Lemmas

We begin with lemmas concerning the difference operator.

Lemma 3.3. *Let $n \in \mathbb{N}$ and $f : S \to B$. Then*

$$\Delta_{2h}^n f(x) = \sum_{i=0}^{n} \binom{n}{i} \Delta_h^n f(x+ih)$$

for all $x, h \in S$.

Proof. We will proceed by induction on n. For $n = 1$,

$$\begin{aligned}\Delta_{2h} f(x) =&f(x+2h) - f(x)\\ =&f(x+h) - f(x) + f(x+2h) - f(x+h)\\ =&\Delta_h f(x) + \Delta_h f(x+h)\end{aligned}$$

for all $x, h \in S$. For the inductive step,

$$
\begin{aligned}
\Delta_{2h}^{k+1} f(x) =& \Delta_{2h}^{k} f(x+2h) - \Delta_{2h}^{k} f(x) \\
=& \Delta_{2h}^{k} f(x+h) - \Delta_{2h}^{k} f(x) + \Delta_{2h}^{k} f(x+2h) - \Delta_{2h}^{k} f(x+h) \\
=& \Delta_h \Delta_{2h}^{k} f(x+h) + \Delta_h \Delta_{2h}^{k} f(x) \\
=& \Delta_{2h}^{k} \Delta_h f(x+h) + \Delta_{2h}^{k} \Delta_h f(x) \\
=& \sum_{i=0}^{k} \binom{k}{i} \Delta_h^k \Delta_h f(x+h+ih) + \sum_{i=0}^{k} \binom{k}{i} \Delta_h^k \Delta_h f(x+ih) \\
=& \sum_{i=0}^{k} \binom{k}{i} \Delta_h^{k+1} f(x+(i+1)h) + \sum_{i=0}^{k} \binom{k}{i} \Delta_h^{k+1} f(x+ih) \\
=& \sum_{i=1}^{k+1} \binom{k}{i-1} \Delta_h^{k+1} f(x+ih) + \sum_{i=0}^{k} \binom{k}{i} \Delta_h^{k+1} f(x+ih) \\
=& \sum_{i=0}^{k+1} \binom{k+1}{i} \Delta_h^{k+1} f(x+ih)
\end{aligned}
$$

for all $x, h \in S$. The last equality results from the identity $\binom{k+1}{i} = \binom{k}{i-1} + \binom{k}{i}$ for all $k \in \mathbb{N}$ and for each integer $i \leq k$.

Hence, the statement is true for all $n \in \mathbb{N}$. □

Lemma 3.4. *Let $n \in \mathbb{N}$ and $f : S \to B$. Then*

$$
\Delta_h^n f(x) = \frac{1}{2^n} \Delta_{2h}^n f(x) - \frac{1}{2^n} \sum_{i=1}^{n} \binom{n}{i} \sum_{l=0}^{i-1} \Delta_h^{n+1} f(x+lh)
$$

for all $x, h \in S$.

Proof. First, observe that

$$
\begin{aligned}
\Delta_h^n f(x+kh) =& \Delta_h^n f(x) + \sum_{l=0}^{k-1} [\Delta_h^n f(x+(1+l)h) - \Delta_h^n f(x+lh)] \\
=& \Delta_h^n f(x) + \sum_{l=0}^{k-1} \Delta_h^{n+1} f(x+lh)
\end{aligned}
$$

for all $x, h \in S$ and for all $k \in \mathbb{N}$. Hence, by Lemma 3.3,

$$\begin{aligned}\Delta_{2h}^n f(x) =& \Delta_h^n f(x) + \sum_{i=1}^{n} \binom{n}{i} \Delta_h^n f(x+ih) \\ =& \Delta_h^n f(x) + \sum_{i=1}^{n} \binom{n}{i} \left[\Delta_h^n f(x) + \sum_{l=0}^{i-1} \Delta_y^{n+1} f(x+ly) \right] \\ =& (2^n) \Delta_h^n f(x) + \sum_{i=1}^{n} \binom{n}{i} \sum_{l=0}^{i-1} \Delta_h^{n+1} f(x+lh).\end{aligned}$$

So

$$\Delta_h^n f(x) = \frac{1}{2^n} \Delta_{2h}^n f(x) - \frac{1}{2^n} \sum_{i=1}^{n} \binom{n}{i} \sum_{l=0}^{i-1} \Delta_h^{n+1} f(x+lh)$$

for all $x, h \in S$. □

3.2 Stability of Fréchet Functional Equation

Now we are ready to solve our problem.

Lemma 3.5. *Let $n \in \mathbb{N}$, $\epsilon \in \mathbb{R}^+$. Assume that $f : S \to B$ satisfies*

$$\|\Delta_h^{n+1} f(x)\| \leq \epsilon$$

for all $x, h \in S$. Then

$$\|\Delta_{h_1} \Delta_{h_2}^n f(x)\| \leq \frac{2^n n}{2^n - 1} \epsilon$$

for all $x, h_1, h_2 \in S$.

Proof. Define a sequence of real numbers (a_m) by $a_1 = M_n \epsilon$ and $a_{m+1} = \frac{1}{2^n} a_m + n\epsilon$ for each $k \in \mathbb{N}$, where M_n is as in Theorem 3.2. We will prove by induction that for every $m \in \mathbb{N}$:

$$\|\Delta_{h_1} \Delta_{h_2}^n f(x)\| \leq a_m \tag{3.2}$$

for every $x, h_1, h_2 \in S$.

The basis step follows from Theorem 3.1. For the inductive step, let $k \in \mathbb{N}$ and assume that inequality (3.2) holds for $m = k$. Lemma 3.4 implies

$$\begin{aligned}
\|\Delta_{h_1}\Delta_{h_2}^n f(x)\| \leq& \frac{1}{2^n}\|\Delta_{h_1}\Delta_{2h_2}^n f(x)\| + \frac{1}{2^n}\sum_{i=1}^{n}\binom{n}{i}\sum_{l=0}^{i-1}\|\Delta_{h_1}\Delta_{h_2}^{n+1} f(x+lh)\| \\
\leq& \frac{1}{2^n}a_k + \frac{1}{2^n}\sum_{i=1}^{n}\binom{n}{i}\sum_{l=0}^{i-1}2\epsilon \\
=& \frac{1}{2^n}a_k + \frac{1}{2^{n-1}}\sum_{i=1}^{n}\binom{n}{i}i\epsilon \\
=& \frac{1}{2^n}a_k + \frac{1}{2^{n-1}}\sum_{i=1}^{n}n\binom{n-1}{i-1}\epsilon \\
=& \frac{1}{2^n}a_k + \frac{1}{2^{n-1}}n2^{n-1}\epsilon \\
=& \frac{1}{2^n}a_k + n\epsilon \\
=& a_{k+1}
\end{aligned}$$

for all $x, h_1, h_2 \in S$. Hence the inequality (3.2) is true for every $m \in \mathbb{N}$.

It is not hard to see that

$$a_m = \frac{1}{2^{(m-1)n}}a_1 + n\left(\sum_{i=0}^{m-2}\frac{1}{2^{in}}\right)\epsilon$$

for every integer $m > 1$. We then obtain

$$\lim_{m\to\infty} a_m = n\left(\sum_{i=0}^{\infty}\frac{1}{2^{in}}\right)\epsilon = \frac{n2^n}{2^n-1}\epsilon.$$

Let $x, h_1, h_2 \in S$. By inequality (3.2),

$$\|\Delta_{h_1}\Delta_{h_2}^n f(x)\| \leq \lim_{m\to\infty} a_m = \frac{n2^n}{2^n-1}\epsilon$$

for all $x, h_1, h_2 \in S$, as required. □

Theorem 3.6. *Let $n \in \mathbb{N}$, $\epsilon \in \mathbb{R}^+$, and $f : S \to B$ be such that*

$$\|\Delta_h^{n+1} f(x)\| \leq \epsilon$$

for all $x, h \in S$. Then

$$\|\Delta^{n+1}_{h_1,h_2,\ldots,h_{n+1}} f(x)\| \leq \left(\prod_{i=1}^{n} \frac{2^i}{2^i - 1}\right) n!\epsilon$$

for all $x, h_1, h_2, \ldots, h_{n+1} \in S$.

Proof. This can be done via induction on n. Lemma 3.5 directly implies the case when $n = 1$. For inductive step, observe that Lemma 3.5 and assumption of the theorem yield

$$\|\Delta^k_{h_2}\Delta_{h_1} f(x)\| \leq \frac{k2^k}{2^k - 1}\epsilon$$

for all $x, h_1, h_2 \in G$. Let $h \in S$ and $f_h = \Delta_h f$. Hence, $\|\Delta^k_{h_1} f_h(x)\| \leq \frac{k2^k}{2^k-1}\epsilon$ for all $h_1 \in S$.

By the induction hypothesis, we obtain

$$\begin{aligned}
\|\Delta^n_{h_1,h_2,\ldots,h_k}\Delta_h f(x)\| =& \|\Delta^n_{h_1,h_2,\ldots,h_k} f_h(x)\| \\
\leq& \left(\prod_{i=1}^{k-1} \frac{2^i}{2^i - 1}\right)(k-1)!\frac{k2^k}{2^k - 1}\epsilon \\
\leq& \left(\prod_{i=1}^{k} \frac{2^i}{2^i - 1}\right) k!\epsilon
\end{aligned}$$

for every $x, h_1, h_2, \ldots, h_k \in S$. Since h is arbitrary, it completes the inductive step. Hence the proof is completed. □

We come to the conclusion of this chapter.

Theorem 3.7. *Let $n \in \mathbb{N}$, $\epsilon \in \mathbb{R}^+$, and $f : S \to B$. Assume that f satisfies the inequality*

$$\|\Delta^{n+1}_h f(x)\| \leq \epsilon$$

for all $x, h \in S$. Then there exist unique $A^1, A^2, \ldots, A^n : S \to B$ and $c \in B$ such that each A^i is a diagonalization of an i-additive function and

$$\left\| f(x) - \left(c + \sum_{i=1}^{n} A^i(x)\right)\right\| \leq 2\left(\prod_{i=1}^{n} \frac{2^i}{2^i - 1}\right) n!\epsilon$$

for all $x \in S$. Furthermore, if S has an identity element 0 then

$$\left\| f(x) - \left(f(0) + \sum_{i=1}^{n} A^i(x) \right) \right\| \le \left(\prod_{i=1}^{n} \frac{2^i}{2^i - 1} \right) n!\epsilon$$

for all $x \in S$.

Proof. The existence follows immediately from Theorems 3.6 and 2.14. The uniqueness follows from Proposition 2.16. □

Remark 3.8. *Note that our new bound*

$$\left(\prod_{i=1}^{n} \frac{2^i}{2^i - 1} \right) n!\epsilon$$

is smaller than $M_n\epsilon$ from Theorem 3.2, at least for some n. For example, one choice for M_1 is $\frac{37}{4}$, and a partial calculation yielded $M_2 > 15$ while

$$\left(\prod_{i=1}^{2} \frac{2^i}{2^i - 1} \right) 2! = \frac{16}{3}.$$

M_3 is also larger than $\left(\prod_{i=1}^{3} \frac{2^i}{2^i - 1} \right) 3! = \frac{128}{7}$.

The author hypothesized that

$$\left(\prod_{i=1}^{n} \frac{2^i}{2^i - 1} \right) n!\epsilon < M_n$$

for all $n \in \mathbb{N}$.

CHAPTER IV
HYERS-ULAM STABILITY OF FRÉCHET FUNCTIONAL EQUATIONS ON CERTAIN GROUPOIDS

In this chapter, let $m > 1$ be an integer, $(G, \circ)$ be a power-associative, mth-power-symmetric groupoid with a left identity e.

The Hyers-Ulam stability of the functional equation

$$\Delta^{n+1}_{h_1,h_2,\ldots,h_{n+1}} f(x) = 0 \tag{1.4}$$

and

$$\Delta^{n+1}_{h} f(x) = 0. \tag{1.5}$$

has been studied by many authors [2, 13, 20]. One of the most general results is Theorem 2.14, which we restate below.

Theorem 4.1. [2] *Let $n \in \mathbb{N}$, $\epsilon \in \mathbb{R}^+$, and $f : S \to B$. Then f satisfies*

$$\|\Delta^{n+1}_{h_1,h_2,\ldots,h_{n+1}} f(x)\| \leq \epsilon$$

for all $x, h_1, h_2, \ldots, h_{n+1} \in S$ if and only if there exist $A^1, A^2, \ldots, A^n : S \to B$ and $c \in B$ such that each A^i is a diagonalization of an i-additive function and

$$\|f(x) - c - \sum_{i=1}^{n} A^i(x)\| \leq 2\epsilon$$

for all $x \in S$. Furthermore, if S has an identity element 0 then

$$\|f(x) - f(0) - \sum_{i=1}^{n} A^i(x)\| \leq \epsilon$$

for all $x \in S$.

We will investigate the stability of eq. (1.4) and eq. (1.5) with a more general domain for f. We adopt the concept of functions with noncommutative domain from a work of Gilányi[10]. He proved the Hyers-Ulam stability of the monomial functional equation

$$\Delta_h^n f(x) = n! f(h) \tag{4.1}$$

when the domain of f is a power-associative, mth-power-symmetric ($m > 1$) groupoid. Since eq. (4.1) and eq. (1.4) are closedly related, it could be speculated that similar assumptions on the domain will also imply the stability of eq. (1.4).

However, power-associativity and power-symmetry are not sufficient for the stability of eq. (1.4), as stated in the next example. Hence our main results also assume the existence of a left identity element.

Example. *Define* $\circ : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ *by*

$$m \circ n = \begin{cases} m & \textit{if} \;\; m \le n \\ m - 1 & \textit{if} \;\; m > n \end{cases}$$

for all $m, n \in \mathbb{N}$. *It is not hard to see that* $(\mathbb{N}, \circ)$ *is a power-associative,* m*th-power-symmetric groupoid for all integer* $m > 1$. *We will show that eq.* (1.4) *is not stable on this structure.*

Define $f : \mathbb{N} \to \mathbb{N}$ *by*

$$f(k) = \frac{k}{(n+1)2^{n+1}}$$

for all $k \in \mathbb{N}$. *Then*

$$\begin{aligned}
\|\Delta^{n+1}_{h_1,h_2,\ldots,h_{n+1}} f(k)\| &= \left\| \sum_{\epsilon_1,\epsilon_2,\ldots,\epsilon_{n+1}} (-1)^{n+1-\epsilon_1-\epsilon_2-\cdots-\epsilon_{n+1}} f(k \circ h_1^{\epsilon_1} \circ h_2^{\epsilon_2} \circ \cdots \circ h_{n+1}^{\epsilon_{n+1}}) \right\| \\
&\leq \sum_{\epsilon_1,\epsilon_2,\ldots,\epsilon_{n+1}} \left\| (-1)^{n+1-\epsilon_1-\epsilon_2-\cdots-\epsilon_{n+1}} \frac{(k \circ h_1^{\epsilon_1} \circ h_2^{\epsilon_2} \circ \cdots \circ h_{n+1}^{\epsilon_{n+1}})}{(n+1)2^{n+1}} \right\| \\
&\leq \sum_{\epsilon_1,\epsilon_2,\ldots,\epsilon_{n+1}} \left\| (-1)^{n+1-\epsilon_1-\epsilon_2-\cdots-\epsilon_{n+1}} \frac{(k \circ h_1^{\epsilon_1} \circ h_2^{\epsilon_2} \circ \cdots \circ h_{n+1}^{\epsilon_{n+1}}) - k}{(n+1)2^{n+1}} \right\| \\
&\quad + \sum_{\epsilon_1,\epsilon_2,\ldots,\epsilon_{n+1}} \left\| (-1)^{n+1-\epsilon_1-\epsilon_2-\cdots-\epsilon_{n+1}} \frac{k}{(n+1)2^{n+1}} \right\| \\
&\leq \sum_{\epsilon_1,\epsilon_2,\ldots,\epsilon_{n+1}} \frac{\left\|(k \circ h_1^{\epsilon_1} \circ h_2^{\epsilon_2} \circ \cdots \circ h_{n+1}^{\epsilon_{n+1}}) - k\right\|}{(n+1)2^{n+1}} + 0
\end{aligned}$$

for all $x, h_1, h_2, \ldots, h_{n+1} \in \mathbb{N}$. *Since each* $\|(k \circ h_1^{\epsilon_1} \circ h_2^{\epsilon_2} \circ \cdots \circ h_{n+1}^{\epsilon_{n+1}}) - k\| \leq n+1$, *the above inequalities yield*

$$\|\Delta^{n+1}_{h_1,h_2,\ldots,h_{n+1}} f(k)\| \leq 1$$

for all $x, h_1, h_2, \ldots, h_{n+1} \in \mathbb{N}$.

Suppose that $g : \mathbb{N} \to \mathbb{N}$ *such that* $\Delta^{n+1}_{h_1,h_2,\ldots,h_{n+1}} g(x) = 0$ *for all* $x, h_1, h_2, \ldots, h_{n+1} \in \mathbb{N}$. *Then*

$$0 = \Delta_k^{n+1} f(k+1) = (-1)^n (g(k+1) - g(k))$$

for all $k \in \mathbb{N}$. *Hence* g *is a constant function and* $\|f(k) - g(k)\|$ *is not bounded. So Fréchet functional equation (1.4) is not stable on this domain.*

4.1 Stability of Nonsymmetric Fréchet Functional Equation

In this section, we will study the stability of eq. (1.4). Before we can solve the problem, we need the following lemma.

Lemma 4.2. *Let $n \in \mathbb{N}$, $\epsilon \in \mathbb{R}^+$, and $f : G \to B$ satisfy*

$$\|\Delta^{n+1}_{h_1,h_2,\dots,h_{n+1}} f(x)\| \le \epsilon$$

for all $x, h_1, h_2, \dots, h_{n+1} \in G$. Suppose that $k \in \mathbb{N}$ such that the limit

$$L(x) := \lim_{s\to\infty} \frac{1}{m^{sk}} f(x^{m^s})$$

exists for every $x \in G$. Then

$$\Delta^{n+1}_{h_1,h_2,\dots,h_{n+1}} L(x) = 0$$

for all $x, h_1, h_2, \dots, h_{n+1} \in G$.

Proof. Given the assumptions,

$$\begin{aligned}
&\|\Delta^{n+1}_{h_1,h_2,\dots,h_{n+1}} L(x)\| \\
&= \left\| \sum_{\epsilon_1,\epsilon_2,\dots,\epsilon_{n+1}\in\{0,1\}} (-1)^{n+1-\epsilon_1-\epsilon_2-\cdots-\epsilon_{n+1}} L(x \circ h_1^{\epsilon_1} \circ h_2^{\epsilon_2} \circ \cdots \circ h_{n+1}^{\epsilon_{n+1}}) \right\| \\
&= \left\| \sum_{\epsilon_1,\epsilon_2,\dots,\epsilon_{n+1}\in\{0,1\}} (-1)^{n+1-\epsilon_1-\epsilon_2-\cdots-\epsilon_{n+1}} \lim_{s\to\infty} \frac{1}{m^{sk}} f((x \circ h_1^{\epsilon_1} \circ h_2^{\epsilon_2} \circ \cdots \circ h_{n+1}^{\epsilon_{n+1}})^{m^s}) \right\| \\
&= \left\| \lim_{s\to\infty} \sum_{\epsilon_1,\epsilon_2,\dots,\epsilon_{n+1}\in\{0,1\}} (-1)^{n+1-\epsilon_1-\epsilon_2-\cdots-\epsilon_{n+1}} \frac{1}{m^{sk}} f(x^{m^s} \circ h_1^{\epsilon_1 m^s} \circ h_2^{\epsilon_2 m^s} \circ \cdots \circ h_{n+1}^{\epsilon_{n+1} m^s}) \right\| \\
&= \lim_{s\to\infty} \left\| \frac{1}{m^{sk}} \sum_{\epsilon_1,\epsilon_2,\dots,\epsilon_{n+1}\in\{0,1\}} (-1)^{n+1-\epsilon_1-\epsilon_2-\cdots-\epsilon_{n+1}} f(x^{m^s} \circ (h_1^{m^s})^{\epsilon_1} \circ (h_2^{m^s})^{\epsilon_2} \circ \cdots \circ (h_{n+1}^{m^s})^{\epsilon_{n+1}}) \right\| \\
&= \lim_{s\to\infty} \frac{1}{m^{sk}} \|\Delta^{n+1}_{h_1^{m^s},h_2^{m^s},\dots,h_{n+1}^{m^s}} f(x^{m^s})\| \\
&\le \lim_{s\to\infty} \frac{1}{m^{sk}} \epsilon \\
&= 0
\end{aligned}$$

for every $x, h_1, h_2, \dots, h_n \in G$. Remind that $x \circ y^0 := x$ and that $(G, \circ)$ is m^sth-power-symmetric for every $s \in \mathbb{N}$. □

Now we approach the problem by considering it on subsets of the domain of f. In the following lemma, for each $x \in G$, denote

$$\begin{aligned} G_x =& \{x^k \mid k \in \mathbb{N}\} \\ \text{and} \quad G_x^0 =& G_x \cup \{e\}. \end{aligned}$$

Also denote $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Lemma 4.3. *Let $n \in \mathbb{N}$, $\epsilon \in \mathbb{R}^+$, $x \in G$, and $f : G \to B$ satisfy*

$$\|\Delta^{n+1}_{h_1,h_2,\dots,h_{n+1}} f(h_0)\| \le \epsilon$$

for all $h_1, h_2, \dots, h_{n+1} \in G_x$ and all $h_0 \in G_x^0$. Then there exists $a_1, a_2, \dots, a_n \in B$ such that

$$\|f(x^k) - \big(f(e) + ka_1 + k^2 a_2 + \cdots + k^n a_n\big)\| \le \epsilon$$

for all $k \in \mathbb{N}$. Furthermore, $a_1, a_2, \dots, a_n$ are defined by

$$\begin{aligned} a_n &= \lim_{k\to\infty} \frac{1}{k^n} f(x^k) \\ a_{n-1} &= \lim_{k\to\infty} \frac{1}{k^{n-1}} \big(f(x^k) - k^n a_n\big) \\ &\vdots \\ a_1 &= \lim_{k\to\infty} \frac{1}{k} \left(f(x^k) - \sum_{i=2}^{n} k^i a_i \right). \end{aligned}$$

Proof. Suppose that all the assumptions are met. Define $F : \mathbb{N}_0 \to B$ by

$$\begin{aligned} F(0) &= f(e) \\ \text{and} \quad F(k) &= f(x^k) \end{aligned}$$

for all $k \in \mathbb{N}$. According to the assumptions,

$$\begin{aligned}
&\|\Delta^{n+1}_{k_1,k_2,\ldots,k_{n+1}} F(k_0)\| \\
=&\left\| \sum_{\epsilon_1,\epsilon_2,\ldots,\epsilon_{n+1}\in\{0,1\}} (-1)^{n+1-\epsilon_1-\epsilon_2-\cdots-\epsilon_{n+1}} F(k_0+\epsilon_1k_1+\epsilon_2k_2+\cdots+\epsilon_{n+1}k_{n+1}) \right\| \\
=&\left\| \sum_{\epsilon_1,\epsilon_2,\ldots,\epsilon_{n+1}\in\{0,1\}} (-1)^{n+1-\epsilon_1-\epsilon_2-\cdots-\epsilon_{n+1}} f(x^{k_0+\epsilon_1k_1+\epsilon_2k_2+\cdots+\epsilon_{n+1}k_{n+1}}) \right\| \\
=&\left\| \sum_{\epsilon_1,\epsilon_2,\ldots,\epsilon_{n+1}\in\{0,1\}} (-1)^{n+1-\epsilon_1-\epsilon_2-\cdots-\epsilon_{n+1}} f(x^{k_0}\circ x^{\epsilon_1k_1}\circ x^{\epsilon_2k_2}\circ\cdots\circ x^{\epsilon_{n+1}k_{n+1}}) \right\| \\
=&\|\Delta^{n+1}_{x^{k_1},x^{k_2},\ldots,x^{k_{n+1}}} f(x^{k_0})\| \\
\leq&\ \epsilon
\end{aligned}$$

for all $k_0, k_1, k_2, \ldots, k_{n+1} \in \mathbb{N}$ and

$$\begin{aligned}
&\|\Delta^{n+1}_{k_1,k_2,\ldots,k_{n+1}} F(0)\| \\
=&\left\| \sum_{\epsilon_1,\epsilon_2,\ldots,\epsilon_{n+1}\in\{0,1\}} (-1)^{n+1-\epsilon_1-\epsilon_2-\cdots-\epsilon_{n+1}} F(\epsilon_1k_1+\epsilon_2k_2+\cdots+\epsilon_{n+1}k_{n+1}) \right\| \\
=&\left\| \sum_{\epsilon_1,\epsilon_2,\ldots,\epsilon_{n+1}\in\{0,1\}} (-1)^{n+1-\epsilon_1-\epsilon_2-\cdots-\epsilon_{n+1}} f(e\circ x^{\epsilon_1k_1}\circ x^{\epsilon_2k_2}\circ\cdots\circ x^{\epsilon_{n+1}k_{n+1}}) \right\| \\
=&\|\Delta^{n+1}_{x^{k_1},x^{k_2},\ldots,x^{k_{n+1}}} f(e)\| \\
\leq&\ \epsilon
\end{aligned}$$

for all $k_1, k_2, \ldots, k_{n+1} \in \mathbb{N}$. So $\|\Delta^{n+1}_{k_1,k_2,\ldots,k_{n+1}} F(k_0)\| \leq \epsilon$ for every $k_0, k_1, k_2, \ldots, k_{n+1} \in \mathbb{N}_0$ (it equals to zero in the case one of the spans is zero).

By Theorem 2.14, There exist $A^1, A^2, \ldots, A^n : \mathbb{N}_0 \to B$ such that each A^i is the diagonalization of an i-additive function and

$$\|F(k) - F(0) - \sum_{i=1}^{n} A^i(k)\| \leq \epsilon$$

for all $k \in \mathbb{N}_0$.

Let $a_i = A^i(1)$ for each $i \in \{1, 2, \ldots, n\}$. Theorem 2.11 implies that

$$\begin{aligned}\|f(x^k) - f(e) - \sum_{i=1}^{n} k^i a_i\| =& \|F(k) - F(0) - \sum_{i=1}^{n} k^i A^i(1)\| \\ =& \|F(k) - F(0) - \sum_{i=1}^{n} A^i(k)\| \\ \leq& \ \epsilon\end{aligned}$$

for all $k \in \mathbb{N}_0$.

Proposition 2.15 yields

$$\begin{aligned}a_n &= \lim_{k\to\infty} \frac{1}{k^n} F(k) = \lim_{k\to\infty} \frac{1}{k^n} f(x^k) \\ a_{n-1} &= \lim_{k\to\infty} \frac{1}{k^{n-1}} (F(k) - k^n a_n) = \lim_{k\to\infty} \frac{1}{k^{n-1}} (f(x^k) - k^n a_n) \\ &\vdots \\ a_1 &= \lim_{k\to\infty} \frac{1}{k} \left(F(k) - \sum_{i=2}^{n} k^i a_i \right) = \lim_{k\to\infty} \frac{1}{k} \left(f(x^k) - \sum_{i=2}^{n} k^i a_i \right)\end{aligned}$$

as desired. □

The next theorem gives the main result of this section.

Theorem 4.4. *Let $n \in \mathbb{N}$, $\epsilon \in \mathbb{R}^+$, and $f : G \to B$ satisfy*

$$\|\Delta^{n+1}_{h_1,h_2,\ldots,h_{n+1}} f(x)\| \leq \epsilon$$

for all $x, h_1, h_2, \ldots, h_{n+1} \in G$. Then there exists a unique function $P : G \to B$ such that $P(e) = 0$,

$$\|f(x) - P(x) - f(e)\| \leq \epsilon$$

for all $x \in G$, and

$$\Delta^{n+1}_{h_1,h_2,\ldots,n_{n+1}} P(x) = 0$$

for all $x, h_1, h_2, \ldots, h_{n+1} \in G$*. Furthermore,* P *is defined by*

$$P(x) = A^1(x) + A^2(x) + \cdots + A^n(x)$$

where

$$\begin{aligned} A^n(x) &= \lim_{k\to\infty} \frac{1}{k^n} f(x^k) \\ A^{n-1}(x) &= \lim_{k\to\infty} \frac{1}{k^{n-1}} \left(f(x^k) - A^n(x^k)\right) \\ &\vdots \\ A^1(x) &= \lim_{k\to\infty} \frac{1}{k} \left(f(x^k) - \sum_{i=2}^{n} A^i(x^k) \right) \end{aligned}$$

for all $x \in G$*.*

Proof. According to lemma 4.3, for each $x \in G$, there exist $a_{x,1}, a_{x,2}, \ldots, a_{x,n} \in B$ such that

$$\|f(x) - f(e) - \sum_{i=1}^{n} a_{x,i}\| \leq \epsilon$$

and

$$\begin{aligned} a_{x,n} &= \lim_{k\to\infty} \frac{1}{k^n} f(x^k) \\ a_{x,n-1} &= \lim_{k\to\infty} \frac{1}{k^{n-1}} (f(x^k) - k^n a_{x,n}) \\ &\vdots \\ a_{x,1} &= \lim_{k\to\infty} \frac{1}{k} \left(f(x^k) - \sum_{i=2}^{n} k^i a_{x,i} \right). \end{aligned}$$

Define $P, A^1, A^2, \ldots, A^n : G \to B$ by $A^i(x) = a_{x,i}$ and

$$P(x) = A^1(x) + A^2(x) + \cdots + A^n(x)$$

for all $x \in G$.

It remains to show that

- $A^i(x^k) = k^i a_{x,i}$ for all $x \in G$, $i \in \{1, 2, \ldots, n\}$, and $k \in \mathbb{N}$, and
- $\Delta^{n+1}_{h_1,h_2,\ldots,h_{n+1}} P(x) = 0$ for all $x, h_1, h_2, \ldots, h_{n+1} \in G$.

To show that $A^i(x^k) = k^i a_{x,i}$ for all $x \in G$, $i \in \{1, 2, \ldots, n\}$, and $k \in \mathbb{N}$, we observe that

$$\begin{aligned} A^n(x^l) &= a_{x^l,n} \\ &= \lim_{k\to\infty} \frac{1}{k^n} f((x^l)^k) \\ &= l^n \lim_{k\to\infty} \frac{1}{(lk)^n} f(x^{lk}) \\ &= l^n a_{x,n}. \end{aligned}$$

and

$$\begin{aligned} A^{n-1}(x^l) &= a_{x^l,n-1} \\ &= \lim_{k\to\infty} \frac{1}{k^{n-1}} (f((x^l)^k) - k^n a_{x^l,n}) \\ &= \lim_{k\to\infty} \frac{1}{k^{n-1}} (f((x^l)^k) - k^n A^n(x^l)) \\ &= l^{n-1} \lim_{k\to\infty} \frac{1}{(lk)^{n-1}} (f(x^{lk}) - k^n l^n a_{x,n}) \\ &= l^{n-1} a_{x,n-1}. \end{aligned}$$

for all $l \in \mathbb{N}$. It can be shown in similar way that

$$A^i(x^k) = k^i a_{x,i}$$

for all $x \in G$, $i \in \{1, 2, \ldots, n-2\}$, and $k \in \mathbb{N}$.

Next, we will show that $\Delta^{n+1}_{h_1,h_2,\ldots,h_{n+1}} P(x) = 0$ for all $x, h_1, h_2, \ldots, h_{n+1} \in G$. By the definition of A^n and Lemma 4.2,

$$\Delta^{n+1}_{h_1,h_2,\ldots,h_{n+1}} A^n(x) = 0$$

for all $x, h_1, h_2, \ldots, h_{n+1} \in G$. So

$$\begin{aligned}\|\Delta^{n+1}_{h_1,h_2,\ldots,h_{n+1}}(f - A^n)(x)\| =& \|\Delta^{n+1}_{h_1,h_2,\ldots,h_{n+1}} f(x)\| - \|\Delta^{n+1}_{h_1,h_2,\ldots,h_{n+1}} A^n(x)\| \\ \leq& \epsilon\end{aligned}$$

for all $x, h_1, h_2, \ldots, h_{n+1} \in G$. Using Lemma 4.2 again yields

$$\Delta^{n+1}_{h_1,h_2,\ldots,h_{n+1}} A^{n-1}(x) = 0$$

for all $x, h_1, h_2, \ldots, h_{n+1} \in G$. It can be shown recursively in similar way that

$$\Delta^{n+1}_{h_1,h_2,\ldots,h_{n+1}} A^{i}(x) = 0$$

for all $x, h_1, h_2, \ldots, h_{n+1} \in G$ and for $i \in \{1, 2, \ldots, n\}$. Hence

$$\begin{aligned}\Delta^{n+1}_{h_1,h_2,\ldots,h_{n+1}} P(x) =& \Delta^{n+1}_{h_1,h_2,\ldots,h_{n+1}} (A^1 + A^2 + \cdots + A^n)(x) \\ =& 0\end{aligned}$$

for all $x, h_1, h_2, \ldots, h_{n+1} \in G$.

For uniqueness of P, let $P^* : G \to B$ be such that

$$\Delta^{n+1}_{h_1,h_2,\ldots,h_{n+1}} P^*(x) = 0$$

for all $x, h_1, h_2, \ldots, h_{h_{n+1}} \in G$ and there exists δ such that

$$\|f(x) - f(e) - P^*(x)\| \leq \delta$$

for all $x \in G$.

Let $x \in G$ and remind that G_x is a commutative semigroup. Observe that the restriction f_x of f on G_x satisfies the assumption of Theorem 2.14. The restrictions

P_x of P and P_x^* of P^* also satisfy

$$\begin{aligned} \Delta^{n+1}_{h_1,h_2,\ldots,h_{n+1}} P_x(y) &= 0, \\ \|f_x(y) - f(e) - P_x(y)\| &\le 2\epsilon, \\ \Delta^{n+1}_{h_1,h_2,\ldots,h_{n+1}} P_x^*(y) &= 0, \\ \text{and} \quad \|f_x(y) - f(x) - P_x^*(y)\| &\le 2\delta \end{aligned}$$

for all $y, h_1, h_2, \ldots, h_{n+1} \in G_x$. Proposition 2.16 then implies that $P_x = P_x^*$, that is, $P(x) = P_x(x) = P_x^*(x) = P^*(x)$.

Since x is arbitrary, we can conclude that $P = P^*$. □

4.2 Stability of Symmetric Fréchet functional equation

The stability results of symmetric version of Fréchet functional equation,

$$\Delta_h^{n+1} f(x) = 0,$$

have mainly been obtained by applying one of certain theorems, such as Djoković's theorem(Theorem 3.1), Kuczma's representation theorem (see Theorem 9.2 of [4]), and our Theorem 3.6, to the stability results of eq. (1.4). However, we do not have such a tool that is applicable for functions on noncommutative domain. Hence we will proceed in a way that is analogous to the previous section.

Lemma 4.5. *Let $n \in \mathbb{N}$, $\epsilon \in \mathbb{R}^+$, and $f : G \to B$ satisfy*

$$\|\Delta_h^{n+1} f(x)\| \le \epsilon$$

for all $x, h \in G$. Suppose that $k \in \mathbb{N}$ such that the limit

$$L(x) := \lim_{s\to\infty} \frac{1}{m^{sk}} f(x^{m^s})$$

exists for every $x \in G$. Then

$$\Delta_h^{n+1} L(x) = 0$$

for all $x, h \in G$.

Proof. We proceed as follows:

$$\begin{aligned}
\|\Delta_h^{n+1} L(x)\| &= \| \sum_{i+0}^{n+1} (-1)^{n+1-i} \binom{n+1}{i} L(x \circ \underbrace{h \circ h \circ \cdots \circ h}_{i \text{ terms}}) \| \\
&= \| \sum_{i+0}^{n+1} (-1)^{n+1-i} \binom{n+1}{i} \lim_{s\to\infty} \frac{1}{m^{sk}} f((x \circ \underbrace{h \circ h \circ \cdots \circ h}_{i \text{ terms}})^{m^s}) \| \\
&= \| \lim_{s\to\infty} \frac{1}{m^{sk}} \sum_{i+0}^{n+1} (-1)^{n+1-i} \binom{n+1}{i} f(x^{m^s} \circ \underbrace{h^{m^s} \circ h^{m^s} \circ \cdots \circ h^{m^s}}_{i \text{ terms}}) \| \\
&= \lim_{s\to\infty} \frac{1}{m^{sk}} \| \sum_{i+0}^{n+1} (-1)^{n+1-i} \binom{n+1}{i} f(x^{m^s} \circ \underbrace{h^{m^s} \circ h^{m^s} \circ \cdots \circ h^{m^s}}_{i \text{ terms}}) \| \\
&= \lim_{s\to\infty} \frac{1}{m^{sk}} \| \Delta_{h^{m^s}}^{n+1} f(x^{m^s}) \| \\
&\leq \lim_{s\to\infty} \frac{1}{m^{sk}} \epsilon \\
&= 0
\end{aligned}$$

for all $x, h \in G$. Remind that $(G, \circ)$ is m^sth-power-symmetric for every $s \in \mathbb{N}$. □

The next lemma allows us to use Lemma 4.3 in the main result of this section. The definitions of G_x and G_x^0 here are the same as in Lemma 4.3.

Lemma 4.6. *Let $n \in \mathbb{N}$, $\epsilon \in \mathbb{R}^+$, and $f : G \to B$ satisfy*

$$\|\Delta_h^{n+1} f(x)\| \leq \epsilon$$

for all $x, h \in G$. Then, for each $x \in G$,

$$\|\Delta_{h_1, h_2, \ldots, h_{n+1}}^{n+1} f(y)\| \leq \left(\prod_{i=1}^{n} \frac{2^i}{2^i - 1} \right) n! \epsilon$$

for all $h_1, h_2, \ldots, h_{n+1} \in G_x$ and for all $y \in G_x^0$.

Proof. Let $x \in G$. Define $F : \mathbb{N}_0 \to B$ by

$$F(0) = f(e)$$
$$\text{and} \quad F(k) = f(x^k)$$

for all $k \in \mathbb{N}$. According to the assumptions,

$$\begin{aligned}
\|\Delta_k^{n+1} F(k_0)\| &= \| \sum_{i=0}^{n+1} (-1)^{n+1-i} \binom{n+1}{i} F(k_0 + ik) \| \\
&= \| \sum_{i=0}^{n+1} (-1)^{n+1-i} \binom{n+1}{i} f(x^{k_0+ik}) \| \\
&= \| \sum_{i=0}^{n+1} (-1)^{n+1-i} \binom{n+1}{i} f(x^{k_0} \circ \underbrace{x^k \circ x^k \circ \cdots \circ x^k}_{i \text{ terms}}) \| \\
&= \| \Delta_{x^k}^{n+1} f(x^{k_0}) \| \\
&= \epsilon
\end{aligned}$$

for all $k_0, k \in \mathbb{N}$. Also,

$$\begin{aligned}
\|\Delta_k^{n+1} F(0)\| &= \| \sum_{i=0}^{n+1} (-1)^{n+1-i} \binom{n+1}{i} F(ik) \| \\
&= \| \sum_{i=0}^{n+1} (-1)^{n+1-i} \binom{n+1}{i} f(x^{ik}) \| \\
&= \| \sum_{i=0}^{n+1} (-1)^{n+1-i} \binom{n+1}{i} f(e \circ \underbrace{x^k \circ x^k \circ \cdots \circ x^k}_{i \text{ terms}} \| \\
&= \| \Delta_{x^k}^{n+1} f(e) \| \\
&= \epsilon
\end{aligned}$$

for all $k \in \mathbb{N}$.

So $\|\Delta_k^{n+1} F(k_0)\| \leq \epsilon$ for every $k_0, k \in \mathbb{N}_0$. Thus Theorem 3.6 yields

$$\|\Delta_{k_1,k_2,\ldots,k_{n+1}}^{n+1} F(k_0)\| \leq \prod_{i=1}^{n} \left(\frac{2^i}{2^i - 1} \right) n!\epsilon$$

for all $k_0, k_1, k_2, \ldots, k_{n+1} \in \mathbb{N}_0$. This implies

$$\|\Delta^{n+1}_{h_1,h_2,\ldots,h_{n+1}} f(y)\| \leq \prod_{i=1}^{n} \left(\frac{2^i}{2^i - 1}\right) n!\epsilon$$

for all $h_1, h_2, \ldots, h_{n+1} \in G_x$ and $y \in G_x^0$. □

The next theorem completes the goal of this section.

Theorem 4.7. *Let $n \in \mathbb{N}$, $\epsilon \in \mathbb{R}^+$, and $f : G \to B$ satisfy*

$$\|\Delta_h^{n+1} f(x)\| \leq \epsilon$$

for all $x, h \in G$. Then there exists a unique function $P : G \to B$ such that $P(e) = 0$,

$$\|f(x) - P(x) - f(e)\| \leq \left(\prod_{i=1}^{n} \frac{2^i}{2^i - 1}\right) n!\epsilon$$

for all $x \in G$, and

$$\Delta_h^{n+1} P(x) = 0$$

for all $x, h \in G$. Furthermore, P is defined by

$$P(x) = A^1(x) + A^2(x) + \cdots + A^n(x)$$

where

$$\begin{aligned}
A^n(x) &= \lim_{s\to\infty} \frac{1}{m^{ns}} f(x^{m^s}) \\
A^{n-1}(x) &= \lim_{s\to\infty} \frac{1}{m^{(n-1)s}} \left(f(x^{m^s}) - A^n(x^{m^s})\right) \\
&\vdots \\
A^1(x) &= \lim_{s\to\infty} \frac{1}{m^s} \left(f(x^{m^s}) - \sum_{i=2}^{n} A^i(x^{m^s})\right)
\end{aligned}$$

for all $x \in G$.

Proof. By to Lemmas 4.6 and 4.3, for each $x \in G$, there exist $a_{x,1}, a_{x,2}, \ldots, a_{x,n} \in B$

such that

$$\|f(x) - f(e) - \sum_{i=1}^{n} a_{x,i}\| \leq \prod_{i=1}^{n} \left(\frac{2^i}{2^i - 1} \right) n!\epsilon$$

and

$$\begin{aligned}
a_{x,n} &= \lim_{k\to\infty} \frac{1}{k^n} f(x^k) \\
a_{x,n-1} &= \lim_{k\to\infty} \frac{1}{k^{n-1}} (f(x^k) - k^n a_{x,n}) \\
&\vdots \\
a_{x,1} &= \lim_{k\to\infty} \frac{1}{k} \left(f(x^k) - \sum_{i=2}^{n} k^i a_{x,i} \right).
\end{aligned}$$

Define $P, A^1, A^2, \ldots, A^n : G \to B$ by $A^i(x) = a_{x,i}$ and

$$P(x) = A^1(x) + A^2(x) + \cdots + A^n(x)$$

for all $x \in G$.

It is now sufficient to show that

- $A^i(x^k) = k^i a_{x,i}$ for all $x \in G$, $i \in \{1, 2, \ldots, n\}$, and $k \in \mathbb{N}$, and
- $\Delta_h^{n+1} P(x) = 0$ for all $x, h \in G$.

The statement $A^i(x^k) = k^i a_{x,i}$ for all $x \in G$, $i \in \{1, 2, \ldots, n\}$, and $k \in \mathbb{N}$ can be shown in identical way as in Theorem 4.4, that is,

$$\begin{aligned}
A^n(x^l) &= a_{x^l,n} \\
&= \lim_{k\to\infty} \frac{1}{k^n} f((x^l)^k) \\
&= l^n \lim_{k\to\infty} \frac{1}{(lk)^n} f(x^{lk}) \\
&= l^n a_{x,n}.
\end{aligned}$$

and

$$\begin{aligned} A^i(x^l) &= a_{x^l,i} \\ &= \lim_{k\to\infty} \frac{1}{k^i}(f((x^l)^k) - \sum_{j=i+1}^{n} k^j a_{x^l,j}) \\ &= \lim_{k\to\infty} \frac{1}{k^i}(f((x^l)^k) - \sum_{j=i+1}^{n} k^j A^n(x^l)) \\ &= l^{n-1} \lim_{k\to\infty} \frac{1}{(lk)^i}(f(x^{lk}) - \sum_{j=i+1}^{n} k^j l^j a_{x,j}) \\ &= l^i a_{x,i}. \end{aligned}$$

for all $l \in \mathbb{N}$ and all $i \in \{1, 2, \ldots, n-1\}$, given the procedure is done in descending order.

Thus we will show that

$$\Delta_h^{n+1} P(x) = 0$$

for all $x, h \in G$. By the definition of A^n and Lemma 4.5,

$$\Delta_h^{n+1} A^n(x) = 0$$

for all $x, h \in G$. So

$$\begin{aligned} \|\Delta_h^{n+1}(f - A^n)(x)\| =& \|\Delta_h^{n+1} f(x)\| - \|\Delta_h^{n+1} A^n(x)\| \\ \leq& \epsilon \end{aligned}$$

for all $x \in G$. Lemma 4.5 again implies

$$\Delta_h^{n+1} A^{n-1}(x) = 0$$

for all $x, h \in G$. Applying Lemma 4.5 recursively yields

$$\Delta_h^{n+1} A^i(x) = 0$$

for all $x, h \in G$ and for $i \in \{1, 2, \ldots, n\}$. Hence

$$\begin{aligned}\Delta_h^{n+1} P(x) =&\Delta_h^{n+1}(A^1 + A^2 + \cdots + A^n)(x)\\ =& \, 0\end{aligned}$$

for all $x, h \in G$.

For uniqueness of P, let $P^* : G \to B$ be such that

$$\Delta_{h_1,h_2,\ldots,h_{n+1}}^{n+1} P^*(x) = 0$$

for all $x, h_1, h_2, \ldots, h_{h_{n+1}} \in G$ and there exists δ such that

$$\|f(x) - f(e) - P^*(x)\| \leq \delta$$

for all $x \in G$.

Let $x \in G$ and remind that G_x is a commutative semigroup under the restriction of $\circ$ on $G_x \times G_x$. Observe that the restriction f_x of f on G_x satisfies the assumption of Theorem 3.7. Similarly, the restrictions P_x of P and P_x^* of P^* on G_x also satisfy

$$\begin{aligned}\Delta_h^{n+1} P_x(y) &= 0\\ \|f_x(y) - f(e) - P_x(y)\| &\leq 2\left(\prod_{i=1}^{n} \frac{2^i}{2^i - 1}\right) n!\epsilon\\ \Delta_h^{n+1} P_x^*(y) &= 0\\ \text{and} \quad \|f_x(y) - f(x) - P_x^*(y)\| &\leq 2\left(\prod_{i=1}^{n} \frac{2^i}{2^i - 1}\right) n!\delta\end{aligned}$$

for all $y \in G_x$. Since P_x is unique, $P(x) = P_x(x) = P_x^*(x) = P^*(x)$.

Since x is arbitrary, we can conclude that $P = P^*$. □

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