การวางนัยทั่วไปของตัววัดการขึ้นต่อกันของชไวเซอร์และวูลฟ์ฟ์

นางสาววรรษมล จันใต้

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต
สาขาวิชาคณิตศาสตร์ ภาควิชาคณิตศาสตร์และวิทยาการคอมพิวเตอร์
คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย
ปีการศึกษา 2558

GENERALIZATIONS OF SCHWEIZER-WOLFF MEASURE OF DEPENDENCE

Miss Wasamon Jantai

A Thesis Submitted in Partial Fulfillment of the Requirements
for the Degree of Master of Science Program in Mathematics
Department of Mathematics and Computer Science
Faculty of Science
Chulalongkorn University
Academic Year 2015

Thesis Title	GENERALIZATIONS OF SCHWEIZER-WOLFF MEASURE OF DEPENDENCE
By	Miss Wasamon Jantai
Field of Study	Mathematics
Thesis Advisor	Associate Professor Songkiat Sumetkijakan, Ph.D.

Accepted by the Faculty of Science, Chulalongkorn University in Partial Fulfillment of the Requirements for the Master's Degree

. . . . Dean of the Faculty of Science

(Associate Professor Polkit Sangvanich, Ph.D.)

THESIS COMMITTEE

. . . . Chairman

(Associate Professor Wicharn Lewkeeratiyutkul, Ph.D.)

. . . . Thesis Advisor

(Associate Professor Songkiat Sumetkijakan, Ph.D.)

. . . . Examiner

(Assistant Professor Nattakarn Chaidee, Ph.D.)

. . . . External Examiner

(Assistant Professor Pattrawut Chansangiam, Ph.D.)

วรรษมล จันใต้ : การวางนัยทั่วไปของตัววัดการขึ้นต่อกันของชไวเซอร์และวูลฟ์ฟ์. (GENERALIZATIONS OF SCHWEIZER-WOLFF MEASURE OF DEPENDENCE) อ.ที่ปรึกษาวิทยานิพนธ์หลัก: รศ.ดร.ทรงเกียรติ สุเมธกิจการ, 73 หน้า.

เราเสนอการวางนัยทั่วไปของตัววัดการขึ้นต่อกันของชไวเซอร์และวูลฟ์ฟ์ในสามรูปแบบ รูปแบบที่หนึ่งเรานิยามฟังก์ชัน $\sigma_\varphi(X,Y)$ ของตัวแปรสุ่ม X และตัวแปรสุ่ม Y ที่มีฟังก์ชันการแจกแจงต่อเนื่อง ซึ่งถูกนิยามเป็นนอร์ม L^1 แบบบรรทัดฐานของ $\varphi(|C_{X,Y}-\Pi|)$ เมื่อ $C_{X,Y}$ เป็นคอปูลาของตัวแปรสุ่มของ X และ Y และให้เงื่อนไขเพียงพอสำหรับฟังก์ชัน φ ที่ทำให้ σ_φ เป็นตัววัดการขึ้นต่อกันที่สอดคล้องกับเซตเดียวกับสมบัติของชไวเซอร์และวูลฟ์ฟ์ รูปแบบที่สองเรานิยาม $\sigma^*_\varphi(X,Y)$ เป็นขอบเขตบนค่าน้อยสุดของ $\sigma_\varphi(f(X),g(Y))$ บนฟังก์ชันหนึ่งต่อหนึ่งที่เมเชอเรเบิลแบบโบเรล f และ g และจากนั้น σ^*_φ กลายเป็นตัววัดการขึ้นต่อกันที่สอดคล้องบนเซตเดียวกับสมบัติของเรนยี รูปแบบที่สามเรานิยามตัววัดการขึ้นต่อกันของชไวเซอร์และวูลฟ์ฟ์ $\sigma_{\frac{M+W}{2}}$ บนเซตของคอปูลาซึ่ง $\sigma_{\frac{M+W}{2}}$ ถูกจัดประเภทเดียวกับตัววัดการขึ้นต่อกันของชไวเซอร์และวูลฟ์ฟ์แบบดั้งเดิมซึ่งถูกนิยามเป็นนอร์ม L^1 แบบบรรทัดฐานของ $C-\Pi$ เมื่อเทียบกับเมเชอร์ที่ถูกเหนี่ยวนำโดย $\frac{M+W}{2}$ เมื่อ C เป็นคอปูลา Π เป็นคอปูลาอิสระ M และ W เป็นคอปูลาทางเดียวร่วมกัน และคอปูลาทางเดียวตรงข้าม ตามลำดับ

ภาควิชา	คณิตศาสตร์และวิทยาการคอมพิวเตอร์	ลายมือชื่อนิสิต	....
สาขาวิชา	คณิตศาสตร์	ลายมือชื่อ อ.ที่ปรึกษาหลัก	....
ปีการศึกษา	2558		

$\#\#$ 5672080023 : MAJOR MATHEMATICS
KEYWORDS : MEASURE OF DEPENDENCE/ SCHWEIZER-WOLFF/RÉNYI'S POSTULATES

WASAMON JANTAI : GENERALIZATIONS OF SCHWEIZER-WOLFF MEASURE OF DEPENDENCE. ADVISOR: ASSOC. PROF. SONGKIAT SUMETKIJAKAN, Ph.D., 73 pp.

We propose the generalizations of Schweizer-Wolff measure of dependence in three ways. Firstly, we define a function $\sigma_\varphi(X,Y)$ for all random variables X and Y of which distribution functions are continuous as the normalized L^1-norm of $\varphi(|C_{X,Y}-\Pi|)$ where $C_{X,Y}$ is the copula of X,Y, and give a set of sufficient conditions for a function φ leading σ_φ to be a measure of dependence satisfying the same set of Schweizer-Wolff's properties. Secondly, we define a function $\sigma^*_\varphi(X,Y)$ as the supremum of $\sigma_\varphi(f(X),g(Y))$ over all injective Borel measurable functions f,g, and σ^*_φ then becomes a measure of dependence satisfying the same set of Rényi's properties. Thirdly, we define a measure of dependence $\sigma_{\frac{M+W}{2}}$ on the class of bivariate copulas, which is classified as the original Schweizer-Wolff measure of dependence, as the normalized L^1-norm of $C-\Pi$ with respect to the measure induced by $\frac{M+W}{2}$ where C is a copula, Π is the independent copula, and M and W are the comonotonic copula and the countermonotonic copula, respectively.

Department:	Mathematics and Computer Science	Student's Signature	....
Field of Study:	Mathematics	Advisor's Signature	....
Academic Year:	2015		

ACKNOWLEDGEMENTS

I would first like to thank my thesis advisor Associate Professor Dr. Songkiat Sumetkijakan for his invaluable advice, patience, motivation and continuous support throughout my research. During my master studies, both his guidance and teaching helped me to overcome many crisis situations and achieve this thesis.

I would also like to acknowledge Associate Professor Dr. Wicharn Lewkeeratiyutkul, the chairman, Assistant Professor Dr. Nattakarn Chaidee and Assistant Professor Dr. Pattrawut Chansangiam, the committee members, for their very insightful comments and hard questions on this thesis.

My sincere thanks also goes to my parents for supporting me throuhtout my life. Also I thank my friends for consistent encouragement me throughout the period of this research. In particular, I am grateful to copula master students for the stimulating discussions.

Last but not least, I would like to express my sincere gratitude to the Development and Promotion of Science and Technology Talents Project (DPST) for financial aid.

CONTENTS

CHAPTER I

INTRODUCTION

One of the interesting topics in probability and statistics is studying a relationship between two random variables or two data sets of which obvious application is in making economic decisions. A tool chosen to study a relation between random variables or data sets is a factor vastly affecting its reliability. When copulas were introduced, they have been studied widely in several branches of mathematics and economics. With abilities to detect independence and dependence structure between two random variables and to be represented via joint distribution function of random variables by Sklar's theorem, copulas have been parts of many popular tools with great potentials in quantifying dependence between random variables. We will study a popular tool called a measure of dependence. Since there are many levels of indicating dependence information between two random vaiables, it leads to many definitions of being a measure of dependence. Only Rényi's definition [19] and Schweizer and Wolff's definition [22] will be discussed in this thesis.

The motivation of this thesis began with studying a mapping σ on the set of all copulas defined in [22] by

$$\sigma(C) = \frac{\int_{I^2} |C - \Pi| d\lambda_2}{\int_{I^2} |M - \Pi| d\lambda_2} \tag{1.1}$$

where I denotes $[0, 1]$, λ_2 denotes the Lebesgue measure on I^2, and Π and M are the independent copula and the comonotonic copula, respectively. In [22], σ was proved to be a measure of dependence according to Schweizer and Wolff's definition. In [12], Edwards and Mikusiński introduced D_4-invariant copulas of which some examples are Π and $\frac{M+W}{2}$ where W is the countermonotonic copula. They showed that a function τ on the class of all copulas defined by

$$\tau(C) = \frac{\int_{I^2} (C - \Pi) dA}{\int_{I^2} (M - \Pi) dA} \tag{1.2}$$

is a measure of concordance if A is D_4-invariant and the converse also holds. Hence, D_4 denotes the dihedral group of the square I^2.

The aim of this research is to generalize σ in three ways. Firstly, we define a function σ_φ on the class of all copulas via

$$\sigma_\varphi(C) = \frac{\int_{I^2} \varphi(|C-\Pi|)d\lambda_2}{\int_{I^2} \varphi(|M-\Pi|)d\lambda_2} \tag{1.3}$$

where φ is a nonnegative real function on I. We give a set of sufficient conditions for φ, which makes σ_φ a measure of dependence in the sense of Schweizer and Wolff. Secondly, we define $\sigma^*_\varphi(X,Y)$ for all random variables X and Y whose distribution functions are continuous by

$$\sigma^*_\varphi(X,Y) = \sup_{f,g\in\mathcal{I}} \sigma_\varphi(C_{f(X),g(Y)}) = \sup_{f,g\in\mathcal{I}} \frac{\int_{I^2} \varphi(|C_{f(X),g(Y)}-\Pi|)d\lambda_2}{\int_{I^2} \varphi(|M-\Pi|)d\lambda_2} \tag{1.4}$$

where $\mathcal{I}$ is the set of all injective Borel measurable functions on $\mathbb{R}$. We show that σ^*_φ is a measure of dependence in the sense of Rényi. Finally, given a copula A, we define σ_A on the class of bivariate copulas by

$$\sigma_A(C) = \frac{\int_{I^2} |C-\Pi|dA}{\int_{I^2} |M-\Pi|dA}. \tag{1.5}$$

Our attempt to characterize all copulas A for which σ_A is a measure of dependence is not successful due to the difficulty in proving that σ_A is bounded by 1. We focused only on $\sigma_{\frac{M+W}{2}}$. We are able to show that $\sigma_{\frac{M+W}{2}}$ is a measure of dependence in the sense of Schweizer and Wolff.

In Chapter II, we will give a brief introduction to copulas. Some background information of a measure of dependence is discussed. Moreover, selected properties in both measure theory and probability theory are also given, some of which are fundamental tools used to prove our main results.

In Chapter III, we give a set of sufficient conditions for φ which makes σ_φ defined in (1.3) a Schweizer and Wolff's measure of dependence:

Theorem 1.1. *The function σ_φ satisfies the following properties for all random variables X and Y whose distribution functions are continuous.*

(i) $\sigma_\varphi(C_{X,Y}) = \sigma_\varphi(C_{Y,X})$.

(ii) $0 \leq \sigma_\varphi(C_{X,Y}) \leq 1$.

(iii) $\sigma_\varphi(C_{X,Y}) = 0$ *if and only if* X *and* Y *are independent.*

(iv) $\sigma_\varphi(C_{X,Y}) = 1$ *if and only if* $Y = f(X)$ *for some strictly monotone function* f.

(v) $\sigma_\varphi(C_{f(X),g(Y)}) = \sigma_\varphi(C_{X,Y})$ *for all strictly monotone functions* f *and* g.

(vi) *If* (X, Y) *and* (X_n, Y_n), $n = 1, 2, \ldots,$ *are pairs of random variables with joint distribution functions* H *and* H_n, *respectively, and if* (X_n, Y_n) *converges in distribution to* (X, Y), *then* $\lim_{n\to\infty} \sigma_\varphi(C_{X_n,Y_n}) = \sigma_\varphi(C_{X,Y})$.

In Chapter IV, we apply the results of σ_φ to prove that σ^*_φ defined in (1.4) is a measure of dependence in the sense of Rényi:

Theorem 1.2. *The function* σ^*_φ *satisfies the following properties for all random variables* X *and* Y *whose distribution functions are continuous.*

(i) $\sigma^*_\varphi(X, Y) = \sigma^*_\varphi(Y, X)$.

(ii) $0 \leq \sigma^*_\varphi(X, Y) \leq 1$

(iii) $\sigma^*_\varphi(X, Y) = 0$ *if and only if* X *and* Y *are independent.*

(iv) $\sigma^*_\varphi(X, Y) = 1$ *if* $Y = f(X)$ *or* $X = g(Y)$ *for some Borel measurable functions* f, g.

(v) $\sigma^*_\varphi(f(X), g(Y)) = \sigma^*_\varphi(X, Y)$ *for all Borel measurable injective functions* f *and* g.

In Chapter V, we investigate the mapping σ_A when A is a D_4-invariant copula. We also show that $\sigma_{\frac{M+W}{2}}$, as defined in (1.5) for $A = \frac{M+W}{2}$, is a measure of dependence in the sense of Schweizer and Wolff:

Theorem 1.3. *The function* $\sigma_{\frac{M+W}{2}}$ *satisfies the following properties for all random variables* X *and* Y *whose distribution functions are continuous.*

(i) $\sigma_{\frac{M+W}{2}}(C_{X,Y}) = \sigma_{\frac{M+W}{2}}(C_{Y,X})$.

(ii) $0 \leq \sigma_{\frac{M+W}{2}}(C_{X,Y}) \leq 1$.

(iii) $\sigma_{\frac{M+W}{2}}(C_{X,Y}) = 0$ *if* X *and* Y *are independent.*

(iv) $\sigma_{\frac{M+W}{2}}(C_{X,Y}) = 1$ *if* $Y = g(X)$ *for some strictly monotone function* g.

(v) $\sigma_{\frac{M+W}{2}}(C_{f(X),g(Y)}) = \sigma_{\frac{M+W}{2}}(C_{X,Y})$ *for all strictly monotone functions* f *and* g.

(vi) If (X, Y) *and* (X_n, Y_n), $n = 1, 2, \ldots,$ *are pairs of random variables whose joint distribution functions are* H *and* H_n, *respectively, and if* (X_n, Y_n) *converges in distribution to* (X, Y), *then* $\lim_{n\to\infty} \sigma_{\frac{M+W}{2}}(C_{X_n,Y_n}) = \sigma_{\frac{M+W}{2}}(C_{X,Y})$.

CHAPTER II

PRELIMINARIES

In this chapter, the main purpose is to give a brief introduction to theory of copulas (see [3], [7], [17]). By Sklar's Theorem, a copula can be viewed as a joint distribution function. Possessing many nice properties, copulas play an important role in financial applications. Moreover, a copula can induce a probability measure on the Borel subsets of $[0,1]^2$ by using Carathéodory-Hahn extension theorem. We will start with some essential definitions and some basic properties from measure theory and probability theory in section 2.1 and 2.2. See [9], [14], [16], [20] for an introduction to measure theory and [1], [15], [18], [24] for an introduction to probability theory.

We use notations $\bar{\mathbb{R}}$ and $\bar{\mathbb{R}}^2$ for $[-\infty,\infty]$ and $[-\infty,\infty]\times[-\infty,\infty]$, respectively. The closed interval $[0,1]$ will be denoted by I, hence the Cartesian product $[0,1]^2$ is denoted by I^2. A *bounded interval* is a set belonging to $\{[a,b],[a,b)(a,b],(a,b) : a,b \in \mathbb{R}\}$. The set $B \subseteq \bar{\mathbb{R}}^2$ is said to be a *closed rectangle* if it is a Cartesian product $[a,b] \times [c,d]$ where $a,b,c,d \in \mathbb{R}$.

2.1 Measure Theory

We begin with fundamental definitions leading to measures, a generalization of functions which can quantify length, area and volume.

Definition 2.1. Let X be a nonempty set. A nonempty collection $\mathcal{F}$ of subsets of X is called an *algebra* if the following properties hold:

(i) If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$.

(ii) It is closed under finite unions.

The algebra $\mathcal{F}$ is called a *σ-algebra* if it is closed under countable unions. Then the ordered pair $(X, \mathcal{F})$ is said to be a *measurable space* and every member of $\mathcal{F}$ is called an *$\mathcal{F}$-measurable set*.

Remark 2.2. Let X be a set and $\mathcal{C} \subseteq \mathcal{P}(X)$. If $\mathcal{F}_\alpha$ is a σ-algebra on X for all $\alpha \in \Lambda$, then $\cap_{\alpha\in\Lambda}\mathcal{F}_\alpha$ is a σ-algebra. Moreover, there exists a smallest σ-algebra on X containing all of the elements in $\mathcal{C}$, which is called the *σ-algebra on X generated by $\mathcal{C}$* and is denoted by $\sigma(\mathcal{C})$.

Definition 2.3. Let X be a nonempty set. A collection $\mathcal{T}$ of subsets of X containing $\varnothing$ and X is called a *topology on X* if the following properties hold:

(i) The union of arbitrary members of $\mathcal{T}$ is in $\mathcal{T}$.

(ii) The intersection of a finite number of members of $\mathcal{T}$ is in $\mathcal{T}$.

Then the ordered pair $(X, \mathcal{T})$ is said to be a *topological space* and every member of $\mathcal{T}$ is called an *open set*. The σ-algebra generated by a topology $\mathcal{T}$ is called the *Borel σ-algebra on X*, denoted by $\mathcal{B}(X)$.

Remark 2.4. Let X be a nonempty set, and $X' \subseteq X$.

(i) If $\mathcal{F}$ is σ-algebra on X, then $\mathcal{F}' = \{A \cap X' : A \in \mathcal{F}\}$ is σ-algebra on X'.

(ii) If the σ-algebra on X is generated by $\mathcal{C}$, then the σ-algebra on X' is generated by $\mathcal{C}' = \{A \cap X' : A \in \mathcal{C}\}$.

Example 2.5. The Borel σ-algebra on $\mathbb{R}^2$ is also generated by $\{[a, b] \times [c, d] : a, b, c, d \in \mathbb{R}$ such that $a \leq b$ and $c \leq d\}$. Consequently, the Borel σ-algebra on I^2 is generated by $\{[a, b] \times [c, d] : a, b, c, d \in I$ such that $a \leq b$ and $c \leq d\}$.

Definition 2.6. Let $(X, \mathcal{F})$ be a measurable space. A nonnegative set function μ on $\mathcal{F}$ is said to be a *measure* if it satisfies the following properties:

(i) $\mu(\varnothing) = 0$.

(ii) (Countably additive property) If $\{A_n\}_{n\in\mathbb{N}}$ is a sequence of disjoint sets in $\mathcal{F}$, then

$$\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n).$$

Then the triple $(X, \mathcal{F}, \mu)$ is called a *measure space.* If a statement P is true on a complement of some measurable set $N \subseteq X$ such that $\mu(N) = 0$, then P is said to be true *μ-almost everywhere,* denoted by μ-a.e.

Definition 2.7. Let X be a nonempty set and $\mathcal{P} \subseteq \mathcal{P}(X)$ be an algebra. Then a function $\mu : \mathcal{P} \to [0, \infty]$ is called a *premeasure* if the following conditions hold:

(i) $\mu(\varnothing) = 0$.

(ii) If $\{A_n\}_{n\in\mathbb{N}}$ is a sequence of elements in $\mathcal{P}$ such that $A_i \cap A_j = \varnothing$ for all $i \neq j$ and $\cup_{n=1}^{\infty} A_n \in \mathcal{P}$, then $\mu(\cup_{n=1}^{\infty} A_k) = \sum_{n=1}^{\infty} \mu(A_n)$.

The premeasure μ is called σ-finite if there is a sequence $\{A_k\}_{k=1}^{\infty}$ of elements in $\mathcal{P}$ such that $\cup_{k=1}^{\infty} A_k = X$ and $\mu(A_k) < \infty$ for all $k \in \mathbb{N}$.

The following is required to define a function quantifying the length, which is called a Lebesgue measure on $\mathbb{R}$.

Definition 2.8. Let $\mathcal{E} = \{(a, b), \varnothing : a, b \in \mathbb{R}\}$. Define m^* on the set of all subsets of $\mathbb{R}$, called *Lebesgue outer measure on* $\mathbb{R}$, by

$$m^*(A) = \inf\{\sum_{n=1}^{\infty} l(E_n) : E_n \in \mathcal{E}, A \subseteq \cup_{n=1}^{\infty} E_n\}$$

where $l(a, b) = b - a$ for all $a, b \in \mathbb{R}$.

Definition 2.9. Let m^* be a Lebesgue outer measure on $\mathbb{R}$ and $B \subseteq \mathbb{R}$. A set B is called *Lebesgue measurable* if

$$m^*(E) = m^*(E \cap B) + m^*(E \cap B^c)$$

for all $E \subseteq \mathbb{R}$.

Theorem 2.10. *Let m^* be Lebesgue outer measure on $\mathbb{R}$ and $\mathcal{E}$ be a collection of all Lebesgue measurable subsets of $\mathbb{R}$. Then the following properties hold:*

(i) $\mathcal{E}$ *is a* σ*-algebra containing* $\mathcal{B}(\mathbb{R})$.

(ii) *The restriction of* m^* *to* $\mathcal{E}$ *is a measure on* $\mathcal{E}$*, called Lebesgue measure on* $\mathbb{R}$.

Definition 2.11. Let X be a set and $\mathcal{P} \subseteq \mathcal{P}(X)$. Then $\mathcal{P}$ is called a *semiring* if the following conditions hold:

(i) If A and B belong to $\mathcal{P}$, then $A \cap B \in \mathcal{P}$.

(ii) If A and B are in $\mathcal{P}$, then $A \setminus B = \cup_{n=1}^{k} C_n$ for some $k \in \mathbb{N}$ such that $C_1, C_2, \ldots, C_k \in \mathcal{P}$.

Theorem 2.12 (Carathéodory-Hahn extension theorem)**.** *Let* X *be a set. If* μ *is a premeasure on a semiring* $\mathcal{P} \subseteq \mathcal{P}(X)$*, then a measure* $\bar{\mu}$ *induced by* μ *is an extension of* μ*. Moreover, if* μ *is* σ*-finite, then* $\bar{\mu}$ *is the unique measure extending* μ.

Example 2.13 ([20])**.** A collection of all products of two bounded intervals of I^2 is a semiring.

Definition 2.14. Let $(X, \mathcal{F})$ and $(Y, \mathcal{F}')$ be measurable spaces. A function $f : X \to Y$ is called $(\mathcal{F}, \mathcal{F}')$*-measurable* if $f^{-1}(E) \in \mathcal{F}$ for all $E \in \mathcal{F}'$. If $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$, then a $(\mathcal{B}(\mathbb{R}), \mathcal{B}(\mathbb{F}))$-measurable function is said to be *Borel measurable.*

Proposition 2.15. *Let* $(X, \mathcal{F})$ *and* $(Y, \mathcal{N})$ *be measurable spaces. If* $f : X \to Y$ *and* $\mathcal{C}$ *generates* $\mathcal{N}$*, then* f *is* $(\mathcal{F}, \mathcal{N})$*-measurable if and only if* $f^{-1}(E) \in \mathcal{F}$ *for all* $E \in \mathcal{C}$.

Note that If f is $(\mathcal{M}, \mathcal{N})$-measurable and g is $(\mathcal{N}, \mathcal{O})$-measurable, then $g \circ f$ is $(\mathcal{M}, \mathcal{O})$-measurable.

Corollary 2.16. *Let* $(X, \mathcal{B}(X))$ *and* $(Y, \mathcal{B}(Y))$ *be topological spaces. If* $f : X \to Y$ *is continuous, then* f *is* $(\mathcal{B}(X), \mathcal{B}(Y))$*-measurable.*

Proposition 2.17. *Let* $(X, \mathcal{F})$ *be a measurable space. If* $f, g : X \to \mathbb{R}$ *are measurable functions, then the functions* $f + g$ *and* fg *are measurable.*

Let $(X, \mathcal{F})$ be a measurable space and $E \subseteq X$. A function $\chi_E : X \to \mathbb{R}$ is said to be the *characteristic function of* E if it is defined by

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E. \end{cases}$$

The characteristic function of E is measurable if and only if $E \in \mathcal{F}$. Then a function $f : X \to \mathbb{R}$ is called a *simple function* if it is a measurable function written by $f = \sum_{i=1}^{n} a_i \chi_{A_i}$ for some real numbers $a_1, a_2, \ldots, a_n$ and for some pairwise disjoint measurable sets $A_1, A_2, \ldots, A_n$ such that $X = \cup_{i=1}^{n} A_i$. If $f, g : X \to \mathbb{R}$ are simple functions, then $f + g, fg$ are also simple functions.

Theorem 2.18. *Let $(X, \mathcal{F})$ be a measurable space, and $f : X \to \mathbb{R}$ nonnegative measurable function. Then there exists a sequence $\{s_n\}_{n\in\mathbb{N}}$ of nonnegative increasing simple functions such that $s_n \leq f$ for all $n \in \mathbb{N}$ and s_n converges pointwise to f.*

Remark 2.19. If $f : I^2 \to I$ is a continuous function and $f = 0$ a.e., then $f = 0$.

The easy way to understand the concept for an integral of a function f with respect to a measure is to view it as the area under the graph of f. In general idea, it is shown as follows: let $(X, \mathcal{F}, \mu)$ be a fixed measure space.

Definition 2.20. Let s be a nonnegative simple function. If $s = \sum_{i=1}^{n} a_i \chi_{A_i}$ where $a_1, a_2, \ldots, a_n$ are nonnegative real values and $A_1, A_2, \ldots, A_n$ such that $X = \cup_{i=1}^{n} A_i$ are pairwise disjoint measurable sets, then the integral of s with respect to the measure μ is defined by

$$\int_X s d\mu = \sum_{i=1}^{n} a_i \mu(A_i).$$

Definition 2.21. Let $f : X \to [0, \infty]$ be a measurable function. The integral of f with respect to the measure μ is defined by

$$\int_X f d\mu = \sup\{\int_X s d\mu : s \text{ is simple and } 0 \leq s \leq f\}.$$

If $E \in \mathcal{F}$, then the integral of a measurable function $f\chi_E$ with respect to the measure μ is defined as

$$\int_E f d\mu = \int_X f \chi_E d\mu.$$

Proposition 2.22. *Let $f : X \to [0,\infty]$ be measurable, and $A \in \mathcal{F}$. If $\int_A f d\mu = 0$, then $f(x) = 0$ μ-a.e. on A.*

The following theorem shows when the order between limit and integral can be changed.

Theorem 2.23 (The Monotone Convergence Theorem)**.** *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of nonnegative measurable functions on X. If*

(i) $f_1(x) \leq f_2(x) \leq \cdots \leq \infty$ for every $x \in X$ and

(ii) $\lim_{n\to\infty} f_n(x) = f(x)$ for each $x \in X$,

then $\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu$.

Theorem 2.24. *If $f, g : X \to [0,\infty]$ are measurable functions, then*

$$\int_X (f+g)d\mu = \int_X f d\mu + \int_X g d\mu.$$

If $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of nonnegative measurable functions on X, then

$$\int_X \sum_{n=1}^{\infty} f_n d\mu = \sum_{n=1}^{\infty} \int_X f_n d\mu.$$

Definition 2.25. Let f be a real valued measurable function on X. Define $f^+ = f\chi_{\{f\geq 0\}}$ and $f^- = -f\chi_{\{f<0\}}$. If $\int_X f^+ d\mu < \infty$ or $\int_X f^- d\mu < \infty$, we will define the integral of f with respect to μ by

$$\int_X f d\mu = \int_X f^+ d\mu - \int_X f^- d\mu.$$

If $\mu(X) < \infty$, then μ is called *finite.*

Remark 2.26. Let $n \in \mathbb{N}$. If $\mu_1, \mu_2, \ldots, \mu_n$ be finite measures on X, then $\mu = \sum_{i=1}^{n} \mu_i$ is also a measure on X. Moreover, $\int_X f d\mu = \sum_{i=1}^{n} \int_X f d\mu_i$ for all nonnegative measurable functions f.

A measurable function f on X is said to be *integrable with respect to* μ if $\int_X |f| d\mu < \infty$.

The following theorem shows that when the order of limit and integral can be switched.

Theorem 2.27 (The Dominated Convergence Theorem (DCT)). *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of integrable functions on X. If*

(i) $\lim_{n\to\infty} f_n(x) = f(x)$ *for each $x \in X$ and*

(ii) $|f_n| \leq g$ for some nonnegative integrable function g for all $n \in \mathbb{N}$,

then f is an integrable function and $\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu$.

Definition 2.28. Let $(X, \mathcal{F})$ be a measurable space. A function $\mu : \mathcal{F} \to [-\infty, \infty]$ is said to be a signed measure if it satisfies the following properties:

(i) $\mu(\varnothing) = 0$

(ii) $\operatorname{Ran} \mu$ is a subset of either $(-\infty, \infty]$ or $[-\infty, \infty)$.

(iii) If $\{A_n\}_n \in \mathbb{N}$ is a sequence of pairwise disjoint measurable sets in $\mathcal{F}$, then

$$\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n).$$

Remark 2.29. If μ_1, μ_2 are finite measures on X, then $\mu = \mu_1 - \mu_2$ is a signed measure on X. Moreover, $\int_X f d\mu = \int_X f d\mu_1 - \int_X f d\mu_2$ for all nonnegative measurable functions f.

For each $n \in \mathbb{N}$, λ_n denotes the Lebesgue measure on $\mathbb{R}^n$.

Theorem 2.30 (Mean Value Theorem). *Let $A \subseteq \mathbb{R}^n, f : A \to \mathbb{R}$ and $0 < \lambda_n(A) < \infty$. If A is connected and f is a continuous bounded function, then*

$$\int_A f d\lambda_n = f(x)\lambda_n(A)$$

for some $x \in A$.

Theorem 2.31. *Let $(X, \mathcal{F}_1, \mu)$ be a measure space, $(Y, \mathcal{F}_2)$ be another measurable space and $g : X \to Y$ be a measurable function. Define a measure ν on Y by $\nu = \mu(g^{-1}(B))$ for all measurable sets $B \subseteq Y$. If $f : Y \to \bar{\mathbb{R}}$ is measurable, then* $\int_X (f \circ g) d\mu = \int_Y f d\nu$.

2.2 Probability Theory

In this section, we will study some properties of a specific case of a measurable space, which is called a probability space.

Definition 2.32. Let $(X, \mathcal{F})$ be a measurable space. A measure $\mathbb{P}$ on X is called a *probability measure* if $\mathbb{P}(X) = 1$.

Then a triple $(X, \mathcal{F}, \mathbb{P})$ is called a *probability space* and every member of $\mathcal{F}$ is called an *event.*

Definition 2.33. Let X be a set and $\mathcal{P}$ be a collection of subsets of X containing $\varnothing$. A collection $\mathcal{P}$ is called a *π-system* if $A \cap B \in \mathcal{P}$ for all $A, B \in \mathcal{P}$. A family $\mathcal{P}$ is said to be a *λ-system* if it satisfies the following properties:

(i) If $A \in \mathcal{P}$, then $A^c \in \mathcal{P}$.

(ii) If $\{A_n\}_{n\in\mathbb{N}}$ is a sequence of pairwise disjoint sets in $\mathcal{P}$, then $\cup_{n=1}^{\infty} A_n \in \mathcal{P}$.

Lemma 2.34. *Let $\mathcal{P}$ be a π-system. If P and Q are probability measures on $(X, \sigma(\mathcal{P}))$ such that $P = Q$ on $\mathcal{P}$, then $P = Q$ on $\sigma(\mathcal{P})$.*

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a fixed probability space.

Definition 2.35. A function $X : \Omega \to \mathbb{R}$ is called a *random variable* if X is a $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable function.

Definition 2.36. Let X be a random variable. A probability measure $\mathbb{P}_X : \mathcal{B}(\mathbb{R}) \to \mathbb{R}$ defined by

$$\mathbb{P}_X(A) := \mathbb{P}(X^{-1}(A))$$

for all $A \in \mathcal{B}(\mathbb{R})$ is called the *probability law of X.*

Definition 2.37. Let X be a random variable on Ω, and

$$F_X(x) := \mathbb{P}(\{\omega : X(\omega) \leq x\})$$

for each $x \in \mathbb{R}$. Then F_X is said to be the *distribution function of X.*

Theorem 2.38. *Let F be the distribution function of a random variable X. Then F has the following properties:*

(i) *F is nondecreasing; that is, if $x_1 < x_2$, then $F(x_1) \leq F(x_2)$.*

(ii) $\lim_{x\to\infty} F(x) = 1$ *and* $\lim_{x\to-\infty} F(x) = 0$.

(iii) *F is a right continuous function; that is* $\lim_{y\to x^+} F(y) = F(x)$.

Definition 2.39. Let X be a random variable. Its distribution function is continuous on $\mathbb{R}$ if and only if $\mathbb{P}(X = x) = 0$ for all $x \in \mathbb{R}$.

Definition 2.40. The random variable X on Ω is called *continuous* if its distribution can be written as follows: for all $x \in \mathbb{R}$,

$$F(x) = \int_{-\infty}^{x} f(u)du$$

for some integrable function $f : \mathbb{R} \to [0, \infty)$, which is called a *probability density function of X.*

Example 2.41. (i) A continuous random variable X is called a *uniform random variable on* $[a, b]$ if its probability density function can be written as follows:

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq x \leq b \\ 0 & \text{otherwise.} \end{cases}$$

The uniform random variable X on $[a, b]$ is denoted by $X \sim U[a, b]$.

(ii) Let $\mu \in \mathbb{R}$ and $\sigma > 0$. A continuous random variable X is called a *normal random variable* with parameters μ and σ^2 if its probability density function can be written as follows:

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

for all $-\infty < x < \infty$. The normal random variable X with parameters μ and σ^2 is denoted by $X \sim \mathrm{N}(\mu, \sigma^2)$. If $X \sim \mathrm{N}(0, 1)$, then it is called a *standard normal random variable.*

Definition 2.42. Fix $n \in \mathbb{N}$. A function $X : \Omega \to \mathbb{R}^n$ is called a *n-dimentional random vector* if X is a $(\mathcal{F}, \mathcal{B}(\mathbb{R}^n))$-measurable function.

Definition 2.43. Fix $n \in \mathbb{N}$. If X is a n-dimentional random vector on Ω, then F_X defined by

$$F_X(x) := \mathbb{P}(\{\omega : X_1(\omega) \le x_1, X_2(\omega) \le x_2, \ldots, X_n(\omega) \le x_n\})$$

for all $x = (x_1, x_2, \ldots, x_n)$, is called the *joint distribution function of the random vector X.*

Theorem 2.44. *Let F_X be a joint distribution function of a random vector X. Then F has the following properties:*

(i) $F_X(\cdot, \cdot, \ldots, \cdot)$ *is nondecreasing and right continous for each of its arguments.*

(ii) $\lim\limits_{x_i \to \infty, i=1,2,\ldots,n} F_X(x_1, x_2, \ldots, x_n) = 1.$

(iii) For each $i = 1, 2, \ldots, n,$ $\lim\limits_{x_i \to -\infty} F_X(x_1, x_2, \ldots, x_n) = 0.$

If $X = (X_1, X_2, \ldots, X_n)$ is a random vector, then, for each i, the *marginal distribution function* of X_i is obtained from F_X by setting the components $x_j = \infty$ for all $i \ne j$.

Definition 2.45. Fix $k \in \mathbb{N}$. Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of k-random vectors associated with distribution functions F_n for all $n \in \mathbb{N}$ and X be another k-random vector associated with a distribution function F. If $\lim\limits_{n\to\infty} F_n(x) = F(x)$ for all continuity points x of F, then we say that X_n *converges in distribution to* X, denoted by $X_n \xrightarrow{d} X$.

Theorem 2.46. *Fix $k \in \mathbb{N}$. Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of k-random vectors associated with distribution functions F_n for all $n \in \mathbb{N}$ and X be another k-random vector associated with a distribution function F. Then $\lim_{n\to\infty} F_n(x) = F(x)$ for all continuity points x of F if and only if*

$$\lim_{n\to\infty} \int_{\mathbb{R}^k} f dF_n = \int_{\mathbb{R}^k} f dF$$

for all $f \in C_b(\mathbb{R}^k)$ where $C_b(\mathbb{R}^k)$ is the collection of all bounded continuous functions on $\mathbb{R}^k$.

2.3 A Copula and Its Fundamental Properties

In the beginning of this section, we will study some basic definitions and some properties leading to being a copula function.

Definition 2.47. A real valued function H is called a 2-*place real function* if its domain is a subset of $\bar{\mathbb{R}}^2$.

Definition 2.48. Let H be a 2-place real function whose domain is the Cartesian product of nonempty subsets S_1, S_2 in $\bar{\mathbb{R}}$. If $B := [x_1, x_2] \times [y_1, y_2]$ is a rectangle whose vertices are in $S_1 \times S_2$; that is, $(x_1, y_1), (x_1, y_2), (x_2, y_1), (x_2, y_2) \in S_1 \times S_2$, then we will define the H-volume of B by

$$V_H(B) = H(x_2, y_2) - H(x_2, y_1) - H(x_1, y_2) + H(x_1, y_1).$$

Definition 2.49. Let H be a 2-place real function. If $V_H(B) \geq 0$ for all rectangles B whose vertices are in $\operatorname{dom} H$, then H is 2-*increasing*.

Lemma 2.50. *Let H be a 2-increasing function whose domain is $S_1 \times S_2 \subseteq \bar{\mathbb{R}}^2$. If $x_1, x_2 \in S_1$ such that $x_1 \leq x_2$, and $y_1, y_2 \in S_2$ such that $y_1 \leq y_2$, then the following holds:*

(i) The function $t \mapsto H(t, y_2) - H(t, y_1)$ is nondecreasing on S_1.

(ii) The function $t \mapsto H(x_2, t) - H(x_1, t)$ is nondecreasing on S_2.

Definition 2.51. Let H be a function whose domain is $S_1 \times S_2 \subseteq \bar{\mathbb{R}}^2$. Suppose a_1 and a_2 are the least elements in S_1 and S_2, respectively. If $H(x, a_2) = 0 = H(a_1, y)$ for all $(x, y) \in S_1 \times S_2$, then H is *grounded*.

Lemma 2.52. *If H is a grounded 2-increasing function whose domain is $S_1 \times S_2 \subseteq \bar{\mathbb{R}}^2$, then H is nondecreasing for each of its arguments.*

Definition 2.53. Let H be a function whose domain is $S_1 \times S_2 \subseteq \bar{\mathbb{R}}^2$. Suppose b_1 and b_2 are the greatest elements in S_1 and S_2, respectively. Then margins F, G of H are defined by

$$F(x) = H(x, b_2) \text{ for all } x \in S_1$$

and

$$G(y) = H(b_1, y) \text{ for all } x \in S_2.$$

Lemma 2.54. *Let H be a grounded 2-increasing function whose domain is $S_1 \times S_2 \subseteq \bar{\mathbb{R}}^2$ and whose margins are F and G. Then*

$$|H(x_2, y_2) - H(x_1, y_1)| \leq |F(x_2) - F(x_1)| + |G(y_2) - G(y_1)|$$

for all $(x_1, y_1), (x_2, y_2) \in S_1 \times S_2$.

Definition 2.55. A function $C : I^2 \to I$ is called a *copula* if it satisfies the following properties:

(i) C is grounded and 2-increasing; that is, $C(u, 0) = C(0, v) = 0$ for all $u, v \in I$ and $V_C(B) \geq 0$ for all rectangles $B \subseteq I^2$.

(ii) $C(u, 1) = u$ and $C(1, v) = v$ for all $u, v \in I$.

Example 2.56. The following functions are all copulas.

(i) $M(u, v) = \min\{u, v\}$ for all $u, v \in I$.

(ii) $\Pi(u, v) = uv$ for all $u, v \in I$.

(iii) $W(u, v) = \max\{u + v - 1, 0\}$ for all $u, v \in I$.

(iv) Fixed $r \in [-1, 1]$. For all $u, v \in I$, $C_r^{Ga}(u, v; r) = \Phi_2(\Phi^{-1}(u), \Phi^{-1}(v); r)$ where $\Phi_2(h, k; r) = \int_{-\infty}^{h} \int_{-\infty}^{k} \frac{1}{2\pi\sqrt{1-r^2}} e^{-\frac{x^2 - 2rxy + y^2}{2(1-r^2)}} dydx$ and $\Phi(h) = \int_{-\infty}^{h} \frac{1}{2\pi} e^{-\frac{x^2}{2}} dx$ for all $h, k \in \mathbb{R}$. It is called a *Gaussian copula*.

Theorem 2.57. *Let C be any copula. Then*

$$W(u, v) \leq C(u, v) \leq M(u, v)$$

for all $u, v \in I$.

Then the copulas M and W are called the *Fréchet-Hoeffding upper bound* and the *Fréchet-Hoeffding lower bound*, respectively.

Theorem 2.58. *Let C be any copula. Then*

$$|C(u_2, v_2) - C(u_1, v_1)| \leq |u_2 - u_1| + |v_2 - v_1|$$

for all $u_1, u_2, v_1, v_2 \in I$. Consequently, C is Lipschitz and then uniformly continuous on I^2.

Definition 2.59. Let C be a copula, and $a \in I$. The functions

$$t \mapsto C(t, a) \text{ and } t \mapsto C(a, t) \text{ for all } t \in I$$

are said to be the *horizontal section of a copula C at a* and the *vertical section of a copula C at a*, respectively.

Corollary 2.60. *The horizontal and vertical sections of a copula C are nondecreasing and uniformly continuous on I.*

The following famous theorem reveals a fact that a copula function is associated with random variables and is indeed a joint distribution function.

Theorem 2.61 (Sklar's Theorem)**.** *For any random variables X and Y, the joint distribution function $F_{X,Y}$ of X, Y can be written in terms of a copula and its marginal distribution functions (F_X and F_Y, respectively) as follows: for all $(u, v) \in \mathbb{R}^2$,*

$$F_{X,Y}(u, v) = C(F_X(u), F_Y(v)) \tag{2.1}$$

for some a copula C. The copula C is unique on I^2 when F_X and F_Y are continuous, otherwise it is unique only on $\operatorname{Ran} F_X \times \operatorname{Ran} F_Y$.

Conversely, for any copula C and any marginal distribution functions F_X and F_Y, if the function $F_{X,Y}$ is defined by (2.1), *then it becomes a joint distribution function whose marginal distribution functions are F_X and F_Y.*

If X, Y are random variables whose their distribution functions are continuous, then a copula C which is associated with X, Y sometimes write $C_{X,Y}$ instead of C.

Remark 2.62. Every copula can be considered as a joint distribution function whose marginal distributions are continuous.

Let C be any copula and define a joint distribution H_C of C as follows:

$$H_C(u,v) = \begin{cases} 0 & \text{if } u < 0 \text{ or } v < 0 \\ C(u,v) & \text{if } (u,v) \in I^2 \\ x & \text{if } v > 1, u \in I \\ y & \text{if } v > 1, u \in I \\ 1 & \text{if } u > 1 \text{ and } v > 1. \end{cases}$$

One can prove that H_C is constructed from uniform random variables on I.

Definition 2.63. A 2-place real function C is said to be a *positively quadrant dependent (PQD)* copula if $C(u,v) \geq \Pi(u,v)$ for all $u,v \in I$. It is called a *negatively quadrant dependent (NQD)* copula if $C(u,v) \leq \Pi(u,v)$ for all $u,v \in I$.

Any copula C induces a probability measure μ_C on the Borel subsets of I^2 via

$$\mu_C([u_1,u_2] \times [v_1,v_2]) = V_C([u_1,u_2] \times [v_1,v_2]),$$

for all rectangles $[u_1,u_2] \times [v_1,v_2] \subseteq I^2$, by using Carathéodory-Hahn extension theorem. Let λ denote Lebesgue measure on I. A measure μ on I^2 is called *doubly stochastic* if $\mu(S \times I) = \mu(I \times S) = \lambda(S)$ for any Borel set S of I. One can prove that μ_C is doubly stochastic. If S is the union of all open subsets A of I^2 with $\mu_C(A) = 0$, then the complement of S is called the *support* of a copula C.

The following theorem shows that dependence structure can be captured by copulas, which is regardless of their marginal distribution functions.

Theorem 2.64 ([22], Theorem 2)**.** *Let X and Y be random variables whose their distribution functions are continuous. Then*

(i) $C_{X,Y} = \Pi$ *if and only if X and Y are independent.*

(ii) $C_{X,Y} = M$ *if and only if $Y = f(X)$ a.s. where f is strictly increasing a.e. on* $\operatorname{Ran} X$.

(iii) $C_{X,Y} = W$ *if and only if* $Y = f(X)$ *a.s. where* f *is strictly decreasing a.e. on* $\operatorname{Ran} X$.

The copula Π, M *and* W *are called the* independent copula, *the* comonotonic copula *and the* countermonotonic copula, *respectively.*

Observe that if X *is a random variable whose distribution is continuous, then a distribution function of* $f(X)$ *is continuous for all injective functions* f. *Let* f, g *be strictly monotone functions on* $\operatorname{Ran} X$ *and* $\operatorname{Ran} Y$, *respectively. Then a copula* $C_{f(X),g(Y)}$ *can be written in the form of* $C_{X,Y}$. *In particular, a copula is invariant under strictly increasing functions, which is shown by the following theorem.*

Theorem 2.65. *Let* X *and* Y *be random variables whose distribution functions are continuous. Then for all* $u, v \in I$,

(i) $C_{f(X),g(Y)}(u,v) = C_{X,Y}(u,v)$ *if* f *and* g *are both strictly increasing a.e. on* $\operatorname{Ran} X$ *and* $\operatorname{Ran} Y$, *respectively.*

(ii) $C_{f(X),g(Y)}(u,v) = u - C_{X,Y}(u, 1-v)$ *if* f *is strictly increasing a.e. on* $\operatorname{Ran} X$ *and* g *is strictly decreasing a.e. on* $\operatorname{Ran} Y$.

(iii) $C_{f(X),g(Y)}(u,v) = v - C_{X,Y}(1-u, v)$ *if* f *is strictly decreasing a.e. on* $\operatorname{Ran} X$ *and* g *is strictly increasing a.e. on* $\operatorname{Ran} Y$.

(iv) $C_{f(X),g(Y)}(u,v) = u + v - 1 + C_{X,Y}(1-u, 1-v)$ *if* f *and* g *are both strictly decreasing a.e. on* $\operatorname{Ran} X$ *and* $\operatorname{Ran} Y$, *respectively.*

We next review some basic definitions of abstract algebra which are necessary to define a D_4-invariant copula. For an introduction to abstract algebra, see [10].

Definition 2.66. Let S be a set. Then a function $* : S \times S \to S$ is called a *binary operation on* S.

Definition 2.67. Let $*$ be a binary operation on S.

(i) If $(a * b) * c = a * (b * c)$ for all $a, b, c \in S$, then $*$ is said to be *associative.*

(ii) If $a * b = b * a$ for all $a, b \in S$, then $*$ is said to be *commutative.*

(iii) An element $e \in S$ is called an *identity element* for $*$ if for all $a \in S$, $a * e = a = e * a$.

Definition 2.68. Let $*$ be a binary operation on a nonempty set S. If $*$ is associate, then the ordered pair $(S, *)$ is a *semigroup.* A semigroup $(S, *)$ equipped with an identity element is called a *monoid.*

Definition 2.69. Let $(G, *)$ be a monoid with an element e. Fixed $a \in G$. An element $a' \in G$ is called an *inverse of a* if $a * a' = e = a' * a$. Then $(G, *)$ is a *group* if every element in G has its inverse. Moreover, a group $(G, *)$ is *abelian* if $*$ is commutative.

Definition 2.70. Let $(G, *)$ be a group with an identity e. The *cardinality* of the set G is the order of a group G, which is denoted by $|G|$. Let $a \in G$. Then the *order* of a, denoted by $o(a)$, is the smallest positive integer $n \in \mathbb{N}$ such that $a^n = e$.

Proposition 2.71. *Let $(G, *)$ be a group. Then the following are true:*

(i) The identity of G is unique.

(ii) For all $a \in G$, the inverse of a is unique, which is denoted by a^{-1}.

Definition 2.72. Let $(G, *)$ be a group and a nonempty set H be a subset of G. If $(H, *)$ is a group, then it is called a *subgroup of G*, denoted by $H \leq G$.

Proposition 2.73. *Let $(G, *)$ be a group. The intersection of subgroups of G is a subgroup of G.*

Definition 2.74. Let $(G, *)$ be a group and $A \subseteq G$ such that $A \neq \varnothing$. Then *the subgroup of G generated by A*, denoted by $\langle A \rangle$, is the intersection of all subgroups of G containing A. Moreover, it is the smallest subgroup of G containing A.

Definition 2.75. For each $n \geq 2$, *the dihedral group of order $2n$* denoted by D_n is a group which is generated by two elements a, b such that the order of a is n, the order of b is 2 and $ba = a^{-1}b$. So we can see it is a group of rotations and reflections of a regular n-gons with n vertices where a is the rotation through angle $\frac{2\pi}{n}$ and b is the reflection.

Example 2.76. *The dihedral group of order* 8 on the unit square $[0,1]^2$, denoted by D_4, is $\{e, r, r^2, r^3, h, hr, hr^2, hr^3\}$ where e is the identity, h is the reflection about $x = \frac{1}{2}$, and r is a $90°$ counterclockwise rotation around the point $(\frac{1}{2}, \frac{1}{2})$.

Definition 2.77 ([12])**.** For $d \in D_4$, a *copula* C^d is defined by

$$C^d(u,v) = \mu_C(d([0,u] \times [0,v])),$$

which has a mass distribution on the the rectangle $d([0,u] \times [0,v])$ with respect to the doubly stochastic measure associated with C.

Example 2.78. For all rectangles $[u_1, u_2] \times [v_1, v_2]$ on I^2,

(i) $hr([u_1, u_2] \times [v_1, v_2]) = [v_1, v_2] \times [u_1, u_2]$

(ii) $hr^2([u_1, u_2] \times [v_1, v_2]) = [u_1, u_2] \times [1 - v_2, 1 - v_1]$

(iii) $h([u_1, u_2] \times [v_1, v_2]) = [1 - u_2, 1 - u_1] \times [v_1, v_2]$

(iv) $r^2([u_1, u_2] \times [v_1, v_2]) = [1 - u_2, 1 - u_1] \times [1 - v_2, 1 - v_1]$.

The following table displays a relationship between C^d and C for all $d \in D_4$.

Copula List		
D_4	$u, v \in I$	Copula
e	$e(u,v) = (u,v)$	$C^e(u,v) = C(u,v)$
r	$r(u,v) = (1-v, u)$	$C^r(u,v) = u - C(1-v, u)$
r^2	$r^2(u,v) = (1-u, 1-v)$	$C^{r^2}(u,v) = u + v - 1 + C(1-u, 1-v)$
r^3	$r^3(u,v) = (v, 1-u)$	$C^{r^3}(u,v) = v - C(v, 1-u)$
h	$h(u,v) = (1-u, v)$	$C^h(u,v) = v - C(1-u, v)$
hr	$hr(u,v) = (v,u)$	$C^{hr}(u,v) = C(v,u)$
hr^2	$hr^2(u,v) = (u, 1-v)$	$C^{hr^2}(u,v) = u - C(u, 1-v)$
hr^3	$hr^3(u,v) = (1-v, 1-u)$	$C^{hr^3}(u,v) = u + v - 1 + C(1-v, 1-u)$

Table 2.1: C^d for $d \in D_4$.

In [12], Edwards, Mikusiński and Taylor introduced a D_4-*invariant* copula defined by

$$C(u,v) = C^d(u,v)$$

for all $(u,v) \in I^2$ and for every $d \in D_4$.

One can see that if C is a D_4-invariant copula then for all $d \in D_4$, then

$$\mu_C([u_1,u_2] \times [v_1,v_2]) = \mu_C(d([u_1,u_2] \times [v_1,v_2]))$$

for all rectangle $[u_1,u_2] \times [v_1,v_2] \subseteq I^2$.

Example 2.79. Π and $\frac{M+W}{2}$ are examples of D_4-invariant copulas.

Next, we will introduce you to a well-known function quantifying the relationship between two random variables X and Y. It is called a measure of dependence. Since there are many definitions of being a measure of dependence, we will focus on Rényi's definition and Schweizer and Wolff's definition.

A history for measures of dependence began in 1959. Initially, Rényi [19] constructed a measure of dependence based on a set of six postulates. It was defined as follows: a real function $\mathcal{R}$ on the Cartesian product of the class $\mathcal{A}$ of all random variables which are not constant with probability one is said to be a *measure of dependence* if the following conditions hold for all $X, Y \in \mathcal{A}$:

(i) $\mathcal{R}(X,Y) = \mathcal{R}(Y,X)$.

(ii) $0 \leq \mathcal{R}(X,Y) \leq 1$.

(iii) $\mathcal{R}(X,Y) = 0$ if and only if X and Y are independent.

(iv) $\mathcal{R}(X,Y) = 1$ if $Y = f(X)$ or $X = g(Y)$ for some Borel measurable functions f on $\operatorname{Ran} X$ and g on $\operatorname{Ran} Y$.

(v) If f and g are Borel measurable injective functions on $\operatorname{Ran} X$ and $\operatorname{Ran} Y$, respectively, then $\mathcal{R}(f(X), g(Y)) = \mathcal{R}(X,Y)$. In other words, $\mathcal{R}$ is invariant under Borel measurable injective transformations.

(vi) If X and Y are jointly normal with correlation coefficient r, then $\mathcal{R}(X,Y) = |r|$.

Afterwards, Schweizer and Wolff [22] proposed a definition for a measure of dependence $\mathcal{R}'$ based on Rényi's definition. The domain of $\mathcal{R}'$ became the set of all random variables of which distribution functions are continuous. Moreover the conditions (iv)-(vi) in Rényi's postulates were adjusted and a continuity property of $\mathcal{R}'$ was added. Then a measure of dependence in sense of Schweizer and Wolff is defined as follows: a real function $\mathcal{R}'$ on the Cartesian product of the collection $\mathcal{A}$ of all random variables whose distribution functions are continuous, is said to be a *measure of dependence* if the following conditions hold for all $X, Y \in \mathcal{A}$:

(i) $\mathcal{R}'(X,Y) = \mathcal{R}'(Y,X)$.

(ii) $0 \leq \mathcal{R}'(X,Y) \leq 1$.

(iii) $\mathcal{R}'(X,Y) = 0$ if and only if X and Y are independent.

(iv) $\mathcal{R}'(X,Y) = 1$ if and only if $Y = g(X)$ for some strictly monotone function g on $\operatorname{Ran} X$.

(v) $\mathcal{R}'(f(X), g(Y)) = \mathcal{R}'(X,Y)$ for all strictly monotone functions f and g on $\operatorname{Ran} X$ and on $\operatorname{Ran} Y$, respectively. Specifically, $\mathcal{R}'$ is invariant under strictly monotone functions.

(vi) If X and Y have a bivariate normal distribution associated with correlation coefficient r, then $\mathcal{R}'(X,Y)$ is a strictly increasing function ϕ of $|r|$.

(vii) Let $(X_n, Y_n)_{n\in\mathbb{N}}$ be a sequence of 2- dimentional random vectors whose joint distribution functions are H_n for all $n \in \mathbb{N}$ and (X,Y) be another pair of random variables whose joint distribution function is H. If $H_n \xrightarrow{d} H$, then

$$\lim_{n\to\infty} \mathcal{R}'(X_n, Y_n) = \mathcal{R}'(X,Y).$$

In [22], the following result was established.

Theorem 2.80 ([22, Theorem 4]). *A function σ on a set of all copulas defined by*

$$\sigma(C) = \frac{\int_{I^2} |C - \Pi| d\lambda_2}{\int_{I^2} |M - \Pi| d\lambda_2}$$

is a Schweizer-Wolff measure of dependence and a function ϕ in (vi) is given by

$$\phi(|r|) = \frac{6}{\pi} \arcsin(\frac{|r|}{2}).$$

All copulas can be approximated by checkerboard copulas. The idea of checkerboard copulas, which can be seen in [25], is shown as follows: let $(X, \mathcal{F}, \mathbb{P})$ be a probability space, and C be a copula associated with uniform random variables U, V on I. Fix $m \in \mathbb{N}$. Denote $I_0 = [0, \frac{1}{m}]$ and $I_i = (\frac{i}{m}, \frac{i+1}{m}]$ for all $i = 1, 2, \ldots, m-1$. Then the following are true.

(i) $\{I_i : i = 0, 1, 2, \ldots, m-1\}$ is a partition of I.

(ii) $\{I_i \times I_j : i, j = 0, 1, 2, \ldots, m-1\}$ is a partition of I^2.

(iii) $\{A_{i,j}\}$ is a partition of X where $A_{i,j} = \{\omega : (U(\omega), V(\omega)) \in I_i \times I_j\}$ for all $i, j = 0, 1, 2, \ldots, m-1$.

By Sklar's Theorem (Theorem 2.61) and conditional probability, we get

$$C(u, v) = \mathbb{P}(U \leq u, V \leq v) = \sum_{i,j} \mathbb{P}(U \leq u, V \leq v | A_{i,j}) \mathbb{P}(A_{i,j})$$

for all $u, v \in I$. Note that $\mathbb{P}(U \leq u, V \leq v | A_{i,j})$ is a joint distribution for all $i, j = 0, 1, 2, \ldots, m-1$. Using Sklar's Theorem again, we have

$$\mathbb{P}(U \leq u, V \leq v | A_{i,j}) = C_{i,j}(F_{i,j}(u), (G_{i,j}(v))$$

where $F_{i,j}(u) = \mathbb{P}(U \leq u | A_{i,j})$ and $G_{i,j}(v) = \mathbb{P}(V \leq v | A_{i,j})$ for all $i, j = 0, 1, 2, \ldots, m-1$. Thus

$$C(u, v) = \mathbb{P}(U \leq u, V \leq v) = \sum_{i,j} \mathbb{P}(A_{i,j}) C_{i,j}(F_{i,j}(u), (G_{i,j}(v)) \tag{2.2}$$

for all $u, v \in I$. Note that $[\mathbb{P}(A_{i,j})]_{m \times m}$ is a doubly stochastic matrix.

Theorem 2.81 ([2], Theorem 3.1)**.** *Fix $m \in \mathbb{N}$. Let $D_{i,j}$ be a copula, $H_{i,j}$ and $L_{i,j}$ be distribution functions over I_i and I_j, respectively, and $a_{i,j} \geq 0$ for all $i, j \in \{0, 1, 2, \ldots, m-1\}$. Then for all $u, v \in I$,*

$$D(u,v) = \sum_{i,j} a_{i,j} D_{i,j}(H_{i,j}(u), L_{i,j}(v)) \tag{2.3}$$

is a copula if and only if the following conditions hold:

(i) for each $h = 0, 1, 2, \ldots, m-1$, $\sum_{j=0}^{m-1} a_{h,j} H_{h,j}(u) = u - \frac{h}{m}$ for all $u \in I_h$,

(ii) for each $k = 0, 1, 2, \ldots, m-1$, $\sum_{j=0}^{m-1} a_{i,k} L_{i,k}(u) = v - \frac{k}{m}$ for all $v \in I_k$.

Remark 2.82 ([2])**.** Let C be a copula. Fix $m \in \mathbb{N}$. If $D_{i,j} = \Pi$, $H_{i,j}$ and $L_{i,j}$ are uniform distribution functions over I_i and I_j and $a_{i,j} = \mathbb{P}(A_{i,j})$ which is defined as (2.3) for all $i, j \in \{0, 1, 2, \ldots, m-1\}$, then the conditions (i) and (ii) hold. Thus for all $u, v \in I$,

$$C_m(u,v) := \sum_{i,j} \mathbb{P}(A_{i,j}) \Pi(H_{i,j}(u), L_{i,j}(v))$$

is a copula, which is called the *checkerboard approximation of C*.

Theorem 2.83 ([2], Theorem 4.2)**.** *If $\{C_m\}_{m \in \mathbb{N}}$ is a sequence of checkerboard copulas of C, then $||C_m - C||_\infty \leq \frac{2}{m}$. Consequently, C_m converges uniformly to C.*

In [5], Darsow et al. introduced a binary operation $*$-product on the class of all copulas defined by

$$A * B(u,v) = \int_0^1 \partial_2 A(u,t) \partial_1 B(t,v) dt$$

for all $u, v \in [0,1]$. Then $(\mathcal{C}_2, *)$ is a monoid with identity M. Moreover, we can prove that the following properties hold:

(i) Π is the zero; that is $\Pi * C = \Pi = C * \Pi$ for all copula C.

(ii) $C * W(u,v) = u - C(u, 1-v)$ for all $u, v \in I$.

(iii) $W * C(u,v) = v - C(1-u, v)$ for all $u, v \in I$.

(iv) $W * W = M$.

Theorem 2.84 ([5], Theorem 2.3)**.** *Let A, B be copulas. If A_n converges uniformly to A, then $A_n * B$ and $B * A_n$ converge uniformly to $A * B$ and $B * A$, respectively.*

For any copula C, a copula C^T defined by $C^T(u,v) = C(v,u)$ for all $u, v \in I$ is called the *transposed copula of* C.

Definition 2.85 ([5])**.** A copula C is said to be *left invertible* if there exists a copula A such that $A * C = M$. Then A is called a *left inverse* of C. Similarly, a copula C is said to be *right invertible* if there is a copula A such that $C * A = M$. Then A is called a *right inverse* of C. A copula C is said to be *invertible* if C is both left invertible and right invertible.

Theorem 2.86 ([5], Theorem 7.3)**.** *If a copula C is left invertible, then its left inverse is unique and equal to C^T. Similarly, If a copula C is right invertible, then its right inverse is unique and equal to C^T.*

Corollary 2.87 ([21], Corollary 4.9)**.** *Let $f, g : \mathbb{R} \to \mathbb{R}$ be Borel measurable functions. Then for all random variables X and Y,*

$$C_{f(X),X} * C_{X,Y} * C_{Y,g(Y)} = C_{f(X),g(Y)}.$$

Definition 2.88 ([23])**.** Let X and Y be two random variables. Y is said to be *completely dependent on* X if there exists a Borel measurable function f such that $P(Y = f(X)) = 1$. They are called *mutually completely dependent (m.c.d.)* if there is a Borel measurable injection f such that $P(Y = f(X)) = 1$.

Lemma 2.89 ([23], Theorem 4.1)**.** *Let X and Y be continuous random variables associated with a copula C. The following statements hold :*

(i) Y is completely dependent on X if and only if C is left invertible.

(ii) X is completely dependent on Y if and only if C is right invertible.

(iii) X and Y are mutually completely dependent if and only if C is invertible.

In 1992, Mikusiński et al. introduced a shuffle of Min, which is constructed from M by cutting the support of M on I^2 into a finite number of vertical strips. Each vertical strip may be flipped with respect to its vertical axis of symmetry. Then all vertical strips will be rearranged to form the I^2.

Corollary 2.90. *Let C be a copula. There exists a sequence of the shuffles of Min converging uniformly to C.*

Remark 2.91. The set of all shuffles of Min is a subset of the class of all invertible copulas.

Definition 2.92 ([4])**.** A measurable function $f : I \to I$ is said to be *measure preserving* if $\lambda(f^{-1}(A)) = \lambda(A)$ for all Borel measurable sets A in I.

Denote $\mathcal{F}$ as the class of all measure preserving functions.

Theorem 2.93 ([4], Theorem 2.2)**.** *If f is a measure preserving function on I, then there exists a sequence of injective piecewise linear measure preserving functions on I which converges to f almost everywhere.*

Theorem 2.94 ([6], Theorem 2.2)**.** *Fix $f, g \in \mathcal{F}$. If a function $C_{f,g}$ is defined by*

$$C_{f,g}(u, v) = \lambda(f^{-1}[0, u] \cap g^{-1}[0, v])$$

for all $u, v \in I$, then $C_{f,g}$ is a copula. Furthermore, if C is any copula, there exist $f, g \in \mathcal{F}$ such that $C = C_{f,g}$.

Let e be the identity function; that is, $e(x) = x$ for all $x \in I$.

Theorem 2.95 ([6], Theorem 2.4)**.** *Let C be left invertible. A function f defined by*

$$f(x) = \inf\{y|\partial_1 C(x, y) = 1\}$$

becomes a measure preserving Borel function and $C = C_{e,f}$.

For each $f \in \mathcal{F}$, if there exists $g \in \mathcal{F}$ such that $g \circ f = e$ a.s. and $f \circ g = e$ a.s., then g is called an *essential inverse of f*.

Corollary 2.96 ([6], Corollary 2.4.1)**.** *A copula C is invertible if and only if $C = C_{f,e}$ for some measure preserving Borel function f whose an essential inverse is g. Consequently, $C_{f,e} = C_{e,g}$.*

Definition 2.97 ([13])**.** If $T : \mathbb{R} \to \mathbb{R}$ is an increasing function with $T(-\infty) = \lim\limits_{x \downarrow -\infty} T(x)$ and $T(\infty) = \lim\limits_{x \uparrow \infty} T(x)$, then the *generalized inverse* $T^{-} : \mathbb{R} \to \bar{\mathbb{R}}$ of T is defined by for all $y \in \mathbb{R}$,

$$T^{-}(y) = \inf\{x \in \mathbb{R} : T(x) \geq y\}.$$

Proposition 2.98 ([13], Proposition 1)**.** *If $T : \mathbb{R} \to \mathbb{R}$ is increasing with $T(-\infty) = \lim\limits_{x \downarrow -\infty} T(x)$ and $T(\infty) = \lim\limits_{x \uparrow \infty} T(x)$, then the following statements hold: for all $x, y \in \mathbb{R}$*

(i) T^{-} is increasing.

(ii) $T^{-}(T(x)) \leq x$ for all $x \in \mathbb{R}$.

(iii) If T is strictly increasing, then $T^{-}(T(x)) = x$ for all $x \in \mathbb{R}$.

(iv) T is continuous if and only if T^{-} is strictly increasing on $[\inf \operatorname{Ran} T, \sup \operatorname{Ran} T]$.

(v) T is strictly increasing if and only if T^{-} is continuous on $RanT$.

Proposition 2.99 ([13], Proposition 2)**.** *Let X be a random variable with a distribution function F, denoted by $X \sim F$, then the following statements hold:*

(i) If F is continuous, then $F(X) \sim U[0,1]$.

(ii) If $U \sim U[0,1]$, then $F^{-}(U) \sim F$.

Corollary 2.100. *Let X and Y be two random variables whose marginal distribution functions are F_X and F_Y, respectively, and joint distribution function is $F_{X,Y}$. Then for all $u, v \in I$,*

$$C_{X,Y}(u,v) = F_{X,Y}(F^{-}{}_X(u), F^{-}{}_Y(v)).$$

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Denote $E(X) = \int_{\Omega} X dP$.

Definition 2.101. Let X and Y be two random variables. Then the *correlation coefficient* $\rho_{X,Y}$ is defined by

$$\rho_{X,Y} = \frac{cov(X,Y)}{\sqrt{var(X)var(Y)}}$$

where $cov(X,Y) = E[(X - E(X))(Y - E(Y))]$ and $var(X) = E(X - E(X))^2$.

Definition 2.102. Let X and Y be random variables. If their joint probability density function $f_{X,Y}$ is defined by for all $x, y \in \mathbb{R}$,

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-r^2}} e^{\frac{-1}{2(1-r^2)}[(\frac{x-\mu_1}{\sigma_1})^2 - 2r(\frac{x-\mu_1}{\sigma_1})(\frac{y-\mu_2}{\sigma_2}) + (\frac{y-\mu_2}{\sigma_2})^2]}, \tag{2.4}$$

then we will say that X and Y have a *bivariate normal distribution with correlation coefficient* r.

Moreover, one can prove that $X \sim \mathrm{N}(\mu_1, \sigma_1^2)$ and $Y \sim \mathrm{N}(\mu_2, \sigma_2^2)$ if X and Y have a bivariate normal distribution of which $f_{X,Y}$ is defined by (2.4).

Proposition 2.103. *X and Y are standard normal random variables with correlation coefficient r if and only if $C_{X,Y}$ is a Gaussian copula with parameter r, denoted by C_r^{Ga}.*

Definition 2.104. Let C_1 and C_2 be copulas. If $C_1(u,v) \leq C_2(u,v)$ for all $u, v \in [0,1]$, then C_1 is said to be *smaller than* C_2, denoted by $C_1 \prec C_2$.

Remark 2.105. Let $\rho_1, \rho_2 \in [-1,1]$. If $\rho_1 \leq \rho_2$, then $C_{\rho_1}^{Ga} \prec C_{\rho_2}^{Ga}$. Moreover, $C_1^{Ga} = M, C_{-1}^{Ga} = W$ and $C_0^{Ga} = \Pi$.

CHAPTER III

GENERALIZATIONS OF SCHWEIZER-WOLFF MEASURE OF DEPENDENCE

Let $\varphi : I \to [0, \infty)$ be a function satisfying the following conditions:

(A) φ is a strictly increasing continuous function.

(B) $\varphi(0) = 0$.

Let us define a function $\sigma_\varphi(X, Y)$ for all random variables X, Y with continuous distribution functions setting by

$$\sigma_\varphi(X, Y) = \sigma_\varphi(C_{X,Y}) = \frac{\int_{I^2} \varphi\big(|C_{X,Y} - \Pi|(u, v)\big) d\lambda_2(u, v)}{\int_{I^2} \varphi\big(|M - \Pi|(u, v)\big) d\lambda_2(u, v)}.$$

For brevity, we will drop the integration variables u and v and write

$$\sigma_\varphi(C_{X,Y}) = \frac{\int_{I^2} \varphi(|C_{X,Y} - \Pi|) d\lambda_2}{\int_{I^2} \varphi(|M - \Pi|) d\lambda_2}.$$

Our goal in this chapter is to show that σ_φ is a measure of dependence in the sense of Schweizer-Wolff.

Let us begin with the existence and boundedness of σ_φ. The main idea of the following lemma was communicated to us by [11].

Lemma 3.1. *If $\varphi : I \to [0, \infty)$ satisfies the conditions A and B, then the following statements hold:*

(i) $\int_{I^2} \varphi(|C - \Pi|) d\lambda_2 \leq \int_{I^2} \varphi(|M - \Pi|) d\lambda_2$ *for all copulas C.*

(ii) $\int_{I^2} \varphi(|M - \Pi|) d\lambda_2 = \int_{I^2} \varphi(|W - \Pi|) d\lambda_2$.

Proof. (i) Let C be a copula. Note that the integral of a measurable function with respect to λ_2 can be transformed into iterated integrals. For each fixed $v \in I$, we denote $C_v(u) = C(u,v)$ for all $u \in I$ and put $A(C_v) = \int_I \varphi\big(|C-\Pi|(u,v)\big)du$. We will show that $A(C_v) \le A(M_v)$.

For each $v \in I$, consider the set $L_v := \{u \in I : C_v(u) < \Pi_v(u)\}$. If L_v is empty, then $C_v \ge \Pi_v$, and hence $0 \le C_v - \Pi_v \le M_v - \Pi_v$ together with φ is increasing, $A(C_v) \le A(M_v)$. If $L_v \ne \varnothing$ then, by the continuity of C_v, it must be open and hence equal to a disjoint union of countably many open intervals, i.e., $L_v = \cup_{i\in\mathbb{N}}(r_i, s_i)$.

For each $i \in \mathbb{N}$, we first define a parallelogram whose farthest corners are $(r_i, r_i v)$ and $(s_i, s_i v)$. Define $f_{i_v}(u) = \max\{u + s_i v - s_i, r_i v\}$ and $g_{i_v}(u) = \min\{r_i v + u - r_i, s_i v\}$ on $[r_i, s_i]$. The graphs of f_{i_v} and g_{i_v} consists of line segments those of W_v and M_v, respectively. See Figure 3.1. Note that $C_v = \Pi_v$ on the boundary of (r_i, s_i) for all $\in \mathbb{N}$. The facts that $V_C([u,s_i]\times[v,1]) \ge 0$ and $C(u,v) \ge r_i v$ for all $u \in [r_i, s_i]$ imply $C_v(u) \ge f_{i_v}(u)$ for all $u \in [r_i, s_i]$. Similarly, using $V_C([r_i,u]\times[v,1]) \ge 0$ and $C(u,v) \le s_i v$ for all $u \in [r_i, s_i]$, we have $C_v \le g_{i_v}$ on $[r_i, s_i]$. Observe that the mapping $u \mapsto s_i - u + r_i$ is a flip with respect to the center of $[r_i, s_i]$, hence for all $u \in [r_i, s_i]$,

$$(\Pi - f_i)(u,v) = (g_i - \Pi)(s_i - u + r_i, v). \tag{3.1}$$

Since φ is increasing, we have

$$\begin{aligned}
\int_{r_i}^{s_i} \varphi\big(|C_v - \Pi_v|(u)\big)du &= \int_{r_i}^{s_i} \varphi\big((\Pi_v - C_v)(u)\big)du \\
&\le \int_{r_i}^{s_i} \varphi\big((\Pi_v - f_{i_v})(u)\big)du \\
&= \int_{r_i}^{s_i} \varphi\big((g_{i_v} - \Pi_v)(s_i - u + r_i)\big)du \\
&= \int_{r_i}^{s_i} \varphi\big((g_{i_v} - \Pi_v)(u)\big)du.
\end{aligned}$$

Define a function H_v on I by

$$H_v(u) = \begin{cases} C_v(u) & \text{if } C_v(u) \ge \Pi_v(u), \\ g_{i_v}(u) & \text{if } C_v(u) < \Pi_v(u) \text{ and } r_i < u < s_i. \end{cases}$$

Then

$$A(C_v) = \sum_{i\in\mathbb{N}} \int_{r_i}^{s_i} \varphi\big((\Pi - C)\big)(u,v)du + \sum_{I\smallsetminus L_v} \int_{s_i}^{r_{i+1}} \varphi\big((C-\Pi)(u,v)\big)du$$
$$\leq A(H_v)$$
$$\leq A(M_v).$$

Therefore, we have proved (i).

(ii) The proof follows from the equation (3.1) when $r_1 = 0$ and $s_1 = 1$.

□

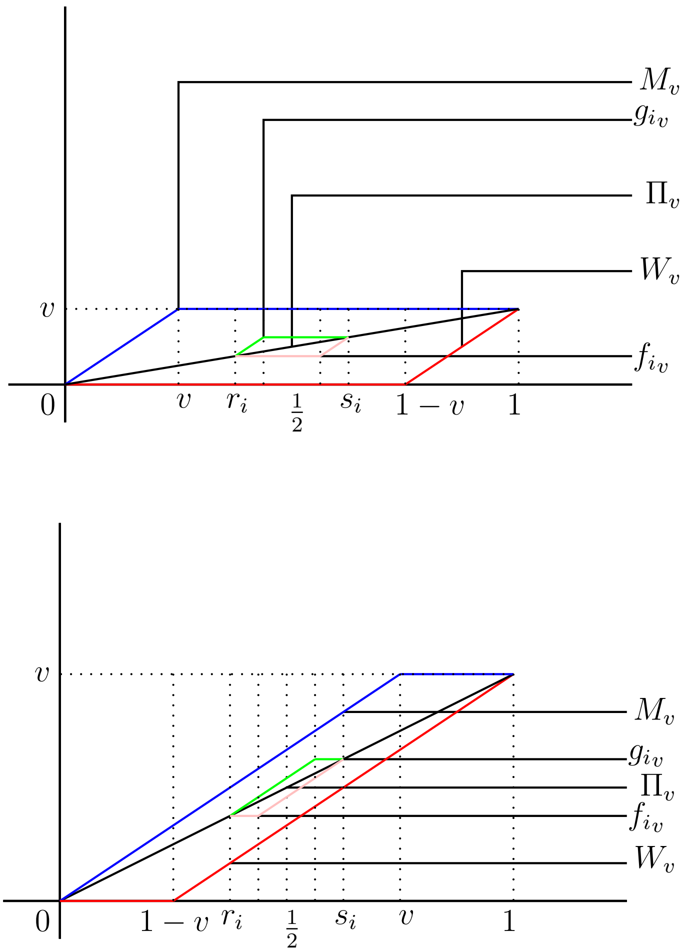


Figure 3.1: Graphs of W_v, M_v, Π_v, f_{i_v} and g_{i_v}.

The following lemma shows that the maximum value of σ_φ occurs exactly when $C = M$ or $C = W$.

Lemma 3.2. *$\sigma_\varphi(C)$ attains its maximum value exactly when $C = M$ or $C = W$.*

Proof. ($\Leftarrow$) This follows from Lemma 3.1.

($\Rightarrow$) Let D be a copula such that $\sigma_\varphi(D)$ is the maximum value. Then

$$\int_{I^2} \varphi(|D - \Pi|)d\lambda_2 = \int_{I^2} \varphi(|M - \Pi|)d\lambda_2 = \int_{I^2} \varphi(|W - \Pi|)d\lambda_2.$$

For any copula C and for each fixed $v \in I$, we define C_v and $A(C_v)$ as we did in Lemma 3.1. By the assumption, we have $A(M_v) = A(D_v) = A(W_v)$ a.e. $v \in I$. Using the continuity of both copulas C and φ and DCT, we have $v \mapsto A(C_v)$ is continuous on I. Then $A(M_v) = A(D_v) = A(W_v)$ for all $v \in [0, 1]$.

Let $v \in I$. Consider the set $L_v := \{u \in I : D_v(u) < \Pi_v(u)\}$. If L_v is an empty set, then $D_v \geq \Pi_v$. Using the fact that φ is strictly increasing, we obtain $D_v = M_v$ a.e. $u \in I$. Because of the continuity of both D and M, we have $D_v = M_v$. If L_v is not an empty set, by the continuity of D_v and Π_v, then L_v is an open set and it is equal to a disjoint union of countably many open intervals $\cup_{i\in\mathbb{N}}(r_i, s_i)$. Define a function H_v as we did in the Lemma 3.1. Then $A(D_v) \leq A(H_v) \leq A(M_v)$, which implies that $A(H_v) = A(M_v)$. Thus $\varphi(|H_v - \Pi_v|) = \varphi(|M_v - \Pi_v|)$ a.e. $u \in I$. Using the fact that φ is strictly increasing, we have $H_v = M_v$ a.e. $u \in I$. Because of the continuity of H and M, we have $H_v = M_v$. If there were $i \in \mathbb{N}$ such that $(r_i, s_i) \neq (0, 1)$, then H_v would not be M_v. Then $(r_i, s_i) = (0, 1)$ for all $i \in \mathbb{N}$. By the assumption, it implies that $D_v = W_v$. If there exists $v \in I$ such that $D_v = M_v$, then, by using the facts that D is Lipschitz continuous and the distance between M_v and W_v is v if $v \leq \frac{1}{2}$ or $1 - v$ if $v > \frac{1}{2}$, $D_v = M_v$ for all $v \in I$. Similarly, if there is $v \in I$ such that $D_v = W_v$, then $D_v = W_v$ for all $v \in I$. $\square$

Theorem 3.3. *The function σ_φ satisfies the following properties for all random variables X, Y with continuous distribution functions.*

(i) $\sigma_\varphi(C_{X,Y}) = \sigma_\varphi(C_{Y,X})$.

(ii) $0 \leq \sigma_\varphi(C_{X,Y}) \leq 1$.

(iii) $\sigma_\varphi(C_{X,Y}) = 0$ *if and only if X and Y are independent.*

(iv) $\sigma_\varphi(C_{X,Y}) = 1$ *if and only if* $Y = f(X)$ *for some strictly monotone function* f.

(v) $\sigma_\varphi(C_{f(X),g(Y)}) = \sigma_\varphi(C_{X,Y})$ *for all strictly monotone functions* f *and* g.

(vi) If (X, Y) *and* (X_n, Y_n), $n = 1, 2, \ldots,$ *are pairs of random variables with joint distribution functions* H *and* H_n, *respectively, and if* $(X_n, Y_n) \xrightarrow{d} (X, Y)$, *then* $\lim_{n\to\infty} \sigma_\varphi(C_{X_n,Y_n}) = \sigma_\varphi(C_{X,Y})$.

Proof. (i) It follows from the fact that $C_{X,Y}(u, v) = C_{Y,X}(v, u)$ for all $u, v \in I$.

(ii) This follows immediately from Lemma 3.1.

(iii) ($\Leftarrow$) By Theorem 2.64 (i), we have $C_{X,Y} = \Pi$. Since $\varphi(0) = 0$, we get $\sigma_\varphi(C_{X,Y}) = 0$.

($\Rightarrow$) By the assumption, we have $\varphi(|C_{X,Y} - \Pi|) = 0$ a.e. As φ is strictly increasing, $C_{X,Y} = \Pi$ a.e. Using the continuity of copulas, we have $C_{X,Y} = \Pi$.

(iv) ($\Leftarrow$) This follows from Theorem 2.64 (ii-iii) and Lemma 3.1.

($\Rightarrow$) This follows from Lemma 3.2.

(v) Let f and g be strictly monotone.

Case 1: Assume that f and g are both strictly increasing. It follows from Theorem 2.65 (i).

Case 2: Assume that f is strictly increasing and g is strictly decreasing. It follows from Theorem 2.65 (ii) and by a change of variable.

Case 3: Assume that f is strictly decreasing and g is strictly increasing. It follows from Theorem 2.65 (iii) and by a change of variable.

Case 4: Assume that f and g are both strictly decreasing. It follows from Theorem 2.65 (iv) and by a change of variable.

(vi) By the assumption, we obtain $\lim_{n\to\infty} H_n(u, v) = H(u, v)$ for all (u, v) at which H is continuous. By Sklar's Theorem and the continuity of the distribution functions of X and Y, we have $\lim_{n\to\infty} C_{X_n,Y_n}(u,v) = C_{X,Y}(u,v)$ for all $u, v \in I$. By using DCT and the continuity of φ, the proof is complete.

$\square$

Given a function φ, the following example shows that $\sigma_\varphi(C_{X,Y})$ is a strictly increasing function of $|r|$ when X and Y have a bivariate normal distribution with correlation coefficient r.

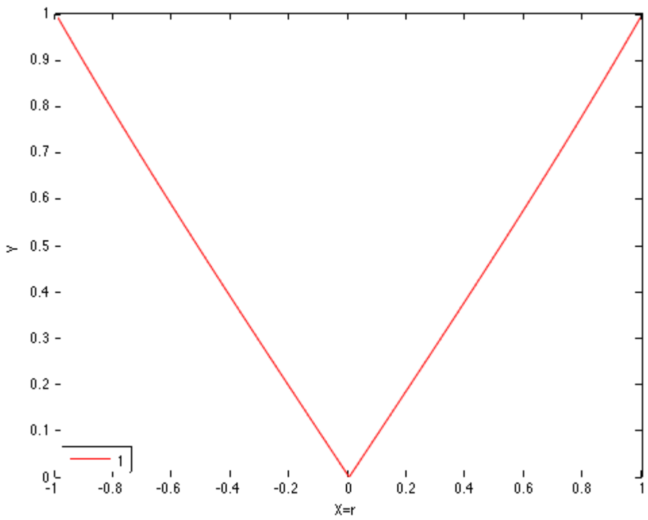
Example 3.4. Define φ_1 and φ_2, both of which satisfy conditions (A) and (B), as follows:

$$\varphi_1(x) = \begin{cases} 2x & \text{if } 0 \le x < \frac{1}{4}, \\ \dfrac{2x+1}{3} & \text{if } \frac{1}{4} \le x \le 1 \end{cases}$$

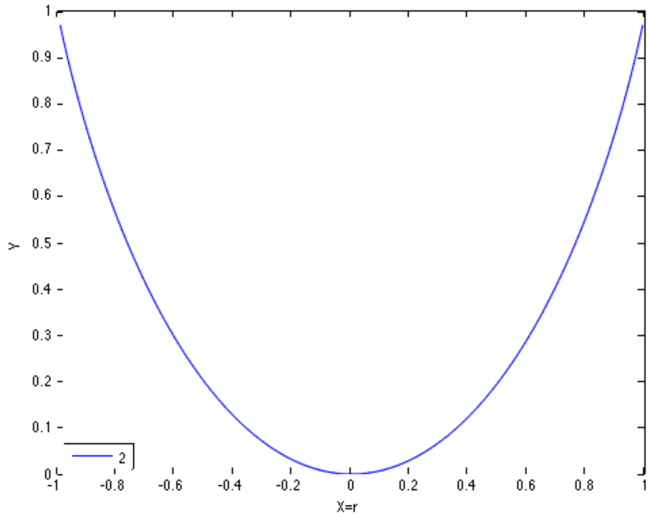
and

$$\varphi_2(x) = \begin{cases} 4x^2 & \text{if } 0 \le x < \frac{1}{4}, \\ 9x - 2 & \text{if } \frac{1}{4} \le x \le \frac{7}{9}, \\ 54x - 37 & \text{if } \frac{7}{9} \le x \le 1. \end{cases}$$

If X and Y have a bivariate normal distribution with correlation coefficient r, then the graphs of $\sigma_{\varphi_i}(C_{X,Y})$ (**Y**-axis) as a function of r (**X**-axis) for $i = 1, 2$ is shown in the following figures. For more details on how the graphs are plotted, see Appendix.



(a)



(b)

Figure 3.2: The graph of $\sigma_{\varphi_1}(X,Y)(a)$ and $\sigma_{\varphi_2}(X,Y)(b)$ for X,Y which are jointly normal with correlation coefficient r.

CHAPTER IV
GENERALIZATIONS OF RÉNYI MEASURE OF DEPENDENCE

Denote the set of all injective Borel measurable functions on $\mathbb{R}$ by $\mathcal{I}$. We define $\sigma_\varphi^*(X,Y)$ for all random variables X and Y whose distribution functions are continuous by

$$\sigma_\varphi^*(X,Y) = \sup_{f,g\in\mathcal{I}} \sigma_\varphi(f(X),g(Y)) = \sup_{f,g\in\mathcal{I}} \frac{\int_{I^2} \varphi(|C_{f(X),g(Y)} - \Pi|)d\lambda_2}{\int_{I^2} \varphi(|M-\Pi|)d\lambda_2} \tag{4.1}$$

where φ satisfies the conditions (A) and (B) in page 30. For all $f,g \in \mathcal{I}$, the copula $C_{f(X),g(Y)}$ still has continuous distribution function. See the paragraph before Theorem 2.65. Let $\mathcal{M}$ be the class of all invertible copulas. In this chapter, we will show that the equation (4.1) can be newly written as

$$\sigma_\varphi^*(X,Y) = \sup_{S_1,S_2\in\mathcal{M}} \frac{\int_{I^2} \varphi(|S_1 * C_{X,Y} * S_2 - \Pi|)d\lambda_2}{\int_{I^2} \varphi(|M-\Pi|)d\lambda_2}. \tag{4.2}$$

Moreover, the function σ_φ^* satisfies all of Rényi's postulates except for (vi).

The following lemmas are required to prove (4.2).

Proposition 4.1. *Let $U \sim U[0,1]$. If f is measure preserving, then*

$$C_{U,f(U)}(x,y) = \lambda([0,x] \cap f^{-1}[0,y])$$

for all $x,y \in I$. Consequently,

$$C_{U,f(U)} = C_{U',f(U')}$$

for all $U' \sim U[0,1]$.

Proof. Using the facts that $U \sim U[0,1]$ and f is measure preserving, we have

$$\mathbb{P}(f(U) \le x) = \mathbb{P}(U \in f^{-1}[0,x]) = \lambda(f^{-1}[0,x]) = x$$

for all $x \in I$. That is $f(U) \sim U[0,1]$. By Sklar's Theorem,

$$C_{U,f(U)}(x,y) = \mathbb{P}(U \leq x, f(U) \leq y) = \mathbb{P}(U \in [0,x] \cap f^{-1}[0,y]) = \lambda([0,x] \cap f^{-1}[0,y])$$

for all $x, y \in I$. Furthermore, we have

$$C_{U,f(U)} = C_{U',f(U')}$$

for all $U' \sim U[0,1]$. □

Lemma 4.2. *Let X be a random variable with a continuous distribution function F. If $\Theta = \{x \in \mathbb{R} | \ F(x) - F(x-h) > 0 \ \ \text{for all } h > 0\}$, then $\mathbb{P}(X \in \Theta) = 1$.*

Proof. Put

$$\Theta = \{x \in \mathbb{R} | \ F(x) - F(x-h) > 0 \ \ \text{for all } h > 0\}$$

and

$$\Theta^c = \{x \in \mathbb{R} | \ F(x) = F(x-h) \text{ for some } h > 0\}.$$

By Proposition 2.98, $F^-(F(X)) \leq X$. Our first claim is that $\{F^-(F(X)) < X\} = \{X \in \Theta^c\}$, which implies that $\{F^-(F(X)) = X\} = \{X \in \Theta\}$, by taking a complement.

Claim that $\{F^-(F(X)) < X\} = \{X \in \Theta^c\}$. Let $\omega \in \{F^-(F(X)) < X\}$. By the definition of F^-, there exists $t \in \mathbb{R}$ such that $F(t) \geq F(X(\omega))$ and $t < X(\omega)$. Since F is increasing, $F(t) \leq F(X(\omega))$. Consequently, $F(t) = F(X(\omega))$ and $\omega \in \{X \in \Theta^c\}$. Conversely, if $\omega \in \{X \in \Theta^c\}$, then there is $h > 0$ such that $F(X(\omega)) = F(X(\omega) - h)$. Since $X(\omega) - h < X(\omega)$, $F^-(F(X(\omega))) < X(\omega)$. It implies $\omega \in \{F^-(F(X)) < X\}$.

Let us focus on proving that $\mathbb{P}(X \in \Theta) = 1$. If $\Theta^c = \varnothing$, the proof is done. If $\Theta^c \neq \varnothing$, our next claim is that Θ^c can be written as disjoint union of a countable collection of intervals. For each fixed $x \in \Theta^c$, there is $h > 0$ such that $F(x) = F(x-h)$. Let $a = \sup\{h > 0 \mid F(x) = F(x-h)\}$ and $b = \sup\{s \geq 0 \mid F(x) = F(x+s)\}$. Then $x \in (x-a, x+b]$. Denote $I_x = (x-a, x+b]$. Claim that $I_x \subseteq \Theta^c$. Let $y \in I_x$ be such that $x \neq y$.

Case 1: $x - a < y < x$. Then we can find $0 < h < a$ such that $y = x - h$. Since h is not an upper bound of I_x, there is an element $h' \in I_x$ such that $h < h'$ and $F(x) = F(x - h')$. Because of monotonicity of F, $F(x) = F(x - h)$. Choose $h^* = h' - h$. Then $F(y) = F(y - h^*)$, which implies $y \in \Theta^c$.

Case 2: $x < y < x + b$. Then there exists $0 < h < b$ such that $y = x + h$. Since h is not an upper bound of I_x, there is an element $h' \in I_x$ such that $h < h'$ and $F(x) = F(x + h')$. Because of monotonicity of F, $F(x) = F(x + h)$. Choose $h^* = h$. Thus $F(y) = F(y - h^*)$, which implies $y \in \Theta^c$.

Case 3: $y = x + b$. Then we choose $h = b$. By left continuity of F, $F(y) = F(x) = F(y - h)$, which implies $y \in \Theta^c$.

Next, we will prove that $x - a \notin \Theta^c$. Suppose $x - a \in \Theta^c$. Then there exists $h > 0$ such that $F(x - a - h) = F(x - a)$. By right continuity of F, we have $F(x - a - h) = F(x - a) = F(x)$. It contradicts the definition of a.

Now, we obtain

$$\Theta^c = \bigcup_{x \in \Theta^c} I_x.$$

Claim that $I_x \cap I_y = \varnothing$ for all $x \neq y$.

Let $(a, b], (c, d] \in \{I_x : x \in \Theta^c\}$ be such that $(a, b] \cap (c, d] \neq \varnothing$. WLOG, assume $c < b$. Since $c \notin \Theta^c$ and $a \notin \Theta^c$, we have $c \leq a$ and $a \leq c$, respectively. Then $a = c$. Claim that $b = d$.

If $b < d$, we have $a < d$. Consequently, $(a, b] \subset (c, d]$. Note that for each $(a, b] \in \{I_x : x \in \Theta^c\}$, $F(y) = F(b)$ for all $y \in (a, b]$. Thus $F(z) = F(b)$ for all $z \in (b, d]$, which contradicts the definition of $(a, b]$. Similarly, if $b > d$, then $F(z) = F(d)$ for all $z \in (d, b]$. It contradicts the definition of $(c, d]$. Thus $b = d$.

Therefore, Θ^c is a disjoint union of a collection of intervals. Note that for each $x \in \Theta^c$, I_x always contains a rational number. So Θ^c is a disjoint union of a countable collection of intervals.

By right continuity of F, we have

$$\begin{aligned}\mathbb{P}(X \in \Theta^c) &= \mathbb{P}(X \in \bigcup_{x \in \Theta^c} I_x) \\ &\leq \sum_{x \in \Theta^c} \mathbb{P}(X \in I_x) \\ &= \sum_{x \in \Theta^c} F(x+b) - F(x-a) \\ &= 0.\end{aligned}$$

Therefore, the proof is done. □

Lemma 4.3. *Let X be a random variable with a continuous distribution function F. Then*

$$\{C_{X,f(X)} | f \in \mathcal{I}\} = \mathcal{M}.$$

Proof. ($\subseteq$) This follows from Theorem 2.89.

($\supseteq$) Let $A \in \mathcal{M}$. By Remark 2.62, there exist $X', Y' \sim U[0,1]$ such that $C_{X',Y'} = A$. Because A is invertible, A is also left invertible. Then there is a measurable function f such that $Y' = f(X')$. Using $f(X'), X' \sim U[0,1]$, we get

$$\lambda(B) = \mathbb{P}(f(X') \in B) = \mathbb{P}(X' \in f^{-1}(B)) = \lambda(f^{-1}(B))$$

for all $B \subseteq \mathcal{B}([0,1])$. Then f is measure-preserving. By Proposition 4.1, $C_{X',f(X')} = C_{U,f(U)}$ for all $U \sim U[0,1]$. Since F is continuous and by Theorem 2.99, $F(X) \sim U[0,1]$. Then $C_{X',f(X')} = C_{F(X),f(F(X))}$. By Proposition 2.98, F^{-} is strictly increasing on I. Note that an identity function is strictly increasing. Using Theorem 2.65 (i), we have

$$C_{F^{-}(F(X)),f(F(X))} = C_{F(X),f(F(X))} = C_{X',f(X')} = A.$$

Our claim is that $C_{X,f(F(X))} = C_{F^{-}(F(X)),f(F(X))}$.

Let $\tilde{F}$ and $\tilde{G}$ be the distributions of $F^{-}(F(X))$ and $f(F(X))$, respectively. Since $F(X) \sim U[0,1]$ and f is measure preserving, we have $f(F(X)) \sim U[0,1]$. Then $\operatorname{Ran} \tilde{G} = I$. Claim that $\tilde{F}(x) = F(x)$ for all $x \in \mathbb{R}$. Let $x \in \mathbb{R}$. Then

$$\{F^{-}(F(X)) \leq x\} = \{F^{-}(F(X)) < X \text{ and } F^{-}(F(X)) \leq x\} \dot\cup \{F^{-}(F(X)) = X \leq x\}.$$

By Lemma 4.2 and $\mathbb{P}(F^-(F(X)) < X) = \mathbb{P}(X \in \Theta^c)$, we get $\tilde{F}(x) = \mathbb{P}\{F^-(F(X)) = X \leq x\} = F(x)$. This implies $\operatorname{Ran}\tilde{F} = I$. Then

$$\begin{aligned} C_{F^-(F(X)),f(F(X))}(\tilde{F}(x), \tilde{G}(y)) &= \mathbb{P}(F^-(F(X)) \leq x, f(F(X)) \leq y) \\ &= \mathbb{P}(X \leq x, f(F(X)) \leq y) \\ &= C_{X,f(F(X))}(F(x), \tilde{G}(y)) \\ &= C_{X,f(F(X))}(\tilde{F}(x), \tilde{G}(y)) \end{aligned}$$

for all $x, y \in \mathbb{R}$. Therefore, $C_{X,f(F(X))} = A$. □

Similarly, if Y is a random variable with a continuous distribution function, then

$$\{C_{f(Y),Y} | f \in \mathcal{I}\} = \mathcal{M}.$$

Thus, we have

$$\{C_{f(Y),Y} | f \in \mathcal{I}\} = \{C_{X,g(X)} | g \in \mathcal{I}\}$$

for all random variables X, Y with continuous distribution functions.

By Corollary 2.87, we have $C_{f(X),g(Y)} = C_{f(X),X} * C_{X,Y} * C_{Y,g(Y)}$ for all Borel measurable functions f, g. Thus $\sigma^*_\varphi(X, Y)$ in (4.1) can be expressed as follows:

$$\sigma^*_\varphi(X, Y) = \sup_{S_1,S_2 \in \mathcal{M}} \frac{\int_{I^2} \varphi(|S_1 * C_{X,Y} * S_2 - \Pi|) d\lambda_2}{\int_{I^2} \varphi(|M - \Pi|) d\lambda_2}. \tag{4.3}$$

The following corollary is key to prove that the maximum value of $\sigma^*_\varphi(C)$ occurs when C is left invertible or right invertible.

Corollary 4.4. *For each copula C, there exists a sequence of invertible copulas $(C_n)_{n\in\mathbb{N}}$ such that C_n converges uniformly to C.*

Proof. Let C be a copula. Since the shuffles of Min are dense in the set of bivariate copulas [Corollary 2.90], there exists a sequence of shuffles of Min converging uniformly to C. Because the shuffles of Min are invertible [Remark 2.91], the proof is complete. □

Theorem 4.5. *The function σ_φ^* satisfies the following properties for all random variables X, Y with continuous distribution functions.*

(i) $\sigma_\varphi^*(X, Y) = \sigma_\varphi^*(Y, X)$.

(ii) $0 \le \sigma_\varphi^*(X, Y) \le 1$

(iii) $\sigma_\varphi^*(X, Y) = 0$ *if and only if* X *and* Y *are independent.*

(iv) $\sigma_\varphi^*(X, Y) = 1$ *if* $Y = f(X)$ *or* $X = g(Y)$ *for some Borel measurable functions* f, g.

(v) $\sigma_\varphi^*(f(X), g(Y)) = \sigma_\varphi^*(X, Y)$ *for all Borel measurable injective functions* f *and* g.

Proof. (i) Because σ_φ is symmetric and the supremum is taken over functions in the same set, we have $\sigma_\varphi^*(X, Y) = \sigma_\varphi^*(Y, X)$ for all X and Y.

(ii) As $0 \le \sigma_\varphi \le 1$ for all X and Y, the proof is done.

(iii) Assume that X and Y are independent. Then $f(X)$ and $g(Y)$ are independent. It implies that $\sigma_\varphi(f(X), g(Y)) = 0$ for all $f, g \in \mathcal{I}$. Thus $\sigma_\varphi^*(X, Y) = 0$. Conversely, if $\sigma_\varphi^*(X, Y) = 0$, then we have $\sigma_\varphi(f(X), g(Y)) = 0$ for all $f, g \in \mathcal{I}$. Choose f and g as the identity function. Then $\sigma_\varphi(X, Y) = 0$. Thus X and Y are independent.

(iv) WLOG, we assume $Y = f(X)$ for some measurable function f. Using Theorem 2.89, $C_{X,Y}$ is left invertible. By Corollary 4.4, we can find a sequence $(C_n)_{n\in\mathbb{N}}$ of invertible copulas such that C_n converges uniformly to $C_{X,Y}$. Then C_n^T also converges uniformly to $C_{X,Y}^T$. By the continuity of the $*$-product in each coordinate, we have that $C_n^T * C_{X,Y}$ converges uniformly to $C_{X,Y}^T * C_{X,Y} = M$. Using DCT and continuity of φ, $\sigma_\varphi^*(C_{X,Y}) = 1$.

(v) Since $\sigma_\varphi^*(f(X), g(Y)) = \sigma_\varphi^*(X, Y)$ for all Borel measurable injective functions f and g, then the proof is done.

$\square$

Example 4.6. Let $1 < p < \infty$, and $\varphi(t) = t^p$ for all $t \in I$. Then $\sigma^*_\varphi(X,Y)$ for all random variables X and Y whose distribution functions are continuous defined by

$$\begin{aligned}\sigma^*_\varphi(X,Y) &= \sup_{f,g\in\mathcal{I}} \left(\frac{\int_{I^2} |C_{f(X),g(Y)} - \Pi|^p d\lambda_2}{\int_{I^2} |M - \Pi|^p d\lambda_2} \right) \\ &= \sup_{S_1,S_2\in\mathcal{M}} \left(\frac{\int_{I^2} |S_1 * C_{X,Y} * S_2 - \Pi|^p d\lambda_2}{\int_{I^2} |M - \Pi|^p d\lambda_2} \right)\end{aligned}$$

is a measure of dependence in the sense of Rényi.

CHAPTER V

D_4-INVARIANT COPULAS

Given a copula A, we first introduce a function σ_A on the class of bivariate copulas defined by

$$\sigma_A(C) = \frac{\int_{I^2} |C - \Pi|(u,v)dA(u,v)}{\int_{I^2} |M - \Pi|(u,v)dA(u,v)}.$$

Schweizer and Wolff proved that σ_Π is a measure of dependence (Theorem 2.80). The aim of this chapter is to show that $\sigma_{\frac{M+W}{2}}$ is a measure of dependence satisfying the same set of properties.

We begin with studying properties of σ_A. When A is a D_4-invariant copula, σ_A is symmetric and invariant under strictly monotone functions. The following lemma is a necessary tool for obtaining symmetric and invariance properties of σ_A.

For any random variables X and Y with continuous distribution functions, a copula of X and Y is denoted by $C_{X,Y}$.

Lemma 5.1. *Let X and Y be random variables with continuous distribution functions. If f and g are strictly monotone on* $\operatorname{Ran} X$ *and* $\operatorname{Ran} Y$, *respectively, then the following statements hold for all copulas A.*

(i) $\int_{I^2} |C^T_{X,Y} - \Pi|(u,v)dA(u,v) = \int_{I^2} |C_{X,Y} - \Pi|(u,v)dA^T(u,v)$.

(ii) $\int_{I^2} |C_{f(X),g(Y)} - \Pi|(u,v)dA(u,v) = \int_{I^2} |C_{X,Y} - \Pi|(u,v)dA^{hr^2}(u,v)$ *if f is strictly increasing on* $\operatorname{Ran} X$ *and g is strictly decreasing on* $\operatorname{Ran} Y$.

(iii) $\int_{I^2} |C_{f(X),g(Y)} - \Pi|(u,v)dA(u,v) = \int_{I^2} |C_{X,Y} - \Pi|dA^h(u,v)$ *if f is strictly decreasing on* $\operatorname{Ran} X$ *and g is strictly increasing on* $\operatorname{Ran} Y$.

(iv) $\int_{I^2} |C_{f(X),g(Y)} - \Pi|(u,v)dA(u,v) = \int_{I^2} |C_{X,Y} - \Pi|(u,v)dA^{r^2}(u,v)$ *if both f and g are strictly decreasing on* $\operatorname{Ran} X$ *and* $\operatorname{Ran} Y$, *respectively.*

Consequently, the following are true: if A is a D_4-invariant copula, then

(i') $\int_{I^2} |C^T_{X,Y} - \Pi|(u,v)dA(u,v) = \int_{I^2} |C_{X,Y} - \Pi|(u,v)dA(u,v)$,

(ii') $\int_{I^2} |C_{f(X),g(Y)} - \Pi|(u,v)dA(u,v) = \int_{I^2} |C_{X,Y} - \Pi|(u,v)dA(u,v)$.

Proof. (i$'$) and (ii$'$) clearly follow from (i–iv). Only (i) needs proof. Using the symmetry of Π, we have

$$\int_{I^2} |C^T_{X,Y} - \Pi|(u,v)dA(u,v) = \int_{I^2} |C_{X,Y} - \Pi|^T(u,v)dA(u,v).$$

We claim that $\int_{I^2} |C_{X,Y} - \Pi|^T(u,v)dA(u,v) = \int_{I^2} |C_{X,Y} - \Pi|(u,v)dA^T(u,v)$.

Define a function $G : \mathbb{R}^2 \to \mathbb{R}^2$ via $G(u,v) = (v,u)$ for all $u, v \in \mathbb{R}$, which is clearly a measurable function. By Theorem 2.31, we obtain that

$$\begin{aligned}\int_{I^2} |C_{X,Y} - \Pi|^T(u,v)dA(u,v) &= \int_{I^2} |C_{X,Y} - \Pi| \circ G(u,v)dA(u,v) \\ &= \int_{I^2} |C_{X,Y} - \Pi| \circ G(u,v)d\mu_A(u,v) \\ &= \int_{I^2} |C_{X,Y} - \Pi|(u,v)d\mu_A \circ G^{-1}(u,v) \\ &= \int_{I^2} |C_{X,Y} - \Pi|(u,v)d\mu_{A^T}(u,v).\end{aligned}$$

By Example 2.78 (i), $\mu_A \circ G^{-1} = \mu_{A^T}$ on the set of all closed rectangles in I^2. Note that a collection of all closed rectangles of I^2 containing $\varnothing$ is a π-system and $\mu_A \circ G^{-1}$ is a probability measure on $\mathcal{B}(I^2)$. By Corollary 2.34, we get $\mu_A \circ G^{-1} = \mu_{A^T}$ on $\mathcal{B}(I^2)$. Hence, the last equation holds and we have proved (i).

To prove (ii), (iii) and (iv), we use the same process as we did in the proof of (i) and define a measurable function $G : \mathbb{R}^2 \to \mathbb{R}^2$ in (ii), (iii) and (iv) by for all $u, v \in \mathbb{R}$, $G(u,v) = (u, 1-v)$, $G(u,v) = (1-u, v)$ and $G(u,v) = (1-u, 1-v)$, respectively. □

Note that $A^T = A^{hr}$ and Lemma 5.1 (i) can also be written as

$$\int_{I^2} |C^{hr} - \Pi|(u,v)dA(u,v) = \int_{I^2} |C - \Pi|(u,v)dA^{hr}(u,v).$$

Next, we determine the value of $\sigma_A(M)$ and $\sigma_A(W)$ by using the following readily verified facts.

Remark 5.2. (i) $(M - \Pi)(u,v) = (\Pi - W)(u, 1-v)$ for all $u, v \in I$.

(ii) $(M - \Pi)(u,v) = (\Pi - W)(1-u, v)$ for all $u, v \in I$.

Corollary 5.3. *For all D_4-invariant copulas A,*

$$\int_{I^2} (M - \Pi)(u,v)dA(u,v) = \int_{I^2} (\Pi - W)(u,v)dA(u,v).$$

Proof. The proof uses Remark 5.2 (i) and Lemma 5.1 (ii). □

The idea of constructing the checkerboard copulas in [12] is shown as follows. Let $n \in \mathbb{N}$ and denote $I_1 = [0, \frac{1}{n}]$ and $I_i = (\frac{i-1}{n}, \frac{i}{n}]$ for all $i = 2, 3, \ldots, n$. Observe that $\{I_r \times I_s : 1 \leq r, s \leq n\}$ is a partition of I^2. Let $p = (p_1, p_2) \in I_i \times I_j$ where $1 < i, j \leq n$. For all $r, s \in \{1, 2, 3, \ldots, n\}$, the probability mass of the checkerboard copula P_p^n and the probability mass of the checkerboard copula N_p^n on $I_r \times I_s$ are defined by

$$\delta^{P_p^n}{}_{r,s} = \begin{cases} 0 & \text{if } (r,s) = (1,j) \text{ or } (r,s) = (i,1) \\ \frac{2}{n^2} & \text{if } (r,s) = (1,1) \text{ or } (r,s) = (i,j) \\ \frac{1}{n^2} & \text{otherwise} \end{cases} \tag{5.1}$$

and

$$\delta^{N_p^n}{}_{r,s} = \begin{cases} \frac{2}{n^2} & \text{if } (r,s) = (1,j) \text{ or } (r,s) = (i,1) \\ 0 & \text{if } (r,s) = (1,1) \text{ or } (r,s) = (i,j) \\ \frac{1}{n^2} & \text{otherwise.} \end{cases} \tag{5.2}$$

Then for all $u, v \in I$,

$$P_p^n(u,v) = \sum_{r=1}^{n} \sum_{s=1}^{n} \delta^{P_p^n}{}_{r,s} \Pi(F_r(u), F_s(v)) \tag{5.3}$$

and

$$N_p^n(u,v) = \sum_{r=1}^{n} \sum_{s=1}^{n} \delta^{N_p^n}{}_{r,s} \Pi(F_r(u), F_s(v)) \tag{5.4}$$

where $F_m(x) = \min(\max(nx - m + 1, 0), 1)$ for all $x \in I$ and for all $m = 1, 2, \ldots, n$.

Remark 5.4. For each $n \in \mathbb{N}$, $\Pi(u,v) = \sum_{r=1}^{n} \sum_{s=1}^{n} \frac{1}{n^2} \Pi(F_r(u), F_s(v))$ for all $u, v \in I$.

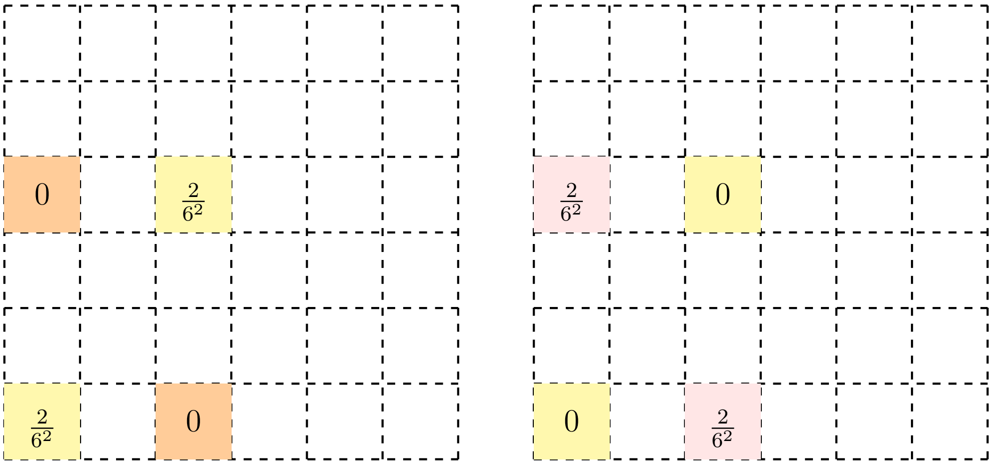


Figure 5.1: Fixed $p \in I_3 \times I_4$, the checkerboard copulas P_p^6 (on the left) and N_p^6 (on the right) whose density in each member of the partition is $\frac{1}{6^2}$ except on $I_1 \times I_1, I_1 \times I_4, I_3 \times I_1$ and $I_3 \times I_4$.

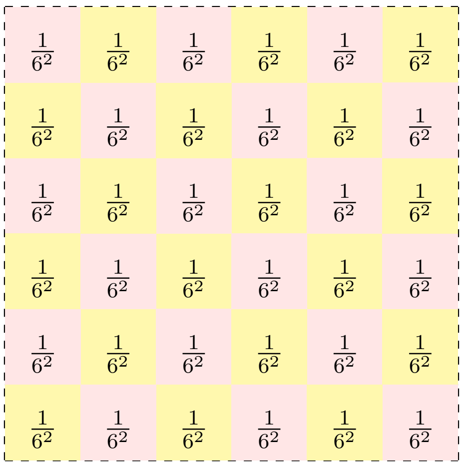


Figure 5.2: The independent copula Π.

Corollary 5.5. *Let $n \in \mathbb{N}$. If $p \in I_i \times I_j$ where $1 < i, j \leq n$, then P_p^n is PQD.*

Proof. Our claim is to prove that $P_p(u, v) \geq \Pi(u, v)$ for all $u, v \in I$. We divide the proof into four steps.

Step 1: Claim that $P_p^n(u, v) \geq \Pi(u, v)$ for all $(u, v) \in \left(\bigcup_{d<i} I_d\right) \times \left(\bigcup_{e<j} I_e\right)$. From the definition of P_p^n in (5.3) and Remark 5.4, if $(u, v) \in I_1 \times I_1$, then it is obvious that $P_p^n(u, v) \geq \Pi(u, v)$. Because the probability masses of both P_p^n and Π on $I_d \times I_e$ are $\frac{1}{n^2}$ for all $d < i$ and $e < j$ such that $(d, e) \neq (1, 1)$, we have $P_p^n(u, v) \geq \Pi(u, v)$ for all $(u, v) \in \left(\bigcup_{d<i} I_d\right) \times \left(\bigcup_{e<j} I_e\right)$.

Step 2: Claim that $P_p^n \geq \Pi$ on $I_1 \times I_j$ and $I_i \times I_1$. Let $(u,v) \in I_1 \times I_j$. Since $v \leq \frac{j}{n}$, we get

$$\begin{aligned}
P_p^n(u,v) &= \sum_{s=1}^{j} \delta^{P_p}{}_{1,s}\Pi(F_1(u), F_s(v)) \\
&= \sum_{s=1}^{j-1} \delta^{P_p}{}_{1,s}\Pi(F_1(u), F_s(v)) \\
&= \delta^{P_p}{}_{1,1}\Pi(F_1(u), F_1(v)) + \sum_{s=2}^{j-1} \delta^{P_p}{}_{1,s}\Pi(F_1(u), F_s(v)) \\
&= \delta^{P_p}{}_{1,1}F_1(u) + \sum_{s=2}^{j-1} \delta^{P_p}{}_{1,s}F_1(u) \\
&= \frac{2}{n^2}F_1(u) + \sum_{s=2}^{j-1} \frac{1}{n^2}F_1(u) \\
&= \frac{j}{n^2}F_1(u) \\
&= \frac{jnu}{n^2} \\
&= \frac{ju}{n} \\
&\geq uv.
\end{aligned}$$

Similarly, if $(u,v) \in I_i \times I_1$, then $P_p^n(u,v) \geq \Pi(u,v)$.

Step 3: Claim that $P_p^n(u,v) \geq \Pi(u,v)$ for all $(u,v) \in \left(\bigcup_{d\leq i} I_d\right) \times \left(\bigcup_{e\leq j} I_e\right)$. We have $P_p^n(u,v) \geq \Pi(u,v)$ for all $(u,v) \in \Big(\left(\bigcup_{d\leq i} I_d\right) \times \left(\bigcup_{e\leq j} I_e\right)\Big) \smallsetminus I_i \times I_j$, which directly follows from Step 1, Step 2 and the probability masses of both P_p^n and Π on $I_d \times I_j$ and $I_i \times I_e$ that are $\frac{1}{n^2}$ for all $1 < d < i$ and $1 < e < j$. As the probability mass of P_p^n on $I_i \times I_j$ is $\frac{2}{n^2}$, we have $P_p^n(u,v) \geq \Pi(u,v)$ for all $(u,v) \in \left(\bigcup_{d\leq i} I_d\right) \times \left(\bigcup_{e\leq j} I_e\right)$.

Step 4: Claim that $P_p^n(u,v) \geq \Pi(u,v)$ for all $(u,v) \in I^2$. It obviously follows from Step 3 and the probability masses of both P_p^n and Π on $\{I_d \times I_e : 1 \leq d, e \leq n\} \smallsetminus \{I_d \times I_e : 1 \leq d \leq i, 1 \leq e \leq j\}$ that are $\frac{1}{n^2}$.

$\square$

Corollary 5.6. *Let $n \in \mathbb{N}$. If $p \in I_i \times I_j$ where $1 < i, j \leq n$, then N_p^n is NQD.*

Proof. The proof is divided into four steps like Lemma 5.5 by swapping the direction of an inequality symbol. It suffices to show that Step 2 holds; that is, $P_p^n \leq \Pi$ on $I_1 \times I_j$ and $I_i \times I_1$. Let $(u,v) \in I_1 \times I_j$. By using $v \leq \frac{j}{n}$, we have

$$\begin{aligned}
P_p(u,v) &= \sum_{s=1}^{j} \delta^{P_p}{}_{1,s}\Pi(F_1(u), F_s(v)) \\
&= \sum_{s=2}^{j} \delta^{P_p}{}_{1,s}\Pi(F_1(u), F_s(v)) \\
&= \delta^{P_p}{}_{1,j}\Pi(F_1(u), F_j(v)) + \sum_{s=2}^{j-1} \delta^{P_p}{}_{1,s}\Pi(F_1(u), F_s(v)) \\
&= \delta^{P_p}{}_{1,1}F_1(u)F_j(v) + \sum_{s=2}^{j-1} \delta^{P_p}{}_{1,s}F_1(u) \\
&= \frac{2}{n^2}F_1(u)(nv - j + 1) + \sum_{s=2}^{j-1} \frac{1}{n^2}F_1(u) \\
&= \frac{2nv - j}{n^2}F_1(u) \\
&= \frac{(2nv - j)(nu)}{n^2} \\
&= u(2v - \frac{j}{n}) \\
&\leq uv.
\end{aligned}$$

Similarly, if $(u,v) \in I_i \times I_1$, then $P_p(u,v) \leq \Pi(u,v)$. $\square$

Lemma 5.7 ([12], Lemma 2.2)**.** $\int_{I^2} AdB = \int_{I^2} BdA$ *for all copulas* A, B.

Denote the set of all PQD copulas and the class of all NQD copulas by $\mathcal{P}$ and $\mathcal{N}$, respectively.

Lemma 5.8. *If for every copula* A, $\int_{I^2} |C - \Pi| dA = \int_{I^2} |C - \Pi| dA^T$ *for all* $C \in \mathcal{P} \cup \mathcal{N}$, *then* $A = A^T$.

Proof. Clearly, $A(u,v) = A^T(u,v)$ on the boundary. Let $p = (p_1, p_2) \in (0,1)^2$. By Lemma 5.7 and the assumption,

$$\int_{I^2} (A - A^T) dC = \int_{I^2} \Pi dA - \int_{I^2} \Pi dA^T$$

for all $C \in \mathcal{P} \cup \mathcal{N}$. Consequently,

$$\begin{aligned} 0 &= \int_{I^2} (A - A^T) d(P_p^n - N_p^n) \\ &= 2\Big(\int_{I_1 \times I_1 \cup I_i \times I_j} (A - A^T) d\Pi - \int_{I_1 \times I_j \cup I_i \times I_1} (A - A^T) d\Pi \Big) \end{aligned}$$

for all $n \in \mathbb{N}$ such that $\frac{1}{n} < \min\{p_1, p_2\}$. Using the mean value theorem (Theorem 2.30), we have

$$\int_{I_r \times I_s} (A - A^T) d\Pi = (A - A^T)(p_{r,s}) \lambda(I_r \times I_s) = \frac{(A - A^T)(p_{r,s})}{n^2}$$

for some $p_{r,s} \in I_r \times I_s$. Then

$$(A - A^T)(p_{1,1}) + (A - A^T)(p_{i,j}) - (A - A^T)(p_{1,j}) - (A - A^T)(p_{i,1}) = 0$$

Note that $\lim_{n\to\infty} p_{1,1} = (0,0)$, $\lim_{n\to\infty} p_{1,j} = (0, p_2)$, $\lim_{n\to\infty} p_{i,1} = (p_1, 0)$ and $\lim_{n\to\infty} p_{i,j} = (p_1, p_2)$. Using the continuity of copulas, we obtain

$$\begin{aligned} 0 &= \lim_{n\to\infty} (A - A^T)(p_{1,1}) + (A - A^T)(p_{i,j}) - (A - A^T)(p_{1,j}) - (A - A^T)(p_{i,1}) \\ &= (A - A^T)(p). \end{aligned}$$

Hence, $A(p) = A^T(p)$. □

Similarly, for every copula A, we can prove that $A = A^h$ if $\int_{I^2} |C - \Pi| dA = \int_{I^2} |C - \Pi| dA^h$ for all $C \in \mathcal{P} \cup \mathcal{N}$.

Given a copula A, the following lemma shows that the symmetric and invariance properties of σ_A leads to the conclusion that A is D_4-invariant.

Lemma 5.9. *Let A be a copula. If σ_A is a measure of dependence in the sense of Schweizer and Wolff, then A is D_4-invariant.*

Proof. Since σ_A is a measure of dependence, σ_A is symmetric and invariant under monotone functions. Using Lemma 5.1 (i, iii) and Lemma 5.8, we have $A = A^T = A^{hr}$ and $A = A^h$. This proof is completed by using the facts that $h(hr) = h^2(r) = r$ and $D_4 = \langle r, h \rangle$. □

We will close this chapter by showing that the converse of Lemma 5.9 still holds if we consider a D_4-invariant copula A as $\frac{M+W}{2}$. The hardest task is to prove that $\sigma_{\frac{M+W}{2}}$ satisfies the boundedness condition. Hence, we will start the process of solving this problem with checkerboard approximations. This can help us see the relationship between the integral with respect to $\frac{M+W}{2}$ and Lebesgue measure on I^2.

Fix $n \in \mathbb{N}$. By Remark 2.82, a checkerboard approximation C_{2^n} of $\dfrac{M+W}{2}$ is shown as follows: for all $u, v \in I$,

$$C_{2^n}(u,v) = \sum_{i=1}^{2^n} \frac{1}{2^{n+1}}[\Pi(F_i(u), F_i(v)) + \Pi(F_i(u), F_{2^n-i+1}(v))] \tag{5.5}$$

where F_i denotes the uniform distribution on $I_i = [\frac{i-1}{2^n}, \frac{i}{2^n}]$ for all $i = 1, 2, \ldots, 2^n$.

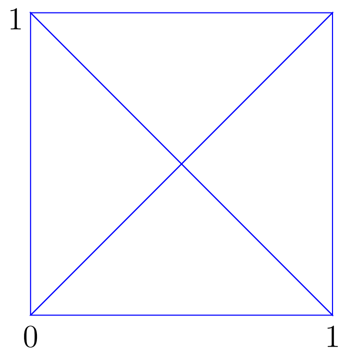


Figure 5.3: The support of $\dfrac{M+W}{2}$ on I^2.

The following lemma gives a relationship between the measure induced by a checkerboard copula and the Lebesgue measure on I^2.

Lemma 5.10. *Let $n \in \mathbb{N}$. Then*

$$\mu_{C_{2^n}} = 2^{n-1} \sum_{i=1}^{2^{n-1}} [\lambda^2_{i,i} + \lambda^2_{i,2^n-i+1} + \lambda^2_{2^n-i+1,i} + \lambda^2_{2^n-i+1,2^n-i+1}]$$

on $\mathcal{B}(I^2)$ where $\lambda^2_{i,j}$ is the Lebesgue measure on $I_i \times I_j$.

Proof. Note that

$$F_i(u) = \begin{cases} 0 & \text{if } u \leq \frac{i-1}{2^n} \\ 2^n u - i + 1 & \text{if } u \in I_i \\ 1 & \text{if } u \geq \frac{i}{2^n}. \end{cases}$$

Since $\mu_{C_{2^n}}(A) = 0$ where $A \subseteq [0,1]^2 \setminus [I_i \cup I_{2^n+1-i}]^2$ and $i = 1, 2, \ldots, 2^{n-1}$, it is enough to consider all closed rectangles $B \subseteq I_i \times I_i, I_i \times I_{2^n+1-i}, I_{2^n+1-i} \times I_i$ or $I_{2^n+1-i} \times I_{2^n+1-i}$ for $i = 1, 2, \ldots, 2^{n-1}$. Then

$$\begin{aligned}
\mu_{C_{2^n}}([a_1, a_2] \times [b_1, b_2]) &= V_{C_{2^n}}([a_1, a_2] \times [b_1, b_2]) \\
&= \frac{1}{2^{n+1}}[C_{2^n}(a_1, b_1) + C_{2^n}(a_2, b_2) - C_{2^n}(a_1, b_2) - C_{2^n}(a_2, b_1)] \\
&= \frac{1}{2^{n+1}}[F_i(a_1)F_i(b_1) + F_i(a_2)F_i(b_2) - F_i(a_1)F_i(b_2) - F_i(a_2)F_i(b_1)] \\
&= \frac{1}{2^{n+1}}[F_i(a_2) - F_i(a_1)][F_i(b_2) - F_i(b_1)] \\
&= \frac{2^{2n}}{2^{n+1}}(a_2 - a_1)(b_2 - b_1) \\
&= 2^{n-1}(a_2 - a_1)(b_2 - b_1) \\
&= 2^{n-1} \int_{I_i \times I_i} \chi_{[a_1,a_2]\times[b_1,b_2]} d\lambda_2 \\
&= 2^{n-1} \lambda^2_{i,i}([a_1, a_2] \times [b_1, b_2]). \qquad (5.6)
\end{aligned}$$

Similarly, we have for all $i = 1, 2, \ldots, 2^{n-1}$,

(i) $\mu_{C_{2^n}}(B) = 2^{n-1}\lambda^2_{i,2^n-i+1}(B)$ for all closed rectangles $B \subseteq I_i \times I_{2^n+1-i}$,

(ii) $\mu_{C_{2^n}}(B) = 2^{n-1}\lambda^2_{2^n-i+1,i}(B)$ for all closed rectangles $B \subseteq I_{2^n+1-i} \times I_i$,

(iii) $\mu_{C_{2^n}}(B) = 2^{n-1}\lambda^2_{2^n-i+1,2^n-i+1}(B)$ for all closed rectangles $B \subseteq I_{2^n+1-i} \times I_{2^n-i+1}$.

Let B be any closed rectangle of I^2. Using (3.6) and (i) - (iii), we have

$$\begin{aligned}
\mu_{C_{2^n}}(B) &= \sum_{i}^{2^{n-1}} \mu_{C_{2^n}}(B \cap R_i) \\
&= 2^{n-1} \sum_{i=1}^{2^{n-1}} \int_{[I_i \cup I_{2^n+1-i}]^2} \chi_B d\lambda_2 \\
&= 2^{n-1} \sum_{i=1}^{2^{n-1}} [\lambda^2_{i,i} + \lambda^2_{i,2^n-i+1} + \lambda^2_{2^n-i+1,i} + \lambda^2_{2^n-i+1,2^n-i+1}](B)
\end{aligned}$$

where $R_i \in \{[I_i \cup I_{2^n+1-i}]^2 : i = 1, 2, \ldots, 2^{n-1}\}$. Note that a collection of all closed rectangles of I^2 is a π-system. By Corollary 2.34, we have

$$\mu_{C_{2^n}} = 2^{n-1} \sum_{i=1}^{2^{n-1}} [\lambda^2_{i,i} + \lambda^2_{i,2^n-i+1} + \lambda^2_{2^n-i+1,i} + \lambda^2_{2^n-i+1,2^n-i+1}]$$

on $\mathcal{B}(I^2)$. Therefore, the proof is complete. □

Corollary 5.11. *Let f be any bounded measurable function. Then*

$$\int_{I^2} f d\mu_{C_{2^n}} = 2^{n-1} \sum_{i=1}^{2^{n-1}} \int_{[I_i \cup I_{2^n-i+1}]^2} f d\lambda_2$$

for all $n \in \mathbb{N}$.

Proof. By Remark 2.26 and Lemma 5.10, we have

$$\begin{aligned}
&\int_{I^2} f d\mu_{C_{2^n}} \\
&= 2^{n-1} \sum_{i=1}^{2^{n-1}} \left(\int_{I^2} f d\lambda^2_{i,i} + \int_{I^2} f d\lambda^2_{i,2^n-i+1} + \int_{I^2} f d\lambda^2_{2^n-i+1,i} + \int_{I^2} f d\lambda^2_{2^n-i+1,2^n-i+1} \right) \\
&= 2^{n-1} \sum_{i=1}^{2^{n-1}} \int_{[I_i \cup I_{2^n+1-i}]^2} f d\lambda_2.
\end{aligned}$$

□

The support of C_{2^3} is illustrated in Figure 5.4.

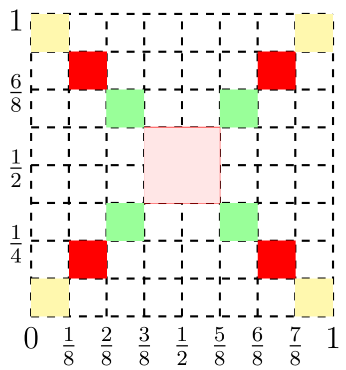


Figure 5.4: The region of integration in case of C_{2^3}.

Next proposition plays a key role in the way to prove boundedness in Theorem 2.80. We will give an original proof and also propose a new approach to prove it.

Proposition 5.12 ([11]). *For any copula C, $\int_{I^2} |C - \Pi| d\lambda_2 \leq \int_{I^2} |M - \Pi| d\lambda_2$.*

Proof. Use Lemma 3.1 when $\varphi(u) = |u|$ for all $u \in I$. $\square$

The following lemma is the crucial ingredient in the new proof of Proposition 5.12. In addition, it is the core of our proof of the boundedness of $\sigma_{\frac{M+W}{2}}$. Indeed, it is bounded by $\sigma_{\frac{M+W}{2}}(C)$ at $C = M$ or $C = W$.

Lemma 5.13. *Let C be any copula. Then*

$$
\begin{aligned}
&|C - \Pi|(u, v) + |C - \Pi|(1 - u, v) + |C - \Pi|(u, 1 - v) + |C - \Pi|(1 - u, 1 - v) \\
&\leq |M - \Pi|(u, v) + |M - \Pi|(1 - u, v) + |M - \Pi|(u, 1 - v) + |M - \Pi|(1 - u, 1 - v)
\end{aligned}
$$

for all $u, v \in I$.

Proof. Let C be a copula. By the symmetry with respect to $u = \frac{1}{2}$ and $v = \frac{1}{2}$ of the sum on both sides, it suffices to show the inequality for $u, v \in [0, \frac{1}{2}]$. If $u \leq v$, then the right-hand side is

$$
\begin{aligned}
&|M - \Pi|(u, v) + |M - \Pi|(1 - u, v) + |M - \Pi|(u, 1 - v) + |M - \Pi|(1 - u, 1 - v) \\
&= u - uv + v - v(1 - u) + u - u(1 - v) + 1 - v - (1 - u)(1 - v) \\
&= 2u.
\end{aligned}
$$

Similarly, if $v < u$, then

$$|M - \Pi|(u, v) + |M - \Pi|(1 - u, v) + |M - \Pi|(u, 1 - v) + |M - \Pi|(1 - u, 1 - v) = 2v.$$

That is, the right-hand side is equal to $2 \min\{u, v\}$. Define $f_1(u, v) = C(u, v)$, $f_2(u, v) = C(1 - u, v)$, $f_3(u, v) = C(u, 1 - v)$ and $f_4(u, v) = C(1 - u, 1 - v)$ for all $u, v \in [0, \frac{1}{2}]$. We also define $\Pi_1(u, v) = \Pi(u, v), \Pi_2(u, v) = \Pi(1 - u, v)$, $\Pi_3(u, v) = \Pi(u, 1 - v)$ and $\Pi_4(u, v) = \Pi(1 - u, 1 - v)$ for all $u, v \in [0, \frac{1}{2}]$. We will need the following properties, which hold for all $u, v \in [0, \frac{1}{2}]$ in the proof.

(i) $C(u, v) \leq M(u, v)$.

(ii) $-C(u, v) \leq 0$.

(iii) $-C(1-u,v) \leq -C(1-u,1-v)-2v+1$ because $V_C([1-u,1]\times[v,1-v]) \geq 0$.

(iv) $-C(u,1-v) \leq -C(u,v)$ as $V_C([0,u]\times[v,1-v]) \geq 0$.

(v) $-C(1-u,1-v) \leq u+v-1$ since $C(1-u,1-v) \geq W(1-u,1-v)$.

Our claim is that for all $u,v \in [0,\frac{1}{2}]$,

$$\sum_{i=1}^{4} |f_i - \Pi_i|(u,v) \leq 2\min\{u,v\}.$$

If $u \leq v$, we will consider all possible cases as follows:

Case 1: $f_i \geq \Pi_i$ for all $i = 1,2,3,4$. Note that the sum of Π at the four vertices of $[u,1-u]\times[v,1-v]$ is equal to 1. Using (i), we have

$$\begin{aligned}
&|C-\Pi|(u,v) + |C-\Pi|(1-u,v) + |C-\Pi|(u,1-v) + |C-\Pi|(1-u,1-v) \\
&= C(u,v) + C(1-u,v) + C(u,1-v) + C(1-u,1-v) - 1 \\
&\leq u+v+u+1-v-1 \\
&= 2u.
\end{aligned}$$

Case 2: $f_i \geq \Pi_i$ for all $i = 1,2,3$ and $f_4 \leq \Pi_4$. Using (i) and (v), we have

$$\begin{aligned}
&|C-\Pi|(u,v) + |C-\Pi|(1-u,v) + |C-\Pi|(u,1-v) + |C-\Pi|(1-u,1-v) \\
&= C(u,v) + C(1-u,v) + C(u,1-v) - C(1-u,1-v) - uv - (1-u)v \\
&\quad - u(1-v) + (1-u)(1-v) \\
&= C(u,v) + C(1-u,v) + C(u,1-v) - C(1-u,1-v) + 1 - 2u - 2v + 2uv \\
&\leq u+v+u+u+v-1+1-2u-2v+2uv \\
&= u+2uv \\
&\leq 2u.
\end{aligned}$$

Case 3: $f_i \geq \Pi_i$ for all $i = 1, 2, 4$ and $f_3 \leq \Pi_3$. Using (iv) and (i), we have

$$
\begin{aligned}
&|C-\Pi|(u,v) + |C-\Pi|(1-u,v) + |C-\Pi|(u,1-v) + |C-\Pi|(1-u,1-v) \\
&= C(u,v) + C(1-u,v) - C(u,1-v) + C(1-u,1-v) - uv - (1-u)v \\
&\quad + u(1-v) - (1-u)(1-v) \\
&= C(u,v) + C(1-u,v) - C(u,1-v) + C(1-u,1-v) + 2u - 1 - 2uv \\
&\leq C(u,v) + C(1-u,v) - C(u,v) + C(1-u,1-v) + 2u - 1 - 2uv \\
&= C(1-u,v) + C(1-u,1-v) + 2u - 1 - 2uv \\
&\leq v + 1 - v + 2u - 1 - 2uv \\
&\leq 2u.
\end{aligned}
$$

Case 4: $f_i \geq \Pi_i$ for all $i = 1, 3, 4$ and $f_2 \leq \Pi_2$. Using (iii) and (i), we have

$$
\begin{aligned}
&|C-\Pi|(u,v) + |C-\Pi|(1-u,v) + |C-\Pi|(u,1-v) + |C-\Pi|(1-u,1-v) \\
&= C(u,v) - C(1-u,v) + C(u,1-v) + C(1-u,1-v) - uv + (1-u)v \\
&\quad - u(1-v) - (1-u)(1-v) \\
&= C(u,v) - C(1-u,v) + C(u,1-v) + C(1-u,1-v) - 2uv + 2v - 1 \\
&\leq C(u,v) - C(1-u,1-v) - 2v + 1 + C(u,1-v) + C(1-u,1-v) \\
&\qquad - 2uv + 2v - 1 \\
&\leq u + u - 2uv \\
&\leq 2u.
\end{aligned}
$$

Case 5: $f_i \geq \Pi_i$ for all $i = 2, 3, 4$ and $f_1 \leq \Pi_1$. Using (ii) and (i), we have

$$
\begin{aligned}
&|C-\Pi|(u,v) + |C-\Pi|(1-u,v) + |C-\Pi|(u,1-v) + |C-\Pi|(1-u,1-v) \\
&= -C(u,v) + C(1-u,v) + C(u,1-v) + C(1-u,1-v) + uv - (1-u)v \\
&\quad - u(1-v) - (1-u)(1-v) \\
&= -C(u,v) + C(1-u,v) + C(u,1-v) + C(1-u,1-v) + 2uv - 1 \\
&\leq 0 + v + u + 1 - v + 2uv - 1 \\
&= u + 2uv \\
&\leq 2u.
\end{aligned}
$$

Case 6: $f_i \geq \Pi_i$ for all $i = 1, 2$ and $f_i \leq \Pi_i$ for all $i = 3, 4$. Using (iv), (v) and (i), we have

$$
\begin{aligned}
&|C - \Pi|(u, v) + |C - \Pi|(1 - u, v) + |C - \Pi|(u, 1 - v) + |C - \Pi|(1 - u, 1 - v) \\
&= C(u, v) + C(1 - u, v) - C(u, 1 - v) - C(1 - u, 1 - v) - uv - (1 - u)v \\
&\quad + u(1 - v) + (1 - u)(1 - v) \\
&= C(u, v) + C(1 - u, v) - C(u, 1 - v) - C(1 - u, 1 - v) - 2v + 1 \\
&\leq C(u, v) + C(1 - u, v) - C(u, v) - C(1 - u, 1 - v) - 2v + 1 \\
&\leq v + u + v - 1 - 2v + 1 \\
&\leq 2u.
\end{aligned}
$$

Case 7: $f_i \geq \Pi_i$ for all $i = 1, 3$ and $f_i \leq \Pi_i$ for all $i = 2, 4$. Using (iii), (v) and (i), we have

$$
\begin{aligned}
&|C - \Pi|(u, v) + |C - \Pi|(1 - u, v) + |C - \Pi|(u, 1 - v) + |C - \Pi|(1 - u, 1 - v) \\
&= C(u, v) - C(1 - u, v) + C(u, 1 - v) - C(1 - u, 1 - v) - uv + (1 - u)v \\
&\quad - u(1 - v) + (1 - u)(1 - v) \\
&= C(u, v) - C(1 - u, v) + C(u, 1 - v) - C(1 - u, 1 - v) - 2u + 1 \\
&\leq C(u, v) - 2C(1 - u, 1 - v) - 2v + 1 + C(u, 1 - v) - 2u + 1 \\
&\leq u + 2u + 2v - 2 - 2v + 1 + u - 2u + 1 \\
&= 2u.
\end{aligned}
$$

Case 8: $f_i \geq \Pi_i$ for all $i = 2, 3$ and $f_i \leq \Pi_i$ for all $i = 1, 4$. Using (ii), (v) and (i), we have

$$
\begin{aligned}
&|C-\Pi|(u,v)+|C-\Pi|(1-u,v)+|C-\Pi|(u,1-v)+|C-\Pi|(1-u,1-v)\\
&=-C(u,v)+C(1-u,v)+C(u,1-v)-C(1-u,1-v)+uv-(1-u)v\\
&\quad -u(1-v)+(1-u)(1-v)\\
&=-C(u,v)+C(1-u,v)+C(u,1-v)-C(1-u,1-v)+4uv-2u-2v+1\\
&\leq 0+v+u+u+v-1+4uv-2u-2v+1\\
&=4uv\\
&\leq 2u.
\end{aligned}
$$

Case 9: $f_i \geq \Pi_i$ for all $i=3,4$ and $f_i \leq \Pi_i$ for all $i=1,2$. Using (iii), (ii) and (i), we have

$$
\begin{aligned}
&|C-\Pi|(u,v)+|C-\Pi|(1-u,v)+|C-\Pi|(u,1-v)+|C-\Pi|(1-u,1-v)\\
&=-C(u,v)-C(1-u,v)+C(u,1-v)+C(1-u,1-v)+uv+(1-u)v\\
&\quad -u(1-v)-(1-u)(1-v)\\
&=-C(u,v)-C(1-u,v)+C(u,1-v)+C(1-u,1-v)+2v-1\\
&\leq -C(u,v)-C(1-u,1-v)-2v+1+C(u,1-v)+C(1-u,1-v)\\
&\quad\ +2v-1\\
&\leq 0-2v+1+u+2v-1\\
&\leq 2u.
\end{aligned}
$$

Case 10: $f_i \geq \Pi_i$ for all $i=1,4$ and $f_i \leq \Pi_i$ for all $i=2,3$. Using (iii) and (iv), we have

$$
\begin{aligned}
&|C-\Pi|(u,v)+|C-\Pi|(1-u,v)+|C-\Pi|(u,1-v)+|C-\Pi|(1-u,1-v)\\
&=C(u,v)-C(1-u,v)-C(u,1-v)+C(1-u,1-v)-uv+(1-u)v\\
&\quad+u(1-v)-(1-u)(1-v)\\
&=C(u,v)-C(1-u,v)-C(u,1-v)+C(1-u,1-v)-4uv+2u+2v-1\\
&\leq C(u,v)-C(1-u,1-v)-2v+1-C(u,v)+C(1-u,1-v)-4uv\\
&\quad\ +2u+2v-1\\
&=2u-4uv\\
&\leq 2u.
\end{aligned}
$$

Case 11: $f_i \geq \Pi_i$ for all $i = 2, 4$ and $f_i \leq \Pi_i$ for all $i = 1, 3$. Using (ii) and (i), we have

$$
\begin{aligned}
&|C-\Pi|(u,v)+|C-\Pi|(1-u,v)+|C-\Pi|(u,1-v)+|C-\Pi|(1-u,1-v)\\
&=-C(u,v)+C(1-u,v)-C(u,1-v)+C(1-u,1-v)+uv-(1-u)v\\
&\quad+u(1-v)-(1-u)(1-v)\\
&=-C(u,v)+C(1-u,v)-C(u,1-v)+C(1-u,1-v)+2u-1\\
&\leq 0+v+0+1-v+2u-1\\
&=2u.
\end{aligned}
$$

Case 12: $f_i \leq \Pi_i$ for all $i = 1, 2, 3$ and $f_4 \geq \Pi_4$. Using (ii), (iii) and (iv), we have

$$
\begin{aligned}
&|C-\Pi|(u,v)+|C-\Pi|(1-u,v)+|C-\Pi|(u,1-v)+|C-\Pi|(1-u,1-v)\\
&=-C(u,v)-C(1-u,v)-C(u,1-v)+C(1-u,1-v)+uv+(1-u)v\\
&\quad+u(1-v)-(1-u)(1-v)\\
&=-C(u,v)-C(1-u,v)-C(u,1-v)+C(1-u,1-v)-2uv+2u+2v-1\\
&\leq 0-C(1-u,1-v)-2v+1-C(u,v)+C(1-u,1-v)-2uv+2u+2v-1\\
&\leq 0+2u-2uv\\
&\leq 2u.
\end{aligned}
$$

Case 13: $f_i \leq \Pi_i$ for all $i = 1, 2, 4$ and $f_3 \geq \Pi_3$. Using (ii), (iii), (i) and (v), we have

$$
\begin{aligned}
&|C - \Pi|(u, v) + |C - \Pi|(1 - u, v) + |C - \Pi|(u, 1 - v) + |C - \Pi|(1 - u, 1 - v) \\
&= -C(u, v) - C(1 - u, v) + C(u, 1 - v) - C(1 - u, 1 - v) + uv + (1 - u)v \\
&\quad - u(1 - v) + (1 - u)(1 - v) \\
&= -C(u, v) - C(1 - u, v) + C(u, 1 - v) - C(1 - u, 1 - v) - 2u + 2uv + 1 \\
&\leq 0 - 2C(1 - u, 1 - v) - 2v + 1 + u - 2u + 2uv + 1 \\
&\leq 2u + 2v - 2 - 2v + 1 + u - 2u + 2uv + 1 \\
&= u + 2uv \\
&\leq 2u.
\end{aligned}
$$

Case 14: $f_i \leq \Pi_i$ for all $i = 1, 3, 4$ and $f_2 \geq \Pi_2$. Using (ii), (i), (iv) and (v), we have

$$
\begin{aligned}
&|C - \Pi|(u, v) + |C - \Pi|(1 - u, v) + |C - \Pi|(u, 1 - v) + |C - \Pi|(1 - u, 1 - v) \\
&= -C(u, v) + C(1 - u, v) - C(u, 1 - v) - C(1 - u, 1 - v) + uv - (1 - u)v \\
&\quad + u(1 - v) + (1 - u)(1 - v) \\
&= -C(u, v) + C(1 - u, v) - C(u, 1 - v) - C(1 - u, 1 - v) - 2v + 2uv + 1 \\
&\leq 0 + v - C(u, v) + u + v - 1 - 2v + 2uv + 1 \\
&\leq v + 0 + u + v - 1 - 2v + 2uv + 1 \\
&= u + 2uv \\
&\leq 2u.
\end{aligned}
$$

Case 15: $f_i \leq \Pi_i$ for all $i = 2, 3, 4$ and $f_1 \geq \Pi_1$. Using (iii), (iv), and (v), we have

$$
\begin{aligned}
&|C - \Pi|(u, v) + |C - \Pi|(1 - u, v) + |C - \Pi|(u, 1 - v) + |C - \Pi|(1 - u, 1 - v) \\
&= C(u, v) - C(1 - u, v) - C(u, 1 - v) - C(1 - u, 1 - v) - uv + (1 - u)v \\
&\quad + u(1 - v) + (1 - u)(1 - v) \\
&= C(u, v) - C(1 - u, v) - C(u, 1 - v) - C(1 - u, 1 - v) - 2uv + 1 \\
&\leq C(u, v) - 2C(1 - u, 1 - v) - 2v + 1 - C(u, v) - 2uv + 1 \\
&\leq 2u + 2v - 2 - 2v - 2uv + 2 \\
&= 2u - 2uv \\
&\leq 2u.
\end{aligned}
$$

Case 16: $f_i \leq \Pi_i$ for all $i = 1, 2, 3, 4$. Using (ii), (iii), (iv) and (v), we have

$$
\begin{aligned}
&|C - \Pi|(u, v) + |C - \Pi|(1 - u, v) + |C - \Pi|(u, 1 - v) + |C - \Pi|(1 - u, 1 - v) \\
&= -C(u, v) - C(1 - u, v) - C(u, 1 - v) - C(1 - u, 1 - v) + uv + (1 - u)v \\
&\quad + u(1 - v) + (1 - u)(1 - v) \\
&= -C(u, v) - C(1 - u, v) - C(u, 1 - v) - C(1 - u, 1 - v) + 1 \\
&\leq 0 - 2(1 - u, 1 - v) - 2v + 1 - C(u, v) + 1 \\
&\leq 2u + 2v - 2 - 2v + 1 + 0 + 1 \\
&= 2u.
\end{aligned}
$$

Hence, $\sum_{i=1}^{4} |f_i - \Pi_i|(u, v) \leq 2u$ for all $u \leq v$. Conversely, if $v < u$, then

$$
\begin{aligned}
&|C - \Pi|(u, v) + |C - \Pi|(1 - u, v) + |C - \Pi|(u, 1 - v) + |C - \Pi|(1 - u, 1 - v) \\
&= |C^T - \Pi|(v, u) + |C^T - \Pi|(v, 1 - u) + |C^T - \Pi|(1 - v, u) + |C^T - \Pi|(1 - v, 1 - u) \\
&\leq 2v.
\end{aligned}
$$

Therefore, the proof is complete. $\square$

From the above lemma, we obtain a new way to prove Proposition 5.12

Proof of Proposition 5.12. If follows from Lemma 5.13 and the simple fact that

$$\int_{I^2} f d\lambda_2 = \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \big(f(u,v) + f(u,1-v) + f(1-u,v) + f(1-u,1-v)\big) dudv$$

for all bounded continuous functions f on I^2. $\square$

There are two ways to prove boundedness of $\sigma_{\frac{M+W}{2}}$, and we thus start with proving via the checkerboard approximation.

Lemma 5.14. *For any copula C, $\int_{I^2} |C - \Pi| d\mu_{C_{2^n}} \leq \int_{I^2} |M - \Pi| d\mu_{C_{2^n}}$.*

Proof. Let C be any copula. Note that for all $i = 1, 2, \ldots, 2^{n-1}$,

$$\begin{aligned}&\int_{[I_i \cup I_{2^n+1-i}]^2} |C - \Pi|(u,v) dudv \\ &= \int_{I_i^2} \big(|C - \Pi|(u,v) + |C - \Pi|(1-u,v) + |C - \Pi|(u,1-v) + |C - \Pi|(1-u,1-v)\big) dudv.\end{aligned}$$

By Lemma 5.13, it results in

$$\int_{[I_i \cup I_{2^n+1-i}]^2} |C - \Pi| d\lambda_2 \leq \int_{[I_i \cup I_{2^n+1-i}]^2} |M - \Pi| d\lambda_2$$

for all $i = 1, 2, \ldots, 2^{n-1}$. This proof is completed by using Corollary 5.11. $\square$

Lemma 5.15. *For any bounded continuous function f on I^2,*

$$\lim_{n \to \infty} \int_{I^2} f d\mu_{C_{2^n}} = \int_{I^2} f d\mu_{\frac{M+W}{2}},$$

where $C_{2^n}(u,v) = \sum_{i=1}^{2^n} \frac{1}{2^{n+1}} [\Pi(F_i(u), F_i(v)) + \Pi(F_i(u), F_{2^n-i+1}(v))]$ *for all* $u, v \in I$.

Proof. Since any copula C induces a probability measure μ_C, $\{\mu_{C_{2^n}}\}_{n \in \mathbb{N}}$ is a sequence of probability measures. From Remark 2.62, every copula C corresponds to uniform random variables X and Y on I whose joint distribution is C, that is

$$\mu_C([u_1, u_2] \times [v_1, v_2]) = \mathbb{P}(u_1 \leq X \leq u_2, v_1 \leq Y \leq v_2).$$

Using Theorem 2.83 and Theorem 2.46, we have $\lim_{n \to \infty} \int_{I^2} f d_{\mu_{C_{2^n}}} = \int_{I^2} f d_{\mu_{\frac{M+W}{2}}}$. $\square$

Corollary 5.16. *$\sigma_{\frac{M+W}{2}}(C)$ gives its maximum if $C = M$ or $C = W$.*

Proof #1. It directly follows from Lemma 5.14, 5.15 and Corollary 5.3. $\square$

Remark 5.17. [17] For all copulas C,

(i) $\int_{I^2} CdM = \int_I C(u,u)du$

(ii) $\int_{I^2} CdW = \int_I C(u,1-u)du$.

Proof #2. A second proof is directly obtained via Lemma 5.13 and Remark 5.17. $\square$

Theorem 5.18. *The function $\sigma_{\frac{M+W}{2}}$ satisfies the following properties for all random variables X and Y whose distribution functions are continuous.*

(i) $\sigma_{\frac{M+W}{2}}(C_{X,Y}) = \sigma_{\frac{M+W}{2}}(C_{Y,X})$.

(ii) $0 \leq \sigma_{\frac{M+W}{2}}(C_{X,Y}) \leq 1$.

(iii) $\sigma_{\frac{M+W}{2}}(C_{X,Y}) = 0$ *if X and Y are independent.*

(iv) $\sigma_{\frac{M+W}{2}}(C_{X,Y}) = 1$ *if $Y = g(X)$ for some strictly monotone function g.*

(v) If $\sigma_{\frac{M+W}{2}}(C_{f(X),g(Y)}) = \sigma_{\frac{M+W}{2}}(C_{X,Y})$ *for all strictly monotone functions f and g.*

(vi) If (X,Y) and (X_n,Y_n), $n = 1,2,\ldots,$ are pairs of random variables whose joint distribution functions are H and H_n, respectively, and if $(X_n,Y_n) \xrightarrow{d} (X,Y)$, then $\lim_{n\to\infty} \sigma_{\frac{M+W}{2}}(C_{X_n,Y_n}) = \sigma_{\frac{M+W}{2}}(C_{X,Y})$.

Proof. (i) Using $C_{X,Y}(u,v) = C_{Y,X}(v,u)$ for all $u,v \in I$, we have

$$\begin{aligned}\int_{I^2} |C_{X,Y} - \Pi|(u,v)d\frac{M+W}{2}(u,v) &= \int_{I^2} |C_{Y,X} - \Pi|(v,u)d\frac{M+W}{2}(u,v)\\ &= \int_{I^2} |C_{Y,X} - \Pi|^T(u,v)d\frac{M+W}{2}(u,v).\end{aligned}$$

Note that $\frac{M+W}{2}$ is D_4-invariant. By Lemma 5.1 (i), the proof is complete.

(ii) This follows from the Lemma 5.14 and Lemma 5.15.

(iii) It directly follows from Theorem 2.64 (i).

(iv) It follows from Theorem 2.64 (ii - iii), Corollary 5.16 and Corollary 5.3.

(v) Let f and g be strictly monotone. Note that $\frac{M+W}{2}$ is D_4-invariant.

Case 1: Assume that f and g are both strictly increasing a.s. It follows from Theorem 2.65 (i).

Case 2: Assume that f is strictly increasing and g is strictly decreasing. It follows from Lemma 5.1 (ii).

Case 3: Assume that f is strictly decreasing and g is strictly increasing. It follows from Lemma 5.1 (iii).

Case 4: Assume that f and g are both strictly decreasing. It follows from Lemma 5.1 (iv).

(vi) By the assumption, we obtain $\lim_{n\to\infty} H_n(u,v) = H(u,v)$ for all (u,v) at which H is continuous. By Sklar's Theorem 2.61 and the continuity of the distribution functions of X and Y, we have C_{X_n,Y_n} converges pointwise to $C_{X,Y}$. Using DCT, we get $\lim_{n\to\infty} \sigma_{\frac{M+W}{2}}(X_n,Y_n) = \sigma_{\frac{M+W}{2}}(X,Y)$.

□

Remark 5.19. The converse of Theorem 5.18 (iii) does not hold because there is an infinity of copulas whose diagonal section and opposite diagonal section are diagonal and opposite diagonal sections of Π. For example, for all $\lambda \in [-1,1]$, a copula C_λ defined by

$$C_\lambda(u,v) = \begin{cases} uv - \lambda(v-u)(u+v-1)\min\{v,1-v\}, & \text{if } (u \le v \wedge u+v \ge 1) \vee \\ & (u \ge v \wedge u+v \le 1) \\ uv - \lambda(u-v)(u+v-1)\min\{u,1-u\}, & \text{otherwise} \end{cases}$$

has both diagonal section and opposite diagonal section as Π. For more details, please see [8].

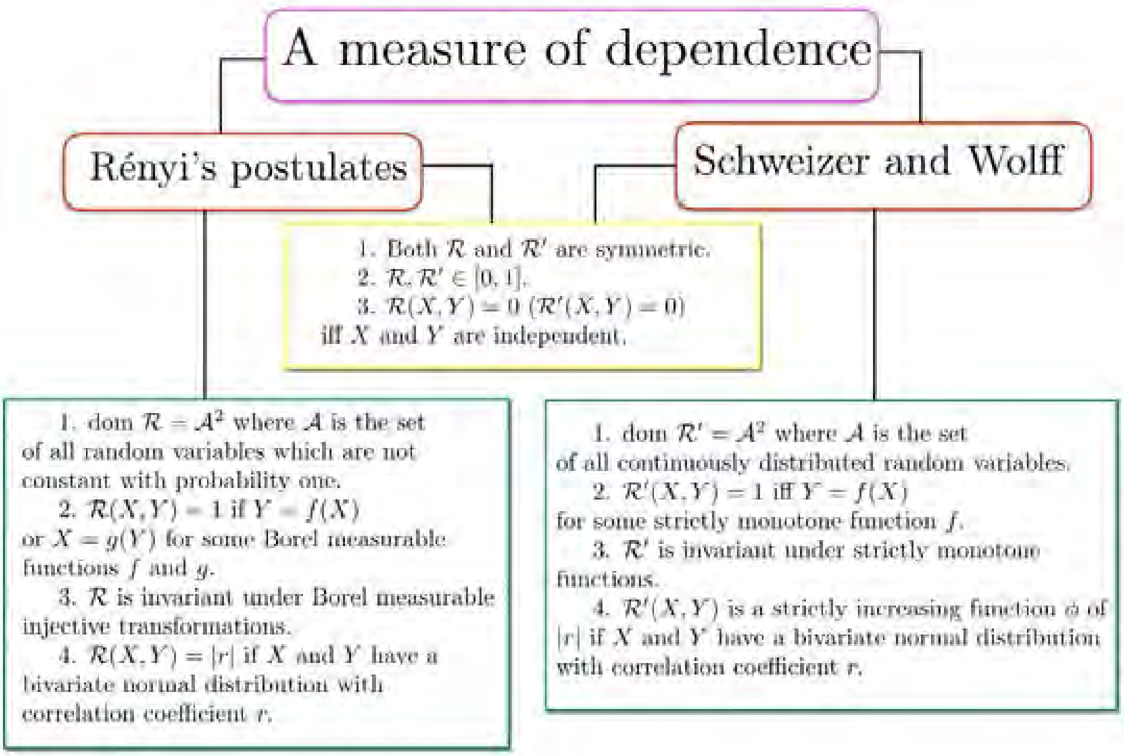
CHAPTER VI

CONCLUSION, DISCUSSION AND FUTURE WORK

In this chapter, we will summarize all of the work which has been done, discuss similarities and differences of our work with an original work [22] and also give directions for future work.

6.1 Summary of the Thesis

We begin with a diagram that compares and contrasts the concept of being a measure of dependence between the original version proposed by Rényi and the one modified by Schweizer-Wolff. Let $\mathcal{R}$ and $\mathcal{R}'$ be measures of dependence according to Rényi's postulates and Schweizer-Wolff, respectively.



In this thesis, we obtain three generalized versions of σ defined in (1.1): σ_φ, σ^*_φ and $\sigma_{\frac{M+W}{2}}$. Recall the functions $\sigma_\varphi, \sigma^*_\varphi$ and $\sigma_{\frac{M+W}{2}}$ as follows: for all random variables X, Y whose distribution functions are continuous,

(i) $\sigma_\varphi(X,Y) = \dfrac{\int_{I^2} \varphi(|C_{X,Y} - \Pi|)d\lambda_2}{\int_{I^2} \varphi(|M - \Pi|)d\lambda_2}$,

(ii) $\sigma^*_\varphi(X,Y) = \sup\limits_{f,g\in\mathcal{I}} \sigma_\varphi(f(X), g(Y)) = \sup\limits_{f,g\in\mathcal{I}} \dfrac{\int_{I^2} \varphi(|C_{f(X),g(Y)} - \Pi|)d\lambda_2}{\int_{I^2} \varphi(|M - \Pi|)d\lambda_2}$,

(iii) $\sigma_{\frac{M+W}{2}}(X,Y) = \dfrac{\int_{I^2} |C_{X,Y} - \Pi| d\frac{M+W}{2}}{\int_{I^2} |M - \Pi| d\frac{M+W}{2}}$.

Then $\sigma_{\frac{M+W}{2}}$ and σ_φ are measures of dependence in the sense of Schweizer and Wolff. Moreover, σ^*_φ is a measure of dependence in the sense of Rényi.

6.2 Discussion

(i) In [22], for each $1 \le p < \infty$, σ^p, which is defined as the normalized L^p-norm of $C - \Pi$ where C is a copula, on the class of all copulas was proved to be a measure of dependence satisfying the same set of Schweizer-Wolff's properties. In our work, σ_φ is a generalization of σ. We are able to prove that σ_φ satisfies all properties in the Schweizer-Wolff's definition except the property (vi).

(ii) For each φ, σ^*_φ satisfies all properties in the Rényi's postulates except (vi). Thus $\{\sigma^*_\varphi : \varphi\}$ is the class of new measures of dependence in the sense of Rényi.

(iii) $\sigma_{\frac{M+W}{2}}$ gives the weaker sense of being a measure of dependence because the minimum value of $\sigma_{\frac{M+W}{2}}(C)$ does not be attained exactly when $C = \Pi$. Moreover, we have no conclusion whether the maximum value of $\sigma_{\frac{M+W}{2}}(C)$ occurs exactly when $C = M$ or $C = W$.

6.3 Future Work

The main result in the chapter V indicates that $\sigma_{\frac{M+W}{2}}$ is almost a measure of dependence in the sense of Schweizer-Wolff. By Lemma 5.10, we see the relationship between the integral with respect to $\frac{M+W}{2}$ as a D_4-invariant copula and the Lebesgue measure on I^2. From this point of view, it is worth investigating the relationship between the integral with respect to all D_4-invariant copulas and the Lebesgue measure on I^2, which is one direction of research so as to classify all copulas A for which σ_A is measure of dependence in the sense of Schweizer-Wolff.

APPENDIX

In this appendix, the primary purpose is to give MATLAB code used in Example 3.4. Before going into the details, we recall the functions φ_1 and φ_2 as follows:

$$\varphi_1(x) = \begin{cases} 2x & \text{if } 0 \leq x < \frac{1}{4} \\ \frac{2}{3}x + \frac{1}{3} & \text{if } \frac{1}{4} \leq x \leq 1 \end{cases}$$

and

$$\varphi_2(x) = \begin{cases} 4x^2 & \text{if } 0 \leq x < \frac{1}{4} \\ 9x - 2 & \text{if } \frac{1}{4} \leq x \leq \frac{7}{9} \\ 54x - 37 & \text{if } \frac{7}{9} \leq x \leq 1. \end{cases}$$

We begin with the MATLAB code of plotting functions σ_{φ_1} on the class of Gaussian copulas. Here is the code.

```
step = 0.01;
range = [0:step:1];
rho = [-0.99:0.01:0.99];
[U,V] = meshgrid(range);

% Define a Gaussian copula C with parameter i
for i = 1 : length(rho)
F = copulacdf('Gaussian',[U(:) V(:)],rho(i));
C(:,:,i) = reshape(F,length(range),length(range));
end

% Define a function varphi_1
PIECE1 = @(x) x<0;
PIECE2 = @(x) x>=0 & x<=0.25;
```

```
PIECE3 = @(x) x>0.25 & x<=1;
PIECE4 = @(x) x>1;
varphi = @(x) PIECE1(x).*(0) + PIECE2(x).*(2*x)
         + PIECE3(x).*((2/3)*x+1/3)+PIECE4(x).*(1);

% Compute the integral of (M-PI)
for i = 1 : length(range)
        for j = 1 : length(range)
        M(i,j) = min(U(i,j),V(i,j));
        PI(i,j) = U(i,j)*V(i,j);
        K=M(i,j)-PI(i,j);
        varphi_ij(i,j)=varphi(K);
        end
end
vol=trapz(range,trapz(range,varphi_ij(:,:)));

% Compute the measure sigma_{\varphi_1}
for i = 1 : length(rho)
        if rho(i)<0
                for j = 1 : length(range)
                        for k = 1 : length(range)
                        K=PI(:,:) -C(:,:,i);
                        varphi_jk(j,k,i)=varphi(K);
                        end
                end
NormalizedVol(i) = trapz(range,trapz(range,
                   varphi_jk(:,:,i)))/vol;
        else
                for j = 1 : length(range)
```

```
                              for k = 1 : length(range)
                              K=C(:,:,i)-PI(:,:);
                              varphi_jk(j,k,i)=varphi(K);
                              end
                    end
NormalizedVol(i) = trapz(range,trapz(range,
                    varphi_jk(:,:,i)))/vol;
          end
end
% Plot a graph of the measure sigma_{\varphi_1}
plot(rho,NormalizedVol(:),'Color','red',
     'LineSmoothing','on'); hold on;
xlabel('X=r')
ylabel('Y')
legend('1','Location','southwest')
```

The following code defines the function φ_2.

```
% Define a function varphi_2
PIECE1 = @(x) x<0;
PIECE2 = @(x) x>=0 & x<=0.25;
PIECE3 = @(x) x>0.25 & x<=7/9;
PIECE4 = @(x) x>7/9 & x<=1;
PIECE5 = @(x) x>1;
varphi = @(x) PIECE1(x).*(0) + PIECE2(x).*(4*x^2)
              +PIECE3(x).*((9)*x-2)
              +PIECE4(x).*((54)*x-37)
              +PIECE5(x).*(1);
```

Using the same code from the above and replacing φ_1's code with φ_2's code, we have the graph of σ_{φ_2}.

REFERENCES

[1] Capasso, V., Bakstein, D.: *An Introduction to Continuous-Time Stochastic Processes: Theory, Models, and Applications to Finance, Biology, and Medicine*, (2nd ed.), Birkhäuser, Boston. (2012).

[2] Chaidee, N., Santiwipanont, T., Sumetkijakan, S.: Patched approximations and their convergence, *Comm. Statist. Theory Methods.* To appear.

[3] Cherubini, U., Luciano, E., Vecchiato, W.: *Copula Method in Finance*, (1st ed.), John Wiley & Son Ltd, Chicheter. (2004).

[4] Chou, S.H., Nguyen, T.T.: On Fréchet theorem in the set of measure preserving functions over the unit interval, *Int. J. Math. Math. Sci.* **13**(2), 373–378 (1990).

[5] Darsow, W.F., Nguyen, B., Olsen, E.T.: Copula and markov processes, *Illinois J. Math.* **36**(4), 600–642 (1992).

[6] Darsow, W.F., Olsen, E.T.: Characterization of idempotent 2-copulas, *Note Mat.* **30**, 147–177 (2010).

[7] Durante, F., Sempi, C.: *Principles of Copula Theory*, CRC Press. (2015).

[8] de Amo, E., De Meyer, H., Carrillo, M.D., Sánchez, J. F.: Characterization of copulas with given diagonal and opposite diagonal sections, *Fuzzy Sets and Systems,* **284**, 63–77 (2016).

[9] Dudley, R.M.: *Real Analysis and Probability*, Cambridge Univ. Press. (2002).

[10] Dummit, D.S., Foote, R.M: *Abstract Algebra*, (3rd ed.), Hoboken, Wiley. (2004).

[11] Durante, F.: *Personal Communication.* (2014).

[12] Edwards, H.H., Mikusiński, P., Taylor, M.D.: Measures of concordance determined by D_4-invariant copulas, *Int. J. Math. Math. Sci.* **70**, 3867–3875 (2004).

[13] Embrechts, P., Hofert, M.: A note on generalized inverses, *Math. Methods Oper. Res.* **77**(3), 423–432 (2013).

[14] Folland, G.B.: *Real Analysis: Modern Techniques and Their Applications*, (2nd ed.), John Wiley & Sons, New York. (1999).

[15] Leonid, K., Yakov, G.S.: *Theory of Probability and Random Processes*, (2nd ed.), Springer, New York. (2012).

[16] Makarov, B., Podkorytov, A.: *Real Analysis: Measures, Integrals and Applications*. Springer, London. (2013).

[17] Nelsen, R.B.: *An Introduction to Copulas*, (2nd ed.), Springer Science & Business Media, New York. (2006).

[18] Polansky, A.M.: *Introduction to Statistical Limit Theory*, CRC Press, Boca Raton. (2011).

[19] Rényi, A.: On measure of dependence, *Acta. Math. Acad. Sci. Hungar.* **10**(3-4), 441–451 (1959).

[20] Royden, H., Fitzpatrick, P.: *Real Analysis*, (4th ed.), Prentice Hall, Boston. (2010).

[21] Ruankong, P., Santiwipanont, T., Sumetkijakan, S.: Shuffles of copulas and a new measure of dependence, *J. Math. Anal. Appl.* **398**(1), 392–402 (2013).

[22] Schweizer, B., Wolff, E.F.: On nonparametric measures of dependence for random variables, *Ann. Statist.* **9**, 879–885 (1981).

[23] Siburg, K.F., Stoimenov, P.A.: A measure of mutual complete dependence, *Metrika.* **71**(2), 239–251 (2010).

[24] Tong, Y.L.: *The Multivariate Normal Distribution*, Springer, New York. (1990).

[25] Zheng, Y., Yang, J., Huang, J.Z.: Approximation of bivariate copulas by patched bivariate Fréchet copulas, *Insurance Math. Econom.* **48**, 246–256 (2011).

VITA

Name	Miss Wasamon Jantai
Date of Birth	24 September 1990
Place of Birth	Mahasarakham, Thailand
Education	B.Sc. Mathematics (First class honors, gold medal), Khon Kaen University, 2013
Scholarship	Development and Promotion of Science and Technology Talents Project (DPST)
Publication	Jantai, W.: On the Length of a Shortest Spanning Walk, *KKU Sci. J.* **41**(1), 112–120 (2013).