CHAPTER II

LITERATURE REVIEW AND THEORETICAL CONSIDERATIONS

The meaning of plasticity

Plasticity is a science describing the behaviour of solid bodies (metal or alloys) in which they deform permanently under the action of external forces, whereas elasticity is the behaviour of solid bodies in which upon removal of the load they return to their original shape. Actually, the elastic body is an idealization since all bodies exhibit more or less plastic behaviour even under smallest loads. For the so called elastic body, however, this permanent deformation caused by small loads is so small and can be neglected for practical purpose. Plasticity theory thus concerns itself with situations in which the loads are sufficiently large so that measurable amounts of permanent deformation occur.

Classes of plasticity

The theories of plasticity fall into two categories: PHYSICAL THEORIES (Structure of metal) and MATHEMATICAL THEORIES.

The physical theories seek to explain why metals flow plastically and what happens to the atoms, crystals and grains of material by looking at materials from a microscopic view point.

The mathematical theories, on the other hand, attempt to formalize and put into useful form the results of macroscopic experiments, without probing very deeply into their physical basis.

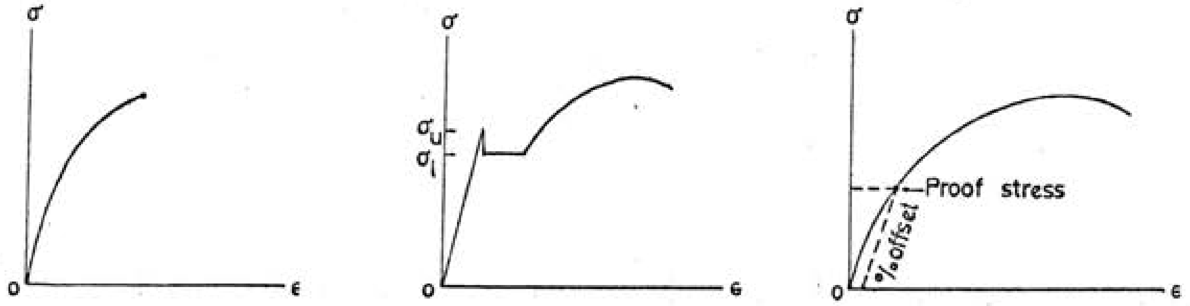
The former theories interest the metal physicist or solid-state physicist, while the latter are the main subject for the structural and machine designer. However, in order to understand the limitations so imposed on the theories, the engineer must have some knowledge of the structure of a metal. Alternatively, the metallurgist must have some knowledge of the mathematical theory if he is to understand the requirement of the engineer.

The Conventional Stress Strain Curves in Simple Tension

The nominal stress is defined as the load divided by the original cross sectional area of the bar, and the conventional or engineering strain as the extension per unit original length. The stress-strain shape varies between the curves represented by Fig. 2a and 2c, according to material, composition mechanical and heat treatments, and methods of testing (the size and shape of the specimen) and recording of load and extension.

Fig. 2.1a shows the stress-strain curve of a material having a coarse grain. Examples of this type of material are cast iron and concrete.

Fig. 2.2b represents a typical curve of a hot rolled structural steel, containing approximately 0.2 per cent carbon. The material is elastic up to point A, known as the upper yield point; and further



(a) Stress-strain curve of brittle material i.e. cast iron, concrete etc.

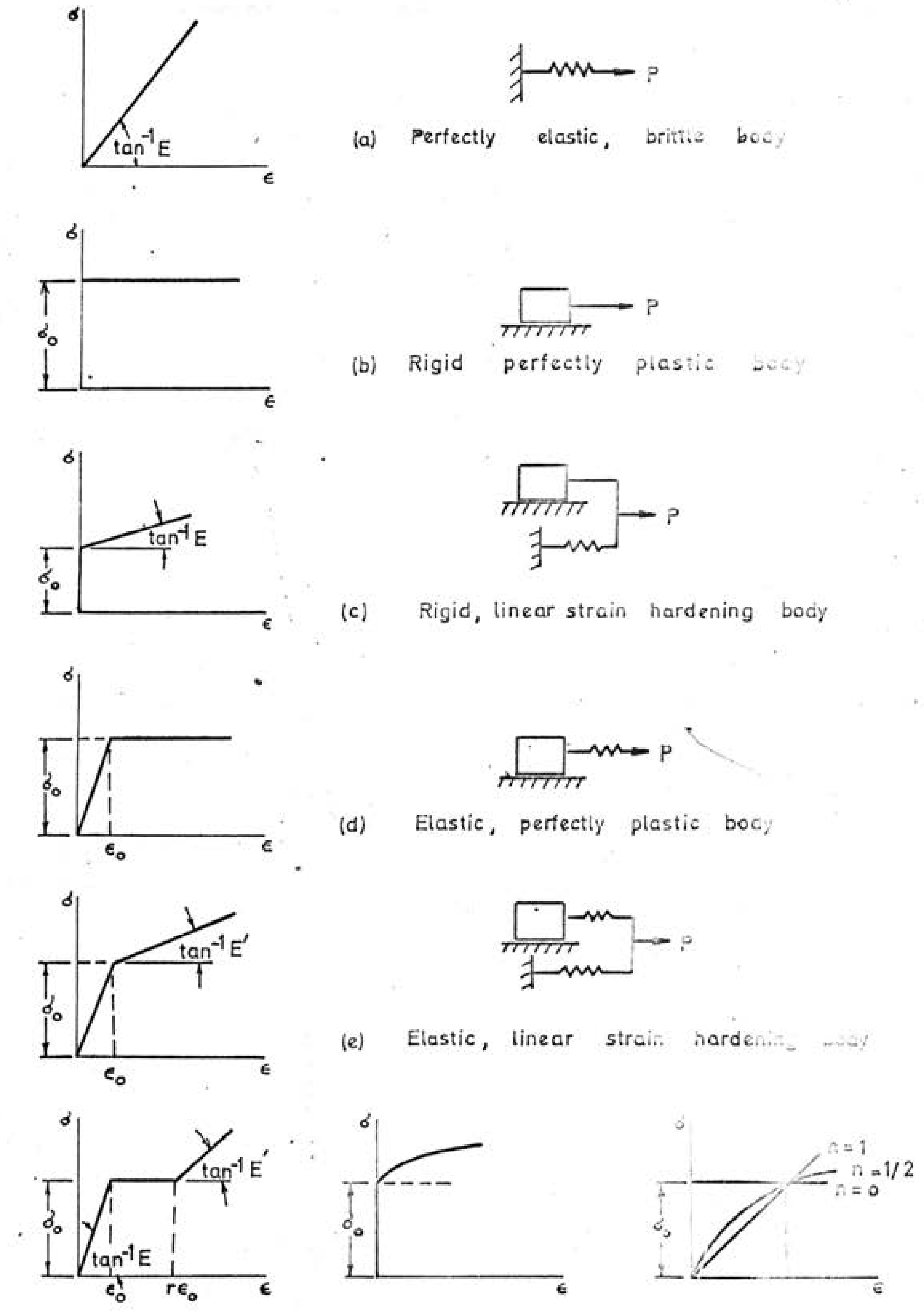
(b) Typical stress-strain curve of hot rolled structural steel containing approximately 0.2 % carbon.

(c) Stress-strain curve of metal or alloy which does not exhibit pronounced yield stress.

Fig. 2.1.- Typical stress-strain curves.

straining is accompanied by a sudden drop in stress marked B on the diagram and known as the lower yield stress. The values of the upper and the lower yield stress vary depending on the rates of application of loads and of extension as well as on the axiality of loading. From B to C the initial plastic zones spread under a constant stress. After C the load increases with strain up to D where knecking begins. The portion CD is called strain hardening or work hardening owing to the fact that the material is able to withstand a greater load despite the uniform reduction in cross sectional area. From D to fracture, the stress is no longer uniform; it indicates the state of instability due to the formation of a kneck at E which gives rise to a triaxial tensile stress system.

For most metal and alloys, the change from purely elastic to elastic-plastic deformation is gradual, otherwise the stress-strain diagram have the same form, Fig.2.1c. In such cases the proof stress, the stress which will cause a permanent strain equal to a specified value is established instead of yield stress. This proof stress finds no application in the mathematical theories.



(a) Perfectly elastic, brittle body

(b) Rigid perfectly plastic body

(c) Rigid, linear strain hardening body

(d) Elastic, perfectly plastic body

(e) Elastic, linear strain hardening body

(f) Elastic, plastic, Linear strain hardening body

(g) Rigid, strain hardening body (Ludwik's expression.)

(h) Power expression $\sigma = \sigma_o \epsilon^n$

Fig. 2.2 — Idealized stress-strain curves

Idealization of Stress Strain Curves, Theirs Empirical Formulas and Dynamic Models

Many attempts are made to idealize the stress-strain relations[1]. Some examples of them as well as corresponding dynamic models[2] are given as follows :

(1) Perfectly elastic body. For small value of strain as usually seen in elastic design, the stress-strain relation is represented by a straight line having a slope of $\tan^{-1} E$ (Fig. 2.2a). This effect is analogous to a spring system where the deflexion of the spring is proportional to the applied force. The stress-strain relation can be expressed mathematically as :

$$\sigma = E\epsilon$$

(2) Rigid perfectly plastic body. In applications where the plastic strains are very large as compared with the elastic strains, the stress-strain relation is represented as shown in Fig. 2.2b. The block shown in Fig. 2.2b. produces the same relation as it is subjected to dry friction and can only move if the force exceeds a certain amount. The stress-strain relation can be expressed as :

$$\sigma = \sigma_0$$

---

[1] W.R. Osgood "Stress Strain Formula", Journal of the Aeronautical Science, Vol. 13 (January,1946), pp. 45-48.

[2] W. Johnson and P.B. Meller, Plasticity for Mechanical Engineers (London; D. Van Nostrand Co., Inc., 1962), pp. 15-22.

These two elementary principles lead to other formulization.

(3) Rigid, linearly strain hardening body. The stress-strain curve of this system (Fig. 2.2c.) may be applied to cold-worked materials and gives an especially good fit for "half-hard" aluminium.

$$\sigma = \sigma_0 + E'\epsilon$$

(4) Elastic, perfectly plastic body. This system gives a good fit for mild steel where strain hardening is not taken into consideration.

$$0 \leq \epsilon \leq \epsilon_0 \qquad \sigma = E\epsilon$$

$$\epsilon > \epsilon_0 \qquad \sigma = \sigma_0$$

(5) Elastic, linear work hardening body. This system (Fig. 2.2e.) is a common method of approximation the true stress-strain curve by using two or more linear expressions between stress and strain.

$$0 \leq \epsilon \leq \epsilon_0 \qquad \sigma = E\epsilon$$

$$\epsilon > \epsilon_0 \qquad \sigma = \sigma_0 + E'(\epsilon - \epsilon_0)$$

(6) Elastic, plastic and linear strain hardening body. The stress-strain curve of this system is illustrated in Fig. 2.2f. If neglecting the strain hardening line, the stress-strain curve has the same shape as shown in Fig. 2.2d.

$$0 \leq \epsilon \leq \epsilon_0 \qquad \sigma = E\epsilon$$

$$\epsilon_0 \leq \epsilon \leq r\epsilon_0 \qquad \sigma = \sigma_0$$

$$\epsilon > r\epsilon_0 \qquad \sigma = \sigma_0 + E'(\epsilon - r\epsilon_0)$$

(7) Ludwik's expression (Fig. 2.2g.)

$$\sigma = \sigma_0 + H\epsilon^n \qquad 0 \leq n \leq 1$$

Where H and n are constants.

The above expression is applicable when the elastic strains in an analysis may be safely neglected. When n = 1, the expression is the case shown in Fig. 2.2c. Again when $\sigma_o = o$, the curves are as given in Fig. 2.2h.

(8) Power expression (Fig. 2.2h.) This expression should not be used for small strains.

$$\sigma = \sigma_o \epsilon^n \quad \text{or,}$$
$$\sigma = K\epsilon_n^{\gamma}$$

Where, ϵ_n = natural or logarithmic strain

γ, n, K = constants

(9) Other expression.

Voce's expression : $\sigma = A + (B - A)(1 - e^{-n})$

Swift's expression : $\sigma = C(A + \epsilon)^n$

Prager's expressin : $\sigma = Y\tanh(\frac{E\epsilon}{Y})$

Where, e = base of natural logarithmic

E = elastic modulus

A, B, C, Y, n = constants

Theory of Elastic-Plastic Bending

General assumptions.

The assumptions made in the following theory are :

(1) Compression and tension are the same.

(2) Plane transverse sections remain plane and mormal to the longitudinal axis after bending.

(3) The effect of shear is neglected.

(4) The cross section of the beam is symmetrical about an axis through its centroid parallel to the plane of bending.

(5) Every layer of the material is free to expand and contact longitudinally and laterally under stress as if separated from the other layer.[3]

General theory of elastic-plastic bending.

Consider the idealized curve of stress-strain as shown in Fig. 2.3a. as general, since the stress-strain shapes in Fig. 2.3b. and 2.3c. are its particular cases.

---

[3] G.H. MacCullough "An Experiment and Analytical Investigation of Creep in Bending", Trans ASME, Vol. 55 (1933), APM 55-9, pp. 55-60.

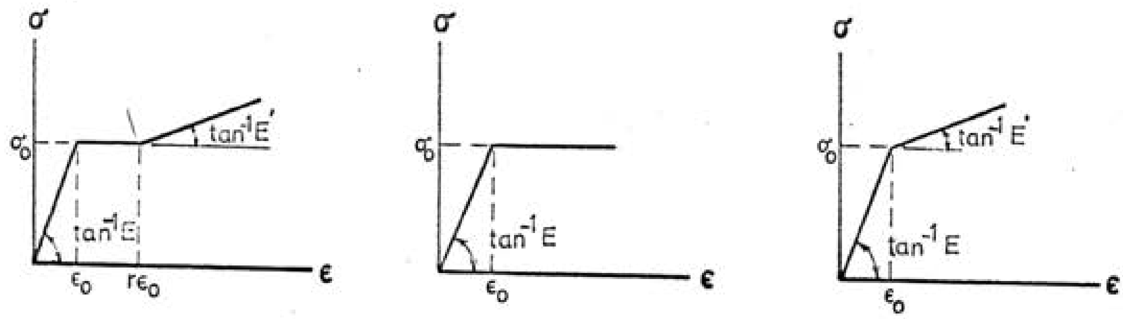


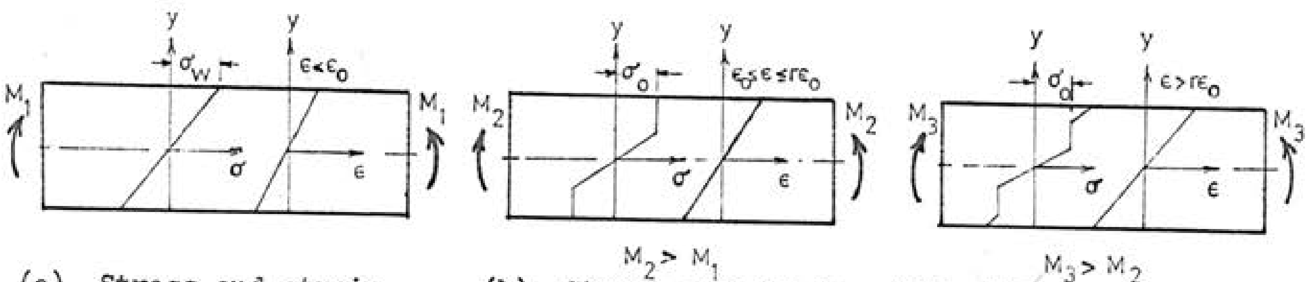
Fig. 2.3. - Representation of stress-strain curves.

For small couples, the entire fibres of the beam is elastic. The stress and strain distribution is shown in Fig. 2.4a.

$$y \leq y_o \qquad \sigma = E\frac{y}{y_o} \qquad \ldots\ldots (1)$$

When the couple is increased some of the outmost fibres strain in elastically while the central portion is still elastic. Fig. 2.4b. shows the stresses in the overstrained fibres equal to a constant value of .

$$y_o \leq y \leq ry_o \qquad \sigma = \sigma_0 \qquad \ldots\ldots (2)$$



(a) Stress and strain distribution in the working state.

(b) Stress and strain distribution in the elastic-plastic state.

(c) Stress and strain distribution in the elastic-plastic-strain hardening state.

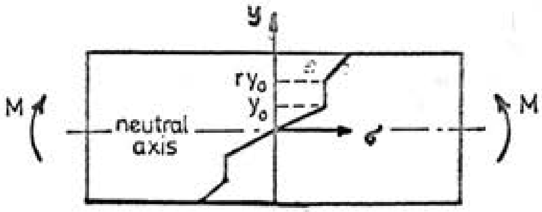
Fig. 2.4. - Distribution of stress and strain across the beam section which is subjected by a couple M.

With further increase in couple, the stress and strain distribution is shown in Fig. 2.4c.

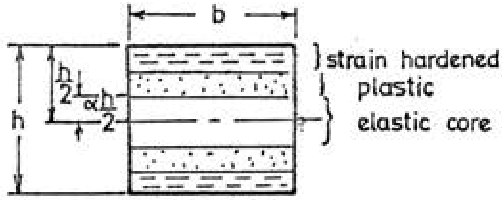
$$r y_0 < y \le \frac{h}{2} \qquad \sigma = \sigma_0 \left(1 - \beta r + \beta \frac{y}{y_0}\right) \qquad \ldots\ldots (3)$$

$$\text{where } \beta = \frac{E'}{E}$$

The resisting moment equation[4,5,6]



(a) Stress distribution.



(b) Rectangular cross section.

Fig. 2.4c. - Distribution of stress across the rectangular beam. (modified)

To evaluate the resisting moment M, it is assumed that the beam is subjected by a couple M which causes compression above the neutral

---

[4] J.F. Baker, M.R. Horne and J. Heyman, The Steel Skeleton, Vol. 2 (Cambridge : Cambridge University Press, 1956), p. 18.

[5] A. Nadai, Theory of Flow and Fracture, Vol. 1 (New York : McGraw-Hill Book Co., Inc., 1950), pp. 364-365.

[6] F.B. Seely and J.O. Smith, Advanced Strength of Material (2nd. ed.; New York : John Wiley and Sons, Inc., 1955), pp. 527-543.

axis and tension below the neutral axis and that the cross section is rectangular.

From statics

$$\int_A \sigma dA = b\int_{-h_2}^{h_1} \sigma dy = 0 \qquad \ldots\ldots \quad (4)$$

$$\int_A \sigma y dA = b\int_{-h_2}^{h_1} \sigma y dy = M \qquad \ldots\ldots \quad (5)$$

The above two equations state that the sum of the internal forces over the entire cross section reduce to a couple equal to the bending moment M.

Since the stress strain in tension is the same as in compression, therefore, the neutral axis lies coincidentally with the centroidal axis. And thus the elastic-plastic resisting moment can be taken as :

$$\frac{1}{2} M = \int_0^{y_0} \sigma_e bydy + \int_{y_0}^{ry_0} \sigma_0 bydy + \int_{ry_0}^{h/2} \sigma_p bydy \qquad \ldots\ldots \quad (6)$$

Substituting equations (1), (2) and (3) into equation (6) :

$$M = \frac{M_0}{2}\left[3(1-\beta r) - \alpha^2(1-\beta r^3) + \frac{2\beta}{\alpha}\right] \qquad \ldots\ldots \quad (7)$$

Where M $(= \frac{\sigma_0 bh^2}{6})$ denotes the bending moment at first yield and $\alpha (= \frac{2y}{h})$ the ratio between the depth of the elastic core and the full depth of the beam.

For the particular case of an elastic, perfectly plastic body (Fig. 2.3b) the resisting moment is :

$$M = \frac{M_0}{2}(3 - \alpha^2) \qquad \ldots\ldots \quad (8)$$

When $\alpha = 0$, known as the PLASTIC HINGE, the whole cross section of the beam is plastic and the carrying moment is 50 per cent greater than the maximum elastic moment.

Now, consider the material which the stress strain relation is approximated by two straight lines (Fig. 2.3c.). The resisting moment is :

$$M = \frac{M_o}{2}\left[(1-\beta)(3-\alpha^2) + \frac{2\beta}{\alpha}\right] \qquad \ldots\ldots (9)$$

Equations (7) and (9) reveal that there is no definite plastic hinge at all and that the moment increases indefinitely with the increase in the external fibre strain until fracture occurs.

General elastic-plastic slope-deflexion equation

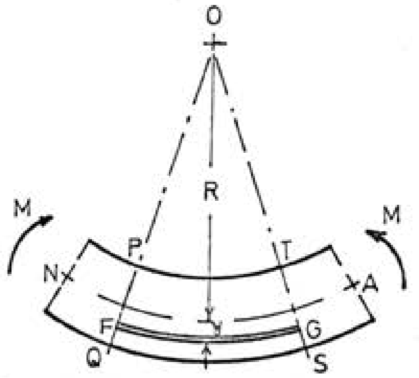


Fig. 2.5 - Deformation of beam subjected to an end couple

Fig. 2.5 shows the deformation of a beam subjected by an end couple M causing a decrease in length of those fibres above the neutral axis and an increase in those below. For any fibre FG distant y from the neutral axis, it can be shown from the geometry of deformation[7] that it has been strained by an amount :

$$\epsilon = \frac{y}{R} \qquad \ldots\ldots (10)$$

Where R is the radius of curvature of the neutral surface.

---

[7] S. Timoshenko, Strength of Material, Part 1 (3rd ed. ; New York : D. Van Nostrand Co., Inc., 1956), p. 93.

This expression is unaffected by the condition of the material, and it is therefore in order to assume that when the end couples are of sufficient magnitude to cause part of the section to become plastic, FG may represent a layer of either ELASTIC or PLASTIC material. And, as in the simple theory of elastic bending the deflexion of an elastic-plastic beam can be obtained with sufficient accuracy by the relation[8]:

$$\frac{1}{R} = \frac{d^2y}{dx^2} = y'' \qquad \ldots\ldots (11)$$

If FG is the elastic material[9]

$$y_e'' = \frac{\epsilon}{y} = \frac{M}{EI} \qquad \ldots\ldots (12)$$

Integrating, $$y_e' = \int \frac{M}{EI}\,dx + C_1 \qquad \ldots\ldots (13)$$

Repeating the process, the expression for the deflexion is obtained :

$$y_e = \frac{M}{EI}\iint (dx)dx + C_1x + C_2 \qquad \ldots\ldots (14)$$

---

[8] Ibid., p. 139.

[9] Ibid., p. 95.

When FG is the plastic material[10]

$$y_p'' = \frac{\epsilon}{y} = \frac{\epsilon_o}{y_o} = \frac{\sigma_o}{Ey_o}$$

Let y_o be a function of x, the longitudinal distance from a given origin, namely

$$y_o = \phi(x)$$

so that

$$y_p'' = \frac{\sigma_o}{E\phi(x)} \qquad \text{..... (15)}$$

$$y_p' = \frac{\sigma_o}{E}\int\frac{dx}{\phi(x)} + C_3 \qquad \text{..... (16)}$$

$$y_p = \frac{\sigma_o}{E}\iint\frac{dx}{\phi(x)}\,dx + C_3x + C_4 \qquad \text{..... (17)}$$

The integrating constants : C_1, C_2, C_3 and C_4 are evaluated from the known boundary conditions.

---

[10] Baker, Horne and Heyman, op.cit., p. 20.

Analysis of the Elastic-Plastic Slope-Deflexion Equation of a Simply Supported Beam Subjected to a Concentrated Load at Mid Span

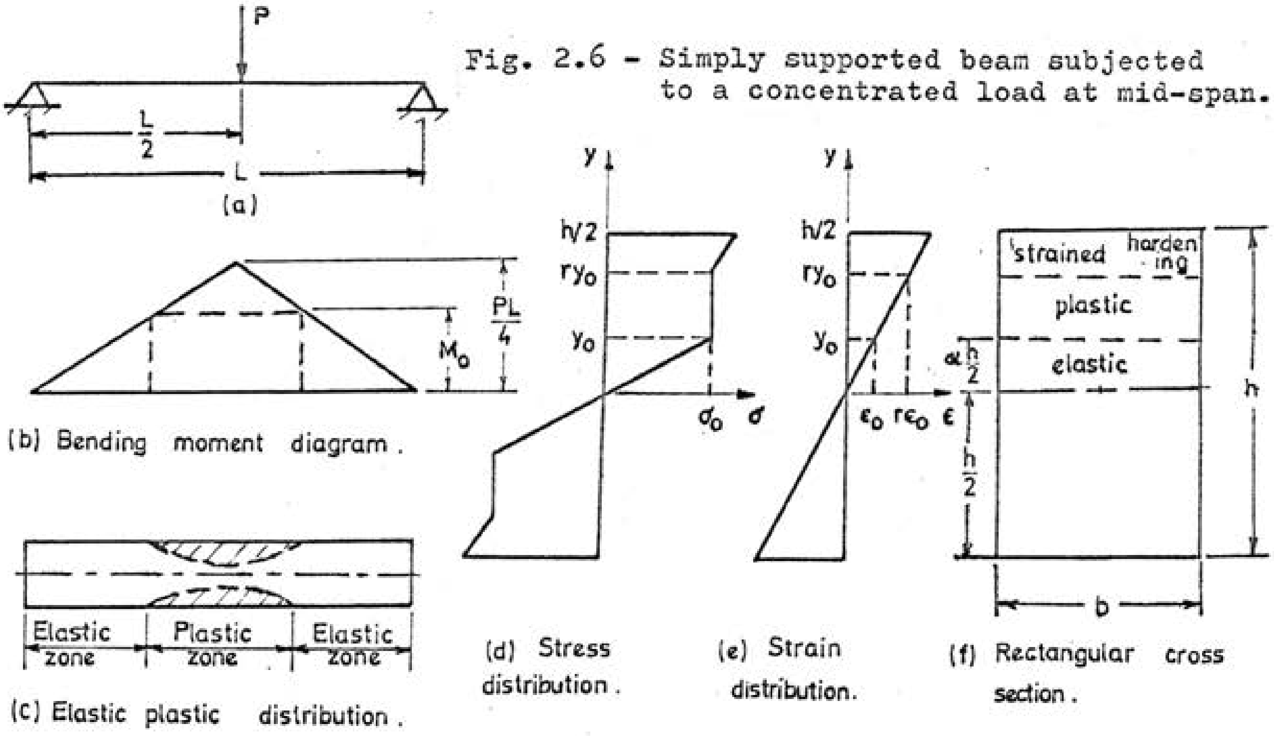


Fig. 2.6 - Simply supported beam subjected to a concentrated load at mid-span.

Fig. 2.6a shows a rectangular simply supported beam OABCD carrying a concentrated load P at mid-span B. The load P is sufficiently large to cause the formation of a plastic zone in the length AC, while OA and CD remain elastic. At this stage the central deflexion of the beam is analysed.

The bending moment at any distance x from the origin O is

$$M_x = \frac{Px}{2} \qquad \text{..... (18)}$$

For equilibrium, the applied bending moment (18) must be equal to the internal-resisting moment (7).

Therefore,

$$\frac{Px}{2} = \frac{M_o}{2}\left[3(1-\beta r) - \alpha^2(1-\beta r^3) + \frac{2\beta}{\alpha}\right]$$

or,

$$x = \frac{M_o}{P}\left[3(1-\beta r) - \alpha^2(1-\beta r^3) + \frac{2\beta}{\alpha}\right] \quad \ldots\ldots (19)$$

The differential relation between x and α is then :

$$\frac{dx}{d\alpha} = -\frac{2M_o}{P}\left[(1-\beta r^3)\alpha + \frac{\beta}{\alpha^2}\right] \quad \ldots\ldots (20)$$

This expression is very convenient in solving the slope-deflexion of beam in the plastic zone.

Analysis in elastic zone : $0 < x \leq \frac{2M_o}{P}$

$$y_e'' = -\frac{Px}{2EI} \quad \ldots\ldots (21)$$

$$y_e' = -\frac{Px^2}{4EI} + C_1 \quad \ldots\ldots (22)$$

$$y_e = -\frac{Px^3}{12EI} + C_1 x + C_2 \quad \ldots\ldots (23)$$

Analysis in plastic work hardening zone : $1 < \alpha < \alpha_o$

$$y_p'' = -\frac{M_o}{EI\alpha} \qquad \text{..... (24)}$$

$$y_p' = -\frac{M_o}{EI}\int\frac{dx}{\alpha} + C_3 \qquad \text{or,}$$

$$= \frac{2M_o^2}{EIP}\left[(1-\beta r^3)\alpha - \frac{\beta}{2\alpha^2}\right] + C_3 \qquad \text{..... (25)}$$

$$y_p = -\frac{4M_o^3}{EIP^2}\left[(1-\beta r^3)^2\frac{\alpha^3}{3} + \frac{\beta^2}{6\alpha^3} + \frac{\beta}{2}(1-\beta r^3)\ln\alpha\right] + C_3x + C_4 \qquad \text{..... (26)}$$

By using boundary conditions :

$x = 0$	$y_e = 0$	
$x = \frac{1}{2}L$, or $\alpha = \alpha_o$	$y_p' = 0$	
$x = \frac{2M_o}{p}$, or $\alpha = 1$	$y_p' = y_e'$, and $y_p = y_e$	..... (27)

The deflexion at any section along AB is :

$$y_x^{AB} = -\frac{4}{3}\frac{M_o^3}{EIP^2}\left\{(1-\beta r^3)^2\alpha^3 + \frac{\beta^2}{2\alpha} + \frac{3}{2}\beta(1-\beta r^3)\ln\alpha\right\}$$
$$- 2\frac{M_o^2}{EIP}\left\{(1-\beta r^3)\alpha_o - \frac{\beta}{2\alpha_o^2}\right\}x$$
$$+ \frac{4}{3}\frac{M_o^3}{EIP^2}\left\{5 + \beta(r^6 + \frac{1}{2}) - \beta(5r^3 + \frac{3}{2})\right\} \qquad \text{..... (28)}$$

Thus, the central deflexion can be taken by letting $x = \frac{1}{2} L$ and $\alpha = \alpha_o$ in the above equation (28), and rearranging :

$$\delta = y_{l/2} = \frac{4}{3}\frac{M_o^3}{EIP^2}\left[\left\{5 + \beta^2(r^6 + \frac{1}{2}) - \beta(5r^3 + \frac{3}{2})\right\}\right.$$

$$- 3\frac{M}{M_o}\left\{(1 - \beta r^3)\alpha_o - \frac{\beta}{2\alpha_o^2}\right\}$$

$$\left. - \left\{(1 - \beta r^3)^2\alpha_o^3 + \frac{\beta^2}{2\alpha_o^3} + \frac{3}{2}\beta(1 - \beta r^3)\ln\alpha_o\right\}\right] \quad ..... (29)$$

For the particular cases of Fig. 2.3b and 2.3c, the central deflexion is computed by substituting $\beta = 0$ and $r = 1$ in equation (29) respectively.

If the stress-strain curve of Fig. 2.3b is considered, the central deflexion is :

$$\delta = \frac{4}{3}\frac{M_o^3}{EIP^2}(5 + \frac{1}{2}\alpha_o^3 - \frac{9}{2}\alpha_o) \quad ..... (30)$$

And, the central deflexion of the stress-strain curve in Fig. 2.3c is :

$$\delta = \frac{4}{3}\frac{M_o^3}{EIP^2}\left[(5 - \frac{13}{2}\beta + \frac{3}{2}\beta^2) - 3\frac{M}{M_o}\left\{(1 - \beta)\alpha_o - \frac{1}{2}\frac{\beta}{\alpha_o^2}\right\}\right.$$

$$\left. - \left\{(1 - \beta)^2\alpha_o^3 + \frac{\beta^2}{2\alpha_o^3} + \frac{3}{2}\beta(1 - \beta)\ln\alpha_o\right\}\right] \quad ..... (31)$$

Rectangular Portal Frame[11,12,13]

In a frame work with rigid joints, points of local maximum bending moment will occur at the joints and under any applied load. In this theory, the material must have a marked yield stress; and when a plastic hinge has developed at a cross section, the moment of resistance at that point remains constant until the whole structure collapse due to formation of required number of plastic hinges.

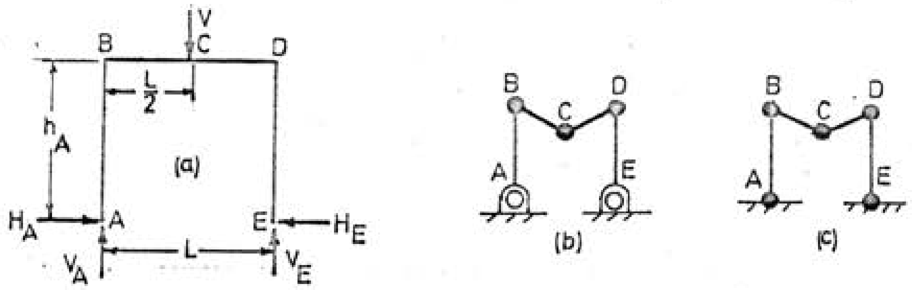


Fig. 2.7. - Portal frame subjected to central vertical load.

Consider a portal of height h and span L as shown in Fig. 2.7a. Under a central vertical load V, plastic hinges may form at BCD (for a frame with pinned bases) or at ABCDE (for a frame with fixed bases).

---

[11] Baker, Horne and Heyman, op.cit., pp. 61-85.

[12] B.G.Neal and P.S. Symonds "The Calculation of Collapse Loads for Framed Structures", J. Instn. Civ. Engrs, Vol.35 (1950), pp. 21-40.

[13] E. Litton "Laboratory Experiments on Plastic Theory", The Engineer, (April 30, 1965), pp. 754-760.

Analysis of collapse loads of portal frame with pinned-bases

(a) Neglecting the effect of elastic deformation: For the mechanism shown in Fig. 2.7b, the distribution of bending moment at collapse can be set out conveniently on a horizontal base line ABCDE (Fig. 2.8) where BCD represents the beam and AB and DE the stanchions.

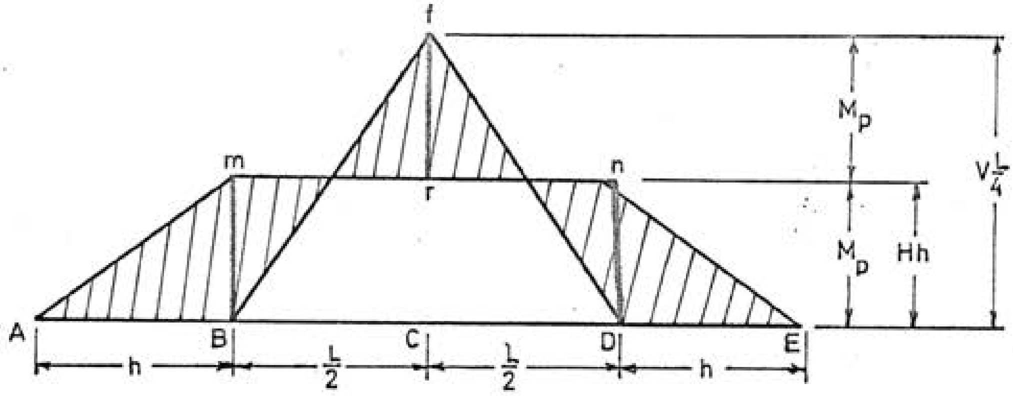


Fig. 2.8. - Bending moment diagram at collapse of portal frame with pinned bases.

BfD is the moment diagram for the vertical loads with the ordinate Cf representing $\frac{VL}{4}$.

To complete the construction for the given mechanism the reactant line due to the springing H_A is superimposed upon those already drawn. The line mn will always be parallel to the base, whilst Bm = Dn = $H_A h$ to the propriate scale. The shaded area in Fig. 2.8 represents the resulting bending moment diagram.

$$M_P^S = Hh \qquad ..... (32)$$

$$M_P^B = \frac{VL}{4} - Hh \qquad \text{..... (33)}$$

Where M_P^S and M_P^B are the full plastic moments of the stanchions and beam respectively. It should be noted that the solution is valid only for $\frac{VL}{4}$ Hh; if this inequality is not fulfilled, the relative proportions of the free bending moment diagram (BfD) will alter, bringing point f below the reactant line mn.

(b) <u>Considering elastic deformation</u> :

from (32) $$H = \frac{M_P^S}{h} \qquad \text{..... (34)}$$

and from (33) $$M_P^B = \frac{VL}{4} - H(h - \delta_C) \qquad \text{..... (35)}$$

where δ_C is the elastic deformation at C

sub. H in (35) and get

$$M_P^B + M_P^S = \frac{VL}{4} + \frac{M_P^S}{h}\delta_C$$

$$= V\left(\frac{L}{4} + \frac{k}{h} M_P^S\right)$$

where k is $1/\delta_C$

therefore $$V = \frac{M_P^B + M_P^S}{\frac{L}{4} + \frac{kM_P^S}{h}}$$

Take the case of a frame of uniform section, i.e. $M_P^B = M_P^S$

$$V = \frac{2M_P}{\frac{L}{4} + \frac{kM_P}{h}} \qquad \text{..... (36)}$$

Analysis of collapse loads of portal frame with fixed bases

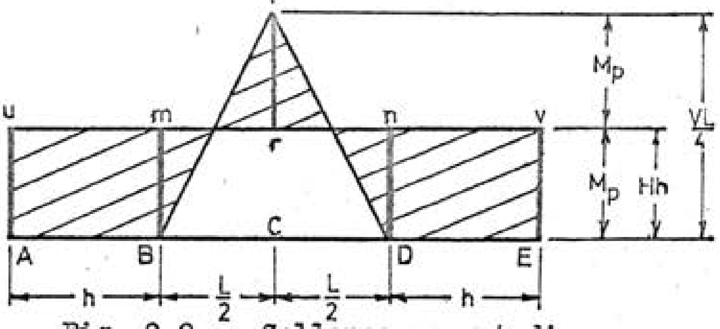


Fig. 2.9. - Collapse moment diagram.

With similar procedure as described in the preceding article, the bending moment diagram at failure is shown in Fig. 2.9.

$$M_P^S = Hh \qquad \text{..... (37)}$$

$$M_P^B = \frac{VL}{8} \qquad \text{..... (38)}$$