CHAPTER II

MEASURE THEORY

This chapter reviews known results on measure theory. The only new knowledge concerns quaternion measures. In this chapter X will denote a non empty set.

2.1 <u>Definition</u> An extended real valued set function μ defined on a set $\mathcal{E}$ of subsets of X is <u>finitely</u> <u>additive</u> if, for every finite, disjoint set $\{E_1,\ldots,E_n\}$ of sets in $\mathcal{E}$ whose uniom is also in $\mathcal{E}$, we have

$$\mu\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} \mu(E_i) .$$

2.2 <u>Definition</u> An extended real valued set function μ defined on a set $\mathcal{E}$ of subsets of X is <u>countably</u> <u>additive</u> if, for every disjoint sequence $(E_n)_{n\in N}$ of sets in $\mathcal{E}$ whose union is also in $\mathcal{E}$, we have

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i) .$$

2.3 <u>Definition</u> A <u>positive</u> <u>measure</u> is an extended real valued, non negative, and countably additive set function μ, defined on a ring $\mathcal{R}$, and such that $\mu(\emptyset) = 0$. If $\mu(E) = 0$ for all $E \in \mathcal{R}$, then μ is called <u>trivial</u>.

2.4 <u>Definition</u> If μ is a positive measure on a ring $\mathcal{R}$, a set E in $\mathcal{R}$ is said to have <u>finite</u> <u>measure</u> if $\mu(E) < \infty$; the measure of E is <u>σ-finite</u> if there exists a sequence

$(E_n)_{n\in N}$ of sets in $\mathcal{R}$ such that

$$E \subseteq \bigcup_{n=1}^{\infty} E_n \text{ and } \mu(E_n)<\infty,\ n=1,2,\ldots .$$

If the measure of every set E in $\mathcal{R}$ is finite (or σ-finite), the measure of μ is called <u>finite</u> (or <u>σ-finite</u>) on $\mathcal{R}$. If $X\in\mathcal{R}$ (i.e. if $\mathcal{R}$ is an algebra and $\mu(X)$ is finite or σ-finite, then μ is called <u>totally finite</u> or <u>totally σ-finite</u> respectively. The positive measure μ is called <u>complete</u> if the conditions

$$E\in\mathcal{R},\ F\subseteq E, \text{ and } \mu(E) = 0$$

imply that $F\in\mathcal{R}$.

2.5 <u>Definition</u> An extended real valued set function μ defined on a set $\mathcal{C}$ of subsets of X is <u>monotone</u> if, whenever $E\in\mathcal{C}$, $F\in\mathcal{C}$, and $E\subseteq F$, then $\mu(E)\le\mu(F)$.

2.6 <u>Definition</u> An extended real valued set function μ defined on a set $\mathcal{C}$ of subsets of X is <u>subtractive</u> if, whenever $E\in\mathcal{C}$, $F\in\mathcal{C}$, $E\subseteq F$, $F\setminus E\in\mathcal{C}$, and $|\mu(E)|<\infty$, then

$$\mu(F\setminus E) = \mu(F)-\mu(E).$$

2.7 <u>Theorem</u> If μ is a positive measure on a ring $\mathcal{R}$, then μ is monotone and subtractive.

<u>Proof</u>[2] If $E\in\mathcal{R}$, $F\in\mathcal{R}$, and $E\subseteq F$, then $F\setminus E\in\mathcal{R}$ and $\mu(F) = \mu(E)+\mu(F\setminus E)$. The fact that μ is monotone follows now from the fact that it is positive; the fact that it is subtractive follows from the fact that $\mu(E)$, if it is finite, may be subtracted from both sides of the last written equation. #

2.8 __Theorem__ If μ is a positive measure on a ring $\mathcal{R}$, if $E \in \mathcal{R}$, and if $(E_i)_{i \in \mathbb{N}}$ is a sequence of sets in $\mathcal{R}$ such that $E \subseteq \bigcup_{i=1}^{\infty} E_i$, then

$$\mu(E) \leq \sum_{i=1}^{\infty} \mu(E_i).$$

__Proof__[2] By Theorem 1.8, there exists a disjoint sequence $(F_i)_{i \in \mathbb{N}}$ in $\mathcal{R}$ such that $\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} F_i$ and $F_i \subseteq E_i$ for all $i \in \mathbb{N}$. Hence $E = \bigcup_{i=1}^{\infty} (E \cap F_i)$ which is a disjoint union of sets in $\mathcal{R}$. Hence $\mu(E) = \sum_{i=1}^{\infty} \mu(E \cap F_i) \leq \sum_{i=1}^{\infty} \mu(F_i) \leq \sum_{i=1}^{\infty} \mu(E_i)$. #

2.8.1 __Theorem__ If μ is a positive measure on a ring $\mathcal{R}$, $E \in \mathcal{R}$, and $(E_i)_{i \in \mathbb{N}}$ is a disjoint sequence of sets in $\mathcal{R}$ such that $\bigcup_{i=1}^{\infty} E_i \subseteq E$, then

$$\sum_{i=1}^{\infty} \mu(E_i) \leq \mu(E).$$

__Proof__[2] For each n, $\bigcup_{i=1}^{n} E_i \in \mathcal{R}$, it follows that

$$\sum_{i=1}^{n} \mu(E_i) = \mu\left(\bigcup_{i=1}^{n} E_i\right) \leq \mu(E).$$

Hence

$$\sum_{i=1}^{\infty} \mu(E_i) \leq \mu(E). \ \#$$

2.9 __Theorem__ If μ is a positive measure on a ring $\mathcal{R}$, and if $(A_n)_{n \in \mathbb{N}}$ is an increasing sequence of sets in $\mathcal{R}$ for which $\bigcup_{n=1}^{\infty} A_n \in \mathcal{R}$, then

$$\lim_{n \to \infty} \mu(A_n) = \mu\left(\bigcup_{n=1}^{\infty} A_n\right).$$

__Proof__[2] Put $B_1 = A_1$ and $B_n = A_n \setminus A_{n-1}$ for $n \geq 2$. Then $B_n \in \mathcal{R}$ for all $n \in \mathbb{N}$ and $B_i \cap B_j = \emptyset$ if $i \neq j$, and $A_n = \bigcup_{i=1}^{n} B_i$ for all $n \in \mathbb{N}$ and $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i$. Hence $\mu(A_n) = \sum_{i=1}^{n} \mu(B_i)$. Thus $\lim_{n \to \infty} \mu(A_n) = \lim_{n \to \infty} \sum_{i=1}^{n} \mu(B_i) = \sum_{i=1}^{\infty} \mu(B_i) = \mu\left(\bigcup_{i=1}^{\infty} B_i\right) = \mu\left(\bigcup_{i=1}^{\infty} A_i\right)$. #

2.10 <u>Theorem</u> If μ is a positive measure on a ring $\mathcal{R}$, and if $(A_n)_{n\in\mathbb{N}}$ is a decreasing sequence of sets in $\mathcal{R}$ for which $\bigcap_{n=1}^{\infty} A_n \in \mathcal{R}$, and $\mu(A_1)<\infty$, then

$$\lim_{n\to\infty}\mu(A_n) = \mu\left(\bigcap_{n=1}^{\infty} A_n\right).$$

<u>Proof</u>[2] Let $C_n = A_1 \setminus A_n$ for all $n\in\mathbb{N}$. Then $C_1 \subseteq C_2 \subseteq \ldots$ and $C_i \in \mathcal{R}$ for all $i\in\mathbb{N}$, also for all $n\in\mathbb{N}$, $\mu(C_n) = \mu(A_1)-\mu(A_n)$. Hence

$$\begin{aligned}
\mu(A_1)-\mu\left(\bigcap_{n=1}^{\infty} A_n\right) &= \mu\left(A_1 \setminus \bigcap_{n=1}^{\infty} A_n\right)\\
&= \mu\left(\bigcup_{n=1}^{\infty}(A_1\setminus A_n)\right)\\
&= \mu\left(\bigcup_{n=1}^{\infty} C_n\right)\\
&= \lim_{n\to\infty}\mu(C_n)\\
&= \lim_{n\to\infty}(\mu(A_1)-\mu(A_n))\\
&= \mu(A_1)-\lim_{n\to\infty}\mu(A_n).
\end{aligned}$$

Since $\mu(A_1)<\infty$, we have

$$\lim_{n\to\infty}\mu(A_n) = \mu\left(\bigcap_{n=1}^{\infty} A_n\right). \#$$

2.11 <u>Definition</u> An extended real valued set function μ^* on a set $\mathcal{C}$ of subsets of X is <u>finitely</u> <u>subadditive</u> if, for every finite set $\{E_1,\ldots,E_n\}$ of sets in $\mathcal{C}$ whose union is also in $\mathcal{C}$, we have

$$\mu^*\left(\bigcup_{i=1}^{n} E_i\right) \le \sum_{i=1}^{n}\mu^*(E_i).$$

2.12 <u>Definition</u> An extended real valued set function μ^* on set $\mathcal{C}$ of subsets of X is <u>countably</u> <u>subadditive</u> if, for every sequence $(E_i)_{i\in\mathbb{N}}$ of sets in $\mathcal{C}$ whose union is also in $\mathcal{C}$, we have

$$\mu^*\left(\bigcup_{i=1}^{\infty} E_i\right) \le \sum_{i=1}^{\infty}\mu^*(E_i).$$

2.13 <u>Definition</u> An <u>outer measure</u> is an extended real valued, non negative, monotone and countably subadditive set function μ^*, defined on a hereditary σ-ring $\mathcal{H}$, and such that $\mu^*(\emptyset) = 0$.

Observe that an outer measure is necessarily finitely subadditive.

2.14 <u>Theorem</u> If μ is a positive measure on a ring $\mathcal{R}$ and if, for every E in $\mathcal{H}(\mathcal{R})$,

$$\mu^*(E) = \inf\left\{\sum_{n=1}^{\infty}\mu(E_n) / \ E_n \in \mathcal{R}, n=1,2,\ldots, E \subseteq \bigcup_{n=1}^{\infty} E_n\right\},$$

then μ^* is an extension of μ to an outer measure on $\mathcal{H}(\mathcal{R})$; if μ is σ-finite, then so is μ^*.

The outer measure μ^* is called the outer measure <u>induced</u> by the positive measure μ.

<u>Proof</u>[2]If $E \in \mathcal{R}$, then $E \subseteq E \cup \emptyset \cup \emptyset \cup \ldots$ and therefore $\mu^*(E) \leq \mu(E) + \mu(\emptyset) + \mu(\emptyset) + \ldots = \mu(E)$. On the other hand if $E \in \mathcal{R}$, $E_n \in \mathcal{R}$, for all $n \in \mathbb{N}$, and $E \subseteq \bigcup_{n=1}^{\infty} E_n$, then, by Theorem 2.8, $\mu(E) \leq \sum_{n=1}^{\infty}\mu(E_n)$, so that $\mu(E) \leq \mu^*(E)$. This proves that μ^* is an extension of μ, i.e., that if $E \in \mathcal{R}$, then $\mu^*(E) = \mu(E)$; it follows in particular that $\mu^*(\emptyset) = 0$.

If $E \in \mathcal{H}(\mathcal{R})$, $F \in \mathcal{H}(\mathcal{R})$, $E \subseteq F$, and $(E_n)_{n \in \mathbb{N}}$ is a sequence of sets in $\mathcal{R}$ which covers F, then $(E_n)_{n \in \mathbb{N}}$ also covers E, and therefore $\mu^*(E) \leq \mu^*(F)$, so μ^* is monotone.

To prove that μ^* is countably subadditive, suppose that E and E_i are sets in $\mathcal{H}(\mathcal{R})$ such that $E \subseteq \bigcup_{i=1}^{\infty} E_i$.

To prove this, let $\varepsilon > 0$ be given, and choose, for each $i = 1,2,\ldots$, a sequence $(E_{ij})_{j\in\mathbb{N}}$ of sets in $\mathcal{R}$ such that

$$E_i \subseteq \bigcup_{j=1}^{\infty} E_{ij} \quad \text{and} \quad \sum_{j=1}^{\infty}\mu(E_{ij}) < \mu^*(E_i) + \frac{\varepsilon}{2^i}.$$

Then, since the sets E_{ij} form a countable set of sets in $\mathcal{R}$ which cover E,

$$\mu^*(E) \leq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\mu(E_{ij}) < \sum_{i=1}^{\infty}\mu^*(E_i) + \varepsilon .$$

The arbitrariness of ε implies that

$$\mu^*(E) \leq \sum_{i=1}^{\infty}\mu^*(E_i).$$

Suppose, finally, that μ is σ-finite and let E be any set in $\mathcal{H}(\mathcal{R})$. Then by the definition of $\mathcal{H}(\mathcal{R})$, there exists a sequence $(E_i)_{i\in\mathbb{N}}$ of sets in $\mathcal{R}$ such that $E \subseteq \bigcup_{i=1}^{\infty} E_i$. Since μ is σ-finite, there exists, for each $i = 1,2,\ldots$, a sequence $(E_{ij})_{j\in\mathbb{N}}$ of sets in $\mathcal{R}$ such that

$$E_i \subseteq \bigcup_{j=1}^{\infty} E_{ij} \quad \text{and} \quad \mu(E_{ij}) < \infty .$$

Consequently

$$E \subseteq \bigcup_{i=1}^{\infty}\bigcup_{j=1}^{\infty} E_{ij} \quad \text{and} \quad \mu^*(E_{ij}) = \mu(E_{ij}) < \infty . \;\#$$

2.15 <u>Definition</u> Let μ^* be an outer measure on a hereditary σ-ring $\mathcal{H}$. A set E in $\mathcal{H}$ is <u>μ^*-measurable</u> if, for every set A in $\mathcal{H}$,

$$\mu^*(A) = \mu^*(A\cap E) + \mu^*(A\cap E^c).$$

2.16 <u>Theorem</u> If μ^* is an outer measure on a hereditary σ-ring $\mathcal{H}$ and if $\overline{\mathcal{S}}$ is the set of all μ^*-measurable sets, then $\overline{\mathcal{S}}$ is a ring.

<u>Proof</u>[2] If E and F are in $\overline{\mathcal{S}}$ and $A \in \mathcal{H}$, then

(1) $$\mu^*(A) = \mu^*(A\cap E) + \mu^*(A\cap E^c),$$

(2) $$\mu^*(A\cap E) = \mu^*(A\cap E\cap F) + \mu^*(A\cap E\cap F^c),$$

(3) $\mu^*(A\cap E^c) = \mu^*(A\cap E^c\cap F)+\mu^*(A\cap E^c\cap F^c)$.

Substituting (2) and (3) into (1) we obtain for all $A\in\mathcal{H}$

(4) $\mu^*(A) = \mu^*(A\cap E\cap F)+\mu^*(A\cap E\cap F^c)+\mu^*(A\cap E^c\cap F) +$
$\mu^*(A\cap E^c\cap F^c)$.

If in equation (4) we replace A by $A\cap(E\cup F)$, the first three terms of the right hand side remain unaltered and the last term drops out; we get that

(5) $\mu^*(A\cap(E\cup F)) = \mu^*(A\cap E\cap F)+\mu^*(A\cap E\cap F^c)+\mu^*(A\cap E^c\cap F)$.

Since $E^c\cap F^c = (E\cup F)^c$, substituting (5) into (4) yields

(6) $\mu^*(A) = \mu^*(A\cap(E\cup F))+\mu^*(A\cap(E\cup F)^c)$,

which proves that $E\cup F\in\overline{\mathcal{S}}$.

If, similarly, we replace A in equation (4) by $A\cap(E\setminus F)^c = A\cap(E^c\cup F)$, we get that

(7) $\mu^*(A\cap(E\setminus F)^c) = \mu^*(A\cap E\cap F)+\mu^*(A\cap E^c\cap F)+\mu^*(A\cap E^c\cap F^c)$.

Since $E\cap F^c = E\setminus F$, Substituting (7) into (4) yields

(8) $\mu^*(A) = \mu^*(A\cap(E\setminus F))+\mu^*(A\cap(E\setminus F)^c)$,

which proves that $E\setminus F\in\overline{\mathcal{S}}$. Since it is clear that $E = \emptyset$ satisfies (1), it follows that $\overline{\mathcal{S}}$ is a ring. #

Observe that if μ^* is an outer measure on a hereditary σ-ring $\mathcal{H}$ and if a set E in $\mathcal{H}$ is such that, for every A in $\mathcal{H}$,

$$\mu^*(A) \geq \mu^*(A\cap E)+\mu^*(A\cap E^c),$$

then E is μ^*-measurable.

2.17 <u>Theorem</u> If μ^* is an outer measure on a hereditary σ-ring $\mathcal{H}$ and if $\overline{\mathcal{S}}$ is the set of all μ^*-measurable sets, then $\overline{\mathcal{S}}$ is a σ-ring. If $A\in\mathcal{H}$ and if $(E_n)_{n\in\mathbb{N}}$ is a disjoint sequence of sets in $\overline{\mathcal{S}}$ with $\bigcup_{n=1}^{\infty}E_n = E$, then

$$\mu^*(A\cap E) = \sum_{n=1}^{\infty}\mu^*(A\cap E_n).$$

Proof[2] Replacing E and F in (5) of Theorem 2.16 by E_1 and E_2 respectively, we see that for all $A\in\mathcal{H}$

$$\mu^*(A\cap(E_1\cup E_2)) = \mu^*(A\cap E_1\cap E_2)+\mu^*(A\cap E_1\cap E_2^c)+\mu^*(A\cap E_1^c\cap E_2)$$
$$= \mu^*(A\cap E_1)+\mu^*(A\cap E_2),$$

since $E_1\cap E_2=\emptyset$. It follows that by mathematical induction that

$$\mu^*(A\cap(\bigcup_{i=1}^{n}E_i)) = \sum_{i=1}^{n}\mu^*(A\cap E_i)$$

for all $n\in\mathbb{N}$. If we write

$$F_n = \bigcup_{i=1}^{n}E_i,\ n = 1,2,\ldots,$$

then it follows from Theorem 2.16 that for all $A\in\mathcal{H}$

$$\mu^*(A) = \mu^*(A\cap F_n)+\mu^*(A\cap F_n^c)$$
$$\geq \sum_{i=1}^{n}\mu^*(A\cap E_i)+\mu^*(A\cap E^c).$$

Since this is true for every n, we obtain for all $A\in\mathcal{H}$

$$\mu^*(A) \geq \sum_{i=1}^{\infty}\mu^*(A\cap E_i)+\mu^*(A\cap E^c)$$
$$\geq \mu^*(A\cap E)+\mu^*(A\cap E^c)$$
$$\geq \mu^*(A),$$

since $A = (A\cap E)\cup(A\cap E^c)$ and μ^* is finitely subadditive. It follows that $E\in\bar{\mathcal{S}}$ (so that, by the way, $\bar{\mathcal{S}}$ is closed under the formation of disjoint countable unions), and also that for all $A\in\mathcal{H}$

(1) $\quad\displaystyle\sum_{i=1}^{\infty}\mu^*(A\cap E_i)+\mu^*(A\cap E^c) = \mu^*(A\cap E)+\mu^*(A\cap E^c).$

Replacing A by $A\cap E$ in (1), we obtain the second assertion of the theorem. (Since $\mu^*(A\cap E^c)$ may be infinite, it is not permissible simply to subtract it from both sides of (1).) Since every countable union of sets in a ring $\mathcal{R}$ may be written as a disjoint countable union of sets in the ring, we see also that $\overline{\mathcal{G}}$ is a σ-ring. #

2.18 Corollary If μ^* is an outer measure on a hereditary σ-ring $\mathcal{H}$ and if $\overline{\mathcal{G}}$ is the set of all μ^*-measurable sets and if $(E_n)_{n\in\mathbb{N}}$ is a disjoint sequence of sets in $\overline{\mathcal{G}}$, then

$$\mu^*\left(\bigcup_{n=1}^{\infty}E_n\right) = \sum_{n=1}^{\infty}\mu^*(E_n).$$

Proof It follows from Theorem 2.17 by replacing A by $\bigcup_{n=1}^{\infty}E_n$. #

2.19 Theorem If μ^* is an outer measure on a hereditary σ-ring $\mathcal{H}$ and if $\overline{\mathcal{G}}$ is the set of all μ^*-measurable sets, then every set of outer measure zero belongs to $\overline{\mathcal{G}}$ and the set function $\bar{\mu}$, defined for E in $\overline{\mathcal{G}}$ by $\bar{\mu}(E) = \mu^*(E)$, is a complete measure on $\overline{\mathcal{G}}$.

Note that the measure $\bar{\mu}$ is called the measure induced by the outer measure μ^*.

Proof[2] If $E\in\mathcal{H}$ and $\mu^*(E) = 0$, then, for every A in $\mathcal{H}$, we have

$$\mu^*(A) = \mu^*(E) + \mu^*(A) \geq \mu^*(A\cap E) + \mu^*(A\cap E^c),$$

so that indeed $E\in\overline{\mathcal{G}}$. By Corollary 2.18 $\bar{\mu}$ is countably additive on $\overline{\mathcal{G}}$, hence $\bar{\mu}$ is a positive measure on $\overline{\mathcal{G}}$. If

$$E\in\overline{\mathcal{G}},\ F\subseteq E,\ \text{and } \bar{\mu}(E) = \mu^*(E) = 0,$$

then $\mu^*(F) = 0$, so that $F\in\overline{\mathcal{G}}$, which proves that $\bar{\mu}$ is complete. #

Next, from Theorem 2.20 to Theorem 2.25, we assume that μ is a positive measure on a ring $\mathcal{R}$, μ^* is the induced outer measure on $\mathcal{H}(\mathcal{R})$, and $\bar{\mu}$ is the positive measure induced by μ^* on the σ-ring $\bar{\mathcal{S}}$ of all μ^*-measurable sets.

2.20 <u>Theorem</u> Every set in $\mathcal{S}(\mathcal{R})$ is μ^*-measurable.

<u>Proof</u> [2] If $E \in \mathcal{R}$, $A \in \mathcal{H}(\mathcal{R})$, and $\varepsilon > 0$, then, by the definition of μ^*, there exists a sequence $(E_n)_{n \in \mathbb{N}}$ of sets in $\mathcal{R}$ such that $A \subseteq \bigcup_{n=1}^{\infty} E_n$ and

$$\mu^*(A) + \varepsilon \geq \sum_{n=1}^{\infty} \mu(E_n).$$

By the addition of μ on $\mathcal{R}$ we have

$$\mu(E_n) = \mu(E_n \cap E) + \mu(E_n \cap E^c),$$

so

$$\begin{aligned} \mu^*(A) + \varepsilon &\geq \sum_{n=1}^{\infty} (\mu(E_n \cap E) + \mu(E_n \cap E^c)) \\ &= \sum_{n=1}^{\infty} \mu(E_n \cap E) + \sum_{n=1}^{\infty} \mu(E_n \cap E^c) \\ &\geq \mu^*(A \cap E) + \mu^*(A \cap E^c), \end{aligned}$$

since

$$A \cap E \subseteq \bigcup_{n=1}^{\infty} (E_n \cap E)$$

and

$$A \cap E^c \subseteq \bigcup_{n=1}^{\infty} (E_n \cap E^c).$$

Since this is true for every ε, it follows that E is μ^*-measurable. In other words, we have proved that $\mathcal{R} \subseteq \bar{\mathcal{S}}$; it follows from the fact $\bar{\mathcal{S}}$ is a σ-ring that $\mathcal{S}(\mathcal{R}) \subseteq \bar{\mathcal{S}}$. #

2.21 <u>Theorem</u> If $E \in \mathcal{H}(\mathcal{R})$, then

$$\begin{aligned} \mu^*(E) &= \inf\{\bar{\mu}(F) / E \subseteq F \in \bar{\mathcal{S}}\} \\ &= \inf\{\bar{\mu}(F) / E \subseteq F \in \mathcal{S}(\mathcal{R})\}. \end{aligned}$$

Equivalent to the statement of Theorem 2.21 is the assertion that the outer measure induced by $\bar{\mu}$ on $\mathcal{S}(\mathcal{R})$ and the outer measure induced by $\bar{\mu}$ on $\bar{\mathcal{S}}$ both coincide with μ^*.

<u>Proof</u>[2] Since, for F in $\mathcal{R}$, $\mu(F) = \bar{\mu}(F)$ (by the definition of $\bar{\mu}$ and Theorem 2.14), it folows that

$$\mu^*(E) = \inf\{\sum_{n=1}^{\infty}\mu(E_n) / E \subseteq \bigcup_{n=1}^{\infty}E_n,\ E_n \in \mathcal{R},\ n=1,2,\ldots\}$$
$$\geq \inf\{\sum_{n=1}^{\infty}\bar{\mu}(E_n) / E \subseteq \bigcup_{n=1}^{\infty}E_n,\ E_n \in \mathcal{S}(\mathcal{R}),\ n=1,2,\ldots\}.$$

Since every sequence $(E_n)_{n\in\mathbb{N}}$ of sets in $\mathcal{S}(\mathcal{R})$ for which

$$E \subseteq \bigcup_{n=1}^{\infty}E_n = F$$

may be replaced by a disjoint sequence with the same property, without increasing the sum of the measures of the terms of the sequence, and since, by the definition of $\bar{\mu}$, $\bar{\mu}(F) = \mu^*(F)$ for F in $\bar{\mathcal{S}}$, it follows that

$$\mu^*(E) \geq \inf\{\bar{\mu}(F) / E \subseteq F \in \mathcal{S}(\mathcal{R})\}$$
$$\geq \inf\{\bar{\mu}(F) / E \subseteq F \in \bar{\mathcal{S}}\}$$
$$\geq \mu^*(E).\ \#$$

2.22 <u>Definition</u> If $E \in \mathcal{H}(\mathcal{R})$ and $F \in \mathcal{S}(\mathcal{R})$, we shall say that F is <u>measurable</u> <u>cover</u> of E if $E \subseteq F$ and if, for every set G in $\mathcal{S}(\mathcal{R})$ for which $G \subseteq F \sim E$, we have $\bar{\mu}(G) = 0$.

2.23 <u>Theorem</u> If a set E in $\mathcal{H}(\mathcal{R})$ is of finite outer measure, then there exists a set F in $\mathcal{S}(\mathcal{R})$ such that $\mu^*(E) = \bar{\mu}(F)$ and such that F is a measurable cover of E.

<u>Proof</u>[2] It follows from Theorem 2.21 that, for every $n = 1,2,\ldots$, there exists a set F_n in $\mathcal{S}(\mathcal{R})$ such that

$$E \subseteq F_n \text{ and } \bar{\mu}(F_n) < \mu^*(E) + \frac{1}{n}.$$

If we write $F = \bigcap_{n=1}^{\infty}F_n$, then

$$E \subseteq F \in \mathcal{S}(\mathcal{R}) \text{ and } \mu^*(E) \leq \bar{\mu}(F) \leq \bar{\mu}(F_n) < \mu^*(E) + \frac{1}{n}.$$

Since n is arbitrary, it follows that $\mu^*(E) = \bar{\mu}(F)$.
If $G \in \mathcal{S}(\mathcal{R})$ and $G \subseteq F \smallsetminus E$, then $E \subseteq F \smallsetminus G$ and therefore

$$\bar{\mu}(F) = \mu^*(E) \leq \mu^*(F \smallsetminus G) = \bar{\mu}(F \smallsetminus G) = \bar{\mu}(F) - \bar{\mu}(G) \leq \bar{\mu}(F);$$

the fact that F is a measurable cover E follows from the finiteness of $\bar{\mu}(F)$.. #

2,24 Theorem If $E \in \mathcal{H}(\mathcal{R})$ and F is a measurable cover of E, then $\mu^*(E) = \bar{\mu}(F)$; if both F_1 and F_2 are measurable covers of E, then $\bar{\mu}(F_1 \Delta F_2) = 0$.

Proof[2] Since the relation $E \subseteq F_1 \cap F_2 \subseteq F_1$ implies that $F_1 \smallsetminus (F_1 \cap F_2) \subseteq F_1 \smallsetminus E$, it follows from the fact that F_1 is a measurable cover of E that

$$\bar{\mu}(F_1 \smallsetminus (F_1 \cap F_2)) = 0.$$

Similarly

$$\bar{\mu}(F_2 \smallsetminus (F_1 \cap F_2)) = 0.$$

Hence

$$\bar{\mu}(F_1 \smallsetminus F_2) = \bar{\mu}(F_2 \smallsetminus F_1) = 0,$$

so

$$\begin{aligned}\bar{\mu}(F_1 \Delta F_2) &= \bar{\mu}((F_1 \smallsetminus F_2) \cup (F_2 \smallsetminus F_1)) \\ &= \bar{\mu}(F_1 \smallsetminus F_2) + \bar{\mu}(F_2 \smallsetminus F_1) \\ &= 0,\end{aligned}$$

also

$$\bar{\mu}(F_1 \smallsetminus F_2) + \bar{\mu}(F_1 \cap F_2) = \bar{\mu}(F_2 \smallsetminus F_1) + \bar{\mu}(F_1 \cap F_2),$$

that is

$$\bar{\mu}(F_1) = \bar{\mu}(F_2).$$

If $\mu^*(E) = \infty$, then the relation $\mu^*(E) = \bar{\mu}(F)$

is trivial; if $\mu^*(E)<\infty$, then it follows from Theorem 2.23 that there exists a measurable cover F_o of E with

$$\bar{\mu}(F_o) = \mu^*(E).$$

By the preceding paragraph, we have that every two measurable covers have the same measure, hence

$$\bar{\mu}(F) = \mu^*(E). \ \#$$

2.25 <u>Theorem</u> If μ on $\mathcal{R}$ is σ-finite, then so are the measures $\bar{\mu}$ on $\mathcal{S}(\mathcal{R})$ and $\bar{\mu}$ on $\bar{\mathcal{S}}$.

<u>Proof</u>[2] According to Theorem 2.14, if μ is σ-finite, then so is μ^*. Hence for every E in $\bar{\mathcal{S}}$ there exists a sequence $(E_i)_{i\in N}$ of sets in $\mathcal{H}(\mathcal{R})$ such that

$$E \subseteq \bigcup_{i=1}^{\infty} E_i,\ \mu^*(E_i)<\infty,\ i=1,2,\ldots .$$

By Theorem 2.23, there exists, for each $i = 1,2,\ldots$, a set F_i in $\mathcal{S}(\mathcal{R})$ such that $\mu^*(E_i) = \bar{\mu}(F_i)$ and $E_i \subseteq F_i$. Then

$$E \subseteq \bigcup_{i=1}^{\infty} F_i,\ \mu^*(E_i) = \bar{\mu}(F_i)<\infty,\ i = 1,2,\ldots . \ \#$$

2.26 <u>Theorem</u> (Carathéodory) If μ is a σ-finite positive measure on a ring $\mathcal{R}$, then there is a unique positive measure $\bar{\mu}$ on the σ-ring $\mathcal{S}(\mathcal{R})$ such that, for E in $\mathcal{R}$, $\bar{\mu}(E) = \mu(E)$; the measure $\bar{\mu}$ is σ-finite.

The measure $\bar{\mu}$ is called the <u>extension</u> of μ; except when it is likely to lead to confusion, we shall write $\mu(E)$ instead of $\bar{\mu}(E)$ even for sets E in $\mathcal{S}(\mathcal{R})$.

<u>Proof</u>[2] The existence of $\bar{\mu}$ (even without the restriction of σ-finiteness) is proved by Theorem 2.19 and Theorem 2.20. If μ is σ-finite on $\mathcal{R}$, then, by

Theorem 2.25 $\bar{\mu}$ is σ-finite on $\mathcal{S}(\mathcal{R})$.

To prove uniqueness, suppose that μ_1 and μ_2 are two positive measures on $\mathcal{S}(\mathcal{R})$ such that $\mu_1(E) = \mu_2(E)$ whenever $E \in \mathcal{R}$. Let A be any fixed set in $\mathcal{R}$ such that $\mu_1(A) < \infty$, so $\mu_2(A) < \infty$. Since $\mathcal{R}$ is a ring, $\mathcal{R} \cap A$ is a ring. By Theorem 1.16, we have $\mathcal{S}(\mathcal{R}) \cap A = \mathcal{S}(\mathcal{R} \cap A)$. Put

$$\mathcal{M} = \{E \in \mathcal{S}(\mathcal{R}) \cap A / \mu_1(E) = \mu_2(E)\}.$$

If $(A_n)_{n \in \mathbb{N}}$ is an increasing sequence of sets in $\mathcal{S}(\mathcal{R}) \cap A$, then, by Theorem 2.9 $\lim_{n\to\infty} \mu_1(A_n) = \mu_1(\bigcup_{n=1}^{\infty} A_n)$ and $\lim_{n\to\infty} \mu_2(A_n) = \mu_2(\bigcup_{n=1}^{\infty} A_n)$. But $\mu_1(A_n) = \mu_2(A_n)$ for all $n \in \mathbb{N}$, so we have $\mu_1(\bigcup_{n=1}^{\infty} A_n) = \mu_2(\bigcup_{n=1}^{\infty} A_n)$, that is $\bigcup_{n=1}^{\infty} A_n \in \mathcal{M}$.
If $(A_n)_{n \in \mathbb{N}}$ is a decreasing sequence of sets in $\mathcal{S}(\mathcal{R}) \cap A$, then, by Theorem 2.10 we get that $\bigcap_{n=1}^{\infty} A_n \in \mathcal{M}$. Hence $\mathcal{M}$ is a monotone class. Since $\mathcal{M}$ contain $\mathcal{R} \cap A$, it follows from Theorem 1.22 that $\mathcal{M}$ contains $\mathcal{S}(\mathcal{R} \cap A)$, i.e., $\mathcal{M}$ contains $\mathcal{S}(\mathcal{R}) \cap A$ and so $\mathcal{M} = \mathcal{S}(\mathcal{R}) \cap A$. Now let $E \in \mathcal{S}(\mathcal{R})$. By Theorem 1.12, we have $E \subseteq \bigcup_{i=1}^{n} E_i$ with $E_i \in \mathcal{R}$. Without loss of generality, we assume $\bigcup_{i=1}^{n} E_i$ is a disjoint union. Since μ is σ-finite on $\mathcal{R}$, there exists, for each $i = 1, 2, \ldots, n$, a sequence $(E_{ij})_{j \in \mathbb{N}}$ of sets in $\mathcal{R}$ such that

$$E_i \subseteq \bigcup_{j=1}^{\infty} E_{ij} \text{ and } \mu(E_{ij}) < \infty .$$

Assume without loss of generality that $\bigcup_{j=1}^{\infty} E_{ij}$ is a disjoint union, then for each $i = 1, 2, \ldots, n$, $E_i = \bigcup_{j=1}^{\infty} (E_i \cap E_{ij})$ is a disjoint union and $\mu(E_i \cap E_{ij}) < \infty$. Let $F_{ij} = E_i \cap E_{ij}$, so $F_{ij} \in \mathcal{R}$ and $\mu(F_{ij}) < \infty$. Then $E \subseteq \bigcup_{i=1}^{n} E_i = \bigcup_{i=1}^{n} \bigcup_{j=1}^{\infty} F_{ij}$ which is a disjoint union, so $E = \bigcup_{i=1}^{n} \bigcup_{j=1}^{\infty} (E \cap F_{ij})$ is a disjoint union.

Since μ_1 and μ_2 are positive measures on $\mathcal{S}(\mathcal{R})$, we have

$$\mu_1(E) = \sum_{i=1}^{n}\sum_{j=1}^{\infty}\mu_1(E\cap F_{ij}) \text{ and } \mu_2(E) = \sum_{i=1}^{n}\sum_{j=1}^{\infty}\mu_2(E\cap F_{ij}).$$

Since for all $A\in\mathcal{R}\ni\mu(A)<\infty$ $\mu_1(E') = \mu_2(E')$ for all $E'\in\mathcal{S}(\mathcal{R})\cap A$, hence $\mu_1(E\cap F_{ij}) = \mu_2(E\cap F_{ij})$. Therefore $\mu_1(E) = \mu_2(E)$. #

2.27 <u>Theorem</u> If μ is a positive measure on a σ-ring $\mathcal{S}$, then the set $\bar{\mathcal{S}}$ of all sets of the form $E\Delta N$, where $E\in\mathcal{S}$ and N is a subset of set of measure zero in $\mathcal{S}$, is a σ-ring, and the set function $\bar{\mu}$ defined by $\bar{\mu}(E\Delta N) = \mu(E)$ is a complete positive measure on $\bar{\mathcal{S}}$.

<u>Proof</u>[2] If $E\in\mathcal{S}$, $N\subseteq A\in\mathcal{S}$, and $\mu(A) = 0$, then the relations

$$E\cup N = (E\setminus A)\Delta[A\cap(E\cup N)]$$

and

$$E\Delta N = (E\setminus A)\cup[A\cap(E\Delta N)]$$

shows that the set $\bar{\mathcal{S}}$ may also be described as the set of all sets of the form $E\cup N$, where $E\in\mathcal{S}$ and N is a subset of a set of measure zero in $\mathcal{S}$. So, clearly, $\bar{\mathcal{S}}$ is closed under the formation of countable unions. Claim that $\bar{\mathcal{S}}$ is closed under the formation of symmetric differences, let $E_1, E_2\in\mathcal{S}$ and N_1, N_2 are subsets of $A\in\mathcal{S}$ with $\mu(A) = 0$. Then $E_1\cup N_1$ and $E_2\cup N_2$ belong to $\bar{\mathcal{S}}$. Subclaim that

$$(E_1\cup N_1)\Delta(E_2\cup N_2) = (E_1\setminus(A\cup E_2))\cup(E_2\setminus(A\cup E_1))\cup[A\cap((E_1\cup N_1)\Delta (E_2\cup N_2))],$$

let $x\in(E_1\cup N_1)\Delta(E_2\cup N_2) = [(E_1\cup N_1)\setminus(E_2\cup N_2)]\cup[(E_2\cup N_2)\setminus(E_1\cup N_1)]$.

If $x \in (E_1 \cup N_1) \smallsetminus (E_2 \cup N_2)$, then $x \in E_1 \cup N_1$ and $x \notin E_2 \cup N_2$, if $x \in A$, so $x \in [A \cap ((E_1 \cup N_1) \Delta (E_2 \cup N_2))]$, if $x \notin A$, so $x \notin N_1$, hence $x \in E_1$ which implies that $x \in (E_1 \smallsetminus (A \cup E_2))$, therefore

$$[(E_1 \cup N_1) \smallsetminus (E_2 \cup N_2)] \subseteq (E_1 \smallsetminus (A \cup E_2)) \cup [A \cap ((E_1 \cup N_1) \Delta (E_2 \cup N_2))]$$
$$\subseteq (E_1 \smallsetminus (A \cup E_2)) \cup (E_2 \smallsetminus (A \cup E_1)) \cup [A \cap ((E_1 \cup N_1) \Delta (E_2 \cup N_2))].$$

Similarly, we have

$$[(E_2 \cup N_1) \smallsetminus (E_1 \cup N_2) \subseteq (E_1 \smallsetminus (A \cup E_2)) \cup (E_2 \smallsetminus (A \cup E_1)) \cup [A \cap ((E_1 \cup N_1) \Delta (E_2 \cup N_2))],$$

so

$$[(E_1 \cup N_1) \Delta (E_2 \cup N_2)] \subseteq (E_1 \smallsetminus (A \cup E_2)) \cup (E_2 \smallsetminus (A \cup E_1)) \cup [A \cap ((E_1 \cup N_1) \Delta (E_2 \cup N_2))].$$

Next, let $x \in (E_1 \smallsetminus (A \cup E_2)) \cup (E_2 \smallsetminus (A \cup E_1)) \cup [A \cap ((E_1 \cup N_1) \Delta (E_2 \cup N_2))]$. If $x \in (E_1 \smallsetminus (A \cup E_2))$, so $x \notin E_2$ and $x \notin N_2$, hence $x \in ((E_1 \cup N_1) \smallsetminus (E_2 \cup N_2))$. Similarly, if $x \in (E_2 \smallsetminus (A \cup E_2))$, then $x \in ((E_2 \cup N_2) \smallsetminus (E_1 \cup N_1))$. Therefore

$$(E_1 \smallsetminus (A \cup E_2)) \cup (E_2 \smallsetminus (A \cup E_1)) \cup [A \cap ((E_1 \cup N_1) \Delta (E_2 \cup N_2))] \subseteq [(E_1 \cup N_1) \Delta (E_2 \cup N_2)].$$

So we have subclaim. It follows that we have the claim. By Remark 1.7.6, we have $\bar{\mathcal{S}}$ is a ring. But $\bar{\mathcal{S}}$ is closed under the formation of countable unions, that is, $\bar{\mathcal{S}}$ is a σ-ring. If

$$E_1 \Delta N_1 = E_2 \Delta N_2$$

where $E_i \in \mathcal{S}$ and N_i is a subset of a set of measure zero in $\mathcal{S}$, $i = 1,2$, then we claim that

$$E_1 \Delta E_2 = N_1 \Delta N_2.$$

Let $x \in E_1 \Delta E_2 = (E_1 \setminus E_2) \cup (E_2 \setminus E_1)$. If $x \in E_1 \setminus E_2$, then $x \in E_1$ and $x \notin E_2$; if $x \in N_1$, then by $E_1 \Delta N_1 = E_2 \Delta N_2$ we get $x \notin N_2$, so $x \in N_1 \setminus N_2 \subseteq N_1 \Delta N_2$; if $x \notin N_1$, then by $E_1 \Delta N_1 = E_2 \Delta N_2$ we get $x \in N_2$, so $x \in N_2 \setminus N_1 \subseteq N_1 \Delta N_2$; hence $E_1 \setminus E_2 \subseteq N_1 \Delta N_2$. Similarly, $E_2 \setminus E_1 \subseteq N_1 \Delta N_2$, so $E_1 \Delta E_2 \subseteq N_1 \Delta N_2$. Similarly, we have $N_1 \Delta N_2 \subseteq E_1 \Delta E_2$. Therefore $E_1 \Delta E_2 = N_1 \Delta N_2$. Since $\mu(N_1 \Delta N_2) = 0$, $\mu(E_1 \Delta E_2) = 0$. It follows that $\mu(E_1) = \mu(E_2)$, and hence that $\bar{\mu}$ as defined by the relations

$$\bar{\mu}(E \Delta N) = \mu(E)$$

is well-defined, also $\bar{\mu}(E \cup N) = \bar{\mu}((E \setminus A) \Delta [A \cap (E \cup N)]) = \mu(E \setminus A) = \mu(E \cap A) + \mu(E \setminus A) = \mu(E)$ whenever $E \in \mathcal{S}$ and $N \subseteq A \in \mathcal{S}$ with $\mu(A) = 0$, therefore

$$\bar{\mu}(E \Delta N) = \bar{\mu}(E \cup N) = \mu(E).$$

Using the union (instead of the symmetric difference) representation of sets in $\bar{\mathcal{S}}$, it is easy to verify that $\bar{\mu}$ is a positive measure. Finally we shall show that $\bar{\mu}$ is complete. To prove this, let $E \in \mathcal{S}$, $N \subseteq A \in \mathcal{S}$ with $\mu(A) = 0$ such that $\bar{\mu}(E \cup N) = 0$. Then $\bar{\mu}(E \cup N) = \mu(E) = 0$. Since $\bar{\mathcal{S}}$ contains all subsets of sets of measure zero in $\mathcal{S}$, we have every subset of $E \cup N$ belongs to $\bar{\mathcal{S}}$. This shows that $\bar{\mu}$ is complete. #

Remark: It is easy to see that the measure $\bar{\mu}$ on $\bar{\mathcal{S}}$ having the property that $\bar{\mu}(N) = 0$ for all $N \subseteq A \in \mathcal{S}$ where $\mu(A) = 0$ is unique.

Let μ be a positive measure on a σ-ring $\mathcal{S}$. Let $\bar{\mathcal{S}} = \{E \cup N / \ E \in \mathcal{S}$ and $N \subseteq A \in \mathcal{S}$ where $\mu(A) = 0\}$. Then Theorem 2.27 shows that $\exists !$ complete positive measure $\bar{\mu}$

on $\bar{\mathcal{S}}$ extending μ. $\bar{\mu}$ is called the <u>completion</u> of μ.

2.28 <u>Theorem</u> If μ is a σ-finite positive measure on a ring $\mathcal{R}$, and if μ^* is the outer measure induced by μ, then the completion of the extension of μ to $\mathcal{S}(\mathcal{R})$ is identical with μ^* on the set of all μ^*-measurable sets.

<u>Proof</u>[2] Let us denote the set of all μ^*-measurable sets by $\mathcal{S}^*$ and the domain of the completion $\bar{\mu}$ of μ by $\bar{\mathcal{S}}$, that is $\bar{\mathcal{S}}$ is the set of all sets of the form $E \Delta N$, where $E \in \mathcal{S}(\mathcal{R})$ and N is a subset of a set of measure zero in $\mathcal{S}(\mathcal{R})$. Since μ^* on $\mathcal{S}^*$ is a complete measure and from the proof of Theorem 2.27 it follows that $\bar{\mathcal{S}}$ is contained in $\mathcal{S}^*$ and that $\bar{\mu}$ and μ^* coincide on $\bar{\mathcal{S}}$. All that we have left to prove is that $\mathcal{S}^*$ is contained in $\bar{\mathcal{S}}$. Claim that if $E_o \in \mathcal{S}^*$ and $\mu^*(E_o) < \infty$, then $E_o \in \bar{\mathcal{S}}$. By Theorem 2.23 there exists a set F in $\mathcal{S}(\mathcal{R})$ such that $\mu^*(E_o) = \mu^*(F)$ and such that F is a measurable cover of E_o. It follows from the finiteness of $\mu^*(E_o)$, and the fact that μ^* is a positive measure on $\mathcal{S}^*$, that $\mu^*(F \setminus E_o) = 0$. By Theorem 2.23 again, there exists a set G in $\mathcal{S}(\mathcal{R})$ such that G is a measurable cover of $F \setminus E_o$ and

$$\mu^*(G) = \mu^*(F \setminus E_o) = 0.$$

The relation

$$E_o = (F \setminus G) \cup (E_o \cap G)$$

exhibits E_o as a union of a set in $\mathcal{S}(\mathcal{R})$ and a set which is a subset of a set of measure zero in $\mathcal{S}(\mathcal{R})$. This shows that $E_o \in \bar{\mathcal{S}}$, and thus we have the claim. Now let $E \in \mathcal{S}^*$. Since μ is a σ-finite positive measure on a ring $\mathcal{R}$, by

Theorem 2.25, we have μ^* on $\mathcal{S}^*$ is σ-finite. Then there exists a sequence $(E_i)_{i\in N}$ of sets in $\mathcal{S}^*$ such that

$$E \subseteq \bigcup_{i=1}^{\infty} E_i \text{ and } \mu^*(E_i) < \infty,\ i = 1,2,\ldots .$$

Without loss of generality we assume $\bigcup_{i=1}^{\infty} E_i$ is a disjoint union. Then $E = \bigcup_{i=1}^{\infty}(E\cap E_i)$ is a disjoint union and $\mu^*(E\cap E_i)<\infty$, $i = 1,2,\ldots$. Hence $E\cap E_i \in \bar{\mathcal{S}}$, therefore $\bigcup_{i=1}^{\infty}(E\cap E_i) \in \bar{\mathcal{S}}$, i.e., $E\in\bar{\mathcal{S}}$. This proves that $\mathcal{S}^*$ is contained in $\bar{\mathcal{S}}$, and thus completes the proof of theorem. #

2.29 Theorem If μ is a σ-finite positive measure on a ring $\mathcal{R}$, then, for every set E of finite measure in $\mathcal{S}(\mathcal{R})$ and for every positive number ε, there exists a set E_o in $\mathcal{R}$ such that $\mu(E \Delta E_o) \le \varepsilon$.

Proof[2] By the definition of μ^*,

$$\mu^*(E) = \inf\Big\{\sum_{i=1}^{\infty}\mu(E_i)/\ E \subseteq \bigcup_{i=1}^{\infty} E_i,\ E_i\in\mathcal{R},\ i = 1,2,\ldots\Big\}.$$

We have μ^* on the set of all μ^*-measurable sets is a complete measure, and $\mathcal{S}(\mathcal{R})$ is a subset of the set of all μ^*-measurable sets, and by Theorem 2.26 we have that there is a unique positive measure $\bar{\mu}$ on $\mathcal{S}(\mathcal{R})$ such that, for E in $\mathcal{R}$, $\bar{\mu}(E) = \mu(E)$, we shall write μ instead of $\bar{\mu}$. Since $E\in\mathcal{S}(\mathcal{R})$, we have $\mu^*(E) = \mu(E)$. Then there exists a sequence $(E_i)_{i\in N}$ of sets in $\mathcal{R}$ such that

$$E \subseteq \bigcup_{i=1}^{\infty} E_i \text{ and } \mu\Big(\bigcup_{i=1}^{\infty} E_i\Big) \le \mu(E)+\frac{\varepsilon}{2}<\infty\ (\because \mu(E)<\infty).$$

Since

$$\lim_{n\to\infty}\mu\Big(\bigcup_{i=1}^{n} E_i\Big) = \mu\Big(\bigcup_{i=1}^{\infty} E_i\Big),$$

there exists a positive integer n such that if

$$E_o = \bigcup_{i=1}^{n} E_i,$$

then

$$\mu\left(\bigcup_{i=1}^{\infty}E_i\right)-\mu(E_o)\le \frac{\varepsilon}{2}.$$

Clearly $E_o \in \mathcal{R}$. Since

$$\mu(E\setminus E_o)\le\mu\left(\bigcup_{i=1}^{\infty}E_i\setminus E_o\right)=\mu\left(\bigcup_{i=1}^{\infty}E_i\right)-\mu(E_o)\le\frac{\varepsilon}{2}$$

and

$$\mu(E_o\setminus E)\le\mu\left(\bigcup_{i=1}^{\infty}E_i\setminus E\right)=\mu\left(\bigcup_{i=1}^{\infty}E_i\right)-\mu(E)\le\frac{\varepsilon}{2},$$

then

$$\mu(E\Delta E_o)=\mu(E\setminus E_o)+\mu(E_o\setminus E)\le\varepsilon.\ \#$$

We have seen that if μ is a positive measure on a σ-ring $\mathcal{S}$, then the set function μ^* (defined for every E in the hereditary σ-ring $\mathcal{H}(\mathcal{S})$ by

$$\mu^*(E)=\inf\{\mu(F)/\ E\subseteq F\in\mathcal{S}\})$$

is an outer measure. Now we shall define the inner measure.

2.30 <u>Definition</u> We define the <u>inner</u> <u>measure</u> μ_* induced by μ; for every E in $\mathcal{H}(\mathcal{S})$ we write

$$\mu_*(E)=\sup\{\mu(F)/\ E\supseteq F\in\mathcal{S}\}.$$

Then μ_* is non negative, monotone, and such that $\mu_*(\emptyset)=0$.

Now from Theorem 2.31 to Theorem 2.39 we assume that μ is a finite positive measure on a σ-ring $\mathcal{S}$, μ^* and μ_* are the outer measure and the inner measure induced by μ, respectively, and $\bar{\mu}$ on $\bar{\mathcal{S}}$ is the completion of μ.

We recall that $\bar{\mu}$ on $\bar{\mathcal{S}}$ coincides with μ^* on the set of all μ^*-measurable sets (by Theorem 2.28).

2.31 <u>Theorem</u> If $E\in\mathcal{H}(\mathcal{S})$, then

$$\mu_*(E)=\sup\{\bar{\mu}(F)/\ E\supseteq F\in\bar{\mathcal{S}}\}.$$

Proof[2] Since $\mathcal{S} \subseteq \bar{\mathcal{S}}$, it is clear from the definition of μ_* that

$$\mu_*(E) \leq \sup\{\bar{\mu}(F) / E \supseteq F \in \bar{\mathcal{S}}\}.$$

By the proof of Theorem 2.27 implies that, for every F in $\bar{\mathcal{S}}$, there is a G in $\mathcal{S}$ with $G \subseteq F$ and $\bar{\mu}(F) = \mu(G)$. Then we have that

$$\{\bar{\mu}(F) / E \supseteq F \in \bar{\mathcal{S}}\} \subseteq \{\mu(G) / E \supseteq G \in \mathcal{S}\},$$

the proof of the theorem is complete. #

2.32 Definition If $E \in \mathcal{H}(\mathcal{S})$ and $F \in \mathcal{S}$, we shall say that F is a measurable kernel of E if $F \subseteq E$ and if, for every set G in $\mathcal{S}$ for which $G \subseteq E \setminus F$, we have $\mu(G) = 0$.

2.33 Theorem Every set E in $\mathcal{H}(\mathcal{S})$ has a measurable kernel.

Proof[2] By Theorem 2.23, let $\hat{E}$ be a measurable cover of E, let N be a measurable cover of $\hat{E} \setminus E$, and write $F = \hat{E} \setminus N$. We have

$$F = \hat{E} \setminus N \subseteq \hat{E} \setminus (\hat{E} \setminus E) = E,$$

and, if G is in $\mathcal{S}$ such that $G \subseteq E \setminus F$, then

$$G \subseteq E \setminus (\hat{E} \setminus N) = E \cap N \subseteq N \setminus (\hat{E} \setminus E).$$

Since N is a measurable cover of $\hat{E} \setminus E$, $\mu(G) = 0$. Hence F is a measurable kernel of E. #

2.34 Theorem If $E \in \mathcal{H}(\mathcal{S})$ and F is a measurable kernel of E, then $\mu(F) = \mu_*(E)$; if both F_1 and F_2 are measurable kernels of E, then $\mu(F_1 \Delta F_2) = 0$.

Proof[2]Since $F \subseteq E$, it is clear that $\mu(F) \leq \mu_*(E)$. If $\mu(F) < \mu_*(E)$, by the definition of $\mu_*(E)$, there exists a set F_o in $\mathcal{G}$ such that $F_o \subseteq E$ and $\mu(F_o) > \mu(F)$. Since

$$F_o = (F_o \cap F) \cup (F_o \setminus F),$$

then

$$\begin{aligned}\mu(F_o) &= \mu(F_o \cap F) + \mu(F_o \setminus F)\\ &\leq \mu(F) + \mu(F_o \setminus F).\end{aligned}$$

Since μ is finite, $\mu(F_o) - \mu(F) \leq \mu(F_o \setminus F)$. We have

$$F_o \setminus F \subseteq E \setminus F \text{ and } \mu(F_o \setminus F) \geq \mu(F_o) - \mu(F) > 0,$$

which contradicts the fact that F is a measurable kernel of E. Hence $\mu(F) \geq \mu_*(E)$, and thus $\mu(F) = \mu_*(E)$.

Since the relation $F_1 \subseteq F_1 \cup F_2 \subseteq E$ implies that $(F_1 \cup F_2) \setminus F_1 \subseteq E \setminus F_1$, it follows from the fact that F_1 is a measurable kernel of E that

$$\mu((F_1 \cup F_2) \setminus F_1) = 0.$$

But $(F_1 \cup F_2) \setminus F_1 = F_2 \setminus F_1$, so $\mu(F_2 \setminus F_1) = 0$. Similarly we have

$$\mu(F_1 \setminus F_2) = 0.$$

Therefore

$$\mu(F_1 \Delta F_2) = 0. \ \#$$

2.35 Theorem If $(E_n)_{n \in \mathbb{N}}$ is a disjoint sequence of sets in $\mathcal{H}(\mathcal{G})$, then

$$\mu_*\left(\bigcup_{n=1}^{\infty} E_n\right) \geq \sum_{n=1}^{\infty} \mu_*(E_n).$$

Proof[2]For each $n \in \mathbb{N}$, let F_n be a measurable kernel of E_n. Then $\bigcup_{n=1}^{\infty} F_n \subseteq \bigcup_{n=1}^{\infty} E_n$ and $\bigcup_{n=1}^{\infty} F_n$ is a disjoint union. Since μ is a positive measure and by Theorem 2.34, it follows that

$$\sum_{n=1}^{\infty}\mu_*(E_n) = \sum_{n=1}^{\infty}\mu(F_n) = \mu(\bigcup_{n=1}^{\infty}F_n) \leq \mu_*(\bigcup_{n=1}^{\infty}E_n). \ \#$$

2.36 Theorem If $A \in \mathcal{H}(\mathcal{G})$ and if $(E_n)_{n\in\mathbb{N}}$ is a disjoint sequence of sets in $\bar{\mathcal{G}}$ with $\bigcup_{n=1}^{\infty}E_n = E$, then

$$\mu_*(A\cap E) = \sum_{n=1}^{\infty}\mu_*(A\cap E_n).$$

Proof[2] If F is a measurable kernel of $A\cap E$, then $\mu_*(A\cap E) = \mu(F)$. Since $F \subsetneq A\cap E = \bigcup_{n=1}^{\infty}(A\cap E_n)$, $F = \bigcup_{n=1}^{\infty}(F\cap A\cap E_n) \subseteq \bigcup_{n=1}^{\infty}(F\cap E_n)$, hence $\mu(F) = \bar{\mu}(F) \leq$ $\bar{\mu}(\bigcup_{n=1}^{\infty}(F\cap E_n)) = \sum_{n=1}^{\infty}\bar{\mu}(F\cap E_n)$ (since $(F\cap E_n)_{n\in\mathbb{N}}$ disjoint) $\leq \sum_{n=1}^{\infty}\mu_*(A\cap E_n)$ (by Theorem 2.31 and $A\cap E_n \supseteq F\cap E_n$).

Thus $\mu_*(A\cap E) \leq \sum_{n=1}^{\infty}\mu_*(A\cap E_n)$. By Theorem 2.35, the proof of the theorem is complete. #

2.37 Theorem If $E \in \bar{\mathcal{G}}$, then

$$\mu^*(E) = \mu_*(E) = \bar{\mu}(E),$$

and, conversely, if $E \in \mathcal{H}(\mathcal{G})$ and

$$\mu^*(E) = \mu_*(E) < \infty,$$

then $E \in \bar{\mathcal{G}}$.

Proof[2] If $E \in \bar{\mathcal{G}}$, then both Theorem 2.21 and Theorem 2.31 we have $\mu^*(E) = \bar{\mu}(E) = \mu_*(E)$. To prove the converse, let A be a measurable kernel of E, so $A \in \mathcal{G}$ and $\mu(A) = \mu_*(E) < \infty$. Since $\mu^*(E) < \infty$, by Theorem 2.23, then there exists a set B in $\mathcal{G}$ such that $\mu^*(E) = \mu(B)$ and $E \subsetneq B$. Thus $A \subsetneq E \subsetneq B$, we have

$$\mu(B \sim A) = \mu(B) - \mu(A) = \mu^*(E) - \mu_*(E) = 0.$$

Since $E = A \cup (E \setminus A)$ and $E \setminus A \subseteq B \setminus A$ with $\mu(B \setminus A) = 0$, by the proof of Theorem 2.27, we have $E \in \overline{\mathcal{S}}$. #

2.38 Theorem If E and F are disjoint sets in $\mathcal{H}(\mathcal{S})$, then

$$\mu_*(E \cup F) \leq \mu_*(E) + \mu^*(F) \leq \mu^*(E \cup F).$$

Proof[2] By Theorem 2.23, there exists a set A in $\mathcal{S}$ such that $\mu^*(F) = \mu(A)$ and A is a measurable cover of F. By Theorem 2.33, let B be a measurable kernel of $E \cup F$, so $B \in \mathcal{S}$ and $\mu(B) = \mu_*(E \cup F)$. Since $F \subseteq A$ and $B \subseteq E \cup F$, $B \setminus A \subseteq E$, it follows that

$$\begin{aligned}\mu_*(E \cup F) &= \mu(B) \leq \mu(B \setminus A) + \mu(A) \quad (\text{since } B \subseteq (B \setminus A) \cup A)\\ &\leq \mu_*(E) + \mu^*(F).\end{aligned}$$

Dually, let A be a measurable kernel of E, so $A \in \mathcal{S}$ and $\mu_*(A) = \mu(E)$; let B be a measurable cover of $E \cup F$, $B \in \mathcal{S}$ and $\mu^*(E \cup F) = \mu(B)$. Since $A \subseteq E$ and $E \cup F \subseteq B$, $F \subseteq B \setminus A$, it follows that

$$\begin{aligned}\mu^*(E \cup F) &= \mu(B) = \mu(A) + \mu(B \setminus A) \quad (\text{since } A \subseteq B)\\ &\geq \mu_*(E) + \mu^*(F). \ \#\end{aligned}$$

2.39 Theorem If $E \in \overline{\mathcal{S}}$, then, for every subset A of X,

$$\mu_*(A \cap E) + \mu^*(A^c \cap E) = \bar{\mu}(E).$$

Proof[2] Applying Theorem 2.38 to $A \cap E$ and $A^c \cap E$, we obtain

$$\mu_*(E) \leq \mu_*(A \cap E) + \mu^*(A^c \cap E) \leq \mu^*(E).$$

Since $E \in \overline{\mathcal{S}}$, we have, by Theorem 2.37, $\mu_*(E) = \mu^*(E) = \bar{\mu}(E)$. #

2.40 <u>Definition</u> Let $\mathfrak{M}$ be a σ-algebra of subsets of X and $E \in \mathfrak{M}$, $(E_n)_{n\in\mathbb{N}}$ in $\mathfrak{M}$ is called a <u>partition</u> of E if $E_i \cap E_j = \emptyset$ whenever $i \neq j$, and if $\bigcup_{n=1}^{\infty} E_n = E$.

2.41 <u>Definition</u> A <u>quaternion</u> <u>measure</u> on a σ-algebra $\mathfrak{M}$ is a quaternion function on $\mathfrak{M}$ such that

$$\mu(E) = \sum_{n=1}^{\infty} \mu(E_n) \qquad (E \in \mathfrak{M})$$

for every partition $(E_n)_{n\in\mathbb{N}}$ of E.

2.42 <u>Definition</u> We define a set function $|\mu|$ on a σ-algebra $\mathfrak{M}$ by

$$|\mu|(E) = \sup\left\{\sum_{n=1}^{\infty} |\mu(E_n)|\right\} \qquad (E \in \mathfrak{M}),$$

the supremum being taken over all partitions $(E_n)_{n\in\mathbb{N}}$ of E. The set function $|\mu|$ is called the <u>total</u> <u>variation</u> <u>of</u> <u>a</u> <u>quaternion</u> <u>measure</u> μ or the <u>total</u> <u>variation</u> <u>quaternion</u> <u>measure</u>. Note that $|\mu|(E) \geq |\mu(E)|$ for all $E \in \mathfrak{M}$.

If μ is a positive measure on a σ-algebra $\mathfrak{M}$, then $\mu = |\mu|$.

If μ is a quaternion measure on a σ-algebra $\mathfrak{M}$ and λ is a positive measure on $\mathfrak{M}$ such that $\lambda(E) > |\mu(E)|$ for all $E \in \mathfrak{M}$, then $\lambda(E) \geq |\mu|(E)$ for all $E \in \mathfrak{M}$. To prove this, let $(E_n)_{n\in\mathbb{N}}$ be a partition of E. Then $\lambda(E) = \sum_{n=1}^{\infty} \lambda(E_n) \geq \sum_{n=1}^{\infty} |\mu(E_n)|$. Since $(E_n)_{n\in\mathbb{N}}$ is an arbitrary partition of E, then

$$\lambda(E) \geq \sup\left\{\sum_{n=1}^{\infty} |\mu(E_n)| \,/\, (E_n)_{n\in\mathbb{N}} \text{ is a partition of } E\right\}$$
$$= |\mu|(E).$$

2.43 Theorem Let μ be a quaternion measure on a σ-algebra $\mathfrak{M}$. Then

(a) $\mu(\emptyset) = 0$.

(b) If $A_1, \ldots, A_n$ are pairwise disjoint members of $\mathfrak{M}$, then $\mu(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} \mu(A_i)$.

(c) For $A, B \in \mathfrak{M}$, $A \subseteq B \rightarrow \mu(B \setminus A) = \mu(B) - \mu(A)$.

(d) If $A_1, A_2, \ldots \in \mathfrak{M}$ such that $A_1 \subseteq A_2 \subseteq \ldots$, then $\mu(A_n) \rightarrow \mu(\bigcup_{i=1}^{\infty} A_i)$ as $n \rightarrow \infty$.

(e) If $A_1, A_2, \ldots \in \mathfrak{M}$ such that $A_1 \supseteq A_2 \supseteq \ldots$, then $\mu(A_n) \rightarrow \mu(\bigcap_{i=1}^{\infty} A_i)$ as $n \rightarrow \infty$.

Proof of (a) Let $A \in \mathfrak{M}$ and take $A_1 = A$, $A_2 = A_3 = A_4 = \ldots = \emptyset$. Then $\bigcup_{i=1}^{\infty} A_i = A$, so $\mu(A) = \mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i) = \mu(A) + \mu(\emptyset) + \mu(\emptyset) + \ldots$. Hence $\mu(\emptyset) = 0$.

Proof of (b) Take $A_{n+1} = A_{n+2} = \ldots = \emptyset$. Then $\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{\infty} A_i$, so $\mu(\bigcup_{i=1}^{n} A_i) = \mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i) = \sum_{i=1}^{n} \mu(A_i)$, since $\mu(A_i) = 0$ for all $i > n$.

Proof of (c) Since $B = (B \setminus A) \cup A$ and by (b), $\mu(B) = \mu(B \setminus A) + \mu(A)$. Since $|\mu(A)| < \infty$, $\mu(B \setminus A) = \mu(B) - \mu(A)$.

Proof of (d) and (e) It is similar to the proof of Theorem 2.9 and Theorem 2.10 respectively. #

2.44 Theorem The total variation $|\mu|$ of a quaternion measure μ on a σ-algebra $\mathfrak{M}$ is a positive measure on $\mathfrak{M}$.

Proof Let $E \in \mathfrak{M}$ and $(E_i)_{i \in \mathbb{N}}$ be an arbitrary partition of E. We must show that $\sum_{i=1}^{\infty} |\mu|(E_i) = |\mu|(E)$. For each $i \in \mathbb{N}$, let $t_i \in \mathbb{R}$ be such that $t_i < |\mu|(E_i)$ and $\sum_{i=1}^{\infty} t_i$ converges. Then for each i there is a partition $(A_{ij})_{j \in \mathbb{N}}$ of E_i such that

$$t_i < \sum_{j=1}^{\infty} |\mu(A_{ij})| .$$

Since $(A_{ij})_{i,j \in \mathbb{N}}$ is a partition of E, it follows that

$$\sum_{i=1}^{\infty} t_i \leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |\mu(A_{ij})| \leq |\mu|(E).$$

Hence we see that

$$\forall t_i < |\mu|(E_i), \quad \sum_{i=1}^{\infty} t_i \leq |\mu|(E) \quad \text{so}$$

(1) $$\sup\left\{\sum_{i=1}^{\infty} t_i / \ t_i < |\mu|(E_i)\right\} \leq |\mu|(E).$$

Claim that $\sum_{i=1}^{\infty} |\mu|(E_i) = \sup\left\{\sum_{i=1}^{\infty} t_i / \ t_i < |\mu|(E_i)\right\}$. Since $\forall t_i < |\mu|(E_i)$ $\sum_{i=1}^{\infty} t_i < \sum_{i=1}^{\infty} |\mu|(E_i)$, $\sup\left\{\sum_{i=1}^{\infty} t_i / \ t_i < |\mu|(E_i)\right\} \leq \sum_{i=1}^{\infty} |\mu|(E_i)$. Let $\varepsilon > 0$ be given. Since $\sup\{t_i / \ t_i < |\mu|(E_i)\} = |\mu|(E_i)$, for each i there is $t_i^* < |\mu|(E_i)$ such that $|\mu|(E_i) - \frac{\varepsilon}{2^i} < t_i^*$. Hence $\sum_{i=1}^{\infty} |\mu|(E_i) - \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i} < \sum_{i=1}^{\infty} t_i^*$. Hence $\sum_{i=1}^{\infty} |\mu|(E_i) - \varepsilon < \sum_{i=1}^{\infty} t_i \leq \sup\left\{\sum_{i=1}^{\infty} t_i / \ t_i < |\mu|(E_i)\right\}$. Since $\varepsilon > 0$ is arbitrary, $\sum_{i=1}^{\infty} |\mu|(E_i) \leq \sup\left\{\sum_{i=1}^{\infty} t_i / \ t_i < |\mu|(E_i)\right\}$. So we have the claim. From (1), we have

(2) $$\sum_{i=1}^{\infty} |\mu|(E_i) \leq |\mu|(E).$$

Let $(A_j)_{j \in \mathbb{N}}$ be an arbitrary partition of E. Then for any fixed j, $(A_j \cap E_i)_{i \in \mathbb{N}}$ is a partition of A_j and for any fixed i, $(A_j \cap E_i)_{j \in \mathbb{N}}$ is a partition of E_i. Hence

$$\begin{aligned}
\sum_{j=1}^{\infty} |\mu(A_j)| &= \sum_{j=1}^{\infty} \left|\sum_{i=1}^{\infty} \mu(A_j \cap E_i)\right| \\
&\leq \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} |\mu(A_j \cap E_i)| \\
&= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |\mu(A_j \cap E_i)| \quad \text{(by Theorem 1.33)} \\
&\leq \sum_{i=1}^{\infty} |\mu|(E_i).
\end{aligned}$$

This holds for any partition $(A_j)_{j\in N}$ of E, we then have

(3) $$\mu(E) \leq \sum_{i=1}^{\infty} |\mu|(E_i).$$

From (2) and (3),

$$|\mu|(E) = \sum_{i=1}^{\infty} |\mu|(E_i).$$

Since $\mu(\emptyset) = 0$, $|\mu|(\emptyset) = 0$. #

2.45 <u>Theorem</u> If μ is a quaternion measure on a σ-algebra $\mathcal{M}$ in a set X, then

$$|\mu|(X) < \infty.$$

<u>Proof</u> <u>Step I</u> Suppose there is $E \in \mathcal{M}$ such that $|\mu|(E) = \infty$. Claim that there exist $A, B \in \mathcal{M}$ such that $E = A \cup B$, $A \cap B = \emptyset$, $|\mu(A)| > 1$ and $|\mu|(B) = \infty$. By the definition of $|\mu|$, for all $t < \infty$ there is a partition $(E_i)_{i\in N}$ of E such that

$$\sum_{i=1}^{\infty} |\mu(E_i)| > t.$$

Let $t = 4(2^5)(1+|\mu(E)|)$. Subclaim that there exists a finite partition $(E_j)_{j=1,2,\ldots,n}$ such that $\sum_{j=1}^{n} |\mu(E_j)| > 4(2^5)(1+|\mu(E)|)$. To prove this, there exist a partition $(F_j)_{j\in N}$ of E such that $\sum_{j=1}^{\infty} |\mu(F_j)| > t+1$. If $(F_j)_{j\in N}$ is a finite partition, we are done. Suppose that $(F_j)_{j\in N}$ is an infinite partition of E. Then there exists $n \in N$ such that $\sum_{j=n}^{\infty} |\mu(E_j)| < 1$, since $\sum_{j=1}^{\infty} |\mu(F_j)|$ converges. Then $\sum_{j=1}^{n-1} |\mu(F_j)| > t$. Let $E_1 = F_1, \ldots, E_{n-1} = F_{n-1}, E_n = \bigcup_{j=n}^{\infty} F_j$. Then $(E_j)_{j=1,\ldots,n}$ is a partition of E and $\sum_{j=1}^{\infty} |\mu(F_j)| = \sum_{j=1}^{n-1} |\mu(F_j)| + |\mu(E_n)| > t$. So we have subclaim. By Theorem 1.3 there exists an $S \subseteq \{1,2,\ldots,n\}$ such that

$$\left|\sum_{j\in S}\mu(E_j)\right| \geq \frac{1}{4(2^5)}\sum_{j=1}^{n}|\mu(E_j)| > 1+|\mu(E)|.$$

Let $A = \bigcup_{j\in S} E_j$. Then $A \subseteq E$ and $|\mu(A)| = \left|\sum_{j\in S}\mu(E_j)\right| > 1+|\mu(E)|$.
≥ 1. Let $B = E \setminus A$. Then $B \in \mathfrak{M}$, $A\cap B = \emptyset$ and $|\mu(B)| =$
$|\mu(E)-\mu(A)| \geq |\mu(A)| - |\mu(E)| > 1+|\mu(E)| - |\mu(E)| = 1$.
Since $|\mu|(E) = |\mu|(A)+|\mu|(B)$, we have

$$|\mu|(A) = \infty \text{ or } |\mu|(B) = \infty,$$

so we have the claim.

<u>Step II</u> Assume $|\mu|(X) = \infty$. Let $B_o = X$. By Step I, there is $A_1, B_1 \in \mathfrak{M}$ such that $A_1 \cap B_1 = \emptyset$, $B_o = A_1 \cup B_1$, $|\mu(A_1)| > 1$ and $|\mu|(B_1) = \infty$. Suppose $n \geq 0$ and B_n is chosen so that $|\mu|(B_n) = \infty$. By Step I there is $A_{n+1}, B_{n+1} \in \mathfrak{M}$ such that $B_n = A_{n+1}\cup B_{n+1}$, $A_{n+1}\cap B_{n+1} = \emptyset$, $|\mu(A_{n+1})| > 1$ and $|\mu|(B_{n+1}) = \infty$
By induction, we obtain disjoint sets $A_1, A_2, \ldots$ with $|\mu(A_n)| > 1$ for all $n \in \mathbb{N}$. Let $C = \bigcup_{n=1}^{\infty} A_n$. Then $C \in \mathfrak{M}$ and $\mu(C) = \sum_{n=1}^{\infty}\mu(A_n)$. But this series cannot converge, since $\mu(A_n)$ does not tend to 0 as $n \to \infty$. This contradiction shows that the theorem must hold. #

2.46 <u>Definition</u> Let μ and λ be quaternion measures on a σ-algebra $\mathfrak{M}$. Define $\mu+\lambda$ and $\alpha\mu$ $(\mu\alpha)$ by

$$(\mu+\lambda)(E) = \mu(E)+\lambda(E)$$

$$(\alpha\mu)(E) = \alpha\mu(E); \quad [(\mu\alpha)(E) = (\mu(E))\alpha]$$

for all $E \in \mathfrak{M}$, $\alpha \in \mathbb{H}$. Then $\mu+\lambda$ and $\alpha\mu$ $(\mu\alpha)$ are quaternion measures on $\mathfrak{M}$. Thus the collection of all quaternion measures on $\mathfrak{M}$ forms a quaternion left(right) vector space.

Let us now specialize and consider a real measure on a σ-algebra $\mathfrak{M}$. Define $|\mu|$ as before, and define

$$\mu^+ = \frac{1}{2}(|\mu| + \mu), \ \mu^- = \frac{1}{2}(|\mu| - \mu).$$

Then both μ^+ and μ^- are positive measures on $\mathfrak{M}$ and they are bounded by Theorem 2.45. Also

$$\mu = \mu^+ - \mu^-, \quad |\mu| = \mu^+ + \mu^-.$$

The measures μ^+ and μ^- are called the positive and negative variations of μ, respectively.

In this chapter, from now, an arbitrary measure means a positive or a quaternion measure.

2.47 Definition Let μ be a positive measure on a σ-algebra $\mathfrak{M}$, and λ be an arbitrary measure on $\mathfrak{M}$. Then λ is said to be absolutely continuous with respect to μ, and write

$$\lambda \ll \mu$$

if for all $E \in \mathfrak{M}$, $\mu(E) = 0 \rightarrow \lambda(E) = 0$.

2.48 Definition Let λ be an arbitrary measure on a σ-algebra $\mathfrak{M}$. If there is $A \in \mathfrak{M}$ such that $\lambda(E) = \lambda(A \cap E)$ for all $E \in \mathfrak{M}$, we say that λ is concentrated on A.

2.49 Proposition λ is concentrated at A iff $\forall E \in \mathfrak{M}\ (E \cap A = \emptyset \rightarrow \lambda(E) = 0)$.

Proof ($\Longrightarrow$) Let $E \in \mathfrak{M}$ and $E \cap A = \emptyset$. Then $\lambda(E) = \lambda(E \cap A) = \lambda(\emptyset) = 0$.

($\Longleftarrow$) Let $E \in \mathfrak{M}$. Then $E = (E \cap A) \cup (E \setminus A)$ which is a disjoint union, so $\lambda(E) = \lambda(E \cap A) + \lambda(E \setminus A)$. Since $(E \setminus A) \cap A = \emptyset$, $\lambda(E \setminus A) = 0$. Hence $\lambda(E) = \lambda(E \cap A)$. Thus λ is concentrated at A. #

2.50 <u>Definition</u> Let λ_1 and λ_2 be arbitrary measures on a σ-algebra $\mathfrak{M}$, and suppose that there exists $A, B \in \mathfrak{M}$ such that $A \cap B = \emptyset$, λ_1 is concentrated on A and λ_2 is concentrated on B. Then we say that λ_1 and λ_2 are <u>mutually singular</u>, and write

$$\lambda_1 \perp \lambda_2 .$$

2.51 <u>Theorem</u> Suppose μ, λ, λ_1 and λ_2 are arbitrary measures on a σ-algebra $\mathfrak{M}$, and μ is positive.

(a) If λ is cocentrated on A, then so is $|\lambda|$.

(b) If $\lambda_1 \perp \lambda_2$, then $|\lambda_1| \perp |\lambda_2|$.

(c) If $\lambda_1 \perp \mu$ and $\lambda_2 \perp \mu$, then $\lambda_1 + \lambda_2 \perp \mu$.

(d) If $\lambda_1 \ll \mu$ and $\lambda_2 \ll \mu$, then $\lambda_1 + \lambda_2 \ll \mu$.

(e) If $\lambda \ll \mu$, then $|\lambda| \ll \mu$.

(f) If $\lambda_1 \ll \mu$ and $\lambda_2 \perp \mu$, then $\lambda_1 \perp \lambda_2$.

(g) If $\lambda \ll \mu$ and $\lambda \perp \mu$, then $\lambda = 0$.

<u>Proof of (a)</u> Let $E \in \mathfrak{M}$ such that $E \cap A = \emptyset$. Assume $(E_j)_{j \in \mathbb{N}}$ is an arbitrary partition of E. Then $E_j \cap A = \emptyset$ for all j, so $\lambda(E_j) = 0$ for all j, hence $\sum_{j=1}^{\infty} |\lambda(E_j)| = 0$. Thus $|\lambda|(E) = 0$. Hence $|\lambda|$ is concentrated at A.

<u>Proof of (b)</u> Obvious.

<u>Proof of (c)</u> By assumption, there exist $A_1, B_1, A_2, B_2 \in \mathfrak{M}$ such that $A_1 \cap B_1 = \emptyset$, $A_2 \cap B_2 = \emptyset$, λ_1 is concentrated on A_1, μ is concentrated on B_1, λ_2 is concentrated on A_2 and μ is concentrated on B_2. Then $\lambda_1 + \lambda_2$ is concentrated on $A_1 \cup A_2$. Claim that μ is concentrated on $B_1 \cap B_2$. To prove this, let $E \in \mathfrak{M}$ be such that $E \cap (B_1 \cap B_2) = \emptyset$. Since

$(E\cap B_1)\cap B_2 = \emptyset$, $\mu(E\cap B_1) = 0$. But μ is concentrated on B_1, then $\mu(E) = \mu(E\cap B_1) = 0$. So we have the claim. Since $(A_1\cup A_2)\cap(B_1\cap B_2) = \emptyset$, we have that $\lambda_1 + \lambda_2 \perp \mu$.

<u>Proof of (d)</u> Let $E\in \mathfrak{M}$ be such that $\mu(E) = 0$. Since $\lambda_1 \ll \mu$ and $\lambda_2 \ll \mu$, $\lambda_1(E) = 0 = \lambda_2(E)$. Hence $(\lambda_1 + \lambda_2)(E) = \lambda_1(E) + \lambda_2(E) = 0$. Then $\lambda_1 + \lambda_2 \ll \mu$.

<u>Proof of (e)</u> Let $E\in \mathfrak{M}$ be such that $\mu(E) = 0$. Since $\lambda \ll \mu$, $\lambda(E) = 0$. Assume $(E_j)_{j\in \mathbb{N}}$ is an arbitrary partition of E. Then $\lambda(E_j) = 0$ for all $j\in \mathbb{N}$, hence $\sum_{j=1}^{\infty} |\lambda(E_j)| = 0$. Hence $|\lambda|(E) = 0$. Hence $|\lambda| \ll \mu$.

<u>Proof of (f)</u> Since $\lambda_2 \perp \mu$, there exist $A, B\in \mathfrak{M}$ such that $A\cap B = \emptyset$, λ_2 and μ is conecntrated on A and B, respectively. Let $E\in \mathfrak{M}$ be such that $E\cap B = \emptyset$. Then $\mu(E) = 0$. Since $\lambda_1 \ll \mu$, $\lambda_1(E) = 0$. Hence λ_1 is concentrated on B. Hence $\lambda_1 \perp \lambda_2$.

<u>Proof of (g)</u> By (f), $\lambda \perp \lambda$. Then there exist $A, B \in \mathfrak{M}$ such that $A\cap B = \emptyset$, λ is cocentrated on A and λ is concentrated on B. Hence $\lambda(A) = 0 = \lambda(B)$. Let $E\in \mathfrak{M}$. Since λ is concentrated on A, $\lambda(E) = \lambda(E\cap A)$. Since $\lambda(A) = 0$, $\lambda(E\cap A) = 0$. Hence $\lambda(E) = 0$. #

2.52 <u>Theorem</u> Suppose μ is a positive measure on a σ-algebra $\mathfrak{M}$ in a set X and λ is a quaternion measure on $\mathfrak{M}$. Then the following are equivalent:

(a) $\lambda \ll \mu$.

(b) For all $\varepsilon > 0$ there is a $\delta > 0$ such that for all

$E \in \mathcal{M}$ $\mu(E) < \delta$ implies that $|\lambda(E)| < \varepsilon$.

Proof Assume (b) holds. Let $E \in \mathcal{M}$ be such that $\mu(E) = 0$. Then $\mu(E) < \delta$ for all $\delta > 0$. Hence $|\lambda(E)| < \varepsilon$ for all $\varepsilon > 0$, so $|\lambda(E)| = 0$, hence $\lambda(E) = 0$. This shows that (b) implies (a).

Suppose (b) is false. Then there exists an $\varepsilon > 0$ such that for each $n \in \mathbb{N}$ there exists $E_n \in \mathcal{M}$ such that $\mu(E_n) < \frac{1}{2^n}$ but $|\lambda(E_n)| \geq \varepsilon$. Put

$$A_n = \bigcup_{i=n}^{\infty} E_i, \quad A = \bigcap_{n=1}^{\infty} A_n$$

Then $A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots$ and $\mu(A_n) \leq \sum_{i=n}^{\infty} \mu(E_i) < \sum_{i=n}^{\infty} \frac{1}{2^i} = \frac{1}{2^{n-1}}$,

so $\mu(A_1) < \frac{1}{2^0} = 1 < \infty$. Thus $0 = \lim_{n \to \infty} \mu(A_n) = \mu(\bigcap_{n=1}^{\infty} A_n) = \mu(A)$. Since $|\lambda|(A_1) < \infty$, $\lim_{n \to \infty} |\lambda|(A_n) = |\lambda|(A)$. Since $|\lambda(E_n)| \geq \varepsilon$, $|\lambda|(E_n) \geq \varepsilon$ for all $n \in \mathbb{N}$, hence $|\lambda|(A_n) \geq |\lambda|(E_n) \geq \varepsilon$ for all $n \in \mathbb{N}$. Then $\lim_{n \to \infty} |\lambda|(A_n) \geq \varepsilon$, hence $|\lambda|(A) \geq \varepsilon$. Hence we do not have $|\lambda| \ll \mu$. By Theorem 2.51(e), we do not have $\lambda \ll \mu$. This proves that (a) implies (b). #

2.53 Theorem Let $\mathcal{M}$ be a σ-algebra of X. Then μ is a quaternion measure on $\mathcal{M}$ if and only if $\mu = \mu_1 + i\mu_2 + j\mu_3 + k\mu_4$ for some real measures μ_l on $\mathcal{M}$ for all $l \leq 4$.

Proof Assume μ is a quaternion measure on $\mathcal{M}$. Then there exist real functions μ_l on $\mathcal{M}$ for all $l \leq 4$ such that $\mu = \mu_1 + i\mu_2 + j\mu_3 + k\mu_4$. Claim that μ_1, μ_2, μ_3 and μ_4 are real measures on $\mathcal{M}$. Let $E \in \mathcal{M}$ and $(E_n)_{n \in \mathbb{N}}$ be an arbitrary partition of E. Then

$$\mu(E) = \sum_{n=1}^{\infty}\mu(E_n) = \sum_{n=1}^{\infty}(\mu_1(E_n)+i\mu_2(E_n)+j\mu_3(E_n)+k\mu_4(E_n)).$$

Assume $\mu(E) = a_1+ia_2+ja_3+ka_4$ for some $a_l \in \mathbb{R}$, $l \leq 4$. Hence

$$a_1+ia_2+ja_3+ka_4 = \lim_{m\to\infty}\sum_{n=1}^{m}(\mu_1(E_n)+i\mu_2(E_n)+j\mu_3(E_n)+k\mu_4(E_n))$$

$$= \lim_{m\to\infty}(\sum_{n=1}^{m}\mu_1(E_n)+i\sum_{n=1}^{m}\mu_2(E_n)+j\sum_{n=1}^{m}\mu_3(E_n)+ k\sum_{n=1}^{m}\mu_4(E_n))$$

By Theorem 1.31, we have

$$\lim_{m\to\infty}\sum_{n=1}^{m}\mu_1(E_n) = a_1, \quad \lim_{m\to\infty}\sum_{n=1}^{m}\mu_2(E_n) = a_2,$$

$$\lim_{m\to\infty}\sum_{n=1}^{m}\mu_3(E_n) = a_3 \text{ and } \lim_{m\to\infty}\sum_{n=1}^{m}\mu_4(E_n) = a_4 .$$

Hence

$$\mu(E) = \sum_{n=1}^{\infty}(\mu_1(E_n)+i\mu_2(E_n)+j\mu_3(E_n)+k\mu_4(E_n))$$

$$= \sum_{n=1}^{\infty}\mu_1(E_n)+i\sum_{n=1}^{\infty}\mu_2(E_n)+j\sum_{n=1}^{\infty}\mu_3(E_n)+k\sum_{n=1}^{\infty}\mu_4(E_n).$$

But

$$\mu(E) = \mu_1(E)+i\mu_2(E)+j\mu_3(E)+k\mu_4(E).$$

Hence

$$\mu_l(E) = \sum_{n=1}^{\infty}\mu_l(E_n)$$

for all $l \leq 4$. Since $\mu(\emptyset) = 0$, $\mu_l(\emptyset) = 0$ for all $l \leq 4$.

So we have the claim.

Conversely, it is clear that if $\mu = \mu_1+i\mu_2+j\mu_3+k\mu_4$ for some real measures μ_l on $\mathcal{M}$ for all $l \leq 4$, then μ is a quaternion measure on $\mathcal{M}$. #