CHAPTER IV

THE CLASSICAL SEQUENCE SPACES

The classical sequence spaces are listed in Chapter I.In this chapter,convergence preserving matrix transformations,limit preserving matrix transformations, summability preserving matrix transformations and sum preserving matrix transformations from each of the classical sequence spaces into ℓ_∞ are characterized.

A necessary condition for an infinite matrice A to map a sequence space X into any sequence space is that each row of A is in the β-dual of X (see Chapter I, page 13), so we have seen that the β-duals of sequence spaces play an important role in studying matrix transformations. The first section of this chapter gives some properties of the β-duals of the classical sequence spaces which will be used for the remaining sections.

4.1 SOME PROPERTIES OF β-DUALS

All of the classical sequence spaces are BK-spaces containing all finite sequences.The norm of the β-dual of a BK-space which contains all finite sequences is given in Chapter I. Theorem 1.7 of Chapter I shows that the β-dual of each of the classical sequence space is one of the classical sequence spaces. We show in this section that if X and Y are classical sequence spaces such that $X^\beta = Y$, then $\|.\|_{X^\beta}$ is equivalent to $\|.\|_Y$. In fact, there are some sequence spaces X with $\|.\|_{X^\beta} = \|.\|_Y$. As a corollary, we give equivalent statements for bounded sets in the β-dual of each of the classical sequence spaces.

Proposition 4.1.1. (i) $c_o^\beta = \ell$ and $\|.\|_{c_o^\beta} \sim \|.\|_\ell$.

(ii) $\ell_p^\beta = \ell_q$ and $\|.\|_{\ell_p^\beta} \sim \|.\|_{\ell_q}$ where $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.

(iii) $\ell^\beta = \ell_\infty$ and $\|.\|_{\ell^\beta} \sim \|.\|_{\ell_\infty}$.

(iv) $bv_o^\beta = bs$ and $\|.\|_{bv_o^\beta} \sim \|.\|_{bs}$.

(v) $cs^\beta = bv$ and $\|.\|_{cs^\beta} \sim \|.\|_{bv}$.

(vi) $\ell_\infty^\beta = \ell$ and $\|.\|_{\ell_\infty^\beta} = \|.\|_\ell$.

(vii) $c^\beta = \ell$ and $\|.\|_{c^\beta} = \|.\|_\ell$.

(viii) $bv^\beta = cs$ and $\|.\|_{bv^\beta} \sim \|.\|_{cs}$.

(ix) $bs^\beta = bv_o$ and $\|.\|_{bs^\beta} \sim \|.\|_{bv_o}$.

Proof: The first part of each of (i)-(ix) follows from Theorem 1.7. By Theorem 1.1 all of the classical sequence spaces are BK-spaces. By Theorem 1.4, the sequence spaces c_o, $\ell_p (1<p<\infty)$, ℓ, bv_o and cs have the AK property. It follows from Theorem 1.8 that $(c_o^\beta, \|.\|_{c_o^\beta})$, $(\ell_p^\beta, \|.\|_{\ell_p^\beta})$, $(\ell^\beta, \|.\|_{\ell^\beta})$, $(bv_o^\beta, \|.\|_{bv_o^\beta})$ and $(cs^\beta, \|.\|_{cs^\beta})$ are BK-spaces. But $c_o^\beta = \ell$, $\ell_p^\beta = \ell_q$ where $\frac{1}{p} + \frac{1}{q} = 1$, $\ell^\beta = \ell_\infty$, $bv_o^\beta = bs$ and $cs^\beta = bv$ and $(\ell, \|\ \|_\ell)$, $(\ell_q, \|.\|_{\ell_q})$, $(\ell_\infty, \|.\|_{\ell_\infty})$, $(bs, \|.\|_{bs})$ and $(bv, \|.\|_{bv})$ are BK-spaces, so we have by Proposition 2.1 that $\|.\|_{c_o^\beta} \sim \|.\|_\ell$, $\|.\|_{\ell_p^\beta} \sim \|.\|_{\ell_q}$,

$\|\cdot\|_{\ell^\beta} \sim \|\cdot\|_{\ell_\infty}$ $\|\cdot\|_{bv_0^\beta} \sim \|\cdot\|_{bs}$ and $\|\cdot\|_{cs^\beta} \sim \|\cdot\|_{bv}$. Hence (i)-(v) hold.

To prove (vi) and (vii), let $X = \ell_\infty$ or c. Then $X^\beta = \ell$. Note that for $(x_k) \in X$, $|x_n| \le \sup_k |x_k| = \|(x_k)\|_X$ for all $n \in \mathbb{N}$. To show that $\|\cdot\|_{X^\beta} = \|\cdot\|_\ell$, let $(y_k) \in X^\beta = \ell$. For each $n \in \mathbb{N}$, let $(z_k^{(n)})_{k=1}^\infty$ be the sequence defined by

$$z_k^{(n)} = \begin{cases} \dfrac{|y_k|}{y_k} & \text{if } y_k \ne 0 \text{ and } 1 \le k \le n , \\ 0 & \text{otherwise.} \end{cases}$$

Then $(z_k^{(n)})_{k=1} \in X$ and $\|(z_k^{(n)})_{k=1}\|_X \le 1$. But $\|(y_k)\|_{X^\beta} = \sup_{\|x\|\le 1} |\sum_{k=1}^\infty x_k y_k|$, so $\|(y_k)\|_{X^\beta} \ge |\sum_{k=1}^\infty z_k^{(n)} y_k| = \sum_{k=1}^n |y_k|$ for every $n \in \mathbb{N}$. Hence $\sum_{k=1}^\infty |y_k| \le \|(y_k)\|_{X^\beta}$. Since $\|(y_k)\|_\ell = \sum_{k=1}^\infty |y_k|$, we have that $\|(y_k)\|_\ell \le \|(y_k)\|_{X^\beta}$. If $(x_k) \in X$ is such that $\|(x_k)\|_X \le 1$, then $|x_k| \le 1$ for all $k \in \mathbb{N}$,so $\|(y_k)\|_\ell = \sum_{k=1}^\infty |y_k| \ge \sum_{k=1}^\infty |x_k||y_k| \ge |\sum_{k=1}^\infty x_k y_k|$. Hence $\|(y_k)\|_{X^\beta} = \sup_{\|x\|\le 1} |\sum_{k=1}^\infty x_k y_k| \le \|(y_k)\|_\ell$. Therefore $\|(y_k)\|_{X^\beta} = \|(y_k)\|_\ell$.

To prove (viii), we need the following inequality:

(*) $\quad |\sum_{k=1}^\infty y_k| \le \|(y_k)\|_{bv_0^\beta}$ for all $(y_k) \in cs$ ($\subseteq bs = bv_0^\beta$).

To show the inequality (*), for each $n \in \mathbb{N}$, let $(u_k^{(n)})_{k=1}^\infty$ be the sequence defined by

$$u_k^{(n)} = \begin{cases} 1 & \text{if } 1 \le k \le n, \\ 0 & \text{if } k > n. \end{cases}$$

Then for every $n \in \mathbb{N}$, $(u_k^{(n)})_{k=1}^\infty \in bv_0$ and $\|(u_k^{(n)})_{k=1}^\infty\|_{bv_0} = 1$. Therefore for $(y_k) \in cs$, $\|(y_k)\|_{bv_0^\beta} = \sup_{\|x\|\le 1} |\sum_{k=1}^\infty x_k y_k| \ge |\sum_{k=1}^\infty u_k^{(n)} y_k| = |\sum_{k=1}^n y_k|$ for every $n \in \mathbb{N}$ which implies that $\|(y_k)\|_{bv_0^\beta} \ge |\sum_{k=1}^\infty y_k|$.

Hence (*) holds. From (iv), $bv_o^\beta = bs$ and there exist a, b > 0 such that

(**) $\quad a\|(x_k)\|_{bs} \le \|(x_k)\|_{bv_o^\beta} \le b\|(x_k)\|_{bs}$ for all $(x_k) \in bv_o^\beta = bs$.

We claim that $a\|(y_k)\|_{cs} \le \|(y_k)\|_{bv^\beta} \le 2b\|(y_k)\|_{cs}$ for all $(y_k) \in bv^\beta = cs$. Let $(y_k) \in bv^\beta = cs$. Since cs is a normed linear subspace of bs, $\|(y_k)\|_{bs} = \|(y_k)\|_{cs}$. Since bv_o is a normed linear subspace of bv, we have that $\|(y_k)\|_{bv^\beta} \ge \|(y_k)\|_{bv_o^\beta}$ (see page 12, Chapter I). It then follows from (**) that $\|(y_k)\|_{bv^\beta} \ge \|(y_k)\|_{bv_o^\beta} \ge a\|(y_k)\|_{bs} = a\|(y_k)\|_{cs}$. Next, let $(x_k) \in bv$ be such that $\|(x_k)\|_{bv} \le 1$. Since $bv = bv_o + \langle e\rangle$, $(x_k) = (z_k) + \alpha e$ for some $(z_k) \in bv_o$ and some scalar α. Then $\|(x_k)\|_{bv} = \|(z_k)\|_{bv_o} + |\alpha|$ (see page 8, Chapter I). It follows from $\|(x_k)\|_{bv} \le 1$ that $\|(z_k)\|_{bv_o} \le 1$ and $|\alpha| \le 1$. Hence

$$
\begin{aligned}
\left|\sum_{k=1}^{\infty} x_k y_k\right| &= \left|\sum_{k=1}^{\infty} z_k y_k + \alpha \sum_{k=1}^{\infty} y_k\right| \\
&\le \left|\sum_{k=1}^{\infty} z_k y_k\right| + |\alpha|\left|\sum_{k=1}^{\infty} y_k\right| \\
&\le \|(y_k)\|_{bv_o^\beta} + \left|\sum_{k=1}^{\infty} y_k\right| \\
&\le \|(y_k)\|_{bv_o^\beta} + \|(y_k)\|_{bv_o^\beta} \quad \text{(from (*))} \\
&= 2\,\|(y_k)\|_{bv_o^\beta} \\
&\le 2b\,\|(y_k)\|_{bs} \qquad \text{(from (**))} \\
&= 2b\,\|(y_k)\|_{cs}
\end{aligned}
$$

which implies that $\|(y_k)\|_{bv^\beta} \le 2b\|(y_k)\|_{cs}$. Therefore we have the claim and hence $\|\cdot\|_{bv^\beta} \sim \|\cdot\|_{cs}$.

Next we shall prove (ix). We have by Theorem 1.7 that $bs^\beta = bv_o$. By (v), $cs^\beta = bv$ and there exists a > 0 such that $a\|x\|_{bv} \le \|x\|_{cs^\beta}$ for all $x \in bv$. We claim that $a\|(y_k)\|_{bv_o} \le \|(y_k)\|_{bs^\beta} \le \|(y_k)\|_{bv_o}$ for all $(y_k) \in bs^\beta$, which implies that $\|\cdot\|_{bs^\beta} \sim \|\cdot\|_{bv_o}$. Since cs is a normed linear subspace of bs, we have that $\|(y_k)\|_{cs^\beta} \le \|(y_k)\|_{bs^\beta}$ for all $(y_k) \in bs^\beta$. Hence we have that $a\|(y_k)\|_{bv_o} = a\|(y_k)\|_{bv} \le \|(y_k)\|_{cs^\beta} \le \|(y_k)\|_{bs^\beta}$ for all $(y_k) \in bs^\beta$. Next, let $(y_k) \in bs^\beta$. Note that for any sequences (u_k), (v_k) we have by induction that

$$\sum_{k=1}^{n} u_k v_k = \sum_{m=1}^{n-1} \Big(\sum_{k=1}^{m} u_k\Big)(v_m - v_{m+1}) + \Big(\sum_{k=1}^{n} u_k\Big)v_n \quad \text{for every } n \in \mathbb{N}.$$

which implies that

$$(*) \quad \Big|\sum_{k=1}^{n} u_k v_k\Big| \le \sum_{m=1}^{n-1} \Big|\sum_{k=1}^{m} u_k\Big||v_m - v_{m+1}| + \Big|\sum_{k=1}^{n} u_k\Big||v_n| \quad \text{for every } n \in \mathbb{N}.$$

Let $(x_k) \in bs$ be such that $\|(x_k)\|_{bs} \le 1$. Then $\left|\sum_{k=1}^{m} x_k\right| \le 1$ for all $m \in \mathbb{N}$. Hence by $(*)$, we have

$$(**) \quad \Big|\sum_{k=1}^{n} x_k y_k\Big| \le \sum_{m=1}^{n-1} |y_m - y_{m+1}| + |y_n| \text{ for every } n \in \mathbb{N}.$$

Since $(y_k) \in bv_o$, $\|(y_k)\|_{bv_o} = \sum_{k=1}^{\infty} |y_k - y_{k+1}|$ and $\lim_{k\to\infty} y_k = 0$. It then follows from $(**)$ that $\left|\sum_{k=1}^{\infty} x_k y_k\right| \le \|(y_k)\|_{bv_o}$. This proves that $\left|\sum_{k=1}^{\infty} x_k y_k\right| \le \|(y_k)\|_{bv_o}$ for every $(x_k) \in bs$ such that $\|(x_k)\|_{bs} \le 1$. Thus $\|(y_k)\|_{bs^\beta} \le \|(y_k)\|_{bv_o}$. Hence we have the claim. #

<u>Corollary 4.1.2</u>. Let S be a nonempty set of sequences. Then:

(i) $S \subseteq c_o^\beta$ and $\sup_{x\in S} \|x\|_{c_o^\beta} < \infty$ if and only if $\sup_{x\in S} \sum_{k=1}^{\infty} |x_k| < \infty$.

(ii) For $1 < p < \infty$, $S \subseteq \ell_p^\beta$ and $\sup_{x\in S} \|x\|_{\ell_p^\beta} < \infty$ if and only if $\sup_{x\in S} \sum_{k=1}^{\infty} |x_k|^q < \infty$ where $\frac{1}{p} + \frac{1}{q} = 1$.

(iii) $S \subseteq \ell^\beta$ and $\sup_{x\in S} \|x\|_{\ell^\beta} < \infty$ if and only if $\sup_{x\in S} (\sup_k |x_k|) < \infty$.

(iv) $S \subseteq bv_o^\beta$ and $\sup_{x\in S} \|x\|_{bv_o^\beta} < \infty$ if and only if $\sup_{x\in S} \Big(\sup_n \Big|\sum_{k=1}^{n} x_k\Big|\Big) < \infty$.

(v) $S \subseteq cs^\beta$ and $\sup_{x\in S} \|x\|_{cs^\beta} < \infty$ if and only if $\sup_{x\in S} \left(\sum_{k=1}^{\infty} |x_k - x_{(k+1)}| + \lim_{n\to\infty} |x_k| \right) < \infty$.

(vi) $S \subseteq \ell_\infty^\beta$ and $\sup_{x\in S} \|x\|_{\ell_\infty^\beta} < \infty$ if and only if $\sup_{x\in S} \sum_{k=1}^\infty |x_k| < \infty$.

(vii) $S \subseteq c^\beta$ and $\sup_{x\in S} \|x\|_{c^\beta} < \infty$ if and only if $\sup_{x\in S} \sum_{k=1}^\infty |x_k| < \infty$.

(viii) If $S \subseteq bv^\beta$, then $\sup_{x\in S} \|x\|_{bv^\beta} < \infty$ if and only if $\sup_{x\in S} (\sup_n |\sum_{k=1}^n x_k|) < \infty$.

(ix) If $S \subseteq bs^\beta$, then $\sup_{x\in S} \|x\|_{bs^\beta} < \infty$ if and only if $\sup_{x\in S} \sum_{k=1}^\infty |x_k - x_{k+1}| < \infty$.

Proof: By the fact that if X is a normed linear space and $\|.\|_1$ and $\|.\|_2$ are equivalent norms on X, then the bounded subsets of $(X, \|.\|_1)$ coincide with the bounded subsets of $(X, \|.\|_2)$, we have that (i)-(ix) hold respectively by (i)-(ix) of Proposition 4.1.1. #

4.2 CONVERGENCE PRESERVING MATRIX TRANFORMATIONS

As mentioned in Chapter I, all of the nine classical sequence spaces are BK-spaces containing all finite sequences. Among these sequence spaces, only c_o, $\ell_p (1 < p < \infty)$, ℓ, bv_o and cs have the AK property. To characterize a convergence preserving matrix transformation from each of these five sequence spaces into ℓ_∞, we need the following two lemmas. The first lemma is used to prove the second one.

Lemma 4.2.1. Let X be a BK-space containing all finite sequences and A an infinite matrix. Then $A: X \longrightarrow \ell_\infty$ if and only if $(A_{nk})_{k=1}^\infty \in X^\beta$ for every $n \in \mathbb{N}$ and $\sup_n \|(A_{nk})_{k=1}^\infty\|_{X^\beta} < \infty$.

Proof: Assume that $A : X \longrightarrow \ell_\infty$. Then $(A_{nk})_{k=1}^\infty \in X^\beta$ for every $n \in \mathbb{N}$.

Since X and ℓ_∞ are BK-spaces, by Theorem 1.9, $\|A\| = \sup_{\|x\|\le 1} \|Ax\|_{\ell_\infty} < \infty$.

Then

$$\begin{aligned}\|A\| &= \sup_{\|x\|\le 1} (\sup_n |\sum_{k=1}^{\infty} A_{nk}x_k|) \\ &= \sup_n(\sup_{\|x\|\le 1} |\sum_{k=1}^{\infty} A_{nk}x_k|) \\ &= \sup_n \|(A_{nk})_{k=1}^{\infty}\|_{X^\beta} .\end{aligned}$$

Hence $\sup_n \|(A_{nk})_{k=1}^{\infty}\|_{X^\beta} < \infty$.

For the converse, assume that $(A_{nk})_{k=1}^{\infty} \in X^\beta$ for every $n \in \mathbb{N}$ and $\sup_n \|(A_{nk})_{k=1}^{\infty}\| < \infty$. Since $(A_{nk})_{k=1}^{\infty} \in X^\beta$ for every $n \in \mathbb{N}$, we have that $A : X \longrightarrow W$. Since for every $y = (y_k) \in X$ with $\|y\|_X \le 1$ and for every $m \in \mathbb{N}$, $|\sum_{k=1}^{\infty} A_{mk}y_k| \le \sup_{\|x\|\le 1} |\sum_{k=1}^{\infty} A_{mk}x_k| = \|(A_{mk})_{k=1}^{\infty}\|_{X^\beta} \le \sup_n \|(A_{nk})_{k=1}^{\infty}\|_{X^\beta} < \infty$, we have that $\sup_n |\sum_{k=1}^{\infty} A_{nk}y_k| < \infty$ for every $y = (y_k) \in X$ such that $\|y\|_X \le 1$. This proves that $Ay \in \ell_\infty$ for every $y \in X$, $\|y\|_X \le 1$. Hence $Ay \in \ell_\infty$ for all $y \in X$, so $A : X \longrightarrow \ell_\infty$. #

<u>Lemma 4.2.2</u>. Let X be a BK-space having the AK property and A an infinite matrix. Then $A:X \longrightarrow \ell_\infty$ and A preserves convergence if and only if

(i) $(A_{nk})_{k=1}^{\infty} \in X^\beta$ for every $n \in \mathbb{N}$ and $\sup_n \|(A_{nk})_{k=1}^{\infty}\|_{X^\beta} < \infty$ and

(ii) $\lim_{n\to\infty} A_{nk}$ exists for every $k \in \mathbb{N}$.

If this is the case, $\lim_{n\to\infty} (Ax)_n = \sum_{k=1}^{\infty} (x_k \lim_{n\to\infty} A_{nk})$ for every convergent sequence $x=(x_k) \in X$.

<u>Proof</u>: First, we note that for $(x_k) \in c$, $|\lim_{k\to\infty} x_k| = \lim_{k\to\infty} |x_k| \le \sup_k |x_k| = \|(x_k)\|_c$. If follows that $(x_k) \longmapsto \lim_{k\to\infty} x_k$ is a continuous linear functional on c. Assume that $A : X \longrightarrow \ell_\infty$ and A preserves convergence. Then (i) holds by Lemma 4.2.1. Since X has the AK property,

$e^{(k)} \in X$ for all $k \in \mathbb{N}$. Hence (ii) holds by Proposition 2.3(i).

Conversely, assume that (i) and (ii) hold. By Lemma 4.2.1, we have that $A : X \longrightarrow \ell_\infty$. Next, let $x = (x_k) \in X \cap c$. Since X has the AK property, $x = \lim_{n\to\infty} \sum_{k=1}^{n} x_k e^{(k)}$ in X. Since X and ℓ_∞ are BK-spaces, by Theorem 1.9, A is a continuous linear transformation on X.This implies that $Ax = \lim_{n\to\infty} \sum_{k=1}^{n} x_k\ Ae^{(k)}$ in ℓ_∞. By (ii), $(A_{nk})_{n=1}^{\infty} \in c$ for every $k \in \mathbb{N}$. But $Ae^{(k)} = (A_{nk})_{n=1}^{\infty}$ for every $k \in \mathbb{N}$, so $\sum_{k=1}^{n} x_k Ae^{(k)} \in c$ for every $n \in \mathbb{N}$. Since c is a closed linear subspace of ℓ_∞, it follows that $\lim_{n\to\infty} \sum_{k=1}^{n} x_k Ae^{(k)} \in c$. Hence $Ax \in c$ and $((Ax)_n)_{n=1}^{\infty} = \lim_{m\to\infty} \sum_{k=1}^{m} x_k (A_{nk})_{n=1}^{\infty}$ in c. Since the map f defined on c by $f((x_k)) = \lim_{k\to\infty} x_k$ is continuous linear functional on c, we have that

$$\begin{aligned}
\lim_{n\to\infty} (Ax)_n &= f\left(((Ax)_n)_{n=1}^{\infty} \right) \\
&= f\left(\lim_{m\to\infty} \sum_{k=1}^{m} x_k\ (A_{nk})_{n=1}^{\infty} \right) \\
&= \lim_{m\to\infty} \sum_{k=1}^{m} x_k f(\ (A_{nk})_{n=1}^{\infty}) \\
&= \lim_{m\to\infty} \sum_{k=1}^{m} (x_k \lim_{n\to\infty} A_{nk}) \\
&= \sum_{k=1}^{\infty} (x_k \lim_{n\to\infty} A_{nk}).
\end{aligned}$$

These prove that A preserves convergence and $\lim_{n\to\infty} (Ax)_n = \sum_{k=1}^{\infty} (x_k \lim_{n\to\infty} A_{nk})$ for every convergent sequence $x = (x_k) \in X$, as required. #

<u>Theorem 4.2.3</u>. For an infinite matrix A, $A: c_o \longrightarrow \ell_\infty$ and A preserves convergence if and only if

(i) $\sup_n \sum_{k=1}^{\infty} |A_{nk}| < \infty$ and

(ii) $\lim_{n\to\infty} A_{nk}$ exists for every $k \in \mathbb{N}$.

<u>Proof</u>: We have by Corollary 4.1.2(i) that $\sup_n \sum_{k=1}^{\infty} |A_{nk}| < \infty$ if and

only if $(A_{nk})_{k=1}^{\infty} \in c_o^{\beta}$ for every $n \in \mathbb{N}$ and $\sup_n \|(A_{nk})_{k=1}^{\infty}\|_{c_o^{\beta}} < \infty$. Hence by Lemma 4.2.2, the theorem is proved. #

<u>Theorem 4.2.4</u>. For an infinite matrix A, $A : \ell_p \longrightarrow \ell_\infty (1 < p < \infty)$ and A preserves convergence if and only if

(i) $\sup_n \sum_{k=1}^{\infty} |A_{nk}|^q < \infty$ where $\frac{1}{p} + \frac{1}{q} = 1$ and

(ii) $\lim_{n\to\infty} A_{nk}$ exists for every $k \in \mathbb{N}$.

<u>Proof</u>: By Corollary 4.1.2 (ii), we have that $\sup_n \sum_{k=1}^{\infty} |A_{nk}|^q < \infty$ where $\frac{1}{p} + \frac{1}{q} = 1$ if and only if $(A_{nk})_{k=1}^{\infty} \in \ell_p^{\beta}$ for every $n \in \mathbb{N}$ and $\sup_n \|(A_{nk})_{k=1}^{\infty}\|_{\ell_p^{\beta}} < \infty$. Hence the theorem is proved by Lemma 4.2.2. #

<u>Theorem 4.2.5</u>. For an infinite matrix A, $A : \ell \longrightarrow \ell_\infty$ and A preserves convergence if and only if

(i) $\sup_{n,k} |A_{nk}| < \infty$ and

(ii) $\lim_{n\to\infty} A_{nk}$ exists for all $k \in \mathbb{N}$.

<u>Proof</u>: It follows from Corollary 4.1.2(iii) that $\sup_{n,k} |A_{nk}| < \infty$ if and only if $(A_{nk})_{k=1}^{\infty} \in \ell^{\beta}$ for every $n \in \mathbb{N}$ and $\sup_n \|(A_{nk})_{k=1}^{\infty}\|_{\ell} < \infty$. Hence the theorem holds from Lemma 4.2.2. #

<u>Theorem 4.2.6</u>. For an infinite matrix A, $A : bv_o \longrightarrow \ell_\infty$ and A preserves convergence if and only if

(i) $\sup_{m,n} |\sum_{k=1}^{m} A_{nk}| < \infty$ and

(ii) $\lim_{n\to\infty} A_{nk}$ exists for every $k \in \mathbb{N}$.

<u>Proof</u>: We have from Corollary 4.1.2(iv) that $\sup_{m,n} |\sum_{k=1}^{m} A_{nk}| < \infty$

if and only if $(A_{nk})_{k=1}^{\infty} \in bv_o^{\beta}$ for every $n \in \mathbb{N}$ and $\sup_n \|(A_{nk})_{k=1}^{\infty}\|_{bv_o^{\beta}} < \infty$. By Lemma 4.2.2, the theorem holds. #

To characterize any convergence preserving matrix $A : cs \longrightarrow \ell_\infty$, we need one more lemma as follows :

Lemma 4.2.7. If A is an infinite matrix such that $\sup_n \sum_{k=1}^{\infty} |A_{nk}-A_{n(k+1)}| < \infty$ and $(A_{nk})_{n=1}^{\infty}$ is bounded for every $k \in \mathbb{N}$, then

$$\sup_n \left(\sum_{k=1}^{\infty} |A_{nk}-A_{n(k+1)}| + \sup_k |A_{nk}| \right) < \infty.$$

Proof: Let $M > 0$ be such that $\sum_{k=1}^{\infty} |A_{nk} - A_{n(k+1)}| < M$ for every $n \in \mathbb{N}$. By assumption, $\sup_n |A_{nk}| < \infty$ for every $k \in \mathbb{N}$. Let $K = \sup_n |A_{n1}|$. We claim that $\sum_{k=1}^{\infty} |A_{nk} - A_{n(k+1)}| + \sup_k |A_{nk}| < 2M + K$ for every $n \in \mathbb{N}$. For $m, n \in \mathbb{N}$, $m > 1$, we have $|A_{n1} - A_{nm}| \le \sum_{k=1}^{m-1} |A_{nk} - A_{n(k+1)}| \le \sum_{k=1}^{\infty} |A_{nk} - A_{n(k+1)}| < M$, so $|A_{nm}| < M + |A_{n1}| \le M + K$ for every $m, n \in \mathbb{N}$. Then $\sup_k |A_{nk}| \le M + K$ for every $n \in \mathbb{N}$. Hence we have the claim. This implies that $\sup_n \left(\sum_{k=1}^{\infty} |A_{nk} - A_{n(k+1)}| + \sup_k |A_{nk}| \right) < \infty$. #

Theorem 4.2.8. For an infinite matrix A, $A : cs \longrightarrow \ell_\infty$ and A preserves convergence if and only if

(i) $\sup_n \sum_{k=1}^{\infty} |A_{nk}-A_{n(k+1)}| < \infty$ and

(ii) $\lim_{n\to\infty} A_{nk}$ exists for every $k \in \mathbb{N}$.

Proof: Recall that $bv = \left\{ (x_k) \mid \sum_{k=1}^{\infty} |x_k-x_{k+1}| < \infty \right\}$ with $\|(x_k)\| = \sum_{k=1}^{\infty} |x_k-x_{k+1}| + \lim_{k\to\infty} |x_k|$. Assume that $A : cs \longrightarrow \ell_\infty$ and A preserves convergence. By Lemma 4.2.2, (ii) holds, $(A_{nk})_{k=1}^{\infty} \in cs^{\beta}$ for every $n \in \mathbb{N}$

and $\sup_n \|(A_{nk})_{k=1}^{\infty}\|_{cs^\beta} < \infty$. We then have by Corollary 4.1.2(v) that

$$\sup_n \left(\sum_{k=1}^{\infty} |A_{nk}-A_{n(k+1)}| + \lim_{k\to\infty} |A_{nk}| \right) < \infty.$$ Then $\sup_n \sum_{k=1}^{\infty} |A_{nk}-A_{n(k+1)}| < \infty$.

Thus (i) holds.

For the converse, assume that (i) and (ii) hold. Then $(A_{nk})_{k=1}^{\infty}$ $\in bv$ for every $n \in \mathbb{N}$. By (i), (ii) and Lemma 4.2.7, we have that

$$\sup_n \left(\sum_{k=1}^{\infty} |A_{nk}-A_{n(k+1)}| + \sup_k |A_{nk}| \right) < \infty .$$ Therefore

$$\sup_n \left(\sum_{k=1}^{\infty} |A_{nk}-A_{n(k+1)}| + \lim_{k\to\infty} |A_{nk}| \right) < \infty$$ since $\lim_{k\to\infty} |A_{nk}| \le \sup_k |A_{nk}|$ $< \infty$ for every $n \in \mathbb{N}$. We then have by Corollary 4.1.2(v) that $(A_{nk})_{k=1}^{\infty}$ $\in cs^\beta$ for every $n \in \mathbb{N}$ and $\sup_n \|(A_{nk})_{k=1}^{\infty}\|_{cs^\beta} < \infty$. By Lemma 4.2.2 , $A : cs \longrightarrow \ell_\infty$ and A preserves convergence. #

By Theorem 1.12 and Theorem 1.13, we have respectively that for an infinite matrix A, $A : \ell_\infty \longrightarrow \ell_\infty$ if and only if $\sup_n \sum_{k=1}^{\infty} |A_{nk}| < \infty$ and $A : c \longrightarrow c$ if and only if $\sup_n \sum_{k=1}^{\infty} |A_{nk}| < \infty$, $\lim_{n\to\infty} A_{nk}$ exists for all $k \in \mathbb{N}$ and $\lim_{n\to\infty} \sum_{k=1}^{\infty} A_{nk}$ exists. It then follows that $A : \ell_\infty \longrightarrow \ell_\infty$ and $A(c) \subseteq c$ if and only if $A : c \longrightarrow c$. It is obvious that $A : c \longrightarrow \ell_\infty$ and A preserves convergence if and only if $A : c \longrightarrow c$. We have the following theorem.

<u>Theorem 4.2.9</u>. If X is ℓ_∞ or c, then for an infinite matrix A, $A:X\longrightarrow \ell_\infty$ and A preserves convergence if and only if

(i) $\sup_n \sum_{k=1}^{\infty} |A_{nk}| < \infty$,

(ii) $\lim_{n\to\infty} A_{nk}$ exists for every $k \in \mathbb{N}$ and

(iii) $\lim_{n\to\infty} \sum_{k=1}^{\infty} A_{nk}$ exists.

We have characterized convergence preserving infinite matrices from bv_o into ℓ_∞ in Theorem 4.2.6. To characterize such matrices from bv into ℓ_∞, we use this characterization, the fact that $bv = bv_o + \langle e\rangle$ and the following lemma.

Lemma 4.2.10. Let X be a sequence space and A an infinite matrix. Then $A:X \longrightarrow \ell_\infty$, A preserves convergence on X and $\lim_{n\to\infty} \sum_{k=1}^{\infty} A_{nk}$ exists if and only if $A : X+\langle e\rangle \longrightarrow \ell_\infty$ and A preserves convergence on $X+\langle e\rangle$.

Proof: Assume that $A:X \longrightarrow \ell_\infty$, A preserves convergence on X and $\lim_{n\to\infty} \sum_{k=1}^{\infty} A_{nk}$ exists. Then $\left(\sum_{k=1}^{\infty} A_{nk} \right)_{n=1}^{\infty} \in c$. But $Ae = \left(\sum_{k=1}^{\infty} A_{nk} \right)_{n=1}^{\infty}$, so $Ae \in c \subseteq \ell_\infty$ which implies that $A : X + \langle e\rangle \longrightarrow \ell_\infty$. By assumption, $A(X \cap c) \subseteq c$. Since $e \in c$, $(X + \langle e\rangle) \cap c = (X \cap c) + \langle e\rangle$. But $A(X \cap c) \subseteq c$ and $Ae \in c$, so we have $A((X + \langle e\rangle) \cap c) = A((X \cap c) + \langle e\rangle) \subseteq c$. Hence $A : X + \langle e\rangle \longrightarrow \ell_\infty$ and A preserves convergence on $X + \langle e\rangle$.

The converse follows from the fact that $e \in c$ and $Ae = \left(\sum_{k=1}^{\infty} A_{nk} \right)_{n=1}^{\infty}$. #

From Theorem 4.2.6 and Lemma 4.2.10, we have the following theorem.

Theorem 4.2.11. For an infinite matrix A, $A : bv \longrightarrow \ell_\infty$ and A preserves convergence if and only if

(i) $\sup_{m,n} \left| \sum_{k=1}^{m} A_{nk} \right| < \infty$,

(ii) $\lim_{n\to\infty} A_{nk}$ exists for all $k \in \mathbb{N}$ and

(iii) $\lim_{n\to\infty} \sum_{k=1}^{\infty} A_{nk}$ exists.

Recall that the Cesaro matrix, finite matrices, scalar matrices, the Borel matrix, Norlund matrices, Kojima matrices and Toeplitz matrices are in the class of K_r-matrices and the Cesaro matrix and row-bounded matrices are in the class of matrices A such that $\sum_{k=1}^{\infty} |A_{nk}-A_{n(k+1)}|$ converges uniformly on n = 1, 2, 3, ... The purpose of the next two theorems is to characterize an infinite matrix A in these two classes such that $A:bs \longrightarrow \ell_\infty$ and A preserves convergence. The general case is still an open problem. We first give necessary and sufficient conditions on an infinite matrix A guaranteeing $A : bs \longrightarrow \ell_\infty$.

Lemma 4.2.12. For an infinite matrix A, $A : bs \longrightarrow \ell_\infty$ if and only if

(i) $\lim_{k\to\infty} A_{nk} = 0$ for all $n \in \mathbb{N}$ and

(ii) $\sup_n \sum_{k=1}^{\infty} |A_{nk}-A_{n(k+1)}| < \infty$.

Proof: Assume that $A : bs \longrightarrow \ell_\infty$. By Lemma 4.2.1, $(A_{nk})_{k=1}^{\infty} \in bs^\beta$ for all $n \in \mathbb{N}$ and $\sup_n \|(A_{nk})_{k=1}^{\infty}\|_{bs^\beta} < \infty$. Hence (ii) holds by Corollary 4.1.2(ix). But $bs^\beta = bv_o$ and $bv_o \subsetneq c_o$, so we have that $(A_{nk})_{k=1}^{\infty} \in c_o$ for all $n \in \mathbb{N}$. Hence (i) holds.

For the converse, assume that (i) and (ii) hold. Then by (i), $(A_{nk})_{k=1}^{\infty} \in c_o$ for all $n \in \mathbb{N}$. From (ii), we have that $(A_{nk})_{k=1}^{\infty} \in bv$ for all $n \in \mathbb{N}$. Hence $(A_{nk})_{k=1}^{\infty} \in bv_o$ for all $n \in \mathbb{N}$ since $bv_o = bv \cap c_o$. Since $bs^\beta = bv_o$, it follows that $(A_{nk})_{k=1}^{\infty} \in bs^\beta$ for all $n \in \mathbb{N}$. By (ii) and Corollary 4.1.2(ix), $\sup_n \|(A_{nk})_{k=1}^{\infty}\|_{bs^\beta} < \infty$. Hence by Lemma 4.2.1, $A : bs \longrightarrow \ell_\infty$. #

To characterize convergence preserving K_r-matrices we need the following two lemmas.

Lemma 4.2.13. $bs \cap c \subseteq c_o$.

Proof: Let $(x_n) \in bs \cap c$. Then $(\sum_{k=1}^{n} x_k)_{k=1}^{\infty}$ is a bounded sequence. First, we assume that (x_n) is a real sequence. If there exists $n_o \in \mathbb{N}$ such that $x_n \geq 0$ for all $n \geq n_o$ or $x_n \leq 0$ for all $n \geq n_o$, then $\sum_{i=1}^{\infty} x_i$ is a convergent series which implies that $(x_n) \in c_o$. For the case that $x_n > 0$ for infinitely many n and $x_n < 0$ for infinitely many n, let (x_{nk}) and (x_{mk}) be subsequences of (x_n) such that $x_{nk} > 0$ for all $k \in \mathbb{N}$ and $x_{mk} < 0$ for all $k \in \mathbb{N}$. Since $(x_n) \in c$, we have that $0 \leq \lim_{k\to\infty} x_{nk} = \lim_{n\to\infty} x_n = \lim_{k\to\infty} x_{mk} \leq 0$. Hence $\lim_{n\to\infty} x_n = 0$, so $(x_n) \in c_o$.

For the general case, let $(x_n) = (y_n) + i(z_n)$ where (x_n) and (y_n) are real sequences. Since $(x_n) \in bs \cap c$, we have that (y_n) and $(z_n) \in bs \cap c$. From the above proof (y_n), $(z_n) \in c_o$ and hence $(x_n) \in c_o$. #

Lemma 4.2.14. Let X be a linear subspace of ℓ_∞ and A a K_r matrix. Then the following statements hold:

(i) $A : X \longrightarrow \ell_\infty$ and

(ii) A is continuos on $(X, \|.\|)$ where $\|.\|$ is the restriction of $\|.\|_{\ell_\infty}$ to X.

Proof: Since A is a K_r-matrix, $\sup_n \sum_{k=1}^{\infty} |A_{nk}| < \infty$. For $x = (x_k) \in X$, we have that $\sum_{k=1}^{\infty} |A_{nk}x_k| \leq \sum_{k=1}^{\infty} |A_{nk}| \sup_k |x_k| < \infty$ for every $n \in \mathbb{N}$. Hence for each $x = (x_k) \in X$, $\sum_{k=1}^{\infty} A_{nk}x_k$ converges for all $n \in \mathbb{N}$, so Ax exists for every $x \in X$. It follows from the above inequality that for $x = (x_k) \in X$,

$$(*) \qquad \left|\sum_{k=1}^{\infty} A_{nk}x_k\right| \le \left(\sup_m \sum_{k=1}^{\infty} |A_{mk}|\right) \|x\| \quad \text{for all } n \in \mathbb{N}.$$

Since for $x = (x_k) \in X$, $Ax = \left(\sum_{k=1}^{\infty} A_{nk}x_k\right)_{n=1}^{\infty}$, the inequality (*) yields that $Ax \in \ell_\infty$ for every $x \in X$. Hence $A : X \longrightarrow \ell_\infty$. It follows from (*) that $\|Ax\|_{\ell_\infty} \le \left(\sup_n \sum_{k=1}^{\infty} |A_{nk}|\right) \|x\|$ which implies that A is continuous on X.#

<u>Theorem 4.2.15</u>. Let A be a K_r matrix.Then the following statements hold.

(i) $A : bs \longrightarrow \ell_\infty$ and

(ii) A preserves convergence if and only if $\lim_{n\to\infty} A_{nk}$ exists for every $k \in \mathbb{N}$.

<u>Proof</u>: By Lemma 4.2.14, we have (i).

To prove (ii), assume that A preserves convergence on bs. Since $e^{(k)} \in bs$ for every $k \in \mathbb{N}$, by Proposition 2.3(i), $\lim_{n\to\infty} A_{nk}$ exists for every $k \in \mathbb{N}$.

Conversely, assume that $\lim_{n\to\infty} A_{nk}$ exists for every $k \in \mathbb{N}$. To show that A preserves convergence, let $(x_k) \in bs \cap c$. By Lemma 4.2.13, $bs \cap c \subseteq c_o$. Let $\|.\|$ be the restriction of $\|.\|_{c_o}$ to $bs \cap c$. Since c_o has the AK property and $\Phi \subseteq bs \cap c$, it follows that $(bs \cap c, \|.\|)$ has the AK property. Then $(x_k) = \lim_{n\to\infty} \sum_{k=1}^{n} x_k e^{(k)}$ in $(bs \cap c, \|.\|)$. By Lemma 4.2.14, A is continuous on $(bs \cap c, \|.\|)$. Then $A((x_k)) = \lim_{n\to\infty} \sum_{k=1}^{n} x_k Ae^{(k)}$ in ℓ_∞. But $\lim_{n\to\infty} A_{nk}$ exists and $Ae^{(k)} = (A_{nk})_{n=1}^{\infty}$ for every $k \in \mathbb{N}$, so $Ae^{(k)} \in c$ for every $k \in \mathbb{N}$ which implies that $\sum_{k=1}^{n} x_k Ae^{(k)} \in c$ for all $n \in \mathbb{N}$. Since c is a closed linear subspace of ℓ_∞, it follows that $\lim_{n\to\infty} \sum_{k=1}^{n} x_k Ae^{(k)}$

$\in c$. Hence $A((x_k)) \in c$, as required. #

The following lemma is required to characterize a convergence preserving matrix transformation $A : bs \longrightarrow \ell_\infty$ where $\sum_{k=1}^{\infty} |A_{nk}-A_{n(k+1)}|$ converges uniformly on $n = 1, 2, 3, \ldots$

<u>Lemma 4.2.16</u>. If $(x_k) \in bs$ and $(y_k) \in bv_o$, then $\sum_{k=1}^{\infty} x_k y_k$ and $\sum_{k=1}^{\infty} (\sum_{i=1}^{k} x_i)(y_k - y_{k+1})$ converge and $\sum_{k=1}^{\infty} x_k y_k = \sum_{k=1}^{\infty} (\sum_{i=1}^{k} x_i)(y_k - y_{k+1})$.

<u>Proof</u>: Since $(x_k) \in bs$, there exists $K > 0$ such that $|\sum_{i=1}^{n} x_i| < K$ for all $n \in \mathbb{N}$. Since $(y_k) \in bv_o$, we have that $\sum_{k=1}^{\infty} |y_k - y_{k+1}| < \infty$ and $\lim_{k\to\infty} y_k = 0$. It follows that $\sum_{k=1}^{\infty} |(\sum_{i=1}^{k} x_i)(y_k - y_{k+1})| \le K \sum_{k=1}^{\infty} |y_k - y_{k+1}| < \infty$ and then $\sum_{k=1}^{\infty} (\sum_{i=1}^{k} x_i)(y_k - y_{k+1})$ converges. We know that $\sum_{k=1}^{n} x_k y_k = \sum_{k=1}^{n-1} (\sum_{i=1}^{k} x_i)(y_k - y_{k+1}) + (\sum_{i=1}^{n} x_i) y_n$ for every $n \in \mathbb{N}$. Since $(\sum_{i=1}^{n} x_i)_{n=1}^{\infty}$ is a bounded sequence and $\lim_{n\to\infty} y_n = 0$, we have that $\lim_{n\to\infty} (\sum_{i=1}^{n} x_i) y_n = 0$. Then $\sum_{k=1}^{\infty} x_k y_k = \lim_{n\to\infty} \sum_{k=1}^{n} x_k y_k = \lim_{n\to\infty} \sum_{k=1}^{n-1} (\sum_{i=1}^{k} x_i)(y_k - y_{k+1}) = \sum_{k=1}^{\infty} (\sum_{i=1}^{k} x_i)(y_k - y_{k+1})$. #

<u>Theorem 4.2.17</u>. Let A be an infinite matrix such that $\sum_{k=1}^{\infty} |A_{nk}-A_{n(k+1)}|$ converges uniformly on $n = 1,2,3,\ldots$. Then $A: bs \longrightarrow \ell_\infty$ and A preserves convergence if and only if

(i) $\lim_{k\to\infty} A_{nk} = 0$ for every $n \in \mathbb{N}$ and

(ii) $\lim_{n\to\infty} A_{nk}$ exists for every $k \in \mathbb{N}$.

<u>Proof</u>: If $A : bs \longrightarrow \ell_\infty$ and A preserves convergence, then (i) and (ii) hold respectively by Lemma 4.2.12 and Proposition 2.3(i).

For the converse, assume that (i) and (ii) hold. We shall show that $\sup_n \sum_{k=1}^{\infty} |A_{nk}-A_{n(k+1)}| < \infty$. Since $\sum_{k=1}^{\infty} |A_{nk}-A_{n(k+1)}|$ converges uniformly on $n = 1, 2, 3, \ldots$, there exists $k_o \in \mathbb{N}$, $k_o > 1$ such that $\sum_{k=k_o}^{\infty} |A_{nk}-A_{n(k+1)}| < 1$ for all $n \in \mathbb{N}$. By (ii), $(A_{nk})_{n=1}^{\infty}$ is bounded for all $k \in \mathbb{N}$. Then there exists $M > 0$ such that $|A_{nk}| < M$ for all $n \in \mathbb{N}$ and $k \in \{1, 2, 3, \ldots, k_o\}$. Hence for each $n \in \mathbb{N}$,

$$\begin{aligned}\sum_{k=1}^{\infty} |A_{nk}-A_{n(k+1)}| &\le \sum_{k=1}^{k_o-1} |A_{nk}-A_{n(k+1)}| + \sum_{k=k_o}^{\infty} |A_{nk}-A_{n(k+1)}| \\ &< 2Mk_o + 1\end{aligned}$$

so $\sup_n \sum_{k=1}^{\infty} |A_{nk}-A_{n(k+1)}| < \infty$. By Lemma 4.2.12, $A : bs \longrightarrow \ell_\infty$.

To show that A preserves convergence, let $x = (x_k) \in bs \cap c$. Then $Ax = (\sum_{k=1}^{\infty} A_{nk}x_k)_{n=1}^{\infty} \in \ell_\infty$. To show that $Ax \in c$, it suffices to show that $(\sum_{k=1}^{\infty} A_{nk}x_k)_{n=1}^{\infty}$ is a Cauchy sequence. Let $\epsilon > 0$ be given. Since $(x_k) \in bs$, there exists $K > 0$ such that $|\sum_{i=1}^{n} x_i| \le K$ for all $n \in \mathbb{N}$. Since $\sum_{k=1}^{\infty} |A_{nk}-A_{n(k+1)}|$ converges uniformly on $n = 1,2,3,\ldots$, there exists $k_o \in \mathbb{N}$, $k_o > 1$

$$(*) \qquad \sum_{k=k_o}^{\infty} |A_{nk}-A_{n(k+1)}| < \frac{\epsilon}{4K} \text{ for all } n \in \mathbb{N}.$$

Since $\sum_{k=1}^{\infty} |A_{nk}-A_{n(k+1)}|$ converges and $\lim_{k\to\infty} A_{nk} = 0$ for every $n \in \mathbb{N}$, we have $(A_{nk})_{k=1}^{\infty} \in bv_o$ for every $n \in \mathbb{N}$. By Lemma 4.2.16, $\sum_{k=1}^{\infty} (\sum_{i=1}^{k} x_i)(A_{nk}-A_{n(k+1)})$ converges and

$$(**) \qquad \sum_{k=1}^{\infty} A_{nk}x_k = \sum_{k=1}^{\infty} (\sum_{i=1}^{k} x_i)(A_{nk}-A_{n(k+1)}) \text{ for every } n \in \mathbb{N}.$$

By (ii), there exists $n_o \in \mathbb{N}$ such that

$$(***) \qquad |A_{nk}-A_{mk}| < \frac{\epsilon}{4K(k_o-1)} \quad \text{for all } m, n \ge n_o \text{ and all } k \in \{1,2,3,\ldots, k_o\}$$

We have from (**) that for $m, n \in \mathbb{N}$,

$$\left|\sum_{k=1}^{\infty} A_{nk}x_k - \sum_{k=1}^{\infty} A_{mk}x_k\right| = \left|\sum_{k=1}^{\infty}\left(\sum_{i=1}^{k} x_i\right)(A_{nk}-A_{n(k+1)}) - \sum_{k=1}^{\infty}\left(\sum_{k=1}^{k} x_i\right)(A_{mk}-A_{m(k+1)})\right|$$

$$= \left|\sum_{k=1}^{\infty}\left(\sum_{i=1}^{k} x_i\right)((A_{nk}-A_{n(k+1)}) - (A_{mk}-A_{m(k+1)}))\right|$$

Then for m, $n \geq n_o$, we obtain the following inequality :

$$\left|\sum_{k=1}^{\infty} A_{nk}x_k - \sum_{k=1}^{\infty} A_{mk}x_k\right| \leq K\sum_{k=1}^{\infty}|(A_{nk}-A_{n(k+1)}) - (A_{mk} - A_{m(k+1)})|$$

$$\leq K\left(\sum_{k=1}^{k_o-1}|A_{nk}-A_{mk}| + \sum_{k=1}^{k_o-1}|A_{n(k+1)} - A_{m(k+1)}| + \sum_{k=k_o}^{\infty}|A_{nk}-A_{n(k+1)}| + \sum_{k=k_o}^{\infty}|A_{mk}-A_{m(k+1)}|\right)$$

$$< K\left(\frac{(k_o-1)\epsilon}{4K(k_o-1)} + \frac{(k_o-1)\epsilon}{4K(k_o-1)} + \frac{\epsilon}{4K} + \frac{\epsilon}{4K}\right)$$

(from (*) and (***))

$$= \epsilon.$$

This proves that $\left(\sum_{k=1}^{\infty} A_{nk}x_k\right)_{n=1}^{\infty}$ is a Cauchy sequence and hence $Ax \in c$. #

4.3 LIMIT PRESERVING MATRIX TRANFORMATIONS

Due to the fact that a limit preserving matrix transformation between sequence spaces is convergence preserving, our main tools in this section are the results of Section 4.2. Also the following two lemmas are required.

Lemma 4.3.1. Let X be an FK-space having the AK property. If there exists a matrix transformation of X into ℓ_∞ which preserves limits , then $X \cap c \subseteq c_o$.

Proof: Let $A : X \longrightarrow \ell_\infty$ be such that A preserves limits. Then $\lim\limits_{n\to\infty} A_{nk} = 0$ for every $k \in \mathbb{N}$ by Proposition 2.3 (ii), so $Ae^{(k)} = (A_{nk})_{n=1}^{\infty} \in c_o$ for all $k \in \mathbb{N}$. Let $x = (x_n) \in X \cap c$. Then $\lim\limits_{n\to\infty} (Ax)_n = \lim\limits_{n\to\infty} x_n$. Since

X has the AK property, $x = \lim_{n\to\infty} \sum_{k=1}^{n} x_k e^{(k)}$ in X. Since X and ℓ_∞ are FK-spaces, A is continuous on X by Theorem 1.9, so $Ax = \lim_{n\to\infty} \sum_{k=1}^{n} x_k Ae^{(k)}$ in ℓ_∞. But $Ae^{(k)} \in c_o$ for all $k \in \mathbb{N}$ and c_o is a closed linear subspace of ℓ_∞, so $Ax \in c_o$ which implies that $\lim_{n\to\infty} x_n = 0$. This proves that $X \cap c \subseteq c_o$. #

It follows from Lemma 4.3.1 that if X is a BK-space having the AK property such that $X \cap c \subsetneq c_o$, then there is no infinite matrix mapping X into ℓ_∞ which preserves limits.

Lemma 4.3.2. Let X be a BK-space having the AK property such that $X \cap c \subsetneq c_o$ and A an infinite matrix. Then $A:X \longrightarrow \ell_\infty$ and A preserves limits if and only if

(i) $(A_{nk})_{k=1}^{\infty} \in X^{\beta}$ for every $n \in \mathbb{N}$ and $\sup_n \|(A_{nk})_{k=1}^{\infty}\|_{X^\beta} < \infty$ and

(ii) $\lim_{n\to\infty} A_{nk} = 0$ for every $k \in \mathbb{N}$.

Proof: Assume that $A : X \longrightarrow \ell_\infty$ and A preserves limits. Then (i) and (ii) hold by Lemma 4.2.1 and Proposition 2.3 (ii), respectively.

Conversely, assume that (i) and (ii) hold. By Lemma 4.2.1 , $A : X \longrightarrow \ell_\infty$. Let $x = (x_n) \in X \cap c$. Since $X \cap c \subsetneq c_o$, $\lim_{n\to\infty} x_n = 0$. By Lemma 4.2.2, $Ax \in c$ and $\lim_{n\to\infty} (Ax)_n = \sum_{k=1}^{\infty} (x_k \lim_{n\to\infty} A_{nk})$. Since $\lim_{n\to\infty} A_{nk} = 0$ for every $k \in \mathbb{N}$, we have that $\lim_{n\to\infty} (Ax)_n = 0 = \lim_{n\to\infty} x_n$. Hence A preserves limits. #

Theorem 4.3.3. For an infinite matrix A, $A:c_o \longrightarrow \ell_\infty$ and A preserves limits if and only if

(i) $\sup_n \sum_{k=1}^{\infty} |A_{nk}| < \infty$ and

(ii) $\lim_{n\to\infty} A_{nk} = 0$ for every $k \in \mathbb{N}$.

Proof: If $A : c_o \longrightarrow \ell_\infty$ and A preserves limits, then (i) holds by Theorem 4.2.3 and (ii) holds by Lemma 4.3.2.

For the converse, assume that (i) and (ii) hold. By Corollary 4.1.2 (i), we have $(A_{nk})_{k=1}^{\infty} \in c_o^{\beta}$ for every $n \in \mathbb{N}$ and $\sup_n \|(A_{nk})_{k=1}^{\infty}\|_{c_o^{\beta}} < \infty$. Since c_o is a subspace of c, we have $c_o \cap c = c_o$. Hence by Lemma 4.3.2, we have that $A : c_o \longrightarrow \ell_\infty$ and A preserves limits. #

Theorem 4.3.4. For an infinite matrix A, $A:\ell_p \longrightarrow \ell_\infty (1 < p < \infty)$ and A preserves limits if and only if

(i) $\sup_n \sum_{k=1}^{\infty} |A_{nk}|^q < \infty$ where $\frac{1}{p} + \frac{1}{q} = 1$ and

(ii) $\lim_{n\to\infty} A_{nk} = 0$ for every $k \in \mathbb{N}$.

Proof: If $A : \ell_p \longrightarrow \ell_\infty$ and A preserves limits, then (i) holds by Theorem 4.2.4 and (ii) holds by Lemma 4.3.2.

For the converse, assume that (i) and (ii) hold. By Corollary 4.1.2(ii), we have that $(A_{nk})_{k=1}^{\infty} \in \ell_p^{\beta}$ for every $n \in \mathbb{N}$ and $\sup_n \|(A_{nk})_{k=1}^{\infty}\|_{\ell_p^{\beta}} < \infty$. Since ℓ_p is a subspace of c_o and c_o is a subspace of c, we have $\ell_p \cap c = \ell_p \subseteq c_o$. Hence by Lemma 4.3.2, $A:\ell_p \longrightarrow \ell_\infty$ and A preserves limits. #

Theorem 4.3.5. For an infinite matrix A, $A:\ell \longrightarrow \ell_\infty$ and A preserves limits if and only if

(i) $\sup_{n,k} |A_{nk}| < \infty$ and

(ii) $\lim_{n\to\infty} A_{nk} = 0$ for all $k \in \mathbb{N}$.

Proof: If $A\ \ell \longrightarrow \ell_\infty$ and A preserves limits, then (i) and (ii) hold respectively by Theorem 4.2.5 and Lemma 4.3.2.

Conversely, assume that (i) and (ii) hold. It follows by Corollary 4.1.2 (iii) $(A_{nk})_{k=1}^{\infty} \in \ell^\beta$ for every $n \in \mathbb{N}$ and $\sup_n \|(A_{nk})_{k=1}^{\infty}\|_{\ell^\beta} < \infty$. Since ℓ is a subspace of c_o and c_o is a subspace of c, $\ell \cap c = \ell \subseteq c_o$. Then by Lemma 4.3.2 we have that $A : \ell \longrightarrow \ell_\infty$ and A preserves limits. #

Theorem 4.3.6. For an infinite matrix A, $A{:}bv_o \longrightarrow \ell_\infty$ and A preserves limits if and only if

(i) $\sup_{m,n} |\sum_{k=1}^{m} A_{nk}| < \infty$ and

(ii) $\lim_{n\to\infty} A_{nk} = 0$ for every $k \in \mathbb{N}$.

Proof: By Theorem 4.2.6 and Lemma 4.3.2, we have that if $A : bv_o \longrightarrow \ell_\infty$ and A preserves limits, then (i) and (ii) hold.

For the converse, assume that (i) and (ii) hold. By Corollary 4.1.2 (iv), we have that $(A_{nk})_{k=1}^{\infty} \in bv_o^\beta$ and $\sup_n \|(A_{nk})_{k=1}^{\infty}\|_{bv_o^\beta} < \infty$. From $bv_o \subseteq c_o \subseteq c$, we have that $bv_o \cap c = bv_o \subseteq c_o$. Hence by Lemma 4.3.2, $A : bv_o \longrightarrow \ell_\infty$ and A preserves limits. #

Theorem 4.3.7. For an infinite matrix A, $A{:}cs \longrightarrow \ell_\infty$ and A preserves limits if and only if

(i) $\sup_n \sum_{k=1}^{\infty} |A_{nk} - A_{n(k+1)}| < \infty$ and

(ii) $\lim_{n\to\infty} A_{nk} = 0$ for every $k \in \mathbb{N}$.

Proof: If $A : cs \longrightarrow \ell_\infty$ and A preserves limits, then (i) and (ii) hold respectively by Theorem 4.2.8 and Lemma 4.3.2.

Conversely, assume that (i) and (ii) hold. We have by (i) that $(A_{nk})_{k=1}^{\infty} \in bv$ for every $n \in \mathbb{N}$, so $(A_{nk})_{k=1}^{\infty} \in c$ for every $n \in \mathbb{N}$. It follows from Lemma 4.2.7 that $\sup_n \left(\sum_{k=1}^{\infty} |A_{nk}-A_{n(k+1)}| + \lim_{k\to\infty} |A_{nk}| \right) < \infty$. By Corollary 4.1.2(v), $(A_{nk})_{k=1}^{\infty} \in cs^\beta$ for every $n \in \mathbb{N}$ and $\sup_n \|(A_{nk})_{k=1}^{\infty}\|_{cs^\beta} < \infty$. Since $cs \cap c = cs \subseteq c_o$, by Lemma 4.3.2 , we have that $A : cs \longrightarrow \ell_\infty$ and A preserves limits. #

It is known from Theorem 1.12 that for an infinite matrix A and for $X = \ell_\infty$ or c, $A : X \longrightarrow \ell_\infty$ if and only if $\sup_n \sum_{k=1}^{\infty} |A_{nk}| < \infty$. Also from Theorem 1.16, $A : c \longrightarrow c$ and A preserves limits if and only if $\sup_n \sum_{k=1}^{\infty} |A_{nk}| < \infty$, $\lim_{n\to\infty} A_{nk} = 0$ for every $k \in \mathbb{N}$ and $\lim_{n\to\infty} \sum_{k=1}^{\infty} A_{nk} = 1$. Then for $X = \ell_\infty$ or c, $A : X \longrightarrow \ell_\infty$ and A preserves limits if and only if $A : c \longrightarrow c$ and A preserves limits. Hence we have the following two theorems.

Theorem 4.3.8. If X is ℓ_∞ or c, then for an infinite matrix A, $A:X\longrightarrow \ell_\infty$ and A preserves limits if and only if

(i) $\sup_n \sum_{k=1}^{\infty} |A_{nk}| < \infty$,

(ii) $\lim_{n\to\infty} A_{nk} = 0$ for every $k \in \mathbb{N}$ and

(iii) $\lim_{n\to\infty} \sum_{k=1}^{\infty} A_{nk} = 1$.

The following lemma will be used to characterize an infinite matrix $A:bv \longrightarrow \ell_\infty$ and A preserves limits. Recall that $bv = bv_o + \langle e\rangle$.

<u>Lemma 4.3.9</u>. Let X be a sequence space an A an infinite matrix. Then $A:X\longrightarrow\ell_\infty$, A preserves limits and $\lim_{n\to\infty} \sum_{k=1}^{\infty} A_{nk} = 1$ if and only if $A:X+\langle e\rangle\longrightarrow\ell_\infty$ and A preserves limits.

<u>Proof</u>: Assume that $A : X \longrightarrow \ell_\infty$, A preserves limits and $\lim_{n\to\infty} \sum_{k=1}^{\infty} A_{nk} = 1$. By Lemma 4.2.10, $A : X + \langle e\rangle \longrightarrow \ell_\infty$ and A preserves convergence. To show that A preserves limits in $X + \langle e\rangle$, let $y = (y_n) \in (X + \langle e\rangle) \cap c$. Since $(X + \langle e\rangle) \cap c = (X \cap c) + \langle e\rangle$, $y = x + \alpha e$ for some $x = (x_n) \in X \cap c$ and some scalar α. Then $y_n = x_n + \alpha$ for all $n \in \mathbb{N}$. Since A preserves limits in X, $\lim_{n\to\infty} (Ax)_n = \lim_{n\to\infty} x_n$. Then for every $n \in \mathbb{N}$, $(Ay)_n = (Ax)_n + \alpha(Ae)_n = (Ax)_n + \alpha \sum_{k=1}^{\infty} A_{nk}$ and hence

$$\begin{aligned}\lim_{n\to\infty} (Ay)_n &= \lim_{n\to\infty} (Ax)_n + \alpha \lim_{n\to\infty} \sum_{k=1}^{\infty} A_{nk} \\ &= \lim_{n\to\infty} x_n + \alpha \\ &= \lim_{n\to\infty} y_n\end{aligned}$$

Therefore A preserves limits in $x + \langle e\rangle$.

The converse follows from the fact that $Ae = (\sum_{k=1}^{\infty} A_{nk})_{n=1}^{\infty}$ and $\lim_{n\to\infty} e_n = 1$. #

<u>Theorem 4.3.10</u>. For an infinite matrix A, $A:bv\longrightarrow\ell_\infty$ and A preserves limits if and only if

(i) $\sup_{m,n} | \sum_{k=1}^{m} A_{nk}| < \infty$,

(ii) $\lim_{n\to\infty} A_{nk} = 0$ for all $k \in \mathbb{N}$ and

(iii) $\lim_{n\to\infty} \sum_{k=1}^{\infty} A_{nk} = 1$.

Proof: This follows from Theorem 4.3.6, Lemma 4.3.9 and the fact that $bv = bv_o + \langle e \rangle$. #

In Section 4.2, we have characterized a convergence preserving matrix transformation $A : bs \longrightarrow \ell_\infty$ which A is in the class of K_r-matrices or when $\sum_{k=1}^{\infty} |A_{nk}-A_{n(k+1)}|$ converges uniformly on $n = 1,2,3,\ldots$ In the next two theorems we characterize such a matrix A when it also preserves limits.

Theorem 4.3.11. Let A be a K_r matrix. Then the following statements hold:

(i) $A : bs \longrightarrow \ell_\infty$ and

(ii) A preserves limits if and only if $\lim_{n\to\infty} A_{nk} = 0$ for every $k \in \mathbb{N}$.

Proof: By Lemma 4.2.14, we have (i).

To prove (ii), assume that A preserves limits. Since $e^{(k)} \in bs$ for every $k \in \mathbb{N}$, by Proposition 2.3 (ii), $\lim_{n\to\infty} A_{nk} = 0$ for every $k \in \mathbb{N}$.

For the converse, assume that $\lim_{n\to\infty} A_{nk} = 0$ for all $k \in \mathbb{N}$. By Theorem 4.2.15 (ii), $A(bs \cap c) \subseteq c$. To show A preserves limits, let $x = (x_k) \in bs \cap c$. By Lemma 4.2.13, $bs \cap c \subseteq c_o$. Then $\lim_{k\to\infty} x_k = 0$. Let $\|.\|$ be the restriction of $\|.\|_{c_o}$ to $bs \cap c$. Since c_o has the AK property, $(bs \cap c, \|.\|)$ has the AK property. Then $x = \lim_{n\to\infty} \sum_{k=1}^{\infty} x_k e^{(k)}$ in $(bs \cap c, \|.\|)$. By Lemma 4.2.14, $A : bs \cap c \longrightarrow \ell_\infty$ is continuous on

$(bs \cap c, \|.\|)$. Then $Ax = \lim_{n\to\infty} \sum_{k=1}^{n} x_k Ae^{(k)}$ in c. Since the linear functional f on c definded by $f((y_n)) = \lim_{n\to\infty} y_n$ is continuous on c, it follows that $f(Ax) = \lim_{n\to\infty} \sum_{k=1}^{n} x_k f(Ae^{(k)})$. But $f(Ae^{(k)}) = \lim_{n\to\infty} (Ae^{(k)})_n = \lim_{n\to\infty} A_{nk} = 0$ for every $k \in \mathbb{N}$, so $f(Ax) = 0$. Since $f(Ax) = \lim_{n\to\infty} (Ax)_n$, we then have that $\lim_{n\to\infty} (Ax)_n = 0 = \lim_{n\to\infty} x_n$. #

Theorem 4.3.12. Let A be an infinite matrix such that $\sum_{k=1}^{\infty} |A_{nk}-A_{n(k+1)}|$ converges uniformly on n = 1,2,3,.... Then $A: bs \longrightarrow \ell_\infty$ and A preserves limits if and only if

(i) $\lim_{k\to\infty} A_{nk} = 0$ for every $n \in \mathbb{N}$ and

(ii) $\lim_{n\to\infty} A_{nk} = 0$ for every $k \in \mathbb{N}$.

Proof: If $A : bs \longrightarrow \ell_\infty$ and A preserves limits, then (i) and (ii) hold respectively by Lemma 4.2.12 and Proposition 2.3 (ii).

For the converse, assume that (i) and (ii) hold. By Theorem 4.2.17, $A : bs \longrightarrow \ell_\infty$ and $A(bs \cap c) \subseteq c$. To show A preserves limits, let $x = (x_k) \in bs \cap c$. Since $bs \cap c \subseteq c_o$, $\lim_{k\to\infty} x_k = 0$. To show $\lim_{n\to\infty} (Ax)_n = 0$, let $\epsilon > 0$ be given. Since $(x_k) \in bs$ there exists $K > 0$ such that $|\sum_{k=1}^{n} x_k| \le K$ for all $n \in \mathbb{N}$. Since $\sum_{k=1}^{\infty} |A_{nk}-A_{n(k+1)}|$ converges uniformly on n = 1, 2, 3, ..., there exists $k_o \in \mathbb{N}$ such that $\sum_{k=k_o}^{\infty} |A_{nk}-A_{n(k+1)}| < \frac{\epsilon}{2K}$ for all $n \in \mathbb{N}$. By (ii), $\lim_{n\to\infty} \sum_{k=1}^{k_o-1} |A_{nk}-A_{n(k+1)}| = 0$, so there exists $n_o \in \mathbb{N}$ such that $\sum_{k=1}^{k_o-1} |A_{nk}-A_{n(k+1)}| < \frac{\epsilon}{2K}$ for all $n \ge n_o$. We have that $(A_{nk})_{k=1}^{\infty} \in bv_o$ for all $n \in \mathbb{N}$ since $\sum_{k=1}^{\infty} |A_{nk}-A_{n(k+1)}|$ converges for all $n \in \mathbb{N}$ and $\lim_{k\to\infty} A_{nk} = 0$ for all $n \in \mathbb{N}$. By Lemma 4.2.16, $\sum_{k=1}^{\infty} A_{nk}x_k = \sum_{k=1}^{\infty} (\sum_{i=1}^{k} x_i)(A_{nk} - A_{n(k+1)})$ for all $n \in \mathbb{N}$. Then for $n \ge n_o$,

$$|\sum_{k=1}^{\infty} A_{nk}x_k| = |\sum_{k=1}^{\infty}(\sum_{i=1}^{k} x_i)(A_{nk}-A_{n(k+1)})|$$
$$\leq K(\sum_{k=1}^{k_o-1}|A_{nk}-A_{n(k+1)}| + \sum_{k=k_o}^{\infty}|A_{nk}-A_{n(k+1)}|)$$
$$< K(\frac{\epsilon}{2K}+\frac{\epsilon}{2K}) = \epsilon$$

Hence $\lim_{n\to\infty}(Ax)_n = 0 = \lim_{n\to\infty} x_n$, as required. #

4.4 SUMMABILITY PRESERVING MATRIX TRANSFORMATIONS

All of the classical sequence spaces which contain cs are ℓ_∞, c, c_o, bs and cs. Then for an infinite matrix A and for X = ℓ_∞, c, c_o, cs or bs , $A : X \longrightarrow \ell_\infty$ and A preserves summability if and only if $A : X \longrightarrow \ell_\infty$ and $A : cs \longrightarrow cs$. By Theorem 1.12, if X = ℓ_∞, c or c_o, $A : X \longrightarrow \ell_\infty$ if and only if $\sup_n \sum_{k=1}^{\infty}|A_{nk}| < \infty$. By Lemma 4.2.12, $A : bs \longrightarrow \ell_\infty$ if and only if $\lim_{k\to\infty} A_{nk} = 0$ for every $n \in \mathbb{N}$ and $\sup_n \sum_{k=1}^{\infty}|A_{nk}-A_{nk(k+1)}| < \infty$. Also by Theorem 1.14, $A : cs \longrightarrow cs$ if and only if $\sup_n \sum_{k=1}^{\infty}|\sum_{i=1}^{n}A_{ik}-\sum_{i=1}^{n}A_{i(k+1)}| < \infty$ and $\sum_{n=1}^{\infty}A_{nk}$ converges for all $k \in \mathbb{N}$. The condition that $\sup_n \sum_{k=1}^{\infty}|\sum_{i=1}^{n}A_{ik} - \sum_{i=1}^{n}A_{i(k+1)}| < \infty$ implies that $\sup_n \sum_{k=1}^{\infty}|A_{nk}-A_{n(k+1)}| < \infty$ since for every $m \in \mathbb{N}$,

$$\sum_{k=1}^{\infty}|A_{mk}-A_{m(k+1)}| = \sum_{k=1}^{\infty}|(\sum_{i=1}^{m}A_{ik} - \sum_{i=1}^{m}A_{i(k+1)}) - (\sum_{i=1}^{m-1}A_{ik} - \sum_{i=1}^{m-1}A_{i(k+1)})|$$
$$\leq \sum_{k=1}^{\infty}|\sum_{i=1}^{m}A_{ik} - \sum_{i=1}^{m}A_{i(k+1)}| + \sum_{k=1}^{\infty}|\sum_{i=1}^{m-1}A_{ik} - \sum_{i=1}^{m-1}A_{i(k+1)}|$$
$$\leq 2\sup_n \sum_{k=1}^{\infty}|\sum_{i=1}^{n}A_{ik} - \sum_{i=1}^{n}A_{i(k+1)}|.$$

Hence, from these facts, the following three theorems are obtained.

<u>Theorem 4.4.1</u>. If X is ℓ_∞, c or c_o, then for an infinite matrix A, $A:X \longrightarrow \ell_\infty$ and A preserves summability if and only if

(i) $\sup_n \sum_{k=1}^{\infty} |A_{nk}| < \infty$,

(ii) $\sup_n \sum_{k=1}^{\infty} | \sum_{i=1}^{n} A_{ik} - \sum_{i=1}^{n} A_{i(k+1)}| < \infty$ and

(iii) $\sum_{n=1}^{\infty} A_{nk}$ converges for all $k \in \mathbb{N}$.

Theorem 4.4.2. For an infinite matrix A, $A: cs \longrightarrow \ell_\infty$ and A preserves summability if and only if

(i) $\sup_n \sum_{k=1}^{\infty} | \sum_{i=1}^{n} A_{ik} - \sum_{i=1}^{n} A_{i(k+1)}| < \infty$ and

(ii) $\sum_{n=1}^{\infty} A_{nk}$ converges for all $k \in \mathbb{N}$.

Theorem 4.4.3. For an infinite matrix A, $A : bs \longrightarrow \ell_\infty$ and A preserves summability if and only if

(i) $\lim_{k\to\infty} A_{nk} = 0$ for all $n \in \mathbb{N}$,

(ii) $\sup_n \sum_{k=1}^{\infty} | \sum_{i=1}^{n} A_{ik} - \sum_{i=1}^{n} A_{i(k+1)}| < \infty$ and

(iii) $\sum_{n=1}^{\infty} A_{nk}$ converges for all $k \in \mathbb{N}$.

Since $\ell \subseteq cs$, we have that for an infinite matrix A, $A : \ell \longrightarrow \ell_\infty$ and A preserves summability if and only if $A : \ell \longrightarrow cs$. A characterization of $A : \ell \longrightarrow cs$ has been given in Theorem 1.15 as follows: $A:\ell \longrightarrow cs$ if and only if $\sup_{n,k} | \sum_{i=1}^{n} A_{ik}| < \infty$ and $\sum_{n=1}^{\infty} A_{nk}$ converges for every $k \in \mathbb{N}$. Hence we have the following theorem.

Theorem 4.4.4. For an infinite matrix A, $A:\ell \longrightarrow \ell_\infty$ and A preserves summability if and only if

(i) $\sup_{n,k} | \sum_{i=1}^{n} A_{ik}| < \infty$ and

(ii) $\sum_{n=1}^{\infty} A_{nk}$ converges for all $k \in \mathbb{N}$.

Next, we shall characterize summability preserving matrix transformations $A : X \longrightarrow \ell_\infty$ where $X = \ell_p(1<p<\infty)$ or bv_o. The spaces $\ell_p(1<p<\infty)$ and bv_o are BK-spaces having the AK property. We first generally characterize summability presserving matrix transformations $A : X \longrightarrow \ell_\infty$ where X is any BK-space with AK property. The following two lemmas are required for this characterization.

Lemma 4.4.5. Let X be a BK-space having the AK property and A an infinite matrix. Then $A : X \longrightarrow cs$ if and only if

(i) $(\sum_{i=1}^{n} A_{ik})_{k=1}^{\infty} \in X^{\beta}$ for every $n \in \mathbb{N}$ and $\sup_n \|(\sum_{i=1}^{n} A_{ik})_{k=1}^{\infty}\|_{X^\beta} < \infty$ and

(ii) $\sum_{n=1}^{\infty} A_{nk}$ converges for every $k \in \mathbb{N}$.

Proof: Let B be the infinite matrix defined by

$$B_{nk} = \begin{cases} 1 & \text{if } 1 \le k \le n, \\ 0 & \text{if } k > n. \end{cases}$$

Then $(BA)_{nk} = \sum_{i=1}^{n} A_{ik}$ for all $n, k \in \mathbb{N}$, so $\left\{ ((BA)_{nk})_{k=1}^{\infty} \mid n \in \mathbb{N} \right\} = \left\{ (\sum_{i=1}^{n} A_{ik})_{k=1}^{\infty} \mid n \in \mathbb{N} \right\}$. By the definition of bs, we have that $B : bs \longrightarrow \ell_\infty$.

Assume that $A : X \longrightarrow cs$. Then (ii) holds by Proposition 2.3(v). Since $cs \subseteq bs$, we have that for every $x \in X$, $B(Ax)$ exists and for $n \in \mathbb{N}$, $(B(Ax))_n = \sum_{k=1}^{\infty} B_{nk}(Ax)_k = \sum_{k=1}^{n} (Ax)_k = \sum_{k=1}^{n} (\sum_{i=1}^{\infty} A_{ki}x_i) = \sum_{i=1}^{\infty} (\sum_{k=1}^{n} A_{ki}\, x_i)$ $= \sum_{i=1}^{\infty} (BA)_{ni}x_i$, we have that $(BA)x = B(Ax)$ for every $x \in X$. Hence

$BA : X \longrightarrow \ell_\infty$. By Lemma 4.2.1, $((BA)_{nk})_{k=1}^{\infty} \in X^\beta$ for every $n \in \mathbb{N}$ and $\sup_n \|((BA)_{nk})_{k=1}^{\infty}\|_{X^\beta} < \infty$. Hence (i) holds.

Conversely, assume that (i) and (ii) hold. By (i), we have that $((BA)_{nk})_{k=1}^{\infty} \in X^\beta$ for every $n \in \mathbb{N}$ and $\sup_n \|((BA)_{nk})_{k=1}^{\infty}\|_{X^\beta} < \infty$. By Lemma 4.2.1, $BA:X \longrightarrow \ell_\infty$. To show that $A : X \longrightarrow cs$, let $x = (x_k) \in X$. Then $(BA)x \in \ell_\infty$. But $(BA)x = (\sum_{k=1}^{\infty} (BA)_{nk}x_k)_{k=1}^{\infty} = (\sum_{k=1}^{\infty} (\sum_{i=1}^{n} A_{ik}x_k))_{n=1}^{\infty}$ $= (\sum_{i=1}^{n} (\sum_{k=1}^{\infty} A_{ik}x_k))_{n=1}^{\infty}$, so we have that $Ax = (\sum_{k=1}^{\infty} A_{nk}x_k)_{n=1}^{\infty} \in bs$. This shows that $A : X \longrightarrow bs$. By Theorem 1.9, $A : X \longrightarrow bs$ is continuous. Since X has the AK property, $x = \lim_{n\to\infty} \sum_{k=1}^{n} x_k e^{(k)}$ in X. Then $Ax = \lim_{n\to\infty} \sum_{k=1}^{n} x_k Ae^{(k)}$ in bs. Since $Ae^{(k)} = (A_{nk})_{n=1}^{\infty}$ for all $k \in \mathbb{N}$, we have from (ii) that $Ae^{(k)} \in cs$ for all $k \in \mathbb{N}$. Since cs is a closed subspace of bs, it follows that $Ax \in cs$. Hence $A : X \longrightarrow cs$, as required. #

<u>Lemma 4.4.6</u>. Let X be a BK-space having the AK property .Then for an infinite matrix A , $A:X \longrightarrow \ell_\infty$ and A preserves summability if and only if

(i) $(A_{nk})_{k=1}^{\infty} \in X^\beta$ for every $n \in \mathbb{N}$ and $\sup_n \|(A_{nk})_{k=1}^{\infty}\|_{X^\beta} < \infty$,

(ii) $(\sum_{i=1}^{n} A_{ik})_{k=1}^{\infty} \in (X \cap cs)^\beta$ for every $n \in \mathbb{N}$ and $\sup_n \|(\sum_{i=1}^{n} A_{ik})_{k=1}^{\infty}\|_{(X\cap cs)^\beta} < \infty$ where the norm of $X \cap cs$ is $\max\{\|.\|_X , \|.\|_{cs}\}$ and

(iii) $\sum_{n=1}^{\infty} A_{nk}$ converges for every $k \in \mathbb{N}$.

<u>Proof</u>: Since cs is a BK-space with AK property, by Proposition 2.2, $X \cap cs$ with norm $\max\{\|.\|_X , \|.\|_{cs}\}$ is a BK-space with AK property. By Lemma 4.2.1, $A : X \longrightarrow \ell_\infty$ if and only if (i) holds. Also,by Lemma 4.4.5, $A : X \cap cs \longrightarrow cs$ if and only if (ii) and (iii) hold. Hence

$A : X \longrightarrow \ell_\infty$ and $A(X \cap cs) \subseteq cs$ if and only if (i),(ii) and (iii) hold. Therefore the lemma is proved. #

Theorem 4.4.7. For an infinite matrix A, $A: bv_o \longrightarrow \ell_\infty$ and A preserves summability if and only if

(i) $\sup_{m,n} \left| \sum_{k=1}^{m} A_{nk} \right| < \infty$,

(ii) $\left(\sum_{i=1}^{n} A_{ik} \right)_{k=1}^{\infty} \in (bv_o \cap cs)^\beta$ for every $n \in \mathbb{N}$ and $\sup_n \left\| \left(\sum_{i=1}^{n} A_{ik} \right)_{k=1}^{\infty} \right\|_{(bv_o \cap cs)^\beta} < \infty$ where the norm of $bv_o \cap cs$ is $\max\{ \|.\|_{bv_o} , \|.\|_{cs} \}$ and

(iii) $\sum_{n=1}^{\infty} A_{nk}$ converges for every $k \in \mathbb{N}$.

Proof: By Corollary 4.1.2(iv), we have that $\sup_{m,n} \left| \sum_{k=1}^{m} A_{nk} \right| < \infty$ if and only if $(A_{nk})_{n=1}^{\infty} \in bv_o^\beta$ for every $n \in \mathbb{N}$ and $\sup_n \|(A_{nk})_{k=1}^{\infty}\|_{bv_o^\beta} < \infty$. Hence by Lemma 4.4.6, the theorem holds. #

Theorem 4.4.8. For an infinite matrix A, $A: \ell_p \longrightarrow \ell_\infty$ and A preserves summability if and only if

(i) $\sup_n \sum_{k=1}^{\infty} |A_{nk}|^q < \infty$ where $\frac{1}{p} + \frac{1}{q} = 1$,

(ii) $\left(\sum_{i=1}^{n} A_{ik} \right)_{k=1}^{\infty} \in (\ell_p \cap cs)^\beta$ for every $n \in \mathbb{N}$ and $\sup_n \left\| \left(\sum_{i=1}^{n} A_{ik} \right)_{k=1}^{\infty} \right\|_{(\ell_p \cap cs)^\beta} < \infty$ where the norm of $\ell_p \cap cs$ is $\max\{ \|.\|_{\ell_p} , \|.\|_{cs} \}$ and

(iii) $\sum_{n=1}^{\infty} A_{nk}$ converges for every $k \in \mathbb{N}$.

Proof: It follows by Corollary 4.1.2(ii) that $\sup_n \sum_{k=1}^{\infty} |A_{nk}|^q < \infty$ where $\frac{1}{p} + \frac{1}{q} = 1$ if and only if $(A_{nk})_{k=1}^{\infty} \in \ell_p^\beta$ for every $n \in \mathbb{N}$ and $\sup_n \|(A_{nk})_{k=1}^{\infty}\|_{\ell_p^\beta} < \infty$. Then by Lemma 4.4.6, the theorem holds. #

The next theorem gives a characterization of summability preserving matrix transformations $A : bv \longrightarrow \ell_\infty$. To obtain this, we use the characterization of such A where $A : bv_o \longrightarrow \ell_\infty$ and the fact that $bv = bv_o + \langle e \rangle$.

<u>Theorem 4.4.9</u>. **For an infinite matrix A, $A : bv \longrightarrow \ell_\infty$ and A preserves summability if and only if**

(i) $\sup_{m,n} \left| \sum_{k=1}^{m} A_{nk} \right| < \infty$,

(ii) $\left(\sum_{i=1}^{n} A_{ik} \right)_{k=1}^{\infty} \in (bv_o \cap cs)^{\beta}$ for every $n \in \mathbb{N}$ and $\sup_n \left\| \left(\sum_{i=1}^{n} A_{ik} \right)_{k=1}^{\infty} \right\|_{(bv_o \cap cs)^{\beta}} < \infty$ where the norm of $bv_o \cap cs$ is $\max\{ \|.\|_{bv_o} , \|.\|_{cs} \}$,

(iii) $\sum_{n=1}^{\infty} A_{nk}$ converges for every $k \in \mathbb{N}$ and

(iv) $\sum_{k=1}^{\infty} A_{nk}$ converges for every $n \in \mathbb{N}$.

<u>Proof</u>: Assume that $A:bv \longrightarrow \ell_\infty$ and $A(bv \cap cs) \subseteq cs$. Then $A:bv_o \longrightarrow \ell_\infty$ and $A(bv_o \cap cs) \subseteq cs$. By Theorem 4.4.7, (i), (ii) and (iii) hold. Since $e \in bv$, Ae exists. But $Ae = \left(\sum_{k=1}^{\infty} A_{nk} \right)_{n=1}^{\infty}$, so (iv) holds.

Conversely, assume that (i)-(iv) hold. The conditions (i), (ii) and (iii) imply by Theorem 4.4.7 that $A:bv_o \longrightarrow \ell_\infty$ and $A(bv_o \cap cs) \subseteq cs$. From (iv), we have that Ae exists. But $Ae = \left(\sum_{k=1}^{\infty} A_{nk} \right)_{n=1}^{\infty}$ and $\infty > \sup_{m,n} \left| \sum_{k=1}^{m} A_{nk} \right| = \sup_n \left(\sup_m \left| \sum_{k=1}^{m} A_{nk} \right| \right) \geq \sup_n \left| \sum_{k=1}^{\infty} A_{nk} \right|$, so $Ae \in \ell_\infty$. Hence $A : bv \longrightarrow \ell_\infty$ since $bv = bv_o + \langle e \rangle$. Since $cs \subseteq c_o$ and $bv \cap c_o = bv_o$, we have that $bv \cap cs = bv_o \cap cs$, so $A(bv \cap cs) \subseteq cs$. Hence the theorem is proved. #

4.5 SUM PRESERVING MATRIX TRANSFORMATIONS

By Theorem 1.17, we have that for a sequence space X containing cs, $A : X \longrightarrow \ell_\infty$ and A preserves sums if and only if $A : X \longrightarrow \ell_\infty$, $\sup_n \sum_{k=1}^{\infty} | \sum_{i=1}^{n} A_{ik} - \sum_{i=1}^{n} A_{i(k+1)}| < \infty$ and $\sum_{n=1}^{\infty} A_{nk} = 1$ for every $k \in \mathbb{N}$. Hence the following theorems are obtained respectively from Theorem 4.4.1 - Theorem 4.4.3.

<u>Theorem 4.5.1</u>. If X is ℓ_∞, c or c_o, then for an infinite matrix A , $A : X \longrightarrow \ell_\infty$ and A preserves sums if and only if

(i) $\sup_n \sum_{k=1}^{\infty} |A_{nk}| < \infty$,

(ii) $\sup_n \sum_{k=1}^{\infty} | \sum_{i=1}^{n} A_{ik} - \sum_{i=1}^{n} A_{i(k+1)}| < \infty$ and

(iii) $\sum_{n=1}^{\infty} A_{nk} = 1$ for all $k \in \mathbb{N}$.

<u>Theorem 4.5.2</u>. For an infinite matrix A, $A : cs \longrightarrow \ell_\infty$ and A preserves sums if and only if

(i) $\sup_n \sum_{k=1}^{\infty} | \sum_{i=1}^{n} A_{ik} - \sum_{i=1}^{n} A_{i(k+1)} | < \infty$ and

(ii) $\sum_{n=1}^{\infty} A_{nk} = 1$ for all $k \in \mathbb{N}$.

<u>Theorem 4.5.3</u>. For an infinite matrix A, $A : bs \longrightarrow \ell_\infty$ and A preserves sums if and only if

(i) $\lim_{k\to\infty} A_{nk} = 0$ for all $n \in \mathbb{N}$,

(ii) $\sup_n \sum_{k=1}^{\infty} | \sum_{i=1}^{n} A_{ik} - \sum_{i=1}^{n} A_{i(k+1)} | < \infty$ and

(iii) $\sum_{n=1}^{\infty} A_{nk} = 1$ for all $k \in \mathbb{N}$.

$\lim_{n\to\infty} \sum_{k=1}^{n} x_k\ f(Ae^{(k)})$. Since $f(Ae^{(k)}) = f(\ (A_{nk})_{n=1}^{\infty}) = \sum_{n=1}^{\infty} A_{nk} = 1$ for all $k \in \mathbb{N}$, we have that $\sum_{n=1}^{\infty} (Ax)_n = \sum_{k=1}^{\infty} x_k$. Hence A preserves sums, as required.

Theorem 4.5.5. For an infinite matrix A, $A : \ell \longrightarrow \ell_\infty$ and A preserves sums if and only if

(i) $\sup_{n,k} |\sum_{i=1}^{n} A_{ik}| < \infty$ and

(ii) $\sum_{n=1}^{\infty} A_{nk} = 1$ for all $k \in \mathbb{N}$.

Proof: Since $\ell \subseteq cs$, $\ell \cap cs = \ell$. Let $\|.\| = \max\{\|.\|_{\ell}\ ,\ \|.\|_{cs}\}$ on ℓ. By Proposition 2.2 $(\ell, \|.\|)$ is a BK-space. By Proposition 2.1, $\|.\| \sim \|.\|_{\ell}$. This implies that $\|.\|_{(\ell,\|.\|)^{\beta}} \sim \|.\|_{(\ell,\|.\|_{\ell})^{\beta}}$ on ℓ^{β}

Assume that $A : \ell \longrightarrow \ell_\infty$ and A preserves sums. By Lemma 4.5.4, (ii) holds and $(\sum_{i=1}^{n} A_{ik})_{k=1}^{\infty} \in \ell^{\beta}$ for every $n \in \mathbb{N}$ and $\sup_n \|(\sum_{i=1}^{n} A_{ik})_{k=1}^{\infty}\|_{(\ell,\|.\|)^{\beta}} < \infty$. But $\|.\|_{(\ell,\|.\|)^{\beta}} \sim \|.\|_{(\ell,\|.\|_{\ell})^{\beta}}$, so $\sup_n \|(\sum_{i=1}^{n} A_{ik})_{k=1}^{\infty}\|_{(\ell,\|.\|_{\ell})^{\beta}} < \infty$. By Corollary 4.1.2 (iii), we have that $\sup_{n,k} |\sum_{i=1}^{n} A_{ik}| < \infty$. Hence (i) holds.

Conversely, assume that (i) and (ii) hold. By Corollary 4.1.2 (iii), $(\sum_{i=1}^{n} A_{ik})_{k=1}^{\infty} \in \ell^{\beta}$ for every $n \in \mathbb{N}$ and $\sup_n \|(\sum_{i=1}^{n} A_{ik})_{k=1}^{\infty}\|_{\ell^{\beta}} < \infty$. Then $\sup_n \|(\sum_{i=1}^{n} A_{ik})_{k=1}^{\infty}\|_{(\ell,\|.\|)^{\beta}} < \infty$. Since for $n,k \in \mathbb{N}$, $A_{nk} = \sum_{i=1}^{n} A_{ik} - \sum_{i=1}^{n-1} A_{ik}$, $\sup_{n,k} |A_{nk}| \le 2 \sup_{n,k} |\sum_{i=1}^{n} A_{ik}| < \infty$. By Corollary 4.1.2(iii), $(A_{nk})_{k=1}^{\infty} \in \ell^{\beta}$ for every $n \in \mathbb{N}$ and $\sup_n \|(A_{nk})_{k=1}^{\infty}\|_{\ell^{\beta}} < \infty$. Hence by Lemma 4.5.4, $A : \ell \longrightarrow \ell_\infty$ and A preserves sums. #

Theorem 4.5.6. For an infinite matrix A, $A : bv_o \longrightarrow \ell_\infty$ and A preserves sums if and only if

(i) $\sup_{m,n} | \sum_{k=1}^{m} A_{nk} | < \infty$

(ii) $(\sum_{i=1}^{n} A_{ik})_{k=1}^{\infty} \in (bv_o \cap cs)^{\beta}$ for every $n \in \mathbb{N}$ and

$\sup_{n} \| (\sum_{i=1}^{n} A_{ik})_{k=1}^{\infty} \|_{(bv_o \cap cs)^{\beta}} < \infty$ where the norm of $bv_o \cap cs$ is

$\max \{ \|.\|_{bv_o} , \|.\|_{cs} \}$ and

(iii) $\sum_{n=1}^{\infty} A_{nk} = 1$ for every $k \in \mathbb{N}$.

Proof: It follows directly from Lemma 4.5.4 and Corollary 4.1.2(iv). #

Theorem 4.5.7. For an infinite matrix A, $A : \ell_p \longrightarrow \ell_{\infty}$ and A preserves sums if and only if

(i) $\sup_{n} \sum_{k=1}^{\infty} |A_{nk}|^{q} < \infty$ where $\frac{1}{p} + \frac{1}{q} = 1$,

(ii) $(\sum_{i=1}^{n} A_{ik})_{k=1}^{\infty} \in (\ell_p \cap cs)^{\beta}$ for every $n \in \mathbb{N}$ and

$\sup_{n} \| (\sum_{i=1}^{n} A_{ik})_{k=1}^{\infty} \|_{(\ell_p \cap cs)^{\beta}} < \infty$ where the norm of $\ell_p \cap cs$ is

$\max\{ \|.\|_{\ell_p} , \|.\|_{cs} \}$ and

(iii) $\sum_{n=1}^{\infty} A_{nk} = 1$ for every $k \in \mathbb{N}$.

Proof: It follows directly from Lemma 4.5.4 and Proposition 4.1.2(ii). #

Theorem 4.5.8. For an infinite matrix A, $A : bv \longrightarrow \ell_{\infty}$ and A preserves sums if and only if

(i) $\sup_{m,n} | \sum_{k=1}^{m} A_{nk} | < \infty$

(ii) $(\sum_{i=1}^{n} A_{ik})_{k=1}^{\infty} \in (bv_o \cap cs)^{\beta}$ for every $n \in \mathbb{N}$ and

$\sup_{n} \| (\sum_{i=1}^{n} A_{ik})_{k=1}^{\infty} \|_{(bv_o \cap cs)^{\beta}} < \infty$ where the norm of $bv_o \cap cs$ is

$\max\{ \|.\|_{bv_o} , \|.\|_{cs} \}$,

(iii) $\sum_{n=1}^{\infty} A_{nk} = 1$ for every $k \in \mathbb{N}$ and

(iv) $\sum_{k=1}^{\infty} A_{nk}$ converges for every $n \in \mathbb{N}$.

<u>Proof</u>: If $A : bv \longrightarrow \ell_\infty$ and A preserves sums, then (i) - (iv) hold by Theorem 4.4.9 and Proposition 2.3 (vi).

For the converse assume that (i) - (iv) hold. Then $A : bv \longrightarrow \ell_\infty$ and $A(bv \cap cs) \subseteq cs$ by Theorem 4.4.9. By Theorem 4.5.6, A preserves sums on bv_o. But $bv_o \cap cs = bv \cap cs$, so A preserves sums on bv. #