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CHAPTER I

PRELIMINARIES

In this chapter we shall give some notations, definitions and theorems used in this thesis. Our notations are

$\mathbb{Z}$ is the set of all integers,

$\mathbb{Z}^+$ is the set of all positive integers,

$\mathbb{Q}$ is the set of all rational numbers,

$\mathbb{Q}^+$ is the set of all positive rational numbers,

$\mathbb{R}$ is the set of all real numbers,

$\mathbb{R}^+$ is the set of all positive real numbers,

$\mathbb{Z}_n$, $n \in \mathbb{Z}^+$, is the set of congruence classes modulo n in $\mathbb{Z}$,

$\mathbb{Z}_0^+ = \mathbb{Z}^+ \cup \{0\}$,

$\mathbb{Q}_0^+ = \mathbb{Q}^+ \cup \{0\}$,

$\mathbb{R}_0^+ = \mathbb{R}^+ \cup \{0\}$,

$[x]$, $x \in \mathbb{R}$, is the largest integer such that $\leq x$.

Note that for $x \in \mathbb{R}$ we can write $x = [x] + r$, where $0 \leq r < 1$.

Definition 1.1 A triple $(S,+,\cdot)$ is said to be a semiring iff S is a set and + (addition) and $\cdot$ (multiplication) are binary operations on S such that,

(a) $(S,+)$ and $(S,\cdot)$ are commutative semigroups,

(b) for all x, y, z $\in$ S, $(x+y)\cdot z = x\cdot z + y\cdot z$.

<u>Example 1.2</u> $\mathbb{Z}$, $\mathbb{Z}^+$, $\mathbb{Z}_0^+$ and $\mathbb{Q}^+$ with the usual addition and multiplication are semirings.

<u>Example 1.3</u> $\mathbb{Z}$, $\mathbb{Z}^+$, $\mathbb{Z}_0^+$ and $\mathbb{Q}^+$ with the usual multiplication and + defined by $x+y = \max\{x,y\}$ for all elements x,y in these sets are semirings. Also, we can define $x+y = \min\{x,y\}$ for all elements x,y and still obtain semirings.

<u>Definition 1.4</u> A semiring $(K,+,\cdot)$ is said to be a <u>ratio semiring</u> iff $(K,\cdot)$ is a group. (In P.Sinutoke's thesis [4], a ratio semiring is called a positive rational domain, P.R.D.).

<u>Example 1.5</u> $\mathbb{Q}^+$ and $\mathbb{Q}_0^+$ with the usual addition and multiplication are ratio semirings.

<u>Example 1.6</u> $\mathbb{Q}^+$ with the usual multiplication and + defined by $x+y = \min\{x,y\}$ $\forall$ $x,y \in \mathbb{Q}^+$ is a ratio semiring. Also we can define $x+y = \max\{x,y\}$ $\forall$ $x,y \in \mathbb{Q}^+$ and still obtain a ratio semiring.

<u>Example 1.7</u> Let $D = \{1\}$ with $1\cdot 1 = 1$ and $1+1 = 1$. Then D is a ratio semiring.

<u>Definition 1.8</u> Let $(S,+,\cdot)$ be a semiring and $T \subseteq S$, then T is said to be a <u>subsemiring</u> of S iff $(T,+,\cdot)$ is a semiring. And T is said to be a <u>ratio subsemiring</u> iff $(T,+,\cdot)$ is a ratio semiring.

<u>Definition 1.9</u> Let S be a semiring. Then $x \in S$ is said to be <u>additively cancellative</u> (A.C.) iff for all $y, z \in S$ $(x+y = x+z \rightarrow y = z)$ And S is said to be <u>additively cancellative</u> (A.C.) iff for all x,y, $z \in S$ $(x+y = x+z \rightarrow y = z)$.

<u>Definition 1.10</u> Let S be a semiring. Then $x \in S$ is said to be

multiplicatively cancellative (M.C.) iff for all y, z ε S (xy = xz imply y = z). And S is said to be multiplicative cancellative (M.C.) iff for all x,y,z ε S (xy = xz imply y = z)

Example 1.11 $\mathbb{Z}^+$ and $\mathbb{Q}^+$ with the usual addition and multiplication are additively cancellative and multiplicatively cancellative.

Definition 1.12 Let S be a semigroup. S is said to be a band iff $x^2 = x$ for all x ε S.

Note that ($\mathbb{Z}$,+), where + defined as in example 1.3 is a band.

Theorem 1.13. There is no finite ratio semiring of order > 1.

See [4], page 5-11.

Proposition 1.14 If D is an infinite ratio semiring then D cannot contain any additive identity.

See [4], page 12.

Proposition 1.15 If D is an infinite ratio semiring then D cannot contain any additive zero.

See [4], page 12.

Theorem 1.16 If S is a semiring then S can be embeded into a ratio semiring iff S is multiplicatively cancellative.

See [4], page 12-14.

Assume that S is multiplicative cancellative. Define a relation $\sim$ on S×S by $(x,y)\sim(x',y')$ iff $xy' = x'y$ $\forall$ $x,y,x',y' \in S$. In theorem 1.16 we obtain that $\sim$ is an equivalence relation.

Let $\alpha, \beta \in \frac{S\times S}{\sim}$. Define + and $\cdot$ on $\frac{S\times S}{\sim}$ in the following way:

Choose $(a,b) \in \alpha$ and $(c,d) \in \beta$. Define $\alpha+\beta = [(ad+bc,bd)]$ and $\alpha \cdot \beta = [(ac,bd)]$. Theorem 1.16 has shown that $(\frac{S \times S}{\sim},+,\cdot)$ is a ratio semiring and S can be embedded into $\frac{S \times S}{\sim}$.

<u>Theorem 1.17</u> If S is a semiring with multiplicative cancellation then $\frac{S \times S}{\sim}$ is the smallest ratio semiring containing S up to isomorphism.

See [4], page 14-15.

<u>Theorem 1.18</u> If D is an infinite ratio semiring, then the smallest ratio subsemiring of D (called the prime ratio semiring of D) is either isomorphic to $\mathbb{Q}^+$ with the usual addition and multiplication or $\{1\}$

See [4], page 15-17

<u>Proposition 1.19</u> Every finite cancellative semigroup is a group.

See [2], page 8.

<u>Theorem 1.20</u> If K is a semifield then either 0 is the additive identity or 0 is the additive zero.

See [4], page 21.

Theorem 1.20 indicates that there are two types of semifields We call a semifield with 0 as its additive identity a <u>semifield of zero type</u> (0-semifield) and a semifield with 0 as its additive zero a <u>semifield of infinity type</u> (∞-semifield).

<u>Remark 1.21</u> Let $K = \{0,1\}$. Define $\cdot$ on K by

$\cdot$	0	1
0	0	0
1	0	1

There are four possible commutative binary operations on K such that K is a semifield :

+	0	1
0	0	1
1	1	0

(1)

+	0	1
0	0	0
1	0	0

(2)

+	0	1
0	0	1
1	1	1

(3)

+	0	1
0	0	0
1	0	1

(4)

Note that Table 1 makes {0,1} into a field, Table 2 makes {0,1} into trivial semifield and Table 4 makes {0,1} into almost trivial semifield.

Theorem 1.22 If K is a semifield of zero type, then the prime semifield of K is either isomorphic to $\mathbb{Q}_0^+$ with the usual addition and multiplication or $\mathbb{Z}_p$ where p is a prime number or the semifield in Table 3.

See [4], page 30-33.

Remark 1.23 Since $\mathbb{Q}^+$ with the usual addition and multiplication, is a ratio semiring, we have $\mathbb{Q}_0^+$ by extending + and $\cdot$ by $x+0 = 0+x = 0$ and $x \cdot 0 = 0 \cdot x = 0 \ \forall \ x \in \mathbb{Q}_0^+$, is a semifield having 0 as its additive zero.

Theorem 1.24 If K is a semifield of infinity type, then the prime semifield of K is either isomorphic to $\mathbb{Q}_0^+$ as in remark 1.23 or the trivial semifield of order 2 or the almost trivial semifield of order 2.

See [4], page 33-35.

Proposition 1.25 Let S be a semiring. Define $B = \{x \in S \mid x \text{ is A.C.}\}$ and $M = \{x \in S \mid x \text{ is M.C.}\}$. Then

1) $B = \Phi$ or B is an additive subsemigroup of S,

2) $M = \Phi$ or M is a multiplicative subsemigroup of S.

Proof 1) Assume that $B \neq \Phi$. Let $x, y \in B$ and $z_1, z_2 \in S$ be such that $(x+y)+z_1 = (x+y)+z_2$. Then $x+(y+z_1) = x+(y+z_2)$. Thus $y+z_1 = y+z_2$ because $x \in B$. Since $y \in B$, so $z_1 = z_2$. Hence $x+y \in B$. Thus B is an additive subsemigroup of S. Therefore $B = \Phi$ or B is an additive subsemigroup of S.

2) Assume that $M \neq \Phi$. Let $x,y \in M$ and $z_1, z_2 \in S$ be such that $(xy)z_1 = (xy)z_2$. Then $x(yz_1) = x(yz_2)$. Thus $yz_1 = yz_2$ because $x \in M$. Since $y \in M$, so $z_1 = z_2$. Thus $xy \in M$. Hence M is a multiplicative subsemigroup of S. Therefore $M = \Phi$ or M is a multiplicative subsemigroup of S. #

Definition 1.26 Let S be a semiring and $d \in S$. Then $x \in S$ is said to be an additive identity of d in S iff $x+d = d$. The set of all additive identity of d in S denoted by $I_S(d)$.

Proposition 1.27 Let S be a semiring and $d \in S$. Then

(1) $I_S(d) = \Phi$ or $I_S(d)$ is a additive subsemigroup of S.

(2) If S is a ratio semiring, then

(2.1) $I_S(1) = \Phi$ or $I_S(1)$ is a subsemiring of S

(2.2) $I_S(d) = I_S(1)\cdot d$ and $I_S(1)\cdot I_S(d) \subseteq I_S(d)$

Proof (1) Assume $I_S(d) \neq \Phi$. Let $x,y \in I_S(d)$. $(x+y)+d = x+(y+d) = x+d = d$ since $x,y \in I_S(d)$. Thus $(x+y)+d = d$. Hence $x+y \in I_S(d)$. Therefore $I_S(d)$ is an additive subsemigroup of S.

(2) Assume that S is a ratio semiring.

(2.1) Suppose that $I_S(1) \neq \Phi$. Then by (1) we get that $I_S(1)$ is an additive subsemigroup of S. To show $I_S(1)$ is a subsemiring of S we only show that $I_S(1)$ is a multiplicative subsemigroup of S. Let $x,y \in I_S(1)$. $xy+1 = xy+(y+1) = (xy+y)+1 = (xy+1y)+1 =$

$(x+1)y+1 = 1y+1 = y+1 = 1$ since $x, y \in I_S(1)$. Thus $xy+1 = 1$. Hence $xy \in I_S(1)$. Thus $I_S(1)$ is a multiplicative subsemigroup of S. Therefore $I_S(1)$ is a subsemiring of S. So we get that $I_S(1) = \Phi$ or $I_S(1)$ is a subsemiring of S.

(2.2) We want to show that $I_S(1) = I_S(1)\cdot d$.

Case 1 $I_S(d) = \Phi$. Claim that $I_S(1) = \Phi$. To prove this suppose not, then $\exists\, x \in I_S(1)$. Thus $x+1 = 1$. Hence $xd+d = d$. Thus $xd \in I_S(d)$, a contradiction. Therefore $I_S(1) = \Phi$. So we have the claim. Thus $I_S(1)\cdot d = \Phi$ Therefore $I_S(d) = I_S(1)\cdot d$.

Case 2 $I_S(d) \neq \Phi$. Let $x \in I_S(d)$, then $x+d = d$. Since $(S,\cdot)$ is a group, $xd^{-1}+1 = 1$. Thus $xd^{-1} \in I_S(1)$. Since $x = (xd^{-1})d$, $x \in I_S(1)\cdot d$. Therefore $I_S(d) \subseteq I_S(1)\cdot d$. Conversely, Let $y \in I_S(1)\cdot d$, then $\exists\, z \in I_S(1)$ such that $y = zd$. Then $y+d = zd+d = (z+1)d = 1d = d$. Thus $y \in I_S(d)$. Therefore $I_S(1)\cdot d \subseteq I_S(d)$

Now we shall show that $I_S(1)\cdot I_S(d) \subseteq I_S(d)$. If $I_S(1) = \Phi$, then $I_S(1)\cdot I_S(d) = \Phi \subseteq I_S(d)$. Suppose that $I_S(1) \neq \Phi$. By case 2.1, we have that $I_S(1)$ is a subsemiring of S, so $I_S(1)\cdot I_S(1) \subseteq I_S(1)$. Then $I_S(1)\cdot I_S(d) = I_S(1)\cdot(I_S(1)\cdot d) = (I_S(1)\cdot I_S(1))\cdot d \subseteq I_S(1)\cdot d = I_S(d)$. Thus $I_S(1)\cdot I_S(d) \subseteq I_S(d)$.   #