CHAPTER II

GENERALIZED SEMIFIELDS

We shall now generalize the concept of semifield by giving a new definition which contains P.Sinutoke's definition as a special case.

<u>Definition 2.1</u> A semiring $(K,+,\cdot)$ is said to be a <u>generalized semifield</u> iff there exists an element a in K such that $(K\setminus\{a\},\cdot)$ is a group.

<u>Example 2.2</u>

(1) $\mathbb{Q}_0^+$ and $\mathbb{R}_0^+$ with the usual addition and multiplication are generalized semifields.

(2) Let D be a ratio semiring. Let a be a symbol not representing any element of D. We can extend + and $\cdot$ to $D^* = D \cup \{a\}$ by $a\cdot x = x\cdot a = a$ and $a+x = x+a = a$ for all $x \in D^*$. Then $(D^*,+,\cdot)$ is a generalized semifield.

(3) Let $K = \{a,e\}$. Define $\cdot$ and + on K by

$\cdot$	a	e
a	e	e
e	e	e

and

+	a	e
a	e	a
e	a	e

Then K is a generalized semifield.

<u>Example 2.3</u> Let $(G,\cdot)$ be a commutative group with zero element ∞. We can define + on G so that $(G,+,\cdot)$ is a generalized semifield by

(1) $x+y = \infty$ for all $x, y \in G$,

(2) $x+y = \infty$ if $x \neq y$ and $x+y = x$ if $x = y$ for all $x, y \in G$.

<u>Example 2.4</u> Let D be a ratio semiring and a a symbol not representing any element of D. Extend + and $\cdot$ from D to $D \cup \{a\}$ by

(1) $ax = xa = a$ for all $x \in D \cup \{a\}$,

(2) $a+x = x+a = 1+x$ for all $x \in D$,

(3) $a+a = 1+1$.

It is easy to show that $D \cup \{a\}$ is a generalized semifield.

<u>Example 2.5</u> Let D be a ratio semiring, a a symbol not representing any element of D and d D. Extend + and $\cdot$ from D to $D \cup \{a\}$ by

(1) $ax = xa = dx$ $\forall x \in D$ and $a^2 = d^2$,

(2) $a+x = x+a = d+x$ $\forall x \in D$,

(3) $a+a = d+d$.

It is easy to show that $D \cup \{a\}$ is a generalized semifield.

From now on the word "semifield" will mean a generalized semifield.

<u>Theorem 2.6</u> Let $(K,+,\cdot)$ be a semifield and a an element in K such that $(K \setminus \{a\},\cdot)$ is a group. Then $ax = a$ for all $x \in K$ or $ax = x$ for all $x \in K$ or $a^2 \neq a$ and $ae \neq a$ where e is the identity of $(K \setminus \{a\},\cdot)$.

<u>Proof</u>. Let e be the identity of $(K \setminus \{a\},\cdot)$. Consider ae and a^2.

<u>Case 1</u> $ae = a$ and $a^2 = a$. Claim that $ax = a$ for all $x \in K$. Let $x \in K \setminus \{a\}$. Suppose that $ax \neq a$. Since $(K \setminus \{a\},\cdot)$ is a group,

$\exists y \in K\setminus\{a\}$ such that $xy = e$, so $a = ae = a(xy) = (ax)y$. Thus $(ax)y = a$, a contradiction since $ax, y \in K\setminus\{a\}$ which is a group. Hence $ax = a$ for all $x \in K$. So we have the claim.

Case 2 $ae = a$ and $a^2 \neq a$. Claim that $ax = a$ for all $x \in K\setminus\{a\}$. Let $x \in K\setminus\{a\}$. Suppose that $ax \neq a$. Then $\exists y \in K\setminus\{a\}$ such that $xy = e$. Then $a = ae = a(xy) = (ax)y$. Thus $(ax)y = a$, a contradiction since $ax, y \in K\setminus\{a\}$ which is a group. Hence $ax = a$ for all $x \in K\setminus\{a\}$. So we have the claim. Since $a^2 \neq a$, $\exists z \in K\setminus\{a\}$ such that $a^2z = e$. Then $e = a^2z = a(az) = aa = a^2$ by the claim. Thus $a^2 = e$. Hence $(K,\cdot)$ is a group. Therefore K is a ratio semiring. Suppose that $|K| > 2$. Let $x \in K\setminus\{a,e\}$. By the claim we get $ax = a$. Then $x = ex = a^2x = a(ax) = aa = a^2 = e$, so $x = e$ which is a contradiction. Thus $|K| = 2$. So we have that K is a ratio semiring of order 2 which contradicts the Theorem 1.13. Therefore this case cannot occur.

Case 3 $ae \neq a$ and $a^2 = a$. Since $(ae)(ae) = a(e(ae)) = a(ae) = (aa)e = ae$, we get that $(ae)(ae) = ae$, so $ae = e$. Let $x \in K\setminus\{a\}$, $ax = a(ex) = (ae)x = ex = x$, so $ax = x$. Thus $ax = x$ for all $x \in K$.

Case 4 $ae \neq a$ and $a^2 \neq a$. So we have the theorem. #

From Theorem 2.6 we see that there are three types of semifields :

(1) Semifields with $ax = a$ for all $x \in K$ (called type I semifields w.r.t. a),

(2) Semifields with $ax = x$ for all $x \in K$ (called type II semifields w.r.t. a),

(3) Semifield with $a^2 \neq a$ and $ae \neq a$ where e is the identity of $(K\setminus\{a\},\cdot)$ (called type III semifield w.r.t. a).

Note that Example 2.2(1),(2) and Example 2.3 are semifields of type I, Example 2.4 is a semifield of type II, Example 2.2(3) and Example 2.5 are semifields of type III.

We see that a semifield of type I is a semifield in P.Sinutoke's definition and it has been already studied in [4].

Proposition 2.7 Let K be a semifield and $a \in K$ be such that $(K \setminus \{a\},\cdot)$ is a group. Then K is a semifield of type III w.r.t. a iff there exists a unique d in $K \setminus \{a\}$ such that $ax = dx$ for all $x \in K$.

Proof Assume that K is a semifield of type III. Let e be the identity of $(K \setminus \{a\},\cdot)$. Then $ae \neq a$ and $a^2 \neq a$. Let $d = ae$, so $d \in K \setminus \{a\}$. Let $x \in K \setminus \{a\}$ then $ax = a(ex) = (ae)x = dx$. Thus $ax = dx$ for all $x \in K \setminus \{a\}$. Since $a^2 \neq a$, $a^2 = a^2 e = a(ae) = ad$, so $aa = ad$. So we get that $ax = dx$ for all $x \in K$. To show the uniqueness of d, let $d' \in K \setminus \{a\}$ be such that $ax = d'x$ for all $x \in K$. Then $d' = d'e = ae = de = d$. Thus $d' = d$.

Conversely assume that $\exists !$ d in $K \setminus \{a\}$ such that $ax = dx$ for all $x \in K$. Then $ae = de = d \neq a$ and $a^2 = aa = da = dd = d^2 \neq a$ since $d \in K \setminus \{a\}$, which is a group. Thus $ae \neq a$ and $a^2 \neq a$. Therefore K is a semifield of type III. #

Remark 2.8 Let $K = \{a,e\}$. Define $\cdot$ and $+$ on K as the following tables.

(1)

$\cdot$	a	e
a	a	a
e	a	e

and

+	a	e
a	a	a
e	a	a

(2)

$\cdot$	a	e
a	a	e
e	e	e

and

+	a	e
a	a	e
e	e	e

Clearly (1) is a semifield of type I w.r.t. a and (2), a semifield of type II w.r.t. a. In these cases there does not exist a unique b in K such that $(K\setminus\{b\},\cdot)$ is a group. However if K is a semifield of type I or type II and $|K| > 2$ then we get uniqueness as the following Theorem shows.

<u>Theorem 2.9</u> Let $(K,+,\cdot)$ be a semifield of type I or type II w.r.t.a of order > 2. If there is an element b in K such that $(K\setminus\{b\},\cdot)$ is a group then b = a.

<u>Proof</u>. Let e denote the identity of $(K\setminus\{a\},\cdot)$. Suppose that $b \neq a$. Since K is a semifield of type I or type II, $a^2 = a$. Thus $a^2 = a \in K\setminus\{b\}$, so a is the identity of $(K\setminus\{b\},\cdot)$. Let $x \in K\setminus\{b,a\}$. Then $\exists\, y \in K\setminus\{b\}$ such that $xy = a$. If $y = a$ then $a = xy = xa = x$, a contradiction. Hence $y \neq a$, so we have that $x \neq a$, $y \neq a$ but $xy = a$ which contradicts the fact that $(K\setminus\{a\},\cdot)$ is a group. Thus b = a. #

<u>Theorem 2.10</u> Let K be a semifield of type III w.r.t. a. If there exists an element b in K such that $(K\setminus\{b\},\cdot)$ is a group then b = a.

<u>Proof</u>. Let e be the identity of $(K\setminus\{a\},\cdot)$ and f be the identity of $(K\setminus\{b\},\cdot)$. Suppose that $b \neq a$. Then $a \in K\setminus\{b\}$, so $af = a$. Since K is a type III semifield, $a^2 \neq a$. Thus e is the only idempotent of K. Since $f^2 = f$, $f = e$. Thus $a = af = ae$, a contradiction. Therefore b = a. #

Let K be a semifield of type I w.r.t. a then by Theorem 1.20, we know that there are two types of semifields of this type. If a is additive identity, then we call K a <u>semifield of zero type</u> and if a is an additive zero, then we call K a <u>semifield of infinity type</u>.

The following theorems will prove the most basic properties of a semifield of type I.

<u>Proposition 2.11</u> Let K be a semifield of type I w.r.t. a and let x, $y \in K$. Then $xy = a$ iff $x = a$ or $y = a$.

<u>Proof</u> Let x, $y \in K$ be such that $xy = a$. Suppose that $x \neq a$, We must show that $y = a$. If $y \neq a$, then we have that $x \neq a$, $y \neq a$ but $xy = a$ which is a contradiction because $(K \setminus \{a\}, \cdot)$ is a group. Hence $y = a$.

Conversely if $x = a$ or $y = a$, then obviously $xy = a$. #

<u>Definition 2.12</u> Let S be a semiring with multiplicative zero a. Then S is said to be <u>zero multiplicatively cancellative</u> (0-M.C.) iff for all $x,y,z \in S$, $xy = xz$ and $x \neq a$ imply $y = z$.

<u>Proposition 2.13</u> If S is a finite 0-M.C. semiring, then S must be a semifield of type I.

<u>Proof</u> Let a be the multiplicative zero of S. Since S is 0-M.C., $(S \setminus \{a\}, \cdot)$ is a finite cancellative semigroup. By Theorem 1.19, $(S \setminus \{a\}, \cdot)$ is a group. So we get $a \in S$ and $(S \setminus \{a\}, \cdot)$ is a group. Therefore S is a semifield and clearly S is a semifield of type I. #

<u>Definition 2.14</u> Let S be a semifield with ∞. Let $y \in S$. Then $z \in S$ is said to be a <u>complement</u> of y iff $y+z = \infty$.

<u>Definition 2.15</u> Let S be a semifield with ∞. Let $y \in S$. Then y is said to be <u>limited</u> iff the only complement of y is ∞. If every non-infinity element of S is limited then S is called <u>limited</u>.

Definition 2.16 Let S be a semifield with ∞. Then S is said to be infinity additively cancellative (∞-A.C.) iff for all $x,y,z \in S$, $x+y = x+z$ and $x \neq \infty$ imply $y = z$.

Proposition 2.17 Let K be a semifield of infinity type. If K is ∞-A.C., then K is limited.

Proof Let $x \in K \setminus \{\infty\}$ and y be a complement of x. Thus $x+y = \infty$. Then $x+y = x+\infty$, so $y = \infty$ since K is ∞-A.C. Hence x is limited. Thus K is limited. #

Corollary 2.18 Let K be a semifield of infinity type. If K is ∞-A.C., then $(K \setminus \{\infty\},+,\cdot)$ is a ratio semiring.

Proof By Proposition 2.17, K is limited. Thus $x+y \neq \infty \; \forall x,y \in K\setminus\{\infty\}$. Hence $(K \setminus \{\infty\},+)$ is a semigroup. Since $(K \setminus \{\infty\},\cdot)$ is a group, so $(K \setminus \{\infty\},+,\cdot)$ is a ratio semiring. #

Definition 2.19 Let K be a semifield of infinity type and let $x \in K$. The core of x, denoted by $Cor(x) = \{y \in K \mid x+y = \infty\}$.

Theorem 2.20 Let K be a semifield of infinity type and e be the identity of $(K \setminus \{\infty\},\cdot)$. Then

(1) $\infty \in Cor(x)$ for all $x \in K$.

(2) For all $x \in K$, $Cor(x)$ is an additive ideal of K.

(3) For all $x, y \in K \setminus \{\infty\}$, $y \in Cor(x)$ iff $yx^{-1} \in Cor(e)$.

(4) For all $x, y \in K$, $x \in Cor(y)$ iff $y \in Cor(x)$.

(5) For all $x \in K \setminus \{\infty\}$, $Cor(x) = Cor(e) \cdot x$.

(6) For all $x \in K$, $x \in Cor(y)$ implies $xz \in Cor(yz)$ for all $z \in K$. The converse is true if $z \neq \infty$.

(7) For all $x,y,z \in K$, $x \in Cor(y+z)$ iff $x+y \in Cor(z)$.

<u>Proof</u> (1) Since $x+\infty = \infty$ for all $x \in K$, $\infty \in Cor(x)$ for all $x \in K$.

(2) Let $x \in K$. Let $y \in Cor(x)$ and $z \in K$. We must show that $y+z \in Cor(x)$. Since $y \in Cor(x)$, $x+y = \infty$. Then $x+(y+z) = (x+y)+z = \infty+z = \infty$. Thus $y+z \in Cor(x)$. Hence $Cor(x)$ is an additive ideal of K.

(3) Let $x,y \in K \setminus \{\infty\}$. Assume $y \in Cor(x)$. Thus $x+y = \infty$. Then $e+yx^{-1} = xx^{-1} + yx^{-1} = (x+y)x^{-1} = \infty x^{-1} = \infty$. Thus $yx^{-1} \in Cor(e)$.

Conversely, assume that $yx^{-1} \in Cor(e)$. Thus $e+yx^{-1} = \infty$. Then $x+y = ex+ey = ex+y(x^{-1}x) = ex+(yx^{-1})x = (e+yx^{-1})x = \infty x = \infty$. Thus $y \in Cor(x)$.

(4) Obvious.

(5) Let $x \in K$. To show $Cor(x) \subseteq Cor(e)\cdot x$, let $y \in Cor(x)$. By (3), $yx^{-1} \in Cor(e)$. Thus $y = (yx^{-1})x \in Cor(e)\cdot x$. Conversely, let $z \in Cor(e)$. Thus $e+z = \infty$. Then $x+zx = (e+z)x = \infty x = \infty$, so $zx \in Cor(x)$. Hence $Cor(e)\cdot x \subseteq Cor(x)$.

(6) Let $x \in K$. Let $y \in K$ be such that $x \in Cor(y)$ and $z \in K$. Thus $y+x = \infty$, so $yz+xz = (y+x)z = \infty z = \infty$. Thus $xz \in Cor(yz)$.

Conversely, assume that $x,y \in K, z \in K \setminus \{\infty\}$ and $xz \in Cor(yz)$. Thus $yz+xz = \infty$. So $y+x = (y+x)zz^{-1} = (yz+xz)z^{-1} = \infty z^{-1} = \infty$. Thus $x \in Cor(y)$.

(7) Let $x,y,z \in K$. $x \in Cor(y+z) \Leftrightarrow (y+z)+x = \infty \Leftrightarrow (z+y)+x = \infty \Leftrightarrow z+(y+x) = \infty \Leftrightarrow z+(x+y) = \infty \Leftrightarrow x+y \in Cor(z)$. #

<u>Theorem 2.21</u> Let K be a semifield of infinity type and let $x, y \in K \setminus \{\infty\}$. Then the cardinality of $Cor(x)$ equals the cardinality of $Cor(y)$ and each one is a multiplicative translate of the other.

<u>Proof</u> Let e be the identity of $(K \setminus \{a\}, \cdot)$

For $z \in Cor(x)$, by Theorem 2.20(5), there is a $u \in Cor(e)$ such that $z = ux$. Define $f: Cor(x) \rightarrow Cor(y)$ by $f(z) = uy$. By Theorem 2.20 (5), $uy \in Cor(y)$. To show f is well-defined, let $z_1 = z_2 \in Cor(x)$. Let $u_1, u_2 \in Cor(e)$ be such that $z_1 = u_1 x$ and $z_2 = u_2 x$. Thus $u_1 x = u_2 x$. Since $x \neq \infty$ so $u_1 = u_2$. Thus $u_1 y = u_2 y$. To show f is one-to-one, let $z_1, z_2 \in Cor(x)$ be such that $f(z_1) = f(z_2)$. Let $u_1, u_2 \in Cor(e)$ be such that $z_1 = u_1 x$ and $z_2 = u_2 x$. Thus $u_1 y = u_2 y$. Since $y \neq \infty$, $u_1 = u_2$. Thus $z_1 = z_2$. To show f is onto, Let $w \in Cor(y)$. Let $v \in Cor(e)$ be such that $w = v \cdot y$. Then $v \cdot x \in Cor(x)$. Thus $f(vx) = vy = w$. Therefore f is one-to-one and onto. To show $Cor(x)$ is a multiplication translate of $Cor(y)$, let $z \in Cor(x)$. By Theorem 2.20(3), $zx^{-1} \in Cor(e)$. By Theorem 2.20(5), $zx^{-1}y \in Cor(y)$. Thus $z = (zx^{-1}y)y^{-1}x \in Cor(y) \cdot y^{-1}x$. Hence $Cor(x) \subseteq Cor(y) \cdot y^{-1}x$. Now let $w \in Cor(y)$. By Theorem 2.20(3), $wy^{-1} \in Cor(e)$. By Theorem 2.20(5), $wy^{-1}x \in Cor(x)$. Thus $Cor(y) \cdot y^{-1}x \subseteq Cor(x)$. Therefore $Cor(x) = Cor(y) \cdot y^{-1}x$. #

<u>Proposition 2.22</u> Let K be a semifield of zero type and let $B = \{x \in K \mid x \text{ is A.C.}\}$. Then B is an ideal of $(K, \cdot)$ and B is an additive subsemigroup of K.

<u>Proof</u> Since $0 \in B$, $B \neq \Phi$. To show B is an ideal of $(K, \cdot)$, let $x \in K$ and $z \in B$. Let $z_1, z_2 \in K$ be such that $zx + z_1 = zx + z_2$. If $x = 0$, then $z_1 = z_2$. Assume that $x \neq 0$. Thus $z + z_1 x^{-1} = (zx + z_1)x^{-1} = (zx + z_2)x^{-1} = z + z_2 x^{-1}$. Since $z \in B$, $z_1 x^{-1} = z_2 x^{-1}$. Thus $z_1 = z_2$, so $zx \in B$. Thus $BK \subseteq B$. Therefore B is an ideal of $(K, \cdot)$. By proposition 1.25, B is an additive subsemigroup of K. #

<u>Proposition 2.23</u> Let K be a semifield of infinity type and let $B = \{x \in K \mid x \text{ is A.C.}\}$. If $x \in B$ and $y \in K \setminus \{\infty\}$, then $xy \in B$.

Proof Let $x \in B$ and $y \in K \setminus \{\infty\}$. Let $z_1, z_2 \in K$ be such that $xy+z_1 = xy+z_2$. Then $x+z_1y^{-1} = (xy+z_1)y^{-1} = (xy+z_2)y^{-1} = x+z_2y^{-1}$. Since $x \in B$, $z_1y^{-1} = z_2y^{-1}$. Thus $z_1 = z_2$, so $xy \in B$. #

Proposition 2.24 Let K be a semifield of type I. Then if one nonzero element of K is additively cancellative, then all nonzero elements are additively cancellative.

Proof Let a be a zero of K and e the identity of $(K \setminus \{a\}, \cdot)$. Let $x \in K \setminus \{a\}$ be additively cancellative. Let y be an element in $K \setminus \{a\}$ and $z_1, z_2 \in K$ be such that $y+z_1 = y+z_2$. Then $e+z_1y^{-1} = (y+z_1)y^{-1} = (y+z_2)y^{-1} = e+z_2y^{-1}$, so $x+z_1y^{-1}x = x+z_2y^{-1}x$. By assumption, $z_1y^{-1}x = z_2y^{-1}x$. Since $y^{-1}x \neq a$, $z_1 \cdot e = z_2 \cdot e$.

If $z_1 = a$, then $a = a \cdot e = z_1 \cdot e = z_2 \cdot e$. Thus $z_2 \cdot e = a$. By Proposition 2.11, $z_2 = a$. Hence $z_1 = z_2$.

If $z_1 \neq a$, then $z_1 = z_1 e = z_2 \cdot e$. Thus $z_2 \cdot e \neq a$. By Proposition 2.11, $z_2 \neq a$. Then $z_1 = z_1 \cdot e = z_2 \cdot e = z_2$. Thus $z_1 = z_2$. Therefore y is additively cancellative. #

Proposition 2.25 Let K be a semifield of zero type. If $x \in K \setminus \{0\}$ is such that x has an additive inverse, then every element in K as an additive inverse and K is a field.

Proof See [4], page 22. #

Proposition 2.26 Let K be a semifield of zero type which is not a field. Then $(K \setminus \{0\}, +, \cdot)$ is a ratio semiring.

Proof Since $(K \setminus \{0\}, \cdot)$ is a group, so we only show that $(K \setminus \{0\}, +)$ is a semigroup. Let $x, y \in K \setminus \{0\}$. Must show that $x+y \neq 0$. Suppose $x+y = 0$. Thus x is a nonzero element which has an additive inverse.

By Proposition 2.25, K is a field, a contradiction. Hence $x+y \neq 0$. Therefore $(K \setminus \{0\},+,\cdot)$ is a ratio semiring. #

Note that $\mathbb{Q}^+ \cup \{0\}$ with the usual addition and multiplication is a semifield of zero type which is not a field and we see that $\mathbb{Q}^+$ is a ratio semiring.

From now we shall study semifields of types II and III.

Prpposition 2.27 Let K be a semifield of type II w.r.t. a and let $x,y \in K$. Then $xy = a$ iff $x = a$ and $y = a$.

Proof Suppose that $xy = a$. If $y = a$, then $a = xy = xa = x$ since K is a semifield of type II w.r.t. a. Thus $a = x = y$.

Conversely if $x = a$ and $y = a$, then obviously $xy = a$. #

Proposition 2.28 Let K be a semifield of type III w.r.t.a. Then $xy \neq a$ for all $x, y \in K$.

Proof Let e be the identity of $(K \setminus \{a\},\cdot)$. We want to show that $xy \neq a$ for all $x, y \in K$. Let $x, y \in K$. Since K is a type III semifield, $a^2 \neq a$ and $ae \neq a$. If $x \neq a$ and $y = a$ then $xy = xa = (xe)a = x(ea) \neq a$ since $x, ea \in K\setminus\{a\}$, which is a group. Hence $xy \neq a$ for all $x, y \in K$. #

Theorem 2.29 Let K be a semifield of type II w.r.t.a. Then $(K \setminus \{a\},+,\cdot)$ is a ratio semiring.

Proof Let e be the identity of $(K \setminus \{a\},\cdot)$. Let $x, y \in K\setminus\{a\}$. Then $xy \in K\setminus\{a\}$. We must show that $x+y \in K\setminus\{a\}$. Suppose not. Then $x+y = a$. So $a = x+y = xe+ye = (x+y)e = ae = e$, a contradiction. Thus $x+y \in K\setminus\{a\}$. Hence $(K\setminus\{a\},+,\cdot)$ is a ratio semiring. #

<u>Theorem 2.30</u> Let K be a semifield of type II w.r.t.a. and e the identity of $(K \setminus \{a\},\cdot)$. Then the following hold :

(1) If a+a = a then (K,+) is a band.

(2) If a+a $\neq$ a then a+a = e+e and for all x, y ε $K \setminus \{a\}$ x+x = y+y iff x = y.

(3) a+x = a or a+x = e+x for all x $\neq$ a.

<u>Proof</u> (1) Suppose that a+a = a. Then x+x = ax+ax = (a+a)x = ax = x for all x ε K. Thus (K,+) is a band.

(2) Suppose that a+a $\neq$ a. Then a+a = (a+a)e = ae+ae = e+e. Thus a+a = e+e. Let x, y ε K $\{a\}$ be such that x+x = y+y. Then (a+a (a+a)x = ax+ax = x+x = y+y = ay+ay = (a+a)y. Thus (a+a)x = (a+a)y. Since a+a $\neq$ a, x = y.

(3) Let x ε $K \setminus \{a\}$. If a+x $\neq$ a, then a+x = (a+x)e = ae+xe = e+x. Thus a+x = e+x. #

<u>Theorem 2.31</u> Let K be a semifield of type II w.r.t.a and e the identity $(K \setminus \{a\},\cdot)$. Define $D = K \setminus \{a\}$ and $S = \{x \in D \mid a+x = a\}$. Then

(1) S = Φ or S is an additive subsemigroup of $I_D(e)$.

(2) If e ε S then S = $I_D(e)$.

(3) $D \setminus S = \Phi$ or $D \setminus S$ is an ideal of (D,+).

<u>Proof</u> (1) Suppose that S $\neq$ Φ. Let x ε S, then a+x = a. Then e = ae = (a+x)e = ae+xe = e+x. Thus x ε $I_D(e)$, so $S \subseteq I_D(e)$. Let x, y ε S. Then a+(x+y) = (a+x)+y = a+y = a, so x+y ε S. Hence S is an additive subsemigroup of $I_D(e)$.

(2) Suppose that $e \in S$. Let $x \in I_D(e)$. Then $e+x = e$. Hence $a = a+e = a+(e+x) = (a+e)+x = a+x$. Thus $a+x = a$, so $x \in S$. Therefore $I_D(e) \subseteq S$. From (1), $S \subseteq I_D(e)$. Hence $S = I_D(e)$.

(3) Suppose that $D \setminus S \neq \Phi$. Let $x \in D \setminus S$, $y \in D$. We want to show that $x+y \in D \setminus S$. By Theorem 2.29, $(K \setminus \{a\},+,\cdot)$ is a ratio semiring, so $(a+x)+y \in K \setminus \{a\}$. Since $a+(x+y) = (a+x)+y$, $a+(x+y) \neq a$. Thus $x+y \in D \setminus S$. #

Proposition 2.32 Let K be a semifield of type II w.r.t.a. If $a+x = a$ for all $x \neq a$ then $|K| = 2$.

Proof Suppose that $a+x = a$ for all $x \neq a$. Let $D = K \setminus \{a\}$ and $S = \{x \in D \mid a+x = a\}$. Then $S = K \setminus \{a\}$. By Theorem 2.31, $S \subseteq I_D(e)$ where e is the identity of $(K \setminus \{a\},\cdot)$. Thus $I_D(e) = K \setminus \{a\}$, so e is an additive zero of $K \setminus \{a\}$ which is a ratio semiring. By Proposition 1.15 and Theorem 1.13, $|K \setminus \{a\}| = 1$. Therefore $|K| = 2$. #

Proposition 2.33 Let K be a finite semifield of type II. Then $|K| = 2$.

Proof By Theorem 2.29, $K \setminus \{a\}$ is a ratio semiring. Since K is finite so is $K \setminus \{a\}$. By Theorem 1.13, $|K \setminus \{a\}| = 1$. Thus $|K| = 2$. #

Proposition 2.34 Let K be a semifield of type II w.r.t. a, $D = K \setminus \{a\}$ and $S = \{x \in D \mid a+x = a\}$. If $y \in D \setminus S$ then y is not A.C..

Proof Let e be the identity of $(K \setminus \{a\},\cdot)$. Suppose that $y \in D \setminus S$. By Theorem 2.30(3), $a+y = e+y$. Since $a \neq e$, y is not A.C.. #

Corollary 2.35 Let K be an infinite semifield of type II w.r.t.a. Then K contains an element $y \in K \setminus \{a\}$ such that y is not A.C..

Proof Let $D = K \setminus \{a\}$ and $S = \{x \in D \mid a+x = a\}$. By Proposition 2.32 we have that $D \setminus S \neq \Phi$. Let $y \in D \setminus S$, then by Proposition 2.34, y is not A.C.. #

Proposition 2.36 Let K be a semifield of type II w.r.t.a and e the identity of $(K \setminus \{a\},\cdot)$. Then K is A.C. iff $K = \{a,e\}$ with $a^2 = a$, $a \cdot e = e \cdot a = e$, $e^2 = e$ and + is defined by

+	a	e
a	e	a
e	a	e

Proof Assume that K is A.C. By Corollary 2.35, $|K| = 2$. Hence $K = \{a,e\}$ with $a^2 = a$, $a \cdot e = e \cdot a = e$ and $e^2 = e$. By Theorem 2.29, $K \setminus \{a\}$ is a ratio semiring, so $e+e = e$. By Proposition 2.34, $a+e = a$. By Theorem 2.30(2), $a+a = a$ or $a+a = e+e$. Thus $a+a = a$ or $a+a = e$. If $a+a = a$, then $a+a = a+e$ but $a \neq e$ which is a contradiction. Thus $a+a = e$. Therefore $K = \{a,e\}$ with the above structure.

Conversely, assume that $K = \{a,e\}$ with the above structure. It is easy to show that K is A.C. since $a+a \neq a+e$ and $e+a \neq e+e$. #

Theorem 2.37 Let K be an infinite semifield of type II. Then K contains no additive zero.

Proof Let $a \in K$ be such that $(K \setminus \{a\},\cdot)$ is a group.

Suppose that z is an additive zero of K. By Theorem 2.29, $K \setminus \{a\}$ is a ratio semiring. Thus $K \setminus \{a\}$ is an infinite ratio semiring. By Proposition 1.15, $z = a$. Then by Proposition 2.32, $|K| = 2$, a contradiction. Hence K contains no additive zero. #

Theorem 2.38 Let K be an infinite semifield of type II. Then K contains no additive identity.

Proof Let $a \in K$ be such that $(K \setminus \{a\}, \cdot)$ is a group.

Suppose that z is an additive identity of K.. By Theorem 2.29, $K \setminus \{a\}$ is a ratio semiring. Thus $K \setminus \{a\}$ is an infinite ratio semiring. By Proposition 1.14, $z = a$. Then $a+x = x$ for all $x \in K$. Let e be the identity of $(K \setminus \{a\}, \cdot)$. By Theorem 2.30(3), $a+x = a$ or $a+x = e+x$ for all $x \neq a$. Thus $x = a+x = e+x$ for all $x \neq a$, so e is an additive identity of $K \setminus \{a\}$ which contradicts Proposition 1.14. Hence K contains no additive identity. #

Theorem 2.39 Let D be a ratio semiring, a a symbol not representing any element in D and let $S \subseteq I_D(1)$ have the property that either $S = \Phi$ or S is an additive subsemigroup of $I_D(1)$ such that $D \setminus S$ is an ideal of $(D,+)$ if D is infinite. Then we can extend the binary operations of D to $K = D \cup \{a\}$ making K into a semifield of type II such that

(1) $ax = xa = x$ for all $x \in K$,

(2) $a+x = x+a = a$ for all $x \in S$ and

$a+x = x+a = 1+x$ for all $x \in D \setminus S$,

(3) $$a+a = \begin{cases} a \text{ or } 1 & \text{if } 1+1 = 1, \\ 1+1 & \text{if } 1+1 \neq 1. \end{cases}$$

Proof Suppose that $S = \Phi$. Extend $+$ and $\cdot$ from D to $K = D \cup \{a\}$ as in (1), (2) and (3) above. Then $ax = xa = x$ for all $x \in K$, $a+x = 1+x$ for all $x \in D$ and

$$a+a = \begin{cases} a \text{ or } 1 & \text{if } 1+1 = 1, \\ 1+1 & \text{if } 1+1 \neq 1. \end{cases}$$

To show K is a semifield, we must show that (a) $(xy)z = x(yz)$ for all $x, y \in K$, (b) $(x+y)+z = x+(y+z)$ for all $x,y, z \in K$ and (c) $(x+y)z = xz+yz$ for all $x, y \in K$. To show (a), we will consider the following cases :

<u>Case 1</u> $z = a$.

$(xy)z = (xy)a = xy = x(ya) = x(yz)$.

<u>Case 2</u> $z \neq a$, $x = a$.

$(xy)z = (ay)z = yz = a(yz) = x(yz)$.

<u>Case 3</u> $z \neq a$, $y = a$.

$(xy)z = (xa)z = xz = x(az) = x(yz)$.

<u>Case 4</u> $z \neq a$, $x \neq a$, $y \neq a$. Then $x,y,z \in D$, so $(xy)z = x(yz)$.

To show (b), we will consider the following cases :

<u>Case 1</u> $x = y = z = a$.

$(x+y)+z = (a+a)+a = a+(a+a) = x+(y+z)$ since $u+v = v+u$ for all $u, v \in K$.

<u>Case 2</u> $x = y = a$, $z \neq a$.

<u>Subcase 2.1</u> $a+a = a$. Then $1+1 = 1$.

$(x+y)+z = (a+a)+z = a+z = 1+z$, $x+(y+z) = a+(a+z) = a+(1+z) = 1+(1+z)$

$= (1+1)+z = 1+z$.

<u>Subcase 2.2</u> $a+a = 1$. Then $1+1 = 1$.

$(x+y)+z = (a+a)+z = 1+z$, $x+(y+z) = a+(a+z) = a+(1+z) = 1+(1+z)$

$= (1+1)+z = 1+z$.

<u>Subcase 2.3</u> $a+a = 1+1$.

$(x+y)+z = (a+a)+z = (1+1)+z$, $x+(y+z) = a+(a+z) = a+(1+z) = 1+(1+z)$

$= (1+1)+z$ since $1, z \in D$.

<u>Case 3</u> $x = z = a$, $y \neq a$.

$(x+y)+z = (a+y)+a = (1+y)+a = (1+y)+1$, $x+(y+z) = a+(y+a) = a+(y+1)$

$= 1+(y+1) = (1+y)+1$ since $1, y \in D$.

<u>Case 4</u> $y = z = a,\ x \neq a$.

$(x+y)+z = (x+a)+a = (a+x)+a = a+(x+a) = a+(a+x) = (a+a)+x = x+(a+a)$

$= x+(y+z)$ by case 2 and case 3 and since $u+v = v+u\ \forall\, u, v \in K$.

<u>Case 5</u> $x = a,\ y \neq a,\ z = a$.

$(x+y)+z = (x+y)+a = (x+y)+1,\ x+(y+z) = x+(y+a) = x+(y+1) = (x+y)+1$

since $x, y, 1 \in D$.

<u>Case 6</u> $x \neq a,\ y = a,\ z \neq a$.

$(x+y)+z = (x+a)+z = (x+1)+z,\ x+(y+z) = x+(a+z) = x+(1+z) = (x+1)+z$

since $x, z, 1 \in D$.

<u>Case 7</u> $x = a,\ y \neq a,\ z \neq a$.

$(x+y)+z = (a+y)+z = (1+y)+z,\ x+(y+z) = a+(y+z) = 1+(y+z) = (1+y)+z$

since $y, z, 1 \in D$.

<u>Case 8</u> $x \neq a,\ y \neq a,\ z \neq a$. Then $x, y, z \in D$, so $(x+y)+z = x+(y+z)$.

To show (c), we will consider the following cases :

<u>Case 1</u> $z = a$.

$(x+y)z = (x+y)a = x+y = xa+ya = xz+yz$.

<u>Case 2</u> $z \neq a,\ x = a,\ y \neq a$.

$(x+y)z = (a+y)z = (1+y)z = 1z+yz = z+yz = az+yz = xz+yz$ since $y, z, 1 \in D$.

<u>Case 3</u> $z \neq a,\ x \neq a,\ y = a$.

$(x+y)z = (x+a)z = (a+x)z = az+xz = xz+az = xz+yz$ by Case 2.

<u>Case 4</u> $z \neq a,\ x = y = a$.

<u>Subcase 4.1</u> $a+a = a$. Then $1+1 = 1$, so $u+u = u$ for all $u \in K$.

$(x+y)z = (a+a)z = az = z = z+z = az+az = xz+yz$.

<u>Subcase 4.2</u> $a+a = 1$. Then $1+1 = 1$, so $u+u = u$ for all $u \neq a$.

$(x+y)z = (a+a)z = 1z = z = z+z = az+az = xz+yz$.

<u>Subcase 4.3</u> $a+a = 1+1$.

$(x+y)z = (a+a)z = (1+1)z = 1z+1z = z+z = az+az = xz+yz$.

<u>Case 5</u> $z \neq a$, $x \neq a$, $y \neq a$. Then $x, y, z \in D$, so $(x+y)z = xz+yz$. Therefore $K = D \cup \{a\}$ is a semifield of type II.

Suppose that $S \neq \Phi$ and $|D| = 1$. Then S is an additive subsemigroup of $I_D(1)$ and $D \setminus S = \Phi$. Thus $S = D = \{1\}$, so $1+1 = 1$. Extend + and $\cdot$ from D to $K = D \cup \{a\}$ as in (1), (2) and (3), then $a^2 = a$, $a1 = 1a = 1$, $1 \cdot 1 = 1$, $a+a = a$ or 1 and $a+1 = 1+a = a$. Thus we have two cases to consider. They are given by the following tables :

(1)

+	a	1
a	a	a
1	a	1

or

(2)

+	a	1
a	1	a
1	a	1

It is easy to show that they are all semifields of type II.

Suppose that S is an additive subsemigroup of $I_D(1)$ and $D \setminus S$ is an ideal of $(D,+)$. Extend + and $\cdot$ from D to $K = D \cup \{a\}$ as in (1), (2) and (3) above. Then $ax = xa = x$ for all $x \in K$, $a+x = a$ for all $x \in S$, $a+x = 1+x$ for all $x \in D \quad S$ and

$$a+a = \begin{cases} a \text{ or } 1 & \text{if } 1+1 = 1, \\ 1+1 & \text{if } 1+1 \neq 1. \end{cases}$$

To show $K = D \cup \{a\}$ is a semifield of type II, we must show that (a′) $(xy)z = x(yz)$ for all $x, y, z \in K$, (b′) $(x+y)+z = x+(y+z)$ for all $x, y, z \in K$ and (c′) $(x+y)z = xz+yz$ for all $x, y, z \in K$. The proof of (a′) is the same as the proof of (a) in the case $S = \Phi$. To show (b′), we will consider the following cases :

<u>Case 1</u> $x = y = z = a$.

$(x+y)+z = (a+a)+a = a+(a+a) = x+(y+z)$ since $u+v = v+u$ for all $u, v \varepsilon K$.

<u>Case 2</u> $x = y = a, z \neq a$.

<u>Subcase 2.1</u> $a+a = a$ and $z \varepsilon S$.

$(x+y)+z = (a+a)+z = a+z = a$, $x+(y+z) = a+(a+z) = a+a = a$.

<u>Subcase 2.2</u> $a+a = a$ and $z \varepsilon D\setminus S$.

Since $z \varepsilon D\setminus S$, which is an ideal of $(D,+)$, $1+z \varepsilon D\setminus S$. Thus $(x+y)+z = (a+a)+z = a+z = 1+z$, $x+(y+z) = a+(a+z) = a+(1+z) = 1+(1+z) = (1+1)+z = 1+z$ (since $1+1 = 1$).

<u>Subcase 2.3</u> $a+a = 1$ and $z \varepsilon S$.

$(x+y)+z = (a+a)+z = 1+z = 1$ (since $z \varepsilon S \subseteq I_D(1)$).

$x+(y+z) = a+(a+z) = a+a = 1$.

<u>Subcase 2.4</u> $a+a = 1$, $z \varepsilon D\setminus S$.

Since $z \varepsilon D\setminus S$, which is an ideal of $(D,+)$, $1+z \varepsilon D\setminus S$. Thus $(x+y)+z = (a+a)+z = 1+z$, $x+(y+z) = a+(a+z) = a+(1+z) = 1+(1+z) = (1+1)+z = 1+z$ (since $1+1 = 1$).

<u>Subcase 2.5</u> $a+a = 1+1$, $z \varepsilon S$.

$(x+y)+z = (a+a)+z = (1+1)+z = 1+(1+z) = 1+1$ since $z \varepsilon S \subseteq I_D(1)$.

$x+(y+z) = a+(a+z) = a+a = 1+1$.

<u>Subcase 2.6</u> $a+a = 1+1$, $z \varepsilon D\setminus S$.

Since $z \varepsilon D\setminus S$, which is an ideal of $(D,+)$, $1+z \varepsilon D\setminus S$. Thus $(x+y)+z = (a+a)+z = (1+1)+z$, $x+(y+z) = a+(a+z) = a+(1+z) = 1+(1+z) = (1+1)+z$.

<u>Case 3</u> $x = z = a, y \neq a$.

<u>Subcase 3.1</u> $y \varepsilon S$.

$(x+y)+z = (a+y)+a = a+a$, $x+(y+z) = a+(y+a) = a+a$.

<u>Subcase 3.2</u> $y \in D \setminus S$.

Since $y \in D \setminus S$, which is an ideal of $(D,+)$, $1+y \in D \setminus S$. Thus

$(x+y)+z = (a+y)+a = (1+y)+a = (1+y)+1$, $x+(y+z) = a+(y+a) = a+(y+1)$

$= 1+(y+1) = (1+y)+1$.

<u>Case 4</u> $y = z = a, x \neq a$.

$(x+y)+z = (x+a)+a = a+(x+a) = a+(a+x) = (a+a)+x = x+(a+a) = x+(y+z)$

by Case 2 and since $u+v = v+u$ for all $u, v \in K$.

<u>Case 5</u> $x \neq a, y \neq a, z = a$.

<u>Subcase 5.1</u> $x, y \in S$. Then

$(x+y)+z = (x+y)+a = a$ since S is an additive subsemigroup of $I_D(1)$.

$x+(y+z) = x+(y+a) = x+a = a$.

<u>Subcase 5.2</u> $x \in S, y \in D \setminus S$.

Since $y \in D \setminus S$, which is an ideal of $(D,+)$, $x+y \in D \setminus S$. Thus

$(x+y)+z = (x+y)+a = (x+y)+1$, $x+(y+z) = x+(y+a) = x+(y+1) = (x+y)+1$

(since $x, y, 1 \in D$).

<u>Subcase 5.3</u> $x \in D \setminus S, y \in S$. Then $x+y \in D \setminus S$. Thus

$(x+y)+z = (x+y)+a = (x+y)+1 = x+(y+1) = x+1$ since $x, y, 1 \in D$ and

$y \in S \subseteq I_D(1)$. $x+(y+z) = x+(y+a) = x+a = x+1$.

<u>Subcase 5.4</u> $x \in D \setminus S, y \in D \setminus S$. Then $x+y \in D \setminus S$. Thus

$(x+y)+z = (x+y)+a = (x+y)+1$, $x+(y+z) = x+(y+a) = x+(y+1) = (x+y)+1$

(since $x, y, 1 \in D$).

<u>Case 6</u> $x = a, y \neq a, z \neq a$.

$(x+y)+z = (a+y)+z = z+(a+y) = z+(y+a) = (z+y)+a = a+(z+y) = a+(y+z)$

$= x+(y+z)$ by Case 5 and since $u+v = v+u$ for all $u, v \in K$.

<u>Case 7</u> $x \neq a, y = a, z \neq a$.

$(x+y)+z = (x+a)+z = (a+x)+z = a+(x+z) = (x+z)+a = x+(z+a) = x+(a+z)$

$= x+(y+z)$ by Case 5, Case 6 and since $u+v = v+u$ $\forall u, v \in K$.

Case 8 $x \neq a$, $y \neq a$, $z \neq a$. Then $x, y, z \in D$, so $(x+y)+z = x+(y+z)$.

To show (c′), we will consider the following cases :

Case 1 $z = a$. $(x+y)z = (x+y)a = x+y = xa+ya = xz+yz$.

Case 2 $z \neq a$, $x = y = a$.

Subcase 2.1 $a+a = a$. Then $1+1 = 1$, so $u+u = u$ for all $u \in K$.

$(x+y)z = (a+a)z = az = z = z+z = az+az = xz+yz$.

Subcase 2.2 $a+a = 1$. Then $1+1 = 1$, so $u+u = u$ for all $u \neq a$.

$(x+y)z = (a+a)z = 1z = z = z+z = az+az = xz+yz$.

Subcase 2.3 $a+a = 1+1$.

$(x+y)z = (a+a)z = (1+1)z = 1z+1z = z+z = az+az = xz+yz$ (since 1 and $z \in D$).

Case 3 $z \neq a$, $x = a$, $y \neq a$.

Subcase 3.1 $y \in S$.

$(x+y)z = (a+y)z = az = z = 1z = (1+y)z = 1z+yz = z+yz = az+yz = xz+yz$

(since $y \in S \subseteq I_D(1)$ and $1, y, z \in D$).

Subcase 3.2 $y \in D \setminus S$.

$(x+y) z = (a+y)z = (1+y)z = 1z+yz = z+yz = az+yz = xz+yz$ (since $1, y, z \in D$).

Case 4 $z \neq a$, $x \neq a$, $y = a$.

$(x+y)z = (x+a)z = (a+x)z = az+xz = xz+az = xz+yz$ by Case 3.

Case 5 $z \neq a$, $x \neq a$, $y \neq a$. Then $x, y, z \in D$, so $(x+y)z = xz+yz$.

Therefore K is a semifield of type II. #

Example 2.40 Let $\mathbb{Q}^+$ have the usual multiplication. Define + on $\mathbb{Q}^+$ by $x+y = \max\{x,y\}$. Then $(\mathbb{Q}^+,+,\cdot)$ is a ratio semiring and we see that

$I_{\mathbb{Q}^+}(1) = \{x \in \mathbb{Q}^+ \mid x \leqslant 1\}$. Let $S = \{x \in \mathbb{Q}^+ \mid x < \frac{1}{2}\}$. Clearly S is an additive subsemigroup of $I_{\mathbb{Q}^+}(1)$ and $\mathbb{Q}^+ \setminus S$ is an ideal of $(\mathbb{Q}^+,+)$. Let a be a symbol not representing any element in $\mathbb{Q}^+$. Extend + and • from $\mathbb{Q}^+$ to $K = \mathbb{Q}^+ \cup \{a\}$ by

(1) $ax = xa = x$ for all $x \in K$,

(2) $a+x = x+a = a$ for all $x \in S$ and

$a+x = x+a = 1+x$ for all $x \in \mathbb{Q}^+ \setminus S$,

(3) $a+a = a$ or 1.

By Theorem 2.39, $K = \mathbb{Q}^+ \cup \{a\}$ is a semifield of type II.

<u>Theorem 2.41</u> Let K be a semifield of type III w.r.t.a. Then $(K \setminus \{a\},+,\cdot)$ is a ratio semiring.

<u>Proof</u> Let e be the identity of $(K \setminus \{a\},\cdot)$. Then $ae \neq a$. Let $x, y \in K \setminus \{a\}$. Then $xy \in K \setminus \{a\}$ because $(K \setminus \{a\},\cdot)$ is a group. We want to show that $x+y \in K \setminus \{a\}$. Suppose that $x+y=a$. Then $a = x+y = xe+ye = (x+y)e = ae$. Thus $ae = a$ which is a contradiction. So $x+y \neq a$. Therefore $(K \setminus \{a\},+,\cdot)$ is a ratio semiring. #

<u>Corollary 2.42</u> Let K be a finite a semifield of type III. Then $|K| = 2$.

<u>Proof</u> Let $a \in K$ be such that $(K \setminus \{a\},\cdot)$ is a group. Then by Theorem 2.41, $K \setminus \{a\}$ is a ratio semiring. Since K is finite, so is $K \setminus \{a\}$. By Theorem 1.13, $|K \setminus \{a\}| = 1$. Hence $|K| = 2$. #

<u>Theorem 2.43</u> Let K be a semifield of type III and $a \in K$, $d \in K \setminus \{a\}$ be such that $(K \setminus \{a\},\cdot)$ is a group and $ax = dx$ for all $x \in K$. Then

(1) If $a+a = a$ then $(K,+)$ is a band.

(2) If $a+a \neq a$ then $a+a = d+d$ and for all $x, y \in K \setminus \{a\}$ $x+x = y+y$ iff $x = y$.

(3) $a+x = a$ or $a+x = d+x$ for all $x \neq a$.

<u>Proof</u> Let e be the identity of $(K \setminus \{a\}, \cdot)$.

(1) If $a+a = a$. Let $x \in K \setminus \{a\}$. Since $ae \neq a$, $\exists y \in K \setminus \{a\}$ such that $(ae)y = e$. Since $(ae)y = a(ey) = ay$, $ay = e$. Then $x+x = ex+ex = (ay)x+(ay)x = a(yx)+a(yx) = (a+a)yx = a(yx) = (ay)x = ex = x$, so $x+x = x$. Hence $(K,+)$ is a band.

(2) If $a+a \neq a$, then $a+a = (a+a)e = ae+ae = de+de = d+d$. Thus $a+a = d+d$. Let $x, y \in K \setminus \{a\}$. Since $ae \neq a$, $\exists z \in K \setminus \{a\}$ such that $(ae)z = e$. Since $(ae)z = a(ez) = az$, $az = e$. Then $x+x = ex+ex = (az)x+(az)x = a(zx)+a(zx) = (a+a)zx$. Thus $x+x = (a+a)zx$. Similarly, $y+y = (a+a)zy$. Thus if $x+x = y+y$, then $(a+a)zx = (a+a)zy$. Since $a+a \neq a$ and $(K \setminus \{a\}, \cdot)$ is a group, $x = y$.

(3) If $a+x \neq a$, then $a+x = (a+x)e = ae+xe = de+x = d+x$. Thus $a+x = d+x$. #

<u>Theorem 2.44</u> Let K be a semifield of type III and $a \in K$, $d \in K \setminus \{a\}$ be such that $(K \setminus \{a\}, \cdot)$ is a group and $ax = dx$ $\forall x \in K$. Let $D = K \setminus \{a\}$ and $S = \{x \in D \mid a+x = a\}$. Then

(1) $S = \Phi$ or S is an additive subsemigroup of $I_D(d)$.

(2) If $d \in S$, then $S = I_D(d)$.

(3) $D \setminus S = \Phi$ or $D \setminus S$ is an ideal of $(D,+)$.

<u>Proof</u> Let e be the identity of $(K \setminus \{a\}, \cdot)$

(1) Suppose that $S \neq \Phi$. Let $x \in S$, then $a+x = a$. Then $d = de$

$= ae = (a+x)e = ae+xe = de+x = d+x$, so $x+d = d$. Thus $x \in I_D(d)$. Hence $S \subseteq I_D(d)$. Let $x, y \in S$, then $a+(x+y) = (a+x)+y = a+y = a$. Thus $x+y \in S$. Hence S is an additive subsemigroup of $I_D(d)$.

(2) Assume that $d \in S$. Then $a+d = a$. Let $x \in I_D(d)$, then $d+x = d$. Then $a = a+d = a+(d+x) = (a+d)+x = a+x$, so $a+x = a$. Thus $x \in S$. Hence $I_D(d) \subseteq S$. From (1), $S \subseteq I_D(d)$, so we get that $S = I_D(d)$.

(3) Suppose that $D \setminus S \neq \Phi$. Let $x \in D \setminus S$ and $y \in D$. By Theorem 2.41, $K \setminus \{a\}$ is a ratio semiring, so $(a+x)+y \in K \setminus \{a\}$. Since $a+(x+y) = (a+x)+y$, $a+(x+y) \neq a$. Thus $x+y \in D \setminus S$. #

<u>Proposition 2.45</u> Let K be a semifield of type III and $a \in K$, $d \in K \setminus \{a\}$ be such that $(K \setminus \{a\}, \cdot)$ is a group and $ax = dx$ for all $x \in K$. If $a+x = a$ for all $x \neq a$ then $|K| = 2$.

<u>Proof</u> Suppose that $a+x = x$ for all $x \neq a$. Let $D = K \setminus \{a\}$ and $S = \{x \in D \mid a+x = a\}$. Then $S = K \setminus \{a\}$. By Theorem 2.44, $S \subseteq I_D(d)$. Thus $I_D(d) = K \setminus \{a\}$, so d is an additive zero of $K \setminus \{a\}$ which is a ratio semiring. By Proposition 1.15 and Theorem 1.13, $|K \setminus \{a\}| = 1$. Therefore $|K| = 2$. #

<u>Theorem 2.46</u> Let K be an infinite semifield of type III. Then K has no additive zero.

<u>Proof</u> Let $a \in K$ be such that $(K \setminus \{a\}, \cdot)$ is a group. Suppose that z is an additive zero of K. By Theorem 2.41, $K \setminus \{a\}$ is a ratio semiring. Thus $K \setminus \{a\}$ is an infinite ratio semiring. By Proposition 1.15, $z = a$. Then by Proposition 2.45, $|K| = 2$, a contradiction. Hence K has no additive zero. #

Theorem 2.47 Let K be an infinite semifield of type III. Then K has no additive identity.

Proof Let $a \varepsilon K$ be such that $(K \setminus \{a\}, \cdot)$ is a group and $d \varepsilon K \setminus \{a\}$ be such that $ax = dx$ for all $x \varepsilon K$.

Suppose that z is an additive identity of K. By Theorem 2.41, $K \setminus \{a\}$ is a ratio semiring. Thus $K \setminus \{a\}$ is an infinite ratio semiring. By Proposition 1.14, $z = a$. Then $a+x = x$ for all $x \varepsilon K$. By Theorem 2.43(3), $a+x = a$ or $a+x = d+x$ for all $x \neq a$. Thus $x = a+x = d+x$ for all $x \neq a$, so d is an additive identity of $K \setminus \{a\}$ which contradicts Proposition 1.14. Hence K has no additive identity. #

Proposition 2.48 Let K be a semifield of type III and $a \varepsilon K$ be such that $(K \setminus \{a\}, \cdot)$ is a group. Let $D = K \setminus \{a\}$ and $S = \{x \varepsilon D \mid a+x = a\}$. If $y \varepsilon D \setminus S$ then y is not A.C..

Proof Let e be the identity of $(K \setminus \{a\}, \cdot)$ and $d \varepsilon K \setminus \{a\}$ such that $ax = dx \; \forall x \varepsilon K$. Assume that $y \varepsilon D \setminus S$. By Theorem 2.43(3), $a+y = d+y$. Since $a \neq d$, y is not A.C.. #

Corollary 2.49 Let K be an infinite semifield of type III. Then K contains an element $y \varepsilon K \setminus \{a\}$ such that y is not A.C..

Proof Let $a \varepsilon K$ be such that $(K \setminus \{a\}, \cdot)$ is a group and let $D = K \setminus \{a\}$, $S = \{x \varepsilon D \mid a+x = a\}$. By Proposition 2.45, we have that $D \setminus S \neq \Phi$. Let $y \varepsilon D \setminus S$, then by Proposition 2.48, we get that y is not A.C.. #

Proposition 2.50 Let K be a semifield of type III and $a \varepsilon K$ be such that $(K \setminus \{a\}, \cdot)$ is a group and e the identity of $(K \setminus \{a\}, \cdot)$. Then K is A.C. iff $K = \{a, e\}$ with $a^2 = e$, $a \cdot e = e \cdot a = e$, $e \cdot e = e$ and + is defined by

+	a	e
a	e	a
e	a	e

Proof Assume that K is A.C. By Corollary 2.49, $|K| = 2$. Hence $K = \{a,e\}$. Since K is a semifield of type III, $a^2 \neq a$ and $a \cdot e = e \cdot a \neq a$. Thus $a^2 = e$ and $a \cdot e = e \cdot a = e$. By Theorem 2.41, $K \setminus \{a\}$ is a ratio semiring, so $e+e = e$. By Proposition 2.48, $a+e = a$. Since $K = \{a,e\}$, either $a+a = a$ or $a+a = e$. If $a+a = a$, then $a+a = a+e$ but $a \neq e$ which is a contradiction. Thus $a+a = e$. So we obtain $K = \{a,e\}$ with the above structure.

Conversely assume that $K = \{a,e\}$ with the above structure. Then K is A.C. since $a+a \neq a+e$ and $e+a \neq e+e$. #

Theorem 2.51 Let D be a ratio semiring, a a symbol not representing any element of D, $d \in D$ and let $S \subseteq I_D(d)$ have the property that either $S = \Phi$ or S is an additive subsemigroup of $I_D(d)$ such that $D \setminus S$ is an ideal of $(D,+)$ if D is infinite. Then we can extend the binary operations of D to $K = D \cup \{a\}$ making K into a semifield of type III such that

(1) $ax = xa = dx$ for all $x \in D$ and $a^2 = d^2$,

(2) $a+x = x+a = a$ for all $x \in S$ and

$a+x = x+a = d+x$ for all $x \in D \setminus S$,

(3) $a+a = \begin{cases} a \text{ or } 1 & \text{if } 1+1 = 1, \\ d+d & \text{if } 1+1 \neq 1. \end{cases}$

Proof Suppose that $S = \Phi$. Extend $+$ and $\cdot$ from D to $K = D \cup \{a\}$ as in (1), (2) and (3) above. Then $ax = xa = dx$ for all $x \in D$, $a^2 = d^2$,

$a+x = d+x$ for all $x \in D$ and $a+a = \begin{cases} a \text{ or } 1 & \text{if } 1+1 = 1, \\ d+d & \text{if } 1+1 \neq 1. \end{cases}$

To show $K = D \cup \{a\}$ is a semifield, we must show that (a) $(xy)z = x(yz)$ for all $x, y, z \in K$, (b) $(x+y)+z = x+(y+z)$ for all $x, y, z \in K$ and (c) $(x+y)z = xz+yz$ for all $x, y, z \in K$. To show (a), we will consider the following cases :

<u>Case 1</u> $x = y = z = a$.

$(xy)z = (aa)a = a(aa) = x(yz)$ (since $uv = vu$ for all $u, v \in K$).

<u>Case 2</u>. $x = y = a$, $z \neq a$.

$(xy)z = (aa)z = (da)z = (dd)z = d(dz) = a(dz) = a(az) = x(yz)$ (since $d, z \in D$).

<u>Case 3</u> $x = z = a$, $y \neq a$.

$(xy)z = (ay)a = (dy)a = (dy)d = d(yd) = a(yd) = a(ya) = x(yz)$ (since $d, y \in D$).

<u>Case 4</u> $y = z = a$, $x \neq a$.

$(xy)z = (xa)a = a(xa) = a(ax) = (aa)x = x(aa) = x(y2)$ (since $uv = vu$ for all $u, v \in K$ and by case 2).

<u>Case 5</u> $x \neq a$, $y \neq a$, $z = a$.

$(xy)z = (xy)a = (xy)d = x(yd) = x(ya) = x(yz)$ (since $x, y, d \in D$).

<u>Case 6</u> $x \neq a$, $y = a$, $z \neq a$.

$(xy)z = (xa)z = (xd)z = x(dz) = x(az) = x(yz)$ (since $x, z, d \in D$).

<u>Case 7</u> $x = a$, $y \neq a$, $z \neq a$.

$(xy)z = (ay)z = (dy)z = d(yz) = a(yz) = x(yz)$ (since $y, z, d \in D$).

<u>Case 8</u> $x \neq a$, $y \neq a$, $z \neq a$. Then $x, y, z \in D$, so $(xy)z = x(yz)$.

To show (b), we will consider the following cases :

<u>Case 1</u> $x = y = z = a$.

$(x+y)+z = (a+a)+a = a+(a+a) = x+(y+z)$ (since $u+v = v+u$).

<u>Case 2</u> $x = y = a$, $z \neq a$.

<u>Subcase 2.1</u> $a+a = a$. Then $1+1 = 1$, so $u+u = u$ for all $u \varepsilon K$.

Thus $(x+y)+z = (a+a)+z = a+z = d+z$, $x+(y+z) = a+(a+z) = a+(d+z)$

$= d+(d+z) = (d+d)+z = d+z$.

<u>Subcase 2.2</u> $a+a = d$. Then $1+1 = 1$, so $u+u = u$ for all

$u \neq a$. Thus $(x+y)+z = (a+a)+z = d+z$, $x+(y+z) = a+(a+z) = a+(d+z)$

$= d+(d+z) = (d+d)+z = d+z$.

<u>Subcase 2.3</u> $a+a = d+d$.

$(x+y)+z = (a+a)+z = (d+d)+z$, $x+(y+z) = a+(a+z) = a+(d+z) = d+(d+z)$

$= (d+d)+z$ (since $d, z \varepsilon D$).

<u>Case 3</u> $x = z = a$, $y \neq a$.

$(x+y)+z = (a+z)+a = (d+y)+a = (d+y)+d$, $x+(y+z) = a+(y+a) = a+(y+d)$

$= d+(y+d) = (d+y)+d$ (since $d, y \varepsilon D$).

<u>Case 4</u> $y = z = a$, $x \neq a$.

$(x+y)+z = (x+a)+a = a+(x+a) = a+(a+x) = (a+a)+x = x+(a+a) = x+(y+z)$

(since $u+v = v+u$ for all $u, v \varepsilon K$ and by case 2).

<u>Case 5</u> $x \neq a$, $y \neq a$, $z = a$.

$(x+y)+z = (x+y)+a = (x+y)+d = x+(y+d) = x+(y+a) = x+(y+z)$

(since $u+v = v+u$ for all $u, v \varepsilon K$ and $x, y, d \varepsilon D$).

<u>Case 6</u> $x = a$, $y \neq a$, $z \neq a$.

$(x+y)+z = (a+y)+z = (d+y)+z = d+(y+z) = a+(y+z) = x+(y+z)$

(since $y, z, d \varepsilon D$).

<u>Case 7</u> $x \neq a$, $y = a$, $z \neq a$.

$(x+y)+z = (x+a)+z = (x+d)+z = x+(d+z) = x+(a+z) = x+(y+z)$

(since $x, z, d \in D$).

<u>Case 8</u> $x \neq a$, $y \neq a$, $z \neq a$. Then $x, y, z \in D$, so $(x+y)+z = x+(y+z)$.

To show (c), we will consider the following cases :

<u>Case 1</u> $x = y = a$.

<u>Subcase 1.1</u> $a+a = a$. Then $1+1 = 1$, so $u+u = u$ for all $u \in K$.

Thus $(x+y)z = (a+a)z = az = az+az = xz+yz$.

<u>Subcase 1.2</u> $a+a = d$. Then $1+1 = 1$, so $u+u = u$ for all $u \neq a$.

Thus $(x+y)z = (a+a)z = dz = dz+dz = az+az = xz+yz$.

<u>Subcase 1.3</u> $a+a = d+d$.

If $z = a$, then $(x+y)z = (a+a)a = (d+d)a = (d+d)d = dd + dd = aa+aa$

$= xz+yz$ (since $d \in D$). If $z \neq a$, then $(x+y)z = (a+a)z = (d+d)z$

$= dz+dz = az+az = xz+yz$ (since $d, z \in D$).

<u>Case 2</u> $x = z = a$, $y \neq a$.

$(x+y)z = (a+y)a = (d+y)a = (d+y)d = dd+yd = aa+ya = xz+yz$.

(since $d, y \in D$).

<u>Case 3</u> $y = z = a$, $x \neq a$.

$(x+y)z = (x+a)a = (x+d)a = (x+d)d = xd+dd = xa+aa = xz+yz$

(since $x, d \in D$).

<u>Case 4</u> $x = a$, $y \neq a$, $z \neq a$.

$(x+y)z = (a+y)z = (d+y)z = dz+yz = az+yz = xz+yz$ (since $d, y, z \in D$).

<u>Case 5</u> $x \neq a$, $y = a$, $z \neq a$.

$(x+y)z = (x+a)z = (x+d)z = xz+dz = xz+az = xz+yz$ (since $x, z, d \in D$).

<u>case 6</u> $x \neq a,\ y \neq a,\ z = a.$

$(x+y)z = (x+y)a = (x+y)d = xd+yd = xa+ya = xz+yz$ (since $x, y, d \in D$).

<u>Case 7</u> $x \neq a,\ y \neq a,\ z \neq a.$ Then $x, y, z \in D$, so $(x+y)z = xz+yz$.

Therefore K is a semifield of type III.

We now assume that $S \neq \Phi$ and $|D| = 1$. Then $D \setminus S = \Phi$. Thus $S = D = \{1\}$. Extend + and • from D to $K = D \cup \{a\}$ as in (1), (2) and (3). Then $a+1 = 1+a = a$. Since $D = \{1\}$ is a ratio semiring, $1+1 = 1$. Thus $a+a = a$ or 1. And by (1), we have that $a \cdot a = 1$, $a \cdot 1 = 1 \cdot a = 1 \cdot 1 = 1$ and $1 \cdot 1 = 1$. So we get that $K = \{a,1\}$ has two structures ;

(1)

•	a	1
a	1	1
1	1	1

and

+	a	1
a	a	a
1	a	1

(2)

•	a	1
a	1	1
1	1	1

and

+	a	1
a	1	a
1	a	1

It is easy to show that they are all semifields.

Suppose that S is an additive subsemigroup of $I_D(d)$ and $D \setminus S$ is an ideal of $(D,+)$. Extend + and • from D to $K = D \cup \{a\}$ as in (1), (2) and (3). Then $a+x = x+a = a$ for all $x \in S$, $a+x = x+a = d+x$ for all $x \in D \setminus S$, $a+a = \begin{cases} a \text{ or } d & \text{if } 1+1 = 1 \\ d+d & \text{if } 1+1 \neq 1 \end{cases}$, $ax = xa = dx\ \forall x \in D$ and $a^2 = d^2$

We must show that $K = D \cup \{a\}$ is a semifield. As in the case $S = \Phi$, we can show that $(xy)z = x(yz)$ for all $x, y, z \in K$. So we have to show that (a) $(x+y)+z = x+(y+z)$ for all $x, y, z \in K$ and (b) $(x+y)z = xz+yz$.

To show (a), we will consider the following cases :

<u>Case 1</u> $x = y = z = a.$

$(x+y)+z = (a+a)+a = a+(a+a) = x+(y+z)$ (since $u+v = v+u$ for all $x \in K$).

<u>Case 2</u> $x = y = a$, $z \neq a$.

<u>Subcase 2.1</u> $a+a = a$. Then $1+1 = 1$, so $u+u = u$ for all $u \in K$.

If $z \in S$, then $(x+y)+z = (a+a)+z = a+z = a$, $x+(y+z) = a+(a+z) = a+a = a$.

If $z \in D\setminus S$, then $d+z \in D\setminus S$. since D S is an ideal of $(D,+)$.

Thus $(x+y)+z = (a+a)+z = a+z = d+z$ and $x+(y+z) = a+(a+z) = a+(d+z)$

$= d+(d+z) = (d+d)+z = d+z$.

<u>Subcase 2.2</u> $a+a = d$. Then $1+1 = 1$, so $u+u = u$ for all $u \neq a$.

If $z \in S$, then $(x+y)+z = (a+a)+z = d+z = d$ (since $z \in S \subseteq I_D(d)$),

$x+(y+z) = a+(a+z) = a+a = d$.

If $z \in D\setminus S$, then $d+z \in D\setminus S$ since D S is an ideal of $(D,+)$. Thus

$(x+y)+z = (a+a)+z = d+z$ and $x+(y+z) = a+(a+z) = a+(d+z) = d+(d+z)$

$= (d+d)+z = d+z$.

<u>Subcase 2.3</u> $a+a = d+d$.

If $z \in S$, then $(x+y)+z = (a+a)+z = (d+d)+z = d+(d+z) = d+d$ (since d,

$z \in D$ and $z \in S \subseteq I_D(d)$).

If $z \in D\setminus S$, then $d+z \in D\setminus S$. Thus $(x+y)+z = (a+a)+z = (d+d)+z$ and

$x+(y+z) = a+(a+z) = a+(d+z) = d+(d+z) = (d+d)+z$ (since $d, z \in D$).

<u>Case 3</u> $x = z = a$, $y \neq a$.

<u>Subcase 3.1</u> $y \in S$.

$(x+y)+z = (a+y)+a = a+a$ and $x+(y+z) = a+(y+a) = a+a$

<u>Subcase 3.2</u> $y \in D\setminus S$.

Since $D\setminus S$ is an ideal of $(D,+)$, $d+y \in D\setminus S$. Thus $(x+y)+z = (a+y)+a$

$= (d+y)+a = (d+y)+d$ and $x+(y+z) = a+(y+a) = a+(y+d) = d+(y+d) = (d+y)+d$

(since $d, y \in D$).

<u>Case 4</u> $y = z = a$, $x \neq a$.

$(x+y)+z = (x+a)+a = a+(x+a) = a+(a+x) = (a+a)+x = x+(a+a) = x+(y+z)$

(by case 2 and the fact that $u+v = v+u$ for all $u, v \in K$).

Case 5 $\quad x \neq a, y \neq a, z = a$.

Subcase 5.1 $\quad x, y \in S$.

Since S is an additive subsemigroup of $I_D(d)$, $x+y \in S$. Thus $(x+y)+z = (x+y)+a = a$ and $x+(y+z) = x+(y+a) = x+a = a$.

Subcase 5.2 $\quad x \in S, y \in D \setminus S$. Then $x+y \in D \setminus S$. Thus $(x+y)+z = (x+y)+a = (x+y)+d$ and $x+(y+z) = x+(y+a) = x+(y+d) = (x+y)+d$ (since $x,y,d \in D$).

Subcase 5.3 $\quad x \in D \setminus S, y \in S$. Then $x+y \in D \setminus S$. Thus $(x+y)+z = (x+y)+a = (x+y)+d = x+(y+d) = x+d$ (since $x,y,d \in D$ and $y \in S \subseteq I_D(d)$). $x+(y+z) = x+(y+a) = x+a = x+d$.

Subcase 5.4 $\quad x,y \in D \setminus S$. Then $x+y \in D \setminus S$. Thus $(x+y)+z = (x+y)+a = (x+y)+d$ and $x+(y+z) = x+(y+a) = x+(y+d) = (x+y)+d$ (since $x,y,d \in D$).

Case 6 $\quad x = a, y \neq a, z \neq a$.

$(x+y)+z = (a+y)+z = z+(a+y) = z+(y+a) = (z+y)+a = a+(z+y) = a+(y+z) = x+(y+z)$ (by case 5 and the fact that $u+v = v+u$ for all $u, v \in K$).

Case 7 $\quad x \neq a, y = a, z \neq a$.

$(x+y)+z = (x+a)+z = (a+x)+z = a+(x+z) = (x+z)+a = x+(z+a) = x+(a+z) = x+(y+z)$ (by case 5, case 6 and the fact that $u+v = v+u$ for all $u, v \in K$).

Case 8 $\quad x \neq a, y \neq a, z \neq a$. Then $x,y,z \in D$, so $(x+y)+z = x+(y+z)$.

To show (b), we will consider the following cases :

Case 1 $\quad x = y = a, z \in K$. The proof is similar to case 1 of the case $S = \Phi$.

Case 2 $x = z = a,\ y \neq a$.

Subcase 2.1 $y \in S$. Since $S \subseteq I_D(d)$, $d+y = d$. Thus

$(x+y)z = (a+y)a = aa = dd = (d+y)d = dd+yd = aa+ya = xz+yz$ (since $d,y \in D$).

Subcase 2.2 $y \in D \setminus S$.

$(x+y)z = (a+y)a = (d+y)a = (d+y)d = dd+yd = aa+ya = xz+yz$ (since $a,y \in D$).

Case 3 $y = z = a,\ x \neq a$.

$(x+y)z = (x+a)a = (a+x)a = aa+xa = xa+aa = xz+yz$ (by case 2 and the commutativity of K).

Case 4 $x \neq a,\ y \neq a,\ z = a$.

$(x+y)z = (x+y)a = (x+y)d = xd+yd = xa+ya = xz+yz)$ (since $x,y,d \in D$).

Case 5 $x \neq a,\ y = a,\ z \neq a$

Subcase 5.1 $x \in S$. Then $x \in I_D(d)$, so $x+d = d$. Thus

$(x+y)z = (x+a)z = az = dz = (x+d)z = xz+dz = xz+az = xz+yz$ (since $x,d,z \in D$).

Subcase 5.2 $x \in D \setminus S$.

$(x+y)z = (x+a)z = (x+d)z = xz+dz = xz+az = xz+yz$ (since $x,d,z \in D$).

Case 6 $x = a,\ y \neq a,\ z \neq a$.

$(x+y)z = (a+y)z = (y+a)z = yz+az = az+yz = xz+yz$

(by case 5 and the commutativity of K).

Case 7 $x \neq a,\ y \neq a,\ z \neq a$. Then $x,y,z \in D$, so $(x+y)z = xz+yz$.

Therefore K is a semifield and clearly it is a type III semifield. #

Example 2.52 Let $\mathbb{Q}^+$ have the usual multiplication. Define + on $\mathbb{Q}^+$ by $x+y = \min\{x,y\}$. Then $(\mathbb{Q}^+ +,\cdot)$ is a ratio semiring. Let $d \in \mathbb{Q}^+$ and a a symbol not representing any element in $\mathbb{Q}^+$. Then $I_{\mathbb{Q}^+}(d) = \{x \in \mathbb{Q}^+ \mid x > d\}$. Let $S = \{x \in \mathbb{Q}^+ \mid x > 2d\}$. Clearly S is an additive subsemigroup of $I_{\mathbb{Q}^+}(d)$ and $\mathbb{Q}^+ \setminus S$ is an ideal of $(\mathbb{Q}^+,+)$. Then by Theorem 2.51, we can extend the binary operations of $\mathbb{Q}^+$ to $K = \mathbb{Q}^+ \cup \{a\}$ making K into a semifield of type III by

(1) $ax = xa = dx$ for all $x \in \mathbb{Q}^+$ and $a^2 = d^2$,

(2) $a+x = x+a = a$ for all $x \in S$ and

$a+x = x+a = d+x$ for all $x \in \mathbb{Q}^+ \setminus S$,

(3) $a+a = d+d$.