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Congruence criterion in neutral geometry

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Congruence Criterion in Neutral Geometry

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Neutral geometry is a geometry based on an axiomatic system for Euclidean geometry without the parallel postulate which is known as the Euclid's fifth postulate. This project studies and establishs certain congruence criterions for triangles in Neutral geometry.

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Contents

Chapter 1

Introduction

In classical geometry, we learnt that two triangles are congruent if they coincide. In addition, we know that it is sufficient that two triangles, whose pair of corresponding sides and their included angle are equal, are congruent. This is famously called side-angle-side congruence criterion. However, there are many more basic congruence criterions in classical geometry. This classical geometry was established by Euclid in his important books called *The Elements*. The Elements consists of 13 books which contain definitions, axioms, theorems concerning various aspects in Geometry. The Elements are widely considered to be the very first Mathematical text books. Later on, the ideas of Euclid have been polished in more contemporary contexts.

Euclidean geometry is the geometry established by Euclid in 300 B.C. when he compiled a body of the Elements. In volume I, Euclid presented five idea axiomatic statements, called Euclidean Postulates, from which propositions and theorems could be deduced. Of these five postulates, the fifth is by far the most complicated and unnatural. For two thousand years, mathematicians attempted to deduce the fifth postulate from the first four simpler postulates. A final progress came when mathematicians abandoned their effort to prove the fifth postulate and instead worked out the consequences of negation.

Consequently, neutral geometry is a geometry which we accept only the first four postulates and if it negates the fifth postulate, then it is called Hyperbolic geometry while it is Euclidean geometry if it satisfies the fifth postulate.

These five postulates were as follows:

1. Every two distinct points lie on a unique line.
2. Any line segment can be indefinitely extended in either direction.
3. There is exactly one circle of any given radius with given center.
4. All right angles are congruent to one another.
5. If two lines intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended far enough.

In this project, we establish more congruent criterion for triangles in Neutral geometry.

Chapter 2

Preliminaries

In this chapter, concepts of Neutral geometry and congruence of triangles are discussed. We start with rudimentary geometry and keep adding more axioms to make it more complex.
We write $\mathcal{S}$ for a set of points, $\mathcal{L}$ for a set of lines and $\mathcal{A}$ for a set of angles.

2.1 Neutral Geometry

Definition 2.1.1. *An **Abstract geometry** consists of a set $\mathcal{S}$, whose elements are called **points**, together with a collection $\mathcal{L}$ of nonempty subsets of $\mathcal{S}$, called **lines**, such that*

1. *For every two points $A, B \in \mathcal{S}$, there exists a line $l \in \mathcal{S}$ with $A, B \in \mathcal{S}$.*
2. *Every line has at least two points.*

Definition 2.1.2. *An Abstract geometry $(\mathcal{S},\mathcal{L})$ is an **Incidence geometry** if*

1. *Every two distinct points in $\mathcal{S}$ lie on a unique line.*
2. *There exist three points in $\mathcal{S}$ which do not lie all on the same line.*

We write $\overleftrightarrow{AB}$ as the line containing A and B.

Definition 2.1.3. *A set of points $\mathcal{T}$ is **collinear** if there is a line containing $\mathcal{T}$ and $\mathcal{T}$ is **non-collinear** if $\mathcal{T}$ is not collinear.*

Definition 2.1.4. *Let l_1 and l_2 be lines in Abstract geometry. Then l_1 is **parallel** to l_2 (written $l_1 \| l_2$) if either $l_1 = l_2$ or $l_1 \cap l_2 = \emptyset$.*

Corollary 2.1.5. *In an Incidence geometry, two lines are either parallel or they intersect at exactly one point.*

Definition 2.1.6. *A **distance function** of $\mathcal{S}$ is a function $d : \mathcal{S} \times \mathcal{S} \to \mathbb{R}$ such that for all $P, Q \in \mathcal{S}$*

1. $d(P, Q) \geq 0$.
2. $d(P, Q) = d(Q, P)$.
3. $d(P, Q) = 0$ *if and only if* $P = Q$.

And we may write PQ instead of $d(P, Q)$.

Definition 2.1.7. *Let l be a line in an Incident geometry equipped with a distance function $(\mathcal{S}, \mathcal{L}, d)$. A function $f_l : l \to \mathbb{R}$ is a **ruler** (or **coordinate system**) for l if f_l is a bijection such that $|f_l(P) - f_l(Q)| = PQ$ for all points $P, Q \in \mathcal{S}$.*

Definition 2.1.8. *An Incidence geometry equipped with a distance function $(\mathcal{S}, \mathcal{L}, d)$ is a **Metric geometry** if every line $l \in \mathcal{L}$ has a ruler.*

Definition 2.1.9. *Let A, B and C be distinct collinear points in a Metric geometry $(\mathcal{S}, \mathcal{L}, d)$. B is **between** A and C(written $A - B - C$) if $AB + BC = AC$.*
*In addition, the **line segment** from A to B is the set*

$$\overline{AB} = \{D \in \mathcal{S} | A - D - B \text{ or } D = A \text{ or } D = B\}$$

*and the **ray** from A to B is the set*

$$\overrightarrow{AB} = \overline{AB} \cup \{D \in \mathcal{S} | A - B - D\}.$$

Theorem 2.1.10. *(Segment Construction). If $\overrightarrow{AB}$ is a ray and $\overline{PQ}$ is a line segment in a Metric geometry, then there is a unique point $C \in \overrightarrow{AB}$ with $PQ = AC$.*

Definition 2.1.11. *Let A, B and C be non-collinear points in a Metric geometry. Then the **angle** ABC is the set*

$$\angle ABC = \overrightarrow{BA} \cup \overrightarrow{BC}$$

*and the **triangle** ABC is the set*

$$\triangle ABC = \overline{AB} \cup \overline{BC} \cup \overline{CA}$$

Definition 2.1.12. *In a Metric geometry $(\mathcal{S}, \mathcal{L}, d)$ and let $\mathcal{T} \subseteq \mathcal{S}$. Then $\mathcal{T}$ is said to be **convex** if every two points $P, Q \in \mathcal{T}$, $\overline{PQ} \subseteq \mathcal{T}$.*

Definition 2.1.13. *A Metric geometry $(\mathcal{S}, \mathcal{L}, d)$ satisfies the **plane separation axiom (PSA)** if every $l \in \mathcal{L}$ there are two subsets H_1 and H_2 of $\mathcal{S}$(called **half planes determined** by l) such that*

1. $H_1 \cup H_2 = S - l$.
2. H_1 *and* H_2 *are disjoint and each is convex.*
3. *For all* $P \in H_1$ *and* $Q \in H_2$, $\overline{PQ} \cap l \neq \emptyset$.

*Furthermore, For each two points A and B, A and B **lie on the same side of** l if they both belong to H_1 or both belong to H_2. And A and B **lie on opposite side of** l if one belongs to H_1 and one belongs to H_2.*

Theorem 2.1.14. *In a Metric geometry $(\mathcal{S}, \mathcal{L}, d)$ which satisfies PSA. Let A and B be two points of $\mathcal{S}$ not on a given line l. Then*

1. A *and* B *lie on opposite sides of* l *if and only if* $\overline{AB} \cap l \neq \emptyset$
2. A *and* B *lie on the same side of* l *if and only if either* $A = B$ *or* $\overline{AB} \cap l = \emptyset$

Theorem 2.1.15. *Let l be a line in a Metric geometry satisfies PSA. If P and Q lie on the same side (opposite sides) of l and Q and R lie on the same side (opposite sides) of l, then P and R lie on the same side of l.*

Theorem 2.1.16. *Let l be a line in a Metric geometry satisfies PSA. If P and Q lie on the same side of l and Q and R lie on the opposite sides of l, then P and R lie on the opposite sides of l.*

Definition 2.1.17. *A Metric geometry satisfies* ***Pasch's Postulate(PP)*** *if for any line l, any $\triangle ABC$, and any point $D \in l$ such that $A - D - B$, then either $\overline{AC} \cap l \neq \emptyset$ or $\overline{BC} \cap l \neq \emptyset$. A* ***Pasch geometry*** *is a metric geometry which satisfies PP.*

Theorem 2.1.18. *Every Pasch geometry satisfies the plane separation axiom.*

Definition 2.1.19. *In a Pasch geometry, the* ***interior of angle*** *$\angle ABC$, written $int(\angle ABC)$, is the intersection of the side of $\overleftrightarrow{AB}$ that contains C with the side of $\overleftrightarrow{BC}$ that contains A. And the* ***interior of triangle*** *$\triangle ABC$, written $int(\triangle ABC)$, is the intersection of the side of $\overleftrightarrow{AB}$ that contain C, the intersection of the side of $\overleftrightarrow{BC}$ that contains A and the intersection of the side of $\overleftrightarrow{AC}$ that contains B.*

Theorem 2.1.20. *(Crossbar Theorem). In a Pasch geometry, if $P \in int(\angle ABC)$, then $\overrightarrow{BP}$ intersects $\overline{AC}$ at a unique point F with $A - F - C$.*

Definition 2.1.21. *Let r_0 be a fixed positive real number. In a Pasch geometry, an* ***angle measure****(or* ***protractor****) based on r_0 is a function m from the set $\mathcal{A}$ of all angles to the set of real number such that*

1. *If $\angle ABC \in \mathcal{A}$, then $0 < m(\angle ABC) < r_0$.*

2. *If $\overline{BC}$ lies in the edge of the half plane H_1 and if θ is a real number between 0 and r_0, then there is a unique ray $\overrightarrow{BA}$ with $A \in H_1$ and $m(\angle ABC) = \theta$.*

3. *If $D \in int(\angle ABC)$, then $m(\angle ABD) + m(\angle DBC) = m(\angle ABC)$.*

Note*: We use $r_0 = 180$, without further assumption.*

Definition 2.1.22. *A* ***Protractor geometry*** *$(\mathcal{S}, \mathcal{L}, d, m)$ is Pasch geometry$(\mathcal{S}, \mathcal{L}, d)$ together with an angle measure m.*

Definition 2.1.23. *In Protractor geometry, a* ***right angle*** *is an angle whose measure is 90. An* ***acute angle*** *is an angle whose measure is less than 90. A* ***obtuse angle*** *is an angle whose measure is more than 90. Two angles are* ***supplementary*** *if the sum of their measures is 180. Two angles are* ***complementary*** *if the sum of their measures is 90.*

Definition 2.1.24. *In Protractor geometry, two angles $\angle ABC$ and $\angle CBD$ form a* ***linear pair*** *if $A - B - D$. Two angles $\angle ABC$ and $\angle A'BC'$ form a* ***vertical pair*** *if $A - B - A'$ and $C - B - C'$.*

Definition 2.1.25. *In a Protractor geometry, two lines l and l' are* ***perpendicular*** *(written $l \perp l'$) if $l \cup l'$ contains a right angle. Two rays or segments are* ***perpendicular*** *if the lines they determine are perpendicular.*

Theorem 2.1.26. *If C and D are points of a Protractor geometry and are on the same side of $\overleftrightarrow{AB}$ and $m(\angle ABC) < m(\angle ABD)$, then $C \in int(\angle ABD)$.*

Theorem 2.1.27. *(Linear pair Theorem). If $\angle ABC$ and $\angle CBD$ form a linear pair in a Protractor geometry, then they are supplementary.*

Definition 2.1.28. *(Segment Congruence and Angle Congruence). In a Metric geometry, $\overline{AB} \cong \overline{CD}$ if $AB = CD$. In a Protractor geometry, $\angle ABC \cong \angle DEF$ if $m(\angle ABC) = m(\angle DEF)$. Sometimes we may write $\angle ABC = \angle DEF$*

Theorem 2.1.29. *(Angle Construction). Given $\angle ABC$ and a ray $\overrightarrow{ED}$ which lies in the edge of half plane H_1, then there exists a unique ray $\overrightarrow{EF}$ with $F \in H_1$ and $\angle ABC \cong \angle DEF$.*

Corollary 2.1.30. *(Vertical pair Theorem). If $\angle ABC$ and $\angle CBD$ form a vertical pair in a Protractor geometry, $\angle ABC \cong \angle CBD$.*

Convention: We may denote $\angle ABC$ by $\angle B$ if there is no confusion.

Definition 2.1.31. *(Triangle Congruence). Let $\triangle ABC$ and $\triangle DEF$ be two triangles in a Protractor geometry and let $f : \{A, B, C\} \to \{D, E, F\}$ be a bijection between the vertices of triangles. f is a* ***congruence*** *if*

$$\overline{AB} \cong \overline{f(A)f(B)},\ \overline{BC} \cong \overline{f(B)f(C)},\ \overline{AC} \cong \overline{f(A)f(C)}$$

and

$$\angle A \cong \angle f(A),\ \angle B \cong \angle f(B),\ \angle C \cong \angle f(C).$$

Two triangles, $\angle ABC$ and $\angle CBD$, are **congruent** *if there is a congruece f. If f is given by $f(A) = D$, $f(B) = E$, and $f(C) = F$, then we write $\triangle ABC \cong \triangle DEF$. However, we can refer $\triangle ABC \cong \triangle DEF$ to other congruence for flexible meaning.*

Definition 2.1.32. *A Protractor geometry satisfies the* ***Side-Angle-Side Axiom(SAS)*** *if whenever $\triangle ABC$ and $\triangle DEF$ are two triangles with $\overline{AB} \cong \overline{DE}$, $\angle B \cong \angle E$, and $\overline{BC} \cong \overline{EF}$, then $\triangle ABC \cong \triangle DEF$.*

Definition 2.1.33. *A* ***Neutral geometry****(or* ***Absolute geometry****) is a Protractor geometry which satisfies* ***SAS****.*

Example of Neutral geometry:

Let $\mathbb{H} = \{(x, y) \in \mathbb{R}^2 | y > 0\}$. We define two types of lines as follows. A **type I** line is any subset of $\mathbb{H}$ of the form

$$_aL = \{(x, y) \in \mathbb{H} | x = a\}$$

where a is a fixed real number. A **types II** line is any subset of $\mathbb{H}$ of the form

$$_cL_r = \{(x, y) \in \mathbb{H} | (x - c)^2 + y^2 = r^2\}$$

where c and r are fixed real numbers with $r > 0$. Let $\mathcal{L}_H$ be the set of all type I and type II lines. Then $\mathcal{H} = (\mathbb{H}, \mathcal{L}_H)$ is an incident geometry. In addition, we define d_H distance and m_H angle measurement for $\mathcal{H}$ by the follows.

Distance: For each $P, Q \in \mathbb{H}$ with $P = (x_1, y_1), B = (x_2, y_2)$, the distance function is given by

$$d_H(P,Q) = \begin{cases} |ln(\frac{y_2}{y_1})| & ; x_1 = x_2, \\ |ln(\frac{\frac{x_1-c+r}{y_1}}{\frac{x_2-c+r}{y_2}})| & ; P, Q \in {}_cL_r. \end{cases}$$

Angle: For each $A, B \in \mathbb{H}$, we first define **Euclidean tangent** to $\overrightarrow{BA}$ at B with $A = (x_A, y_A), B = (x_B, y_B)$ by

$$T_{BA} == \begin{cases} (0, y_A - y_B) & ; \overleftrightarrow{AB} \text{ is a type I line,} \\ (y_B, c - x_B) & ; \overleftrightarrow{AB} \text{ is a type II line } {}_cL_r, x_B < x_A, \\ -(y_B, c - x_B) & ; \overleftrightarrow{AB} \text{ is a type II line } {}_cL_r, x_B > x_A. \end{cases}$$

Then, the angle measurement is given by

$$m_H(\angle ABC) = cos^{-1}(\frac{T_{BA} \cdot T_{BC}}{||T_{BA}||\ ||T_{BC}||})$$

for any $\angle ABC$. Hence, $\mathcal{H}$ is a neutral geometry.

Proposition 2.1.34. *The Hyperbolic Plane $\mathcal{H}$ is a neutral geometry.*

Definition 2.1.35. *In a Metric geometry, $\overline{AB} < \overline{CD}(\overline{AB} \leq \overline{CD})$ if $AB < CD$ $(AB \leq CD)$. In a Protractor geometry, $\angle ABC < \angle DEF(\angle ABC \leq \angle DEF)$ if $m(\angle ABC) < m(\angle DEF)(m(\angle ABC) \leq m(\angle DEF))$.*

Theorem 2.1.36. *(Exterior Angle Theorem). Given $\triangle ABC$ in a Neutral geometry, if $A - C - D$, then $\angle BCD > \angle A$ and $\angle BCD > \angle B$.*

Corollary 2.1.37. *In a Neutral geometry, there is exactly one line through a given point P perpendicular to a given line l.*

Theorem 2.1.38. *(Triangle Inequality). In a Neutral geometry, the length of one side of a triangle is strictly less than the sum of the lengths of other two sides.*

Theorem 2.1.39. *(Open Mouth Theorem). In a Neutral geometry, given two triangles $\triangle ABC$ and $\triangle DEF$ with $\overline{AB} \cong \overline{DE}$ and $\overline{BC} \cong \overline{EF}$, if $\angle B > \angle E$ then $\overline{AC} > \overline{DF}$.*

Definition 2.1.40. *A triangle in a Protractor geometry is a* ***right triangle*** *if it has a right angle. A side opposite to a right angle in a right triangle is called a* ***hypotenuse****.*

Theorem 2.1.41. *In a Neutral geometry, there is only one right angle and one hypotenuse for each right triangle. The remaining angles are acute, and the hypotenuse is the longest side of triangle. The other sides is called* ***legs*** *of triangle.*

Corollary 2.1.42. *(Perpendicular Distance Theorem). In a Neutral geometry, if l is a line, $Q \in l$, and $P \notin l$, then*

$$PQ \leq PR \text{ for all } R \in l \text{ if and only if } \overleftrightarrow{PQ} \perp l.$$

Definition 2.1.43. *Given three distinct lines l, l_1, and l_2, we say that l is a* ***transversal*** *of l_1 and l_2 if l intersects both l_1 and l_2, but in different points.*

Definition 2.1.44. *Assume that the line* $\overleftrightarrow{GH}$ *is transversal to* $\overleftrightarrow{AC}$ *and* $\overleftrightarrow{DF}$ *in a metric geometry and that* $A-B-C$*,* $D-E-F$*,* $G-B-E-F$*,* A *and* D *are on same side of* $\overleftrightarrow{GH}$*, then*

$\angle ABE$ *and* $\angle FEB$ *are a pair of* ***alternate interior angles***

,and

$\angle ABG$ *and* $\angle DEB$ *are a pair of* ***corresponding angles****.*

Theorem 2.1.45. *Let* l *and* n *be two lines in a Neutral geometry. If there is a transversal of* l *and* n *with a pair of alternate interior angles congruent, then* l *and* n *have a common perpendicular.*

Theorem 2.1.46. *In a Neutral geometry, if* l *and* n *have a common perpendicular, then* $l \| n$*. In particular, if there is a transversal of* l *and* n *with a pair of alternate interior angles congruent, then* $l \| n$*.*

Definition 2.1.47. *If* l *is the unique perpendicular to* $\overleftrightarrow{AB}$ *through the vertex* C *of* $\triangle ABC$ *in a protractor geometry and if* $l \cup AB = \{D\}$*, then* $\overline{CD}$ *is the* ***altitude*** *from* C *perpendicular to* D*.*

Definition 2.1.48. *In metric geometry, a* ***median*** *of a triangle is a line segment joining a vertex to the midpoint of the opposite side.*

Theorem 2.1.49. *In a protractor geometry, every angle* $\angle ABC$ *has a unique ray* $\overrightarrow{BD}$ *with* $D \in int(\angle ABC)$ *and* $m(\angle ABD) = m(\angle DBC)$*. The ray is called an* ***angle bisector*** *of a triangle.*

Definition 2.1.50. *The* ***perimeter*** *of a triangle in metric geometry is the sum of three lengths.*

Definition 2.1.51. *If C is a point in a metric geometry $(\mathcal{S}, \mathcal{L}, d)$ and if $r > 0$, then*

$$\mathcal{C} = \mathcal{C}_r(C) = \{P \in \mathcal{S} | PC = r\}$$

Theorem 2.1.52. *(Two circle Theorem). In a neutral geometry, if $\mathcal{C}_1 = \mathcal{C}_b(A)$, $\mathcal{C}_2 = \mathcal{C}_a(B)$, $AB = c$, and if each of a, b, c is less than the sum of the other two, then $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect in exactly two points, and these points are on opposite sides of $\overleftrightarrow{AB}$.*

2.2 Basic Triangle Congruence Criterions

Here, we give congruence criterions for triangles. We also discuss whether the two criterion become equivalent.

Theorem 2.2.1. *(**Angle-Side-Angle, ASA**). In a Neutral geometry, if $\triangle ABC$ and $\triangle DEF$ are given with $\angle B \cong \angle E$, $\overline{AB} \cong \overline{DE}$, and $\angle A \cong \angle D$, then $\triangle ABC \cong \triangle DEF$.*

Theorem 2.2.2. *A protractor geometry which satisfies **ASA** is a neutral geometry.*

Hence, the ***SAS*** axiom can be replaced by the ***ASA*** axiom.

Theorem 2.2.3. *(**Side-Side-Side, SSS**). In a Neutral geometry, if $\triangle ABC$ and $\triangle DEF$ are given with $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, and $\overline{AC} \cong \overline{DF}$, then $\triangle ABC \cong \triangle DEF$.*

Theorem 2.2.4. *Protractor geometry in which **SSS**, the Triangle Inequality, and the Two circle Theorem are satisfied is a neutral geometry.*

Unfortunately, it is still unknown if protractor geometries which satisfies ***SSS*** is Neutral geometry.

Theorem 2.2.5. *(**Side-Side-Side, SAA**). In a Neutral geometry, if $\triangle ABC$ and $\triangle DEF$ are given with $\overline{AB} \cong \overline{DE}$, $\angle A \cong \angle D$, and $\angle C \cong \angle F$, then $\triangle ABC \cong \triangle DEF$.*

Theorem 2.2.6. *(**Hypotenuse-Leg, HL**). In a Neutral geometry, if $\triangle ABC$ and $\triangle DEF$ are right triangles with right angles at C and F, and if $\overline{AB} \cong \overline{DE}$ and $\overline{AC} \cong \overline{DF}$, then $\triangle ABC \cong \triangle DEF$.*

Chapter 3

Certain congruence criterions

Theorem 3.1. *Let $\triangle ABC$ and $\triangle A'B'C'$ be triangles in Neutral Geometry. If $\angle ABC = \angle A'B'C'$, $\angle CAB = \angle C'A'B'$ and $\triangle ABC$ has the same perimeter as $\triangle A'B'C'$, then $\triangle ABC \cong \triangle A'B'C'$.*

Proof. If $AB = A'B'$, then $\triangle ABC \cong \triangle A'B'C'$ by ASA. Without loss of generality, we will assume that $AB < A'B'$.

Construct a point D between A and B such that $BD = B'A'$. Next, draw a unique ray $\overrightarrow{DE}$ with the half plane of C and $\angle BDE = \angle B'A'C'$.

Then, we know that $\overleftrightarrow{DF} // \overleftrightarrow{AC}$ as a pair of corresponding angles is congruent. Since $\overleftrightarrow{DF}$ intersects $\overline{AB}$ which D is neither A nor B, $\overrightarrow{DE}$ meet $\overline{BC}$ at a point F.

Thus, $\triangle DBF \cong \triangle A'B'C'$ by ASA.

Now, we may join a segment $\overline{AF}$ and consider perimeter of $\triangle DBF$. Since $DF < AC + AD + FC$, $(BD + BF) + DF < (BD + AD) + (BF + FC) + AC = AB + BC + AC$ which is a contradiction. $\square$

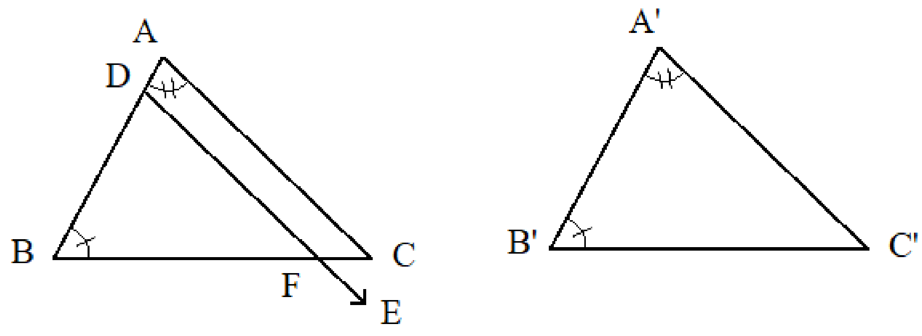

Figure 3.1: For the theorem 3.1

Theorem 3.2. *Let $\triangle ABC$ and $\triangle A'B'C'$ be triangles in Neutral Geometry. Let $\overline{AE}, \overline{A'E'}$ be the altitudes of $\triangle ABC$ and $\triangle A'B'C'$ respectively, and F, F' be midpoints of $\overline{BC}$ and $\overline{B'C'}$ respectively. If $BC = B'C', AE = A'E'$ and $AF = A'F'$, then $\triangle ABC \cong \triangle A'B'C'$.*

Proof. Suppose that $E = F$. It follows that $A'F' = AF = AE = A'E'$. Since $A'E'$ is perpendicular to $B'C'$, we have $E' = F'$. Consequently, $\triangle AEC \cong \triangle A'E'C'$ and $\triangle AEB \cong \triangle A'E'B'$. Thus $\triangle ABC \cong \triangle A'B'C'$.

Now we suppose that $E \neq F$, it follows that $\triangle AEF \cong \triangle A'E'F'$ by HS. Without loss of generality, let E, C be between of $\overleftrightarrow{AF}$ and E', C' be between of $\overleftrightarrow{A'F'}$ because B, C be between of $\overleftrightarrow{AF}$ and B', C' be between of $\overleftrightarrow{A'F'}$. Since $\angle AFE = \angle A'F'E'$, $\angle AFC = \angle A'F'C'$, so $\triangle AFC \cong \triangle A'F'C'$ by SAS. Thus $\angle ACF = \angle A'C'F'$. Hence $\triangle ABC \cong \triangle A'B'C'$ by SAS again.

□

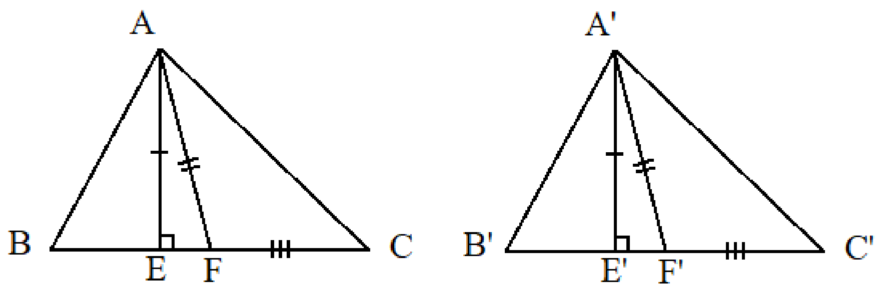

Figure 3.2: For the theorem 3.2

Theorem 3.3. *Let $\triangle ABC$ and $\triangle A'B'C'$ be triangles in Neutral Geometry. Let $\overline{AE}, \overline{A'E'}$ be the altitudes of $\triangle ABC$ and $\triangle A'B'C'$ respectively, and F, F' be points in $\overline{AC}$ and $\overline{A'C'}$ respectively such that $\overrightarrow{BF}$ and $\overrightarrow{B'F'}$ are angle bisectors.*
If $\angle ABC = \angle A'B'C'$, $BE = B'E'$ and $BF = B'F'$, then $\triangle ABC \cong \triangle A'B'C'$.

Proof. Suppose that $E = F$. It follows that $A'F' = AF = AE = A'E'$. Since $A'E'$ is perpendicular to $B'C'$, we have $E' = F'$. Consequently, $\triangle BEC \cong \triangle B'E'C'$ and $\triangle AEB \cong \triangle A'E'B'$. Thus $\triangle ABC \cong \triangle A'B'C'$.

Now we suppose that $E \neq F$, it follows that $\triangle AEF \cong \triangle A'E'F'$ by HS. Without loss of generality, let E, C be between of $\overleftrightarrow{AF}$ and E', C' be between of $\overleftrightarrow{A'F'}$ because B, C be between of $\overleftrightarrow{AF}$ and B', C' be between of $\overleftrightarrow{A'F'}$. Since $\angle AFE = \angle A'F'E'$, $\angle AFC = \angle A'F'C'$. By the assumption, we have $\angle FAC = \angle F'A'C'$, so $\triangle AFC \cong \triangle A'F'C'$ by ASA. Thus $\angle ACF = \angle A'C'F'$. Hence $\triangle ABC \cong \triangle A'B'C'$ by ASA again. $\square$

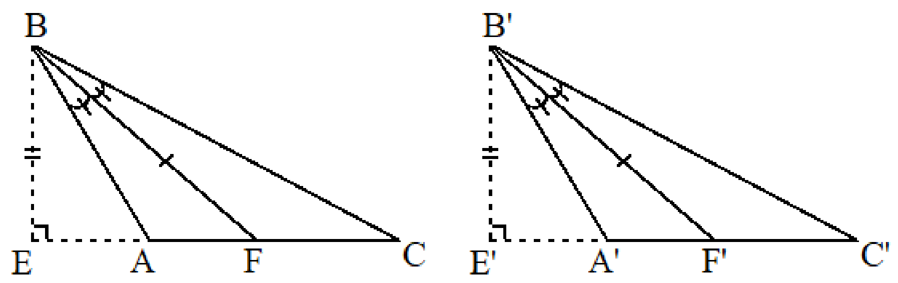

Figure 3.3: For the theorem 3.3

Theorem 3.4. *Let $\triangle ABC$ and $\triangle A'B'C'$ be triangles in Neutral Geometry, and D, D' be midpoints of $\overline{AC}$ and $\overline{A'C'}$ respectively. If $BD = B'D'$, $\angle DBC = \angle D'B'C'$, and $BC = B'C'$, then $\triangle ABC \cong \triangle A'B'C'$.*

Proof. First we know that $\triangle DBC \cong \triangle D'B'C'$ by SAS. Hence, $\angle BCA = \angle B'C'A'$ and $DC = D'C'$. Since D, D' are midpoints of $\overline{AC}$ and $\overline{A'C'}$ respectively, we have $AC = A'C'$. Thus, $\triangle ABC \cong \triangle A'B'C'$ by SAS again. $\square$

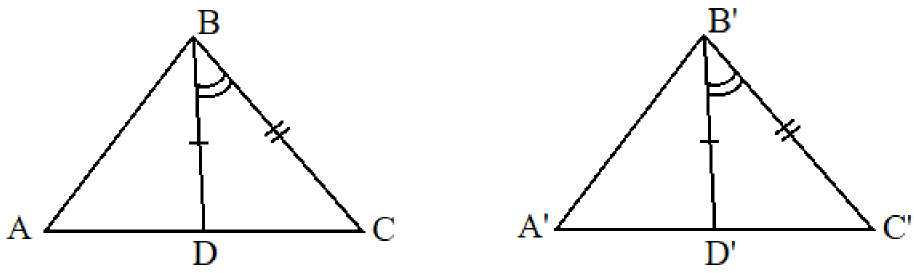

Figure 3.4: For the theorem 3.4

For convenience, We will denote Median-Angle-Side (MAS) for two triangles $\triangle ABC$ and $\triangle A'B'C'$ such that $BD = B'D'$, $\angle DBC = \angle D'B'C'$, and $BC = B'C'$ where D and D' are midpoints of $\overline{AC}$ and $\overline{A'C'}$ respectively.

Theorem 3.5. *A protractor geometry which satisfies MAS is a neutral geometry. That is, SAS is equivalent to MAS in a protractor geometry.*

Proof. Let $\triangle ABC$ and $\triangle A'B'C'$ be two triangles in a protractor geometry with $AB = A'B'$, $\angle ABC = \angle A'B'C'$, and $BC = B'C'$. Construct 2 points D, D' such that $D - A - C$, $D' - A' - C'$ with $DA = AC$, $D'A' = A'C'$. Consequently, $\overline{BA}$, $\overline{B'A'}$ are both medians of $\triangle DBC$ and $\triangle D'B'C'$ respectively. Thus, $\triangle DBC \cong \triangle D'B'C'$ by the assumption. Then, we have $DC = D'C'$ and $\angle DCB = \angle D'C'B'$, so $AC = A'C'$ and $\angle ACB = \angle A'C'B'$. Similarly, we obtain $\angle CAB = \angle C'A'B'$ by constructing 2 points E, E' which $A - C - E$, $A' - C' - E'$ and $CE = AC$, $C'E' = A'C'$. □

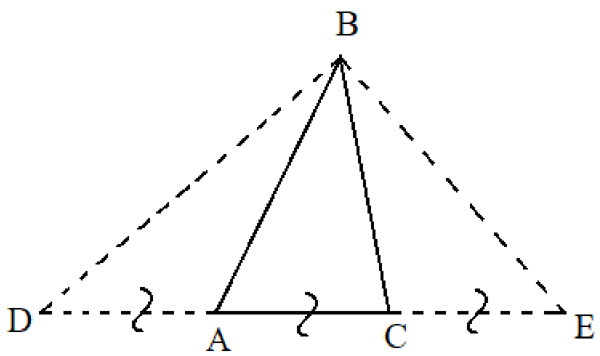

Figure 3.5: For the theorem 3.5

References

[1] I.L. Heiberg *Euclid's element of geometry* by Richard Fitzpatrick, 2007.

[2] Millman RS, Parker GD. *Geometry a metric approach with models*. 2^{nd} Ed. New York: Spinger-Verlag, 1991.

[3] Brannan DA, Esplen MF, Gray JJ. *Geometry*. 1^{st} Ed. United Kingdom: The Open University; 1999.

Appendix

The Project Proposal of Course 2301399 Project Proposal

First Semester, Academic Year 2018

Title (Thai)	เกณฑ์การเท่ากันทุกประการในเรขาคณิตสัมบูรณ์
Title (English)	Congruence Criterion in Neutral Geometry
Advisor	Associate Professor Nataphan Kitisin, Ph.D.
By	Mr. Silipong Thongmeepun ID 5833543023
	Mathematics, Department of Mathematics and Computer Science

Background and Rationale

Euclid laid the foundation of Geometry in his important books called *The Elements* [1]. The Elements consists of 13 books which contain definitions, axioms, theorems concerning various aspects in Geometry. The Elements are widely considered to be the very first Mathematical text books. Later on, the ideas of Euclid have been polished in more contemporary contexts. We give a more refined version of Euclid's idea which leads to the notion of Neutral Geometry as follows.

Neutral Geometry is defined in [2] as the followings. An ordered pair $(\mathcal{S}, \mathcal{L})$ which consists a set of points $\mathcal{S}$ and a collection $\mathcal{L}$ of subsets of $\mathcal{L}$ which are called line. $(\mathcal{S}, \mathcal{L})$ is an **abstract geometry** if every line contains at least two points, and for any two points, there exists a line containing them.

An abstract geometry is an **incident geometry** if for any two distinct points, there exists a unique line containing them, and there exist three points such that no line containing all of them.

A **distance function** on a set is a function $d : \mathcal{S} \times \mathcal{S} \to \mathbb{R}$ such that $d(P, Q) \geq 0$, $d(P, Q) = d(Q, P)$, and $d(P, Q) = 0$ if and only if $P = Q$ for all points P,Q in $\mathcal{S}$. An incident geometry equipped with distance function is a **metric geometry** if every line l has a bijection $f : l \to \mathbb{R}$ such that $|f(P) - f(Q)| = d(P, Q)$ for any two points P,Q in $\mathcal{S}$. We will denote $|PQ|$ for $d(P, Q)$.

A metric geometry is a **Pasch geometry** if for any triangle $\triangle ABC$ and line l such that $l \cap \overline{AB} \neq \emptyset$, then either $l \cap \overline{AC} \neq \emptyset$ or $l \cap \overline{BC} \neq \emptyset$.

An **angle measure** is a function m from the set of all angle to $(0, 180)$ such that if $D \in int(A\hat{B}C)$, then $m(A\hat{B}D) + m(D\hat{B}C) = m(A\hat{B}C)$, and for all $r \in (0, 180)$, there is $C \notin \overleftrightarrow{AB}$ where **m**$(A\hat{B}C) = r$. A **Pasch geometry** equipped with angle measure is called **Protractor geometry**. we will denote $A\hat{B}C$ for $m(A\hat{B}C)$.

Two triangles in Protractor geometry are **congruent** if three sides and three angles in one triangle are congruent to the others in another triangle.
Protractor geometry satisfies **SAS axiom(Side-Angle-Side)** if for any $\triangle ABC$ and $\triangle DEF$ such that $|AB| = |DE|$, $|BC| = |EF|$ and $A\hat{B}C = D\hat{E}F$, then $\triangle ABC$ and $\triangle DEF$ are congruent. **Neutral geometry** is a protractor geometry which satisfying SAS axiom. However, it can be show that in Protractor Geometry SAS Axiom is equivalent to **ASA(Angle-Side-Angle) and AAS(Angle-Angle-Side)**.

In Neutral Geometry, **SSA (Side-Side-Angle)** does not always hold.
However, SSA in Neutral Geometry can be satisfied provided the triangle is acute.

Therefore, it is an interesting problem to find the criterion that can guarantee the congruence of two triangles.

Objectives

Establish certain congruence criterions in Neutral Geometry.

Project Activities

1. Review the elementary knowledge in Modern Geometry.
2. Study the congruence criterions.
3. Conjecture congruence criterions and try to prove or disprove.
4. Investigate whether the criterion is equivalent to SAS axiom.
5. Write a report.

Activities Table

Project ActivitiesProject Ac-tivitiesProject Activitiespt<=-Project Activitiespt>=	August 2018 - April 2019								
	Aug	Sep	Oct	Nov	Dec	Jan	Feb	Mar	Apr
1. Review the elementary knowledge in Modern Geometry.									
2. Study the congruence criterions.									
3. Conjecture congruence criterions and try to prove or disprove.									
4. Investigate whether the criterion is equivalent to SAS axiom.									
5. Write a report.									

Benefits

To obtain certain congruence criterions in Neutral Geometry.

Equipment

1. Computer
2. Printer
3. Stationary
4. Paper

References

[1] I.L. Heiberg *Euclid's element of geometry* by Richard Fitzpatrick, 2007.

[2] Millman RS, Parker GD. *Geometry a metric approach with models*. 2nd Ed. New york: Spinger-Verlag, 1991.

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