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Numerical methods for a stochastic generalized logistic model

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NUMERICAL METHODS FOR A STOCHASTIC GENERALIZED LOGISTIC MODEL

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In this project, we study the logistic model which has been developed in the form of a stochastic differential equation model. We will compare the performance of six numerical methods, namely Euler-Maruyama, two-point simplified weak Euler, three-point simplified weak Euler, simplified order 2.0 weak Taylor, explicit order 2.0 weak and order 2.0 weak predictor-corrector with the generalized logistic stochastic differential equation model in terms of the weak order of convergence and the run time.

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CHAPTER I

INTRODUCTION

The logistic model has been widely used for a long time because of its ease of use and effectiveness. The logistic model is used in many fields such as economics, ecology, demography, sociology and biomathematics. The logistic model [6] generally has the form

$$dX_t = rX_t\left(1 - \frac{X_t}{K}\right)dt, \tag{1.1}$$

where X_t is the target process,
K is the carrying capacity,
r is the intrinsic growth rate.

This model is an ordinary differential equation (ODE). Here, X_t represents the number of the target population, and $\frac{dX_t}{dt}$ is the population growth rate. The carrying capacity means the maximum population size, and the parameter r affects the overall population growth rate. A solution of this ODE has an S-shaped curve. Figure 1.1 shows an example of a solution of the logistic model.

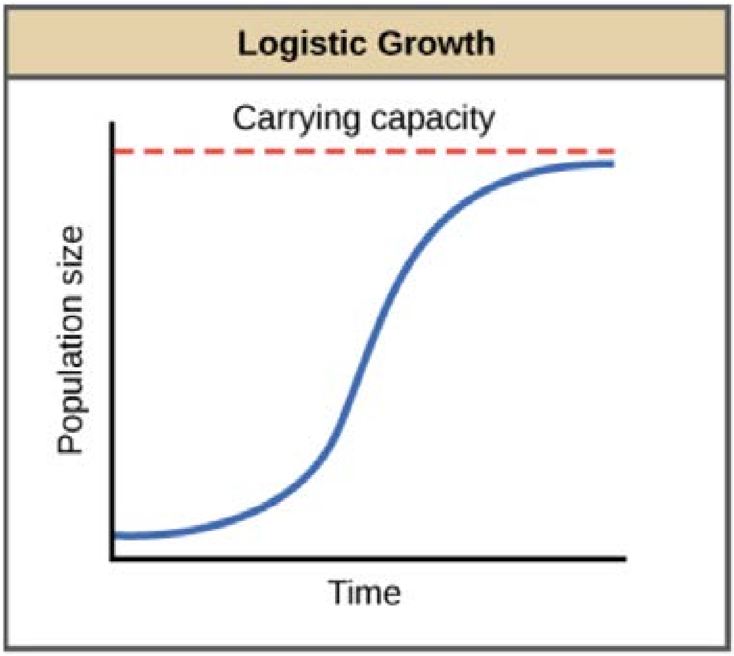

Figure 1.1: A general form of a logistic model.

Image credit: Environmental limits to population growth: Figure 1.1, by OpenStax College, Biology, CC BY 4.0.

The logistic model has been developed as a stochastic differential equation (SDE) model. The so-called generalized logistic SDE (GLSDE) has the form

$$dX_t = rX_t\left(1 - \left(\frac{X_t}{K}\right)^m\right)dt + \sigma X_t dW_t\,, \tag{1.2}$$

where X_t is the target process,
K is the carrying capacity,
r is the intrinsic growth rate,
m and σ are parameters in the model,
W_t is the Wiener process.

The parameter $m \in (0, \infty)$ affects the S-shaped curve of the GLSDE model. Figure 1.2-1.4 illustrate such effect, and we can see that the higher m, the more S-shaped curve.

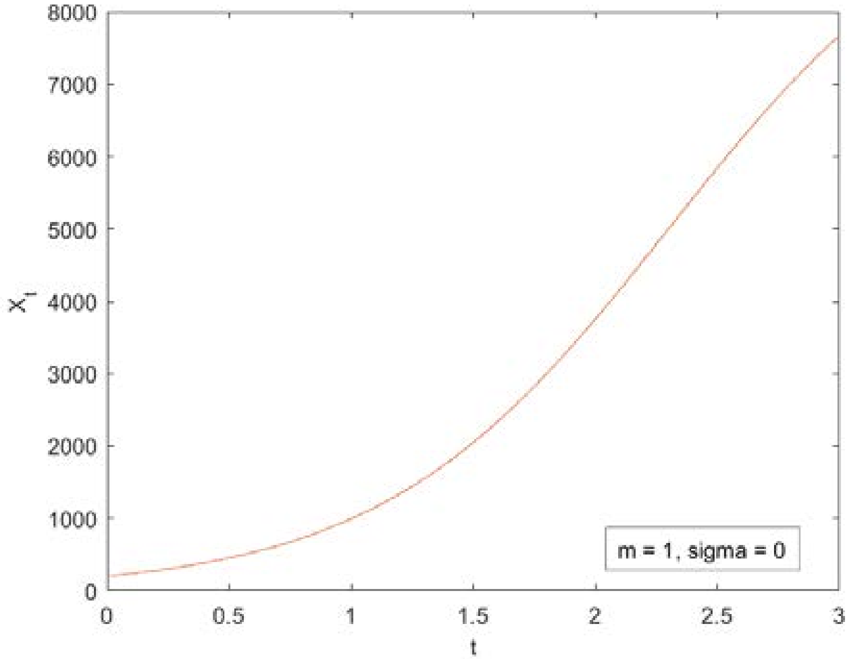

Figure 1.2: A solution of GLSDE with $m = 1, \sigma = 0, K = 10,000, r = 1.7$.

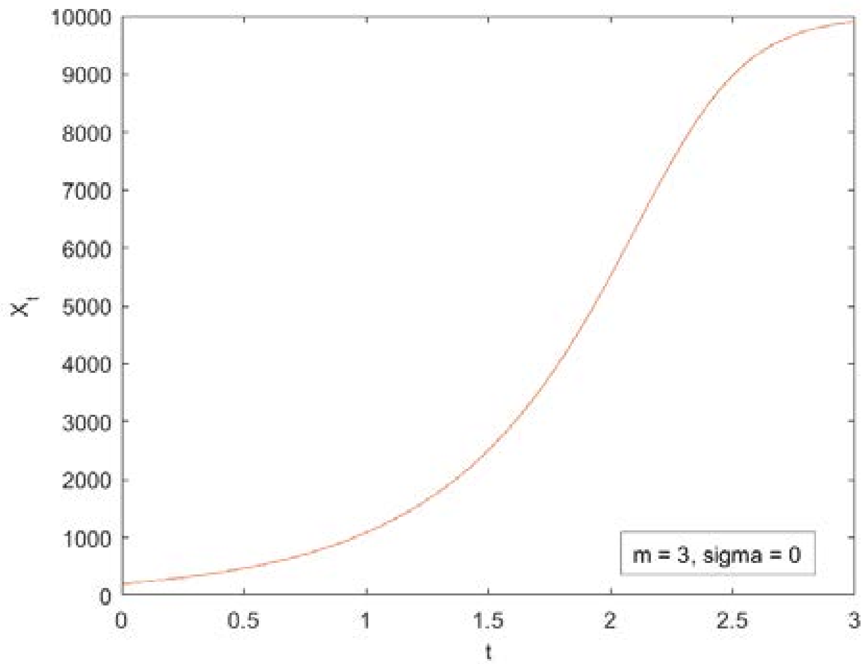

Figure 1.3: A solution of GLSDE with $m = 3, \sigma = 0, K = 10,000, r = 1.7$.

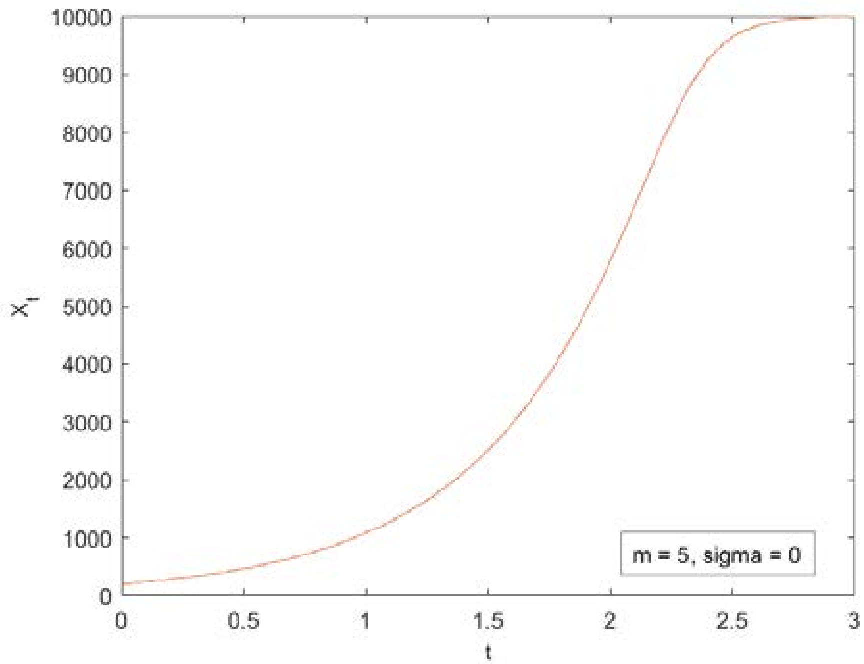

Figure 1.4: A solution of GLSDE with $m = 5, \sigma = 0, K = 10,000, r = 1.7$.

When the parameter $\sigma = 0$, this model is just an ODE. The term $\sigma X_t dW_t$ adds stochasticity to the model and is interpreted as the Ito stochastic integral. The noise in the target process vary accordingly to the target process itself multiplied by the parameter σ.

The explicit solution of the GLSDE is known [5] and has the form

$$X_t = K\Phi_t^{-1/m} \left\{ \left(\frac{X_0}{K} \right)^{-m} + rm \int_0^t \Phi_s^{-1} ds \right\}^{-1/m}, \tag{1.3}$$

where $\Phi_t = e^{\left(-rm + \frac{\sigma^2 m}{2}\right)t - \sigma m W_t}$ and $X_0 \in \mathbb{R}$.

In chapter 2, we talk about the background knowledge of SDEs, numerical methods and weak order of convergence. In chapter 3, we simulate the GLSDE using six numerical methods: Euler-Maruyama, two-point simplified weak Euler, three-point simplified weak Euler, simplified order 2.0 weak Taylor, explicit order 2.0 weak and order 2.0 weak predictor-corrector. We also numerically find the weak orders of convergence and run time of these six methods for the GLSDE model by approximating the explicit solution with harmonic mean and arithmetic mean. In chapter 4 and chapter 5, we show results and conclusion, respectively.

CHAPTER II

PRELIMINARIES

In this chapter, we describe the background knowledge and the concept for our project which are stochastic differential equations, numerical methods and weak order of convergence.

2.1 Introduction to stochastic differential equations

Definition 2.1 Standard Brownian motion.A scalar *standard Brownian motion* or *standard Wiener process*, over $[0, T]$ is a collection of random variables $\{W_t\}_{t \in [0,T]}$ that satisfies the following four conditions.

1. $W_0 = 0$ (with probability 1).
2. For $0 \le s < t \le T$, the random variable given by the increment $W_t - W_s$ is normally distributed with mean zero and variance $t - s$. This is equivalent to $W_t - W_s \sim \sqrt{t-s}\, N(0, 1)$, where $N(0, 1)$ denotes a normally distributed random variable with zero mean and unit variance.
3. For $0 \le s < t \le u < v \le T$, the increments $W_t - W_s$ and $W_v - W_u$ are independent.
4. $\{W_t\}_{t \in [0,T]}$ has continuous sample paths.

Definition 2.2 Stochastic differential equations. An SDE generally has the form

$$\begin{aligned} dX_t &= f(X_t)dt + g(X_t)dW_t, \qquad 0 \le t \le T \\ X_0 &= x, \end{aligned} \tag{2.1}$$

where f and g are scalar functions of X_t , W_t is a Wiener process, and $x \in \mathbb{R}$.
The equation (2.1) is understood to be the differential form of the integral equation

$$X_t - X_0 = \int_0^t f(X_s)\, ds + \int_0^t g(X_s)\, dW_s \tag{2.2}$$

where the first term on the right-hand side of the above equation is interpreted as the ordinary Riemann integral, and the second term is the *Itô* integral [4].

2.2 Numerical Methods

Numerical methods used in this project are based on [3]. Assume that we have an SDE

$$dX_t = f(X_t)dt + g(X_t)dW_t.$$

2.2.1 Euler-Maruyama (EM) method

The simplest numerical approximation for solving an SDE is the Euler-Maruyama method. We need to discretize the time domain [0,T] into N equidistance subintervals. Let $t_n = n\Delta$ for all $n = 0, 1, .., N$ where $\Delta = \frac{T}{N}$. We denote x_n to be the numerical solution at time step t_n using the *Euler-Maruyama scheme*

$$x_{n+1} = x_n + f(x_n)\Delta + g(x_n)\,\Delta W_n\,, \tag{2.3}$$

where ΔW_n is normally distributed with mean 0 and variance Δ and $x_0 = X_0$.

2.2.2 Two-point simplified weak Euler (SE) method

This method is similar to the EM method but it will be much easier. We can replace the Wiener increments ΔW_n in (2.3) by another random variable $\Delta\widetilde{W}_n$ with similar moment properties. We can thus obtain a simpler scheme by choosing more easily generated noise increments. This leads to the *simplified weak Euler scheme*

$$x_{n+1} = x_n + f(x_n)\,\Delta + g(x_n)\,\Delta\widetilde{W}_n \tag{2.4}$$

where $\Delta\widetilde{W}_n$ satisfies the condition

$$\left|E\Big(\Delta\widetilde{W}_n\Big)\right| + \left|E\Big((\Delta\widetilde{W}_n)^3\Big)\right| + \left|E\Big((\Delta\widetilde{W}_n)^2\Big) - \Delta\right| \leq K\Delta^2 \tag{2.5}$$

for some constant K. A very easy example of such $\Delta\widetilde{W}_n$ in (2.4) is a two-point distributed random variable with

$$P(\Delta\widetilde{W}_n = \pm\sqrt{\Delta}) = \frac{1}{2}.$$

We will call the scheme (2.4) with this two-point distribution $\Delta\widetilde{W}_n$ the two-point simplified weak Euler method.

2.2.3 Three-point simplified weak Euler (WE) method

The scheme of this method is the same as the *two-point simplified weak Euler method* but $\Delta\widehat{W}_n$ is a three-point distributed random variable. This leads to the *three-point simplified weak Euler scheme*

$$x_{n+1} \;=\; x_n \;+\; f(x_n)\,\Delta \;+\; g(x_n)\,\Delta\widehat{W}_n\,,$$

where $\Delta\widehat{W}_n$ is a three-point distributed random variable with

$$P(\Delta\widehat{W}_n = \pm\sqrt{3\Delta}) = \frac{1}{6}, \qquad P(\Delta\widehat{W}_n = 0) = \frac{2}{3}.$$

Note that this distribution satisfies the condition (2.5).

2.2.4 Simplified order 2.0 weak Taylor (ST) method

We have the *simplified order 2.0 weak Taylor scheme*

$$\begin{aligned} x_{n+1} =& \, x_n \;+\; f(x_n)\,\Delta \;+\; g(x_n)\,\Delta\widehat{W}_n \;+\; \frac{1}{2}\,g(x_n)\,g'(x_n)\,\Big\{(\Delta\widehat{W}_n)^2 \;-\; \Delta\Big\} \\ &+\; \frac{1}{2}\left\{f'(x_n)\,g(x_n) \;+\; f(x_n)\,g'(x_n) \;+\; \frac{1}{2}\,g''(x_n)\,g(x_n)^2\right\}\,\Delta\widehat{W}_n\,\Delta \\ &+\; \frac{1}{2}\left\{f(x_n)\,f'(x_n) \;+\; \frac{1}{2}\,f''(x_n)\,g(x_n)^2\right\}\,\Delta^2 \end{aligned} \tag{2.6}$$

where $\Delta\widehat{W}_n$ satisfies the condition

$$\begin{aligned} &\left|E\Big(\Delta\widehat{W}_n\Big)\right| + \left|E\Big((\Delta\widehat{W}_n)^3\Big)\right| + \left|E\Big((\Delta\widehat{W}_n)^5\Big)\right| \\ &\qquad + \left|E\Big((\Delta\widehat{W}_n)^2\Big) - \Delta\right| + \left|E\Big((\Delta\widehat{W}_n)^4\Big) - 3\Delta^2\right| \le K\Delta^3 \end{aligned} \tag{2.7}$$

for some constant K. The random variable $\Delta W_n \sim N(0,\Delta)$ certainly satisfies the moment condition (2.7), and so does a three-point distributed random variable $\Delta\widehat{W}_n$ with

$$P(\Delta\widehat{W}_n = \pm\sqrt{3\Delta}) = \frac{1}{6}, \qquad P(\Delta\widehat{W}_n = 0) = \frac{2}{3}.$$

2.2.5 Explicit order 2.0 weak (EW) method

Platen proposed the following *explicit order 2.0 weak scheme*

$$
\begin{aligned}
x_{n+1} \,=\, x_n \,+\, & \frac{1}{2}\, \big(f(\bar{\Upsilon}) \,+\, f(x_n)\big)\, \Delta \\
+\, & \frac{1}{4}\, \big(g(\bar{\Upsilon}^+) \,+\, g(\bar{\Upsilon}^-) \,+\, 2\, g(x_n)\big)\, \Delta \widehat{W_n} \\
+\, & \frac{1}{4}\, \big(g(\bar{\Upsilon}^+) \,-\, g(\bar{\Upsilon}^-)\big) \left\{ \left(\Delta \widehat{W_n}\right)^2 \,-\, \Delta \right\} \Delta^{-1/2}
\end{aligned}
\tag{2.8}
$$

with supporting values

$$
\bar{\Upsilon} \,=\, x_n \,+\, f(x_n)\, \Delta \,+\, g(x_n)\, \Delta \widehat{W_n},
$$

and

$$
\begin{aligned}
\bar{\Upsilon}^+ \,&=\, x_n \,+\, f(x_n)\, \Delta + \, g(x_n)\, \sqrt{\Delta}, \\
\bar{\Upsilon}^- \,&=\, x_n \,+\, f(x_n)\, \Delta - \, g(x_n)\, \sqrt{\Delta},
\end{aligned}
$$

where $\widehat{W_n}$ is a three-point distributed random variable with

$$
P(\Delta \widehat{W_n} = \pm\sqrt{3\Delta}) = \frac{1}{6}, \qquad P(\Delta \widehat{W_n} = 0) = \frac{2}{3}.
$$

Note that this distribution satisfies the condition (2.7).

2.2.6 Order 2.0 weak predictor-corrector (PC) method

The *order 2.0 weak predictor-corrector method* has corrector scheme

$$
x_{n+1} = x_n \,+ \frac{1}{2} \left\{ f(\bar{x}_{n+1}) + f(x_n) \right\} \Delta + \Psi_n \tag{2.9}
$$

with

$$
\begin{aligned}
\Psi_n = g(x_n)\, \Delta \widehat{W_n} \,+\, & \frac{1}{2}\, g(x_n)\, g'(x_n) \left\{ (\Delta \widehat{W_n})^2 \,-\, \Delta \right\} \\
+\, \frac{1}{2} & \left(f(x_n)\, g'(x_n) \,+\, \frac{1}{2}\, g(x_n)^2\, g''(x_n) \right) \Delta \widehat{W_n} \Delta,
\end{aligned}
$$

and predictor

$$\bar{x}_{n+1} = x_n + f(x_n)\,\Delta + \Psi_n + \frac{1}{2}\,f'(x_n)\,g(x_n)\,\Delta\widehat{W}_n\Delta$$
$$+ \frac{1}{2}\left(f(x_n)\,f'(x_n) + \frac{1}{2}\,f''(x_n)\,g(x_n)^2\right)\Delta^2,$$

where $\widehat{W}_n$ is a three-point distributed random variable with

$$P(\Delta\widehat{W}_n = \pm\sqrt{3\Delta}) = \frac{1}{6}, \qquad P(\Delta\widehat{W}_n = 0) = \frac{2}{3}.$$

2.3 Weak order of convergence

For a numerical method with the numerical solution x_n to the exact solution X_{t_n}, it is said to have *weak order of convergence* equal to α, if there exists a constant C such that

$$|E[X_{t_n}] - E[x_n]| \leq C\Delta^{\alpha}$$

for all $n = 1, 2, \ldots, N$ and for all sufficiently small Δ. The left-hand side is called the *weak error*, $e_n^{\Delta} = |E[X_{t_n}] - E[x_n]|$. $E[X_{t_n}]$ is the expected value of the exact solution and $E[x_n]$ is the expected value of the numerical solution.

Normally, Δ is a small number, so, the greater α is, the smaller e_n will be. Thus, a *high* weak order of convergence indicates that the corresponding numerical method has *high* accuracy. According to the theory, the EM, SE and WE methods have weak order of convergence equal to 1. The ST, EW and PC methods have weak order of convergence equal to 2. In this work, we will numerically test the performance of these six numerical methods.

We can use the Monte Carlo approach to numerically find the weak order of convergence for a numerical method. First, we select many time steps Δ. For each Δ, we simulate numerical solutions using the numerical method. Then, we find the average of the numerical solutions to approximate $E[x_n]$ so that we can find the weak error $e_n^{\Delta} = |E[X_{t_n}] - E[x_n]|$ provided that $E[X_{t_n}]$ is known or can be approximated. Now, let's consider the weak error. Taking the logarithm of both sides of the inequality

$$e_n^{\Delta} \leq C\Delta^{\alpha},$$

we have

$$log\, e_n^{\Delta} \leq log\, C + \alpha\, log\Delta.$$

In practice, α is approximately the slope of the graph between the size of $log\Delta$ and the logarithm of the weak error, $log\, e_n^{\Delta}$. Thus, we can find the line of best fit for the regression model

$$log\, e_n^{\Delta} \approx log\, C + \alpha\, log\Delta,$$

and the weak order of convergence for this numerical method is approximately the slope of the line of best fit.

CHAPTER III

METHODOLOGY

In this chapter, we use MATLAB to simulate the GLSDE for six numerical methods: Euler-Maruyama, two-point simplified weak Euler, three-point simplified weak Euler, simplified order 2.0 weak Taylor, explicit order 2.0 weak and order 2.0 weak predictor-corrector. We also estimate the explicit solution by using harmonic mean and arithmetic mean in order to find the mean value of GLSDE and weak order of convergence of each method.

3.1 The simulation of the GLSDE

In this section, we set the parameters as follows $X_0 = 200$, $K = 10000$, $r = 1.7$ (these parameters are used in [1]), $m = 3$ and $\sigma = 0.3$. For each of the ten stepsizes $\Delta = 2^{-14}, 2^{-13}, 2^{-12}, ..., 2^{-5}$, we generate 100,000 discretized sample paths over $[0, 1]$. Let N be the number of discretization intervals, so $N = \frac{T}{\Delta}$. Next, we use the EM, SE, WE, ST, EW and PC schemes described in chapter 2 to simulate sample paths for the process X_t from (1.2). For every method, we first set $x_0 = X_0$. For $n = 0, 1, \ldots, N-1$, we will update x_{n+1} recursively according to each of the six numerical methods.

3.1.1 Euler - Maruyama (EM) method

The EM scheme for X_t has the form

$$x_{n+1} = x_n + r x_n \left(1 - \left(\frac{x_n}{K}\right)^m\right) \Delta + \sigma x_n \Delta W_n \,,$$

where $\Delta W_n \sim N(0, \Delta)$.

3.1.2 Two-point simplified weak Euler (SE) method

The SE scheme for X_t has the form

$$x_{n+1} = x_n + r x_n \left(1 - \left(\frac{x_n}{K}\right)^m\right) \Delta + \sigma x_n \Delta \widetilde{W}_n \,,$$

where $\Delta \widetilde{W}_n$ is a two-point distributed random variable with

$$P(\Delta \widetilde{W}_n = \pm\sqrt{\Delta}) = \frac{1}{2}.$$

3.1.3 **Three-point simplified weak Euler (WE) method**

The WE scheme for X_t has the form

$$x_{n+1} = x_n + rx_n \left(1 - \left(\frac{x_n}{K}\right)^m\right) \Delta + \sigma x_n \Delta\widehat{W}_n \,,$$

where $\Delta\widehat{W}_n$ is a three-point distributed random variable with

$$P(\Delta\widehat{W}_n = \pm\sqrt{3\Delta}) = \frac{1}{6}, \qquad P(\Delta\widehat{W}_n = 0) = \frac{2}{3}.$$

3.1.4 **Simplified order 2.0 weak Taylor (ST) method**

$$f(x_n) = rx_n \left(1 - \left(\frac{x_n}{K}\right)^m\right) \qquad g(x_n) = \sigma x_n \,,$$

$$f'(x_n) = r \left(1 - (m+1) \left(\frac{x_n}{K}\right)^m\right) \qquad g'(x_n) = \sigma \,,$$

$$f''(x_n) = \frac{-rm(m+1)}{x_n} \left(\frac{x_n}{K}\right)^m \qquad g''(x_n) = 0.$$

The ST scheme for X_t has the form

$$\begin{aligned} x_{n+1} &= x_n + rx_n \left(1 - \left(\frac{x_n}{K}\right)^m\right) \Delta + \sigma x_n \Delta\widehat{W}_n + \frac{1}{2}\sigma^2 x_n \left\{(\Delta\widehat{W}_n)^2 - \Delta\right\} \\ &+ \frac{1}{2}\left[rx_n\sigma \left(2 - (m+2)\left(\frac{x_n}{K}\right)^m\right)\right] \Delta\widehat{W}_n \Delta \\ &+ \frac{1}{2}\Bigg(r^2 x_n \left[1 - (m+2)\left(\frac{x_n}{K}\right)^m + (m+1)\left(\frac{x_n}{K}\right)^{2m}\right] \\ &\qquad - \frac{1}{2} rm(m+1)\sigma^2 x_n \left(\frac{x_n}{K}\right)^m \Bigg)\Delta^2, \end{aligned}$$

where $\Delta\widehat{W}_n$ is a three-point distributed random variable with

$$P(\Delta\widehat{W}_n = \pm\sqrt{3\Delta}) = \frac{1}{6}, \qquad P(\Delta\widehat{W}_n = 0) = \frac{2}{3}.$$

3.1.5 **Explicit order 2.0 weak (EW) method**

$$
\begin{aligned}
f(\bar{\Upsilon}) =& r\left(x_n + rx_n\left(1-\left(\frac{x_n}{K}\right)^m\right)\Delta + \sigma x_n \Delta\widehat{W}_n\right) \\
& \left(1-\left[\frac{x_n + rx_n\left(1-\left(\frac{x_n}{K}\right)^m\right)\Delta + \sigma x_n \Delta\widehat{W}_n}{K}\right]^m\right), \\
g(\bar{\Upsilon}^+) =& \sigma\left(x_n + rx_n\left(1-\left(\frac{x_n}{K}\right)^m\right)\Delta + \sigma x_n\sqrt{\Delta}\right), \\
g(\bar{\Upsilon}^-) =& \sigma\left(x_n + rx_n\left(1-\left(\frac{x_n}{K}\right)^m\right)\Delta - \sigma x_n\sqrt{\Delta}\right),
\end{aligned}
$$

The EW scheme for X_t has the form

$$
\begin{aligned}
x_{n+1} = x_n + \frac{1}{2}\Bigg[& r\left(x_n + rx_n\left(1-\left(\frac{x_n}{K}\right)^m\right)\Delta + \sigma x_n \Delta\widehat{W}_n\right) \\
& \cdot\left(1-\left[\frac{x_n + rx_n\left(1-\left(\frac{x_n}{K}\right)^m\right)\Delta + \sigma x_n \Delta\widehat{W}_n}{K}\right]^m\right) \\
& + rx_n\left(1-\left(\frac{x_n}{K}\right)^m\right)\Bigg]\Delta \\
& + \sigma x_n\left(1+\frac{r}{2}\left(1-\left(\frac{x_n}{K}\right)^m\right)\Delta\right)\Delta\widehat{W}_n \\
& + \frac{1}{2}\sigma^2 x_n\sqrt{\Delta}\left\{(\Delta\widehat{W}_n)^2 - \Delta\right\}\Delta^{-1/2},
\end{aligned}
$$

where $\Delta\widehat{W}_n$ is a three-point distributed random variable with

$$
P(\Delta\widehat{W}_n = \pm\sqrt{3\Delta}) = \frac{1}{6}, \qquad P(\Delta\widehat{W}_n = 0) = \frac{2}{3}.
$$

3.1.6 **Order 2.0 weak predictor-corrector (PC) method**

$$
\begin{aligned}
f(x_n) &= rx_n\left(1-\left(\frac{x_n}{K}\right)^m\right) & g(x_n) &= \sigma x_n\,, \\
f'(x_n) &= r\left(1-(m+1)\left(\frac{x_n}{K}\right)^m\right) & g'(x_n) &= \sigma\,, \\
f''(x_n) &= \frac{-rm(m+1)}{x_n}\left(\frac{x_n}{K}\right)^m & g''(x_n) &= 0\,,
\end{aligned}
$$

$$
\Psi_n = \sigma x_n \Delta\widehat{W}_n + \frac{1}{2}\sigma^2 x_n\left\{(\Delta\widehat{W}_n)^2 - \Delta\right\} + \frac{1}{2}\sigma r x_n\left(1-\left(\frac{x_n}{K}\right)^m\right)\Delta\widehat{W}_n\Delta,
$$

$$\bar{x}_{n+1} = x_n + rx_n\left(1-\left(\frac{x_n}{K}\right)^m\right)\Delta + \Psi_n + \frac{1}{2}\sigma r x_n\left(1-(m+1)\left(\frac{x_n}{K}\right)^m\right)\Delta\widehat{W}_n\Delta$$
$$+\frac{1}{2}\Bigg(r^2x_n\left(1-(m+2)\left(\frac{x_n}{K}\right)^m+(m+1)\left(\frac{x_n}{K}\right)^{2m}\right)$$
$$-\frac{1}{2}rm(m+1)\sigma^2x_n\left(\frac{x_n}{K}\right)^m\Bigg)\Delta^2,$$

The PC scheme for X_t has the form

$$x_{n+1} = x_n + \frac{1}{2}\left[r\bar{x}_{n+1}\left(1-\left(\frac{\bar{x}_{n+1}}{K}\right)^m\right) + rx_n\left(1-\left(\frac{x_n}{K}\right)^m\right)\right]\Delta$$
$$+\sigma x_n\Delta\widehat{W}_n + \frac{1}{2}\sigma^2x_n\left\{(\Delta\widehat{W}_n)^2-\Delta\right\}$$
$$+\frac{1}{2}\sigma r x_n\left(1-\left(\frac{x_n}{K}\right)^m\right)\Delta\widehat{W}_n\Delta,$$

where $\Delta\widehat{W}_n$ is a three-point distributed random variable with

$$P(\Delta\widehat{W}_n = \pm\sqrt{3\Delta}) = \frac{1}{6}, \qquad P(\Delta\widehat{W}_n = 0) = \frac{2}{3}.$$

3.2 Mean value of GLSDE

In this section, we will approximate the mean value of the explicit solution from (1.3) to numerically find weak order of convergence. We use harmonic mean and arithmetic mean to approximate the explicit solution.

3.2.1 Harmonic mean

From (1.3), we have the problem in the integral part of the solution. We can approximate

$$\int_{(i-1)\Delta}^{i\Delta}\frac{1}{\Phi_s}ds \approx \int_{(i-1)\Delta}^{i\Delta}\frac{1}{\frac{\Phi_{(i-1)\Delta}+\Phi_{i\Delta}}{2}}ds.$$

Hence,

$$
\begin{aligned}
\int_0^T \Phi_s^{-1} ds &= \int_0^T \frac{1}{\Phi_s} ds \\
&= \int_0^{\Delta} \frac{1}{\Phi_s} ds + \int_{\Delta}^{2\Delta} \frac{1}{\Phi_s} ds + \cdots + \int_{(N-1)\Delta}^{N\Delta} \frac{1}{\Phi_s} ds \\
&\approx \int_0^{\Delta} \frac{1}{\frac{\Phi_0+\Phi_\Delta}{2}} ds + \int_{\Delta}^{2\Delta} \frac{1}{\frac{\Phi_\Delta+\Phi_{2\Delta}}{2}} ds + \cdots + \int_{(N-1)\Delta}^{N\Delta} \frac{1}{\frac{\Phi_{(N-1)\Delta}+\Phi_{N\Delta}}{2}} ds \\
&= \frac{2}{\Phi_0+\Phi_\Delta} \int_0^{\Delta} 1\, ds + \cdots + \frac{2}{\Phi_{(N-1)\Delta}+\Phi_{N\Delta}} \int_{(N-1)\Delta}^{N\Delta} 1\, ds \\
&= \Delta \left(\frac{2}{\Phi_0+\Phi_\Delta} + \frac{2}{\Phi_\Delta+\Phi_{2\Delta}} + \frac{2}{\Phi_{2\Delta}+\Phi_{3\Delta}} + \cdots + \frac{2}{\Phi_{(N-1)\Delta}+\Phi_{N\Delta}} \right) \\
&= 2\Delta \sum_{i=1}^{N} \frac{1}{\Phi_{(i-1)\Delta}+\Phi_{i\Delta}}
\end{aligned}
$$

Therefore, we can estimate the explicit solution of (1.3). Note that in this project, we concentrate on the final time T, so we want to find $E[X_T]$.

From (1.3), we have

$$
X_T \approx K\Phi_T^{-1/m} \left\{ \left(\frac{X_0}{K} \right)^{-m} + 2rm\Delta \sum_{i=1}^{N} \frac{1}{\Phi_{(i-1)\Delta}+\Phi_{i\Delta}} \right\}^{-1/m},
$$

where $\Phi_T = e^{\left(-rm+\frac{\sigma^2 m}{2}\right)T-\sigma m W_T}$ and X_0 is given; hence.

$$
E[X_T] \approx E\left[K\Phi_T^{-1/m} \left\{ \left(\frac{X_0}{K} \right)^{-m} + 2rm\Delta \sum_{i=1}^{N} \frac{1}{\Phi_{(i-1)\Delta}+\Phi_{i\Delta}} \right\}^{-1/m} \right]. \tag{3.1}
$$

3.2.2 Arithmetic mean

Also, we can approximate

$$
\int_{(i-1)\Delta}^{i\Delta} \frac{1}{\Phi_s} ds \approx \int_{(i-1)\Delta}^{i\Delta} \frac{\frac{1}{\Phi_{(i-1)\Delta}} + \frac{1}{\Phi_{i\Delta}}}{2} ds.
$$

Hence,

$$\begin{aligned}
\int_0^T \Phi_s^{-1} ds &= \int_0^T \frac{1}{\Phi_s} ds \\
&= \int_0^{\Delta} \frac{1}{\Phi_s} ds + \int_{\Delta}^{2\Delta} \frac{1}{\Phi_s} ds + \cdots + \int_{(N-1)\Delta}^{N\Delta} \frac{1}{\Phi_s} ds \\
&\approx \int_0^{\Delta} \frac{\frac{1}{\Phi_0} + \frac{1}{\Phi_\Delta}}{2} ds + \cdots + \int_{(N-1)\Delta}^{N\Delta} \frac{\frac{1}{\Phi_{(N-1)\Delta}} + \frac{1}{\Phi_{N\Delta}}}{2} ds \\
&= \frac{1}{2}\left(\frac{1}{\Phi_0} + \frac{1}{\Phi_\Delta}\right) \int_0^{\Delta} 1\, ds + \cdots + \frac{1}{2}\left(\frac{1}{\Phi_{(N-1)\Delta}} + \frac{1}{\Phi_{N\Delta}}\right) \int_{(N-1)\Delta}^{N\Delta} 1\, ds \\
&= \frac{\Delta}{2}\left[\left(\frac{1}{\Phi_0} + \frac{1}{\Phi_\Delta}\right) + \left(\frac{1}{\Phi_\Delta} + \frac{1}{\Phi_{2\Delta}}\right) + \cdots + \left(\frac{1}{\Phi_{(N-1)\Delta}} + \frac{1}{\Phi_{N\Delta}}\right)\right] \\
&= \frac{\Delta}{2} \sum_{i=1}^{N} \left(\frac{1}{\Phi_{(i-1)\Delta}} + \frac{1}{\Phi_{i\Delta}}\right).
\end{aligned}$$

From (1.3), we have

$$X_T \approx K\Phi_T^{-1/m} \left\{ \left(\frac{X_0}{K}\right)^{-m} + rm\frac{\Delta}{2} \sum_{i=1}^{N} \left(\frac{1}{\Phi_{(i-1)\Delta}} + \frac{1}{\Phi_{i\Delta}}\right) \right\}^{-1/m},$$

where $\Phi_T = e^{\left(-rm+\frac{\sigma^2 m}{2}\right)T - \sigma m W_T}$ and X_0 is given; hence.

$$E[X_T] \approx E\left[K\Phi_T^{-1/m} \left\{ \left(\frac{X_0}{K}\right)^{-m} + rm\frac{\Delta}{2} \sum_{i=1}^{N} \left(\frac{1}{\Phi_{(i-1)\Delta}} + \frac{1}{\Phi_{i\Delta}}\right) \right\}^{-1/m} \right]. \quad (3.2)$$

3.3 Weak order of convergence

For each of the 6 numerical methods, we simulate 100,000 discretized sample paths for the process X_t with ten stepsizes $\Delta = 2^{-14}, 2^{-13}, 2^{-12}, ..., 2^{-5}$ and fixed parameters $X_0 = 200,\ K = 10,000,\ r = 1.7, m = 3, \sigma = 0.3, T = 1$. We will measure the weak error at the final time T. To do this, we need to approximate $E[X_T]$ and $E[x_N]$.

As for $E[x_N]$, we can find the average value of the numerical solution at time T, x_N, from the 100,000 simulated sample paths. Regarding $E[X_T]$, we use the approximated formula (3.1) and (3.2) from section 3.2 to approximately find $E[X_T]$. For both harmonic mean and arithmetic mean approximation, we can find the average value of the approximated exact solution at time T from the corresponding 100,000 simulated

Wiener sample paths. Once we have the approximated $E[x_N]$ and $E[X_T]$, we can numerically find the weak order of convergence of the corresponding numerical method for the GLSDE model.

CHAPTER IV

RESULT

This chapter presents our main results. We will divide our selected methods into 2 groups. Group A consists of the EM, SE, and WE methods whose theoretical weak order of convergence is 1. Group B consists of the ST, EW, and PC methods whose theoretical weak order of convergence is 2. We will compare their run time and numerical weak orders of convergence for the process X_t for each group.

4.1 The simulation of GLSDE

In this section, we would like to show some sample paths for the GLSDE using the six numerical methods. We set $X_0 = 200,\ K = 10,000,\ r = 1.7,\ m = 3,\ \sigma = 0.3,$ $T = 1$, and $\Delta = 2^{-7}$. We compute 100 discretized sample path for the process X_t over $[0, 1]$ and N is the number of discretization intervals where $N = \dfrac{T}{\Delta}$. Figure 4.1 - 4.6 show results of the sample paths for the process X_t using the six numerical methods.

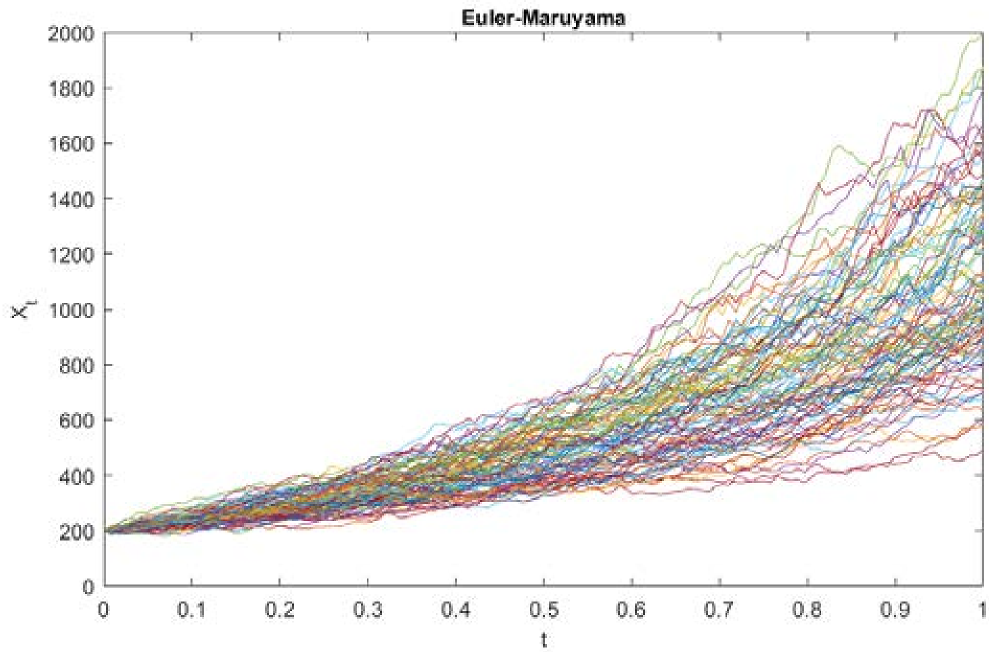

Figure 4.1: Simulated sample paths of GLSDE using EM method.

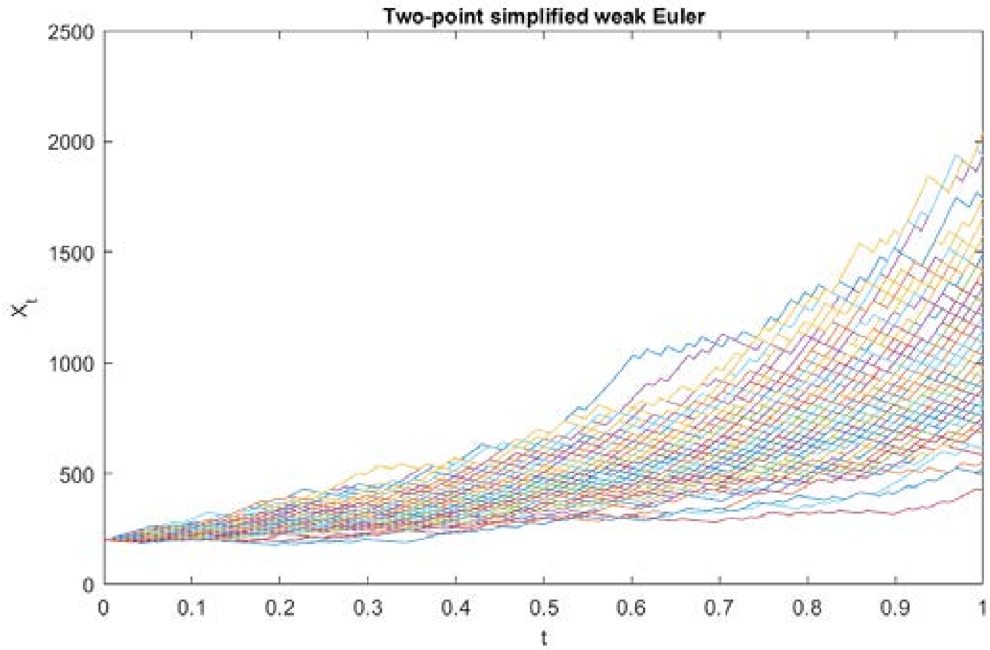

Figure 4.2: Simulated sample paths of GLSDE using SE method.

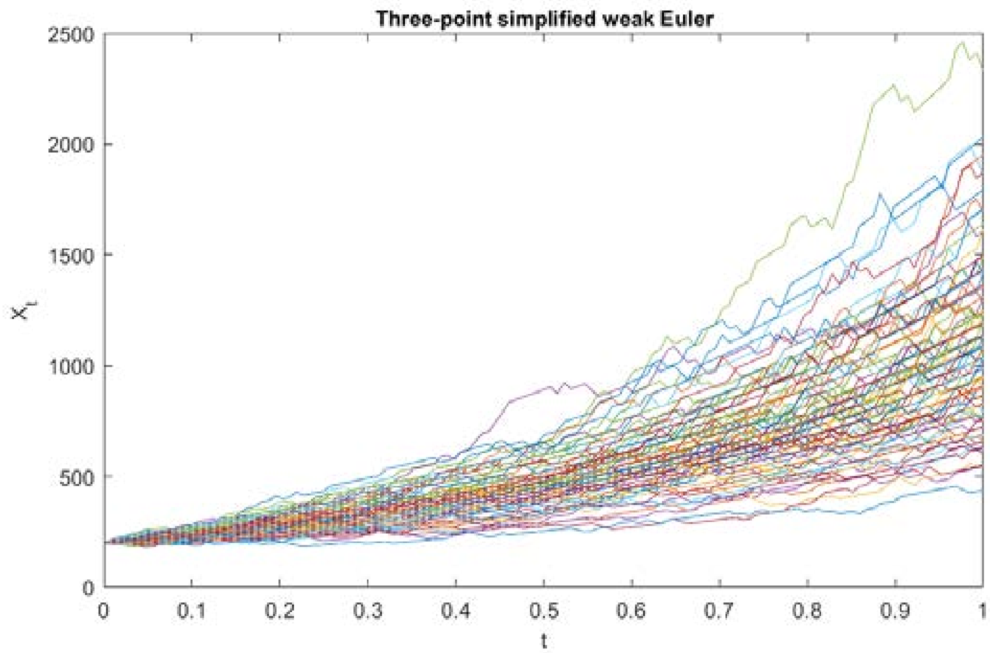

Figure 4.3: Simulated sample paths of GLSDE using WE method.

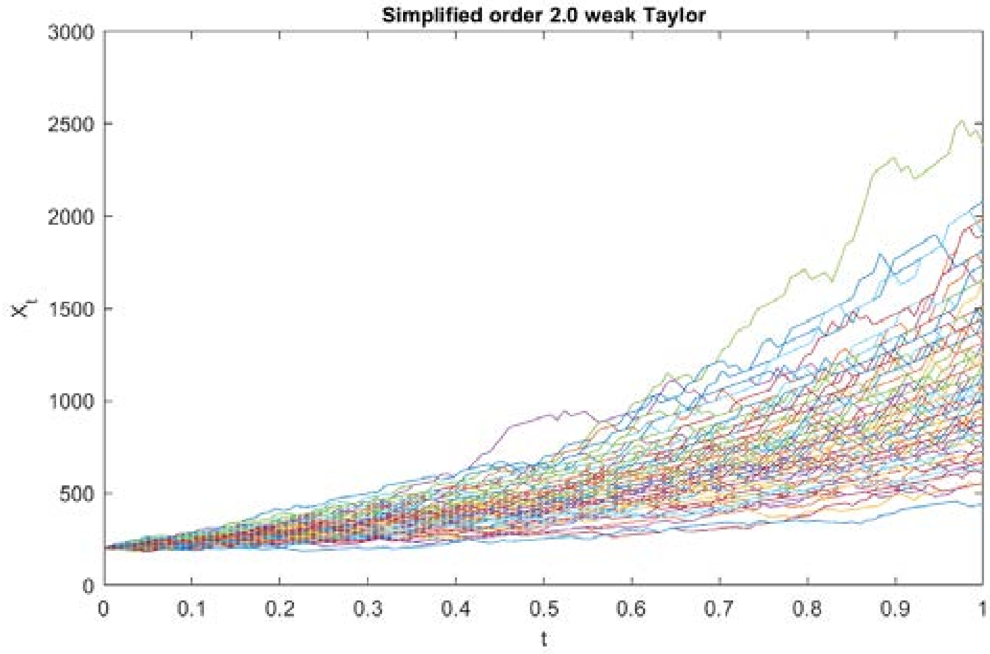

Figure 4.4: Simulated sample paths of GLSDE using ST method.

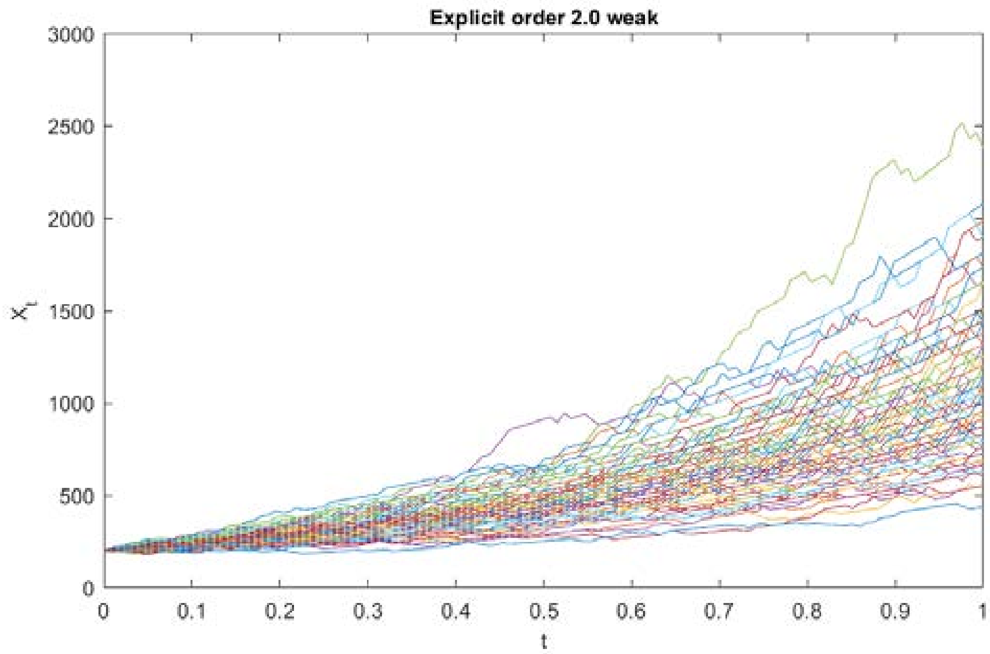

Figure 4.5: Simulated sample paths of GLSDE using EW method.

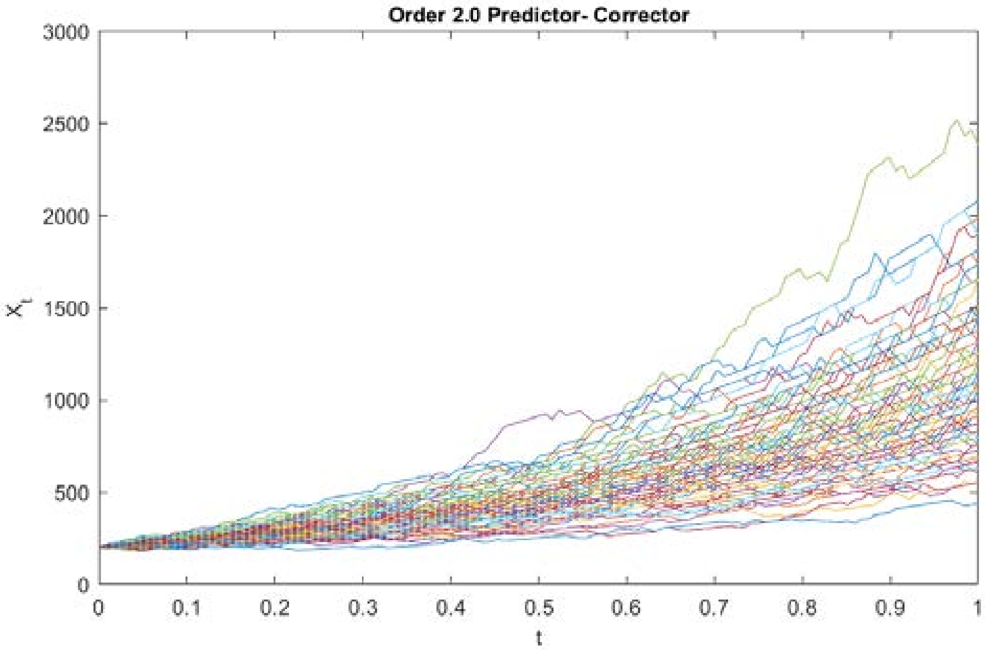

Figure 4.6: Simulated sample paths of GLSDE using PC method.

4.2 Performance comparison

In this section, we compare the performance of the six numerical methods. We set the parameters as follows $X_0 = 200,\ K = 10,000,\ r = 1.7,$
$m = 3,\ \sigma = 0.3$. We compute 100,000 discretized sample paths over $[0, 1]$ with ten stepsizes $\Delta = 2^{-14}, 2^{-13}, 2^{-12}, ..., 2^{-5}$ and N is the number of discretization intervals where $N = \dfrac{T}{\Delta}$.

The results of the approximation of the exact solution using the harmonic mean and arithmetic mean are shown in Figure 4.7 - 4.13 and Figure 4.14 - 4.20, respectively. All figures are shown in the log-log scale of Δ and the weak error, and the blue lines with markers indicate the weak error for each Δ. Figure 4.7-4.9 and 4.14-4.16 have the green dashed reference line with slope 1, and Figure 4.10-4.12 and 4.17-4.19 have the blue dashed reference line with slope 2.

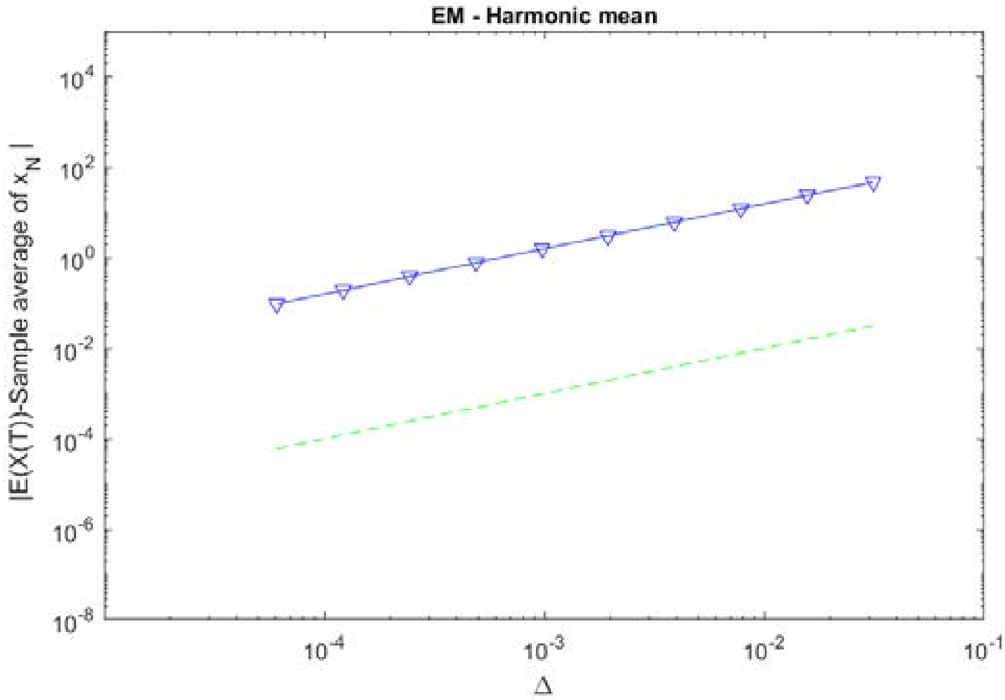

Figure 4.7: The weak error plot for EM with harmonic approximation.

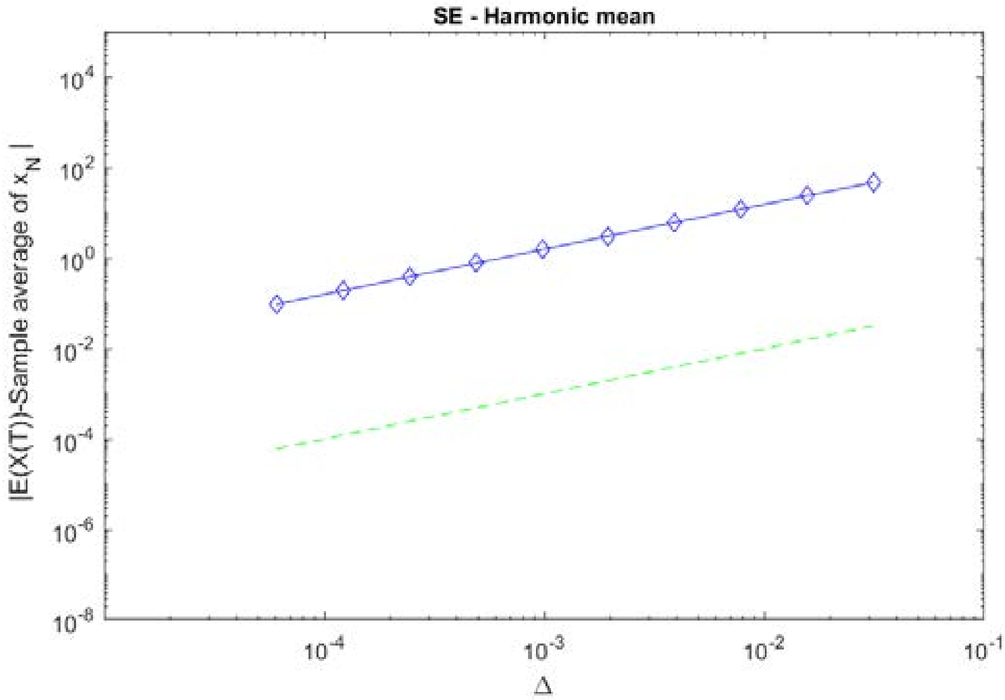

Figure 4.8: The weak error plot for SE with harmonic approximation.

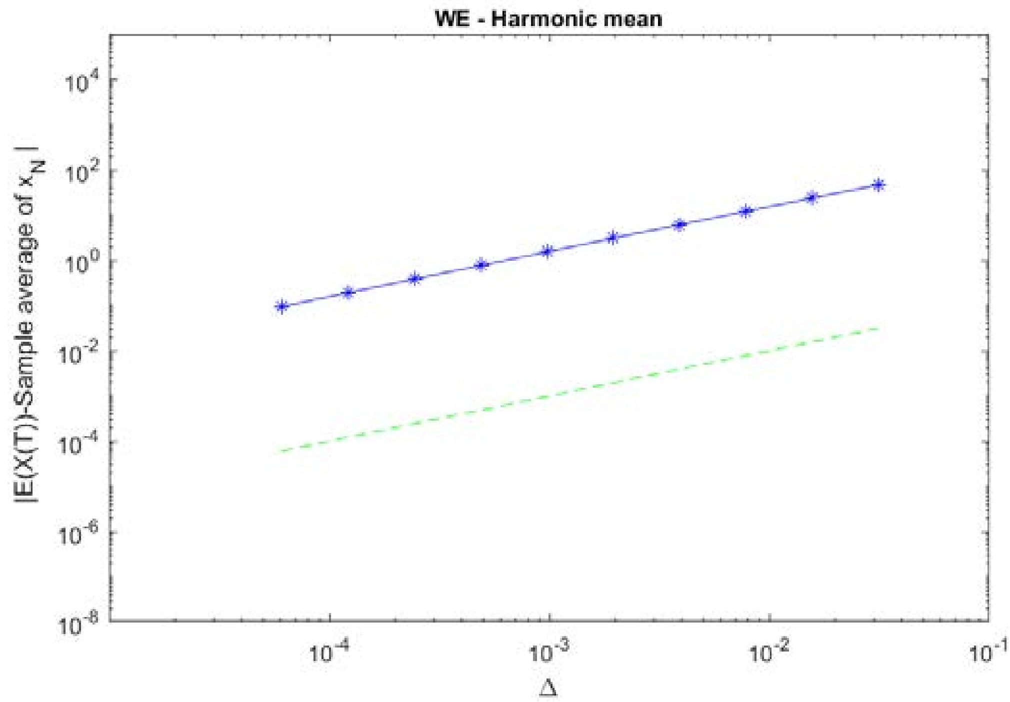

Figure 4.9: The weak error plot for WE with harmonic approximation.

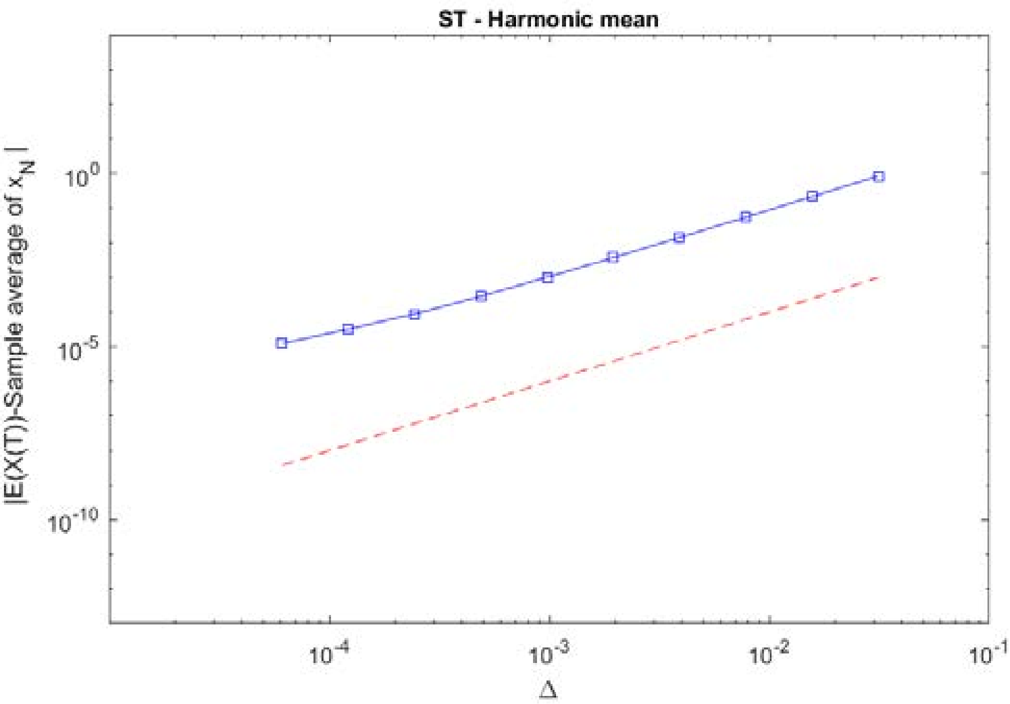

Figure 4.10: The weak error plot for ST with harmonic approximation.

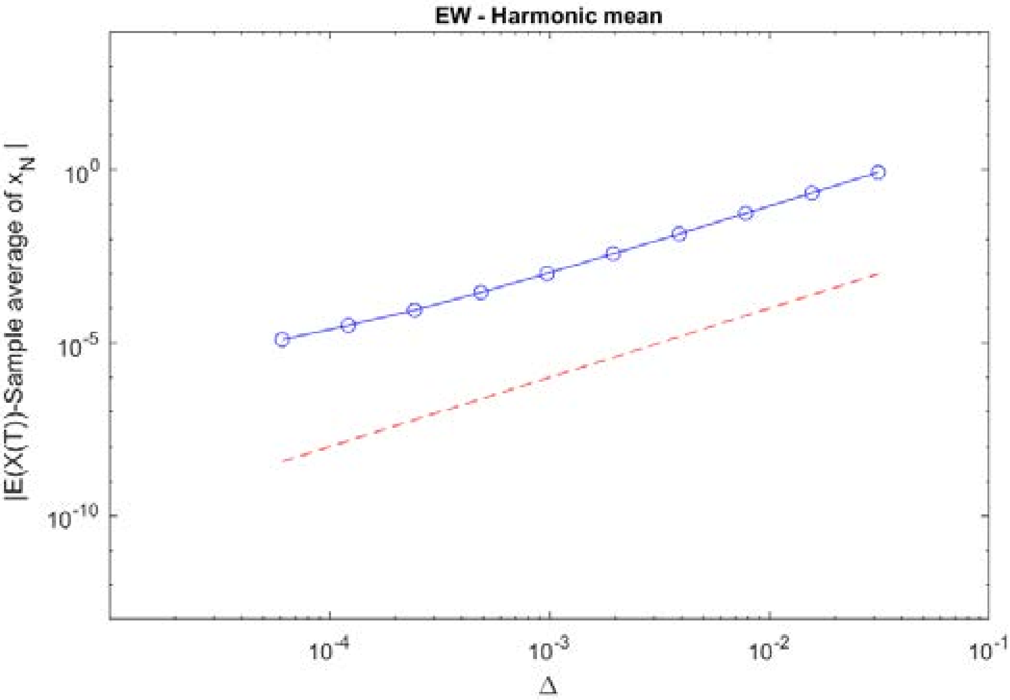

Figure 4.11: The weak error plot for EW with harmonic approximation.

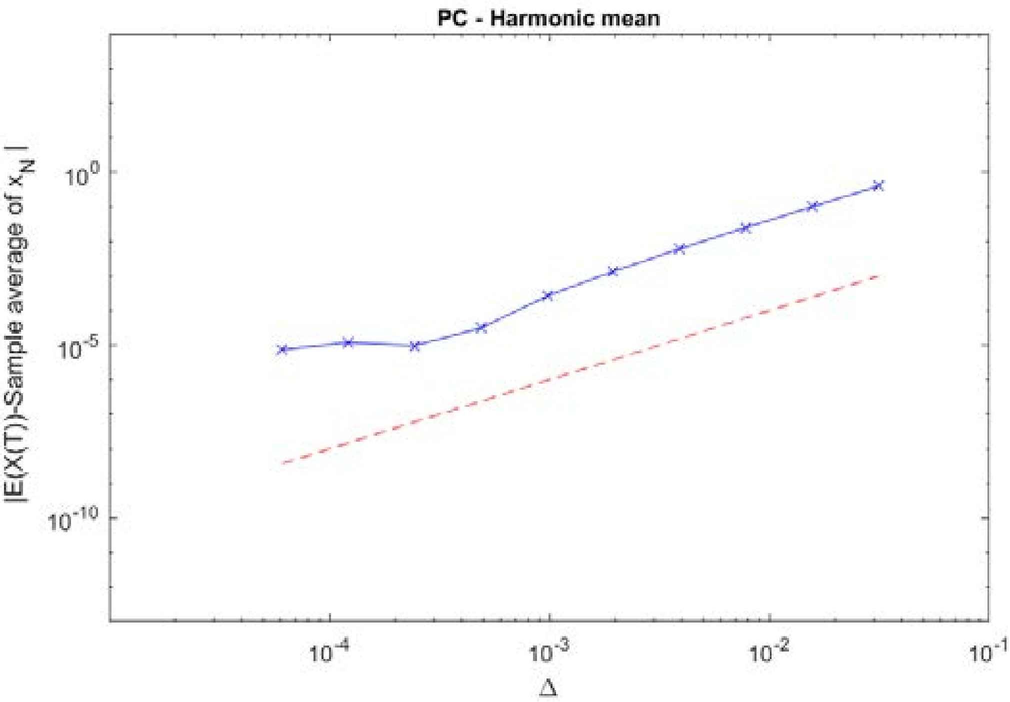

Figure 4.12: The weak error plot for PC with harmonic approximation.

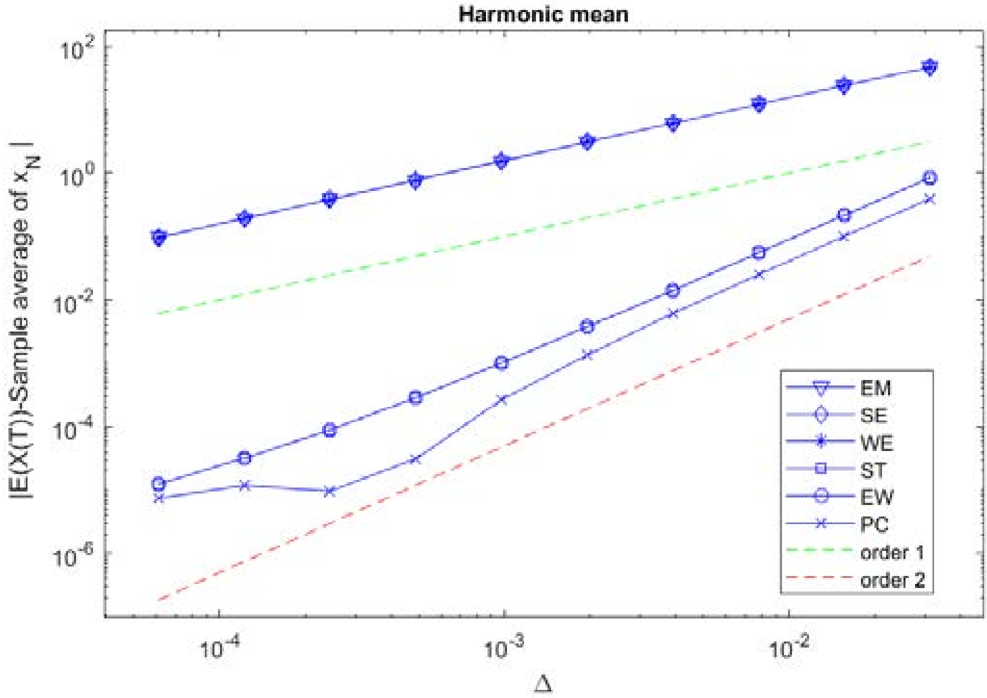

Figure 4.13: The weak error plot for six methods with harmonic approximation.

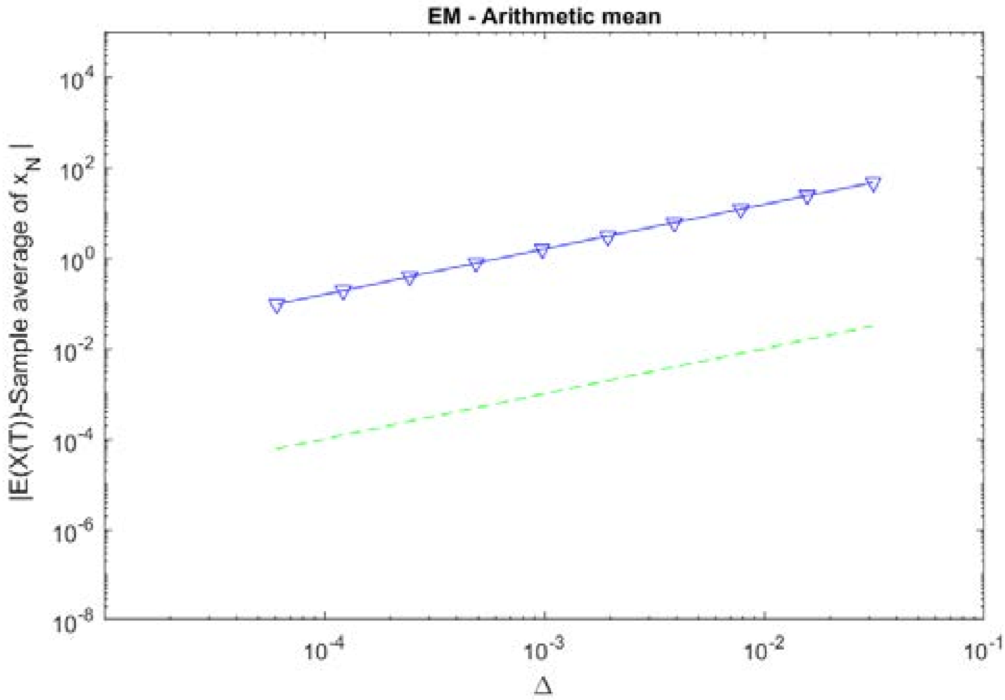

Figure 4.14: The weak error plot for EM with arithmetic approximation.

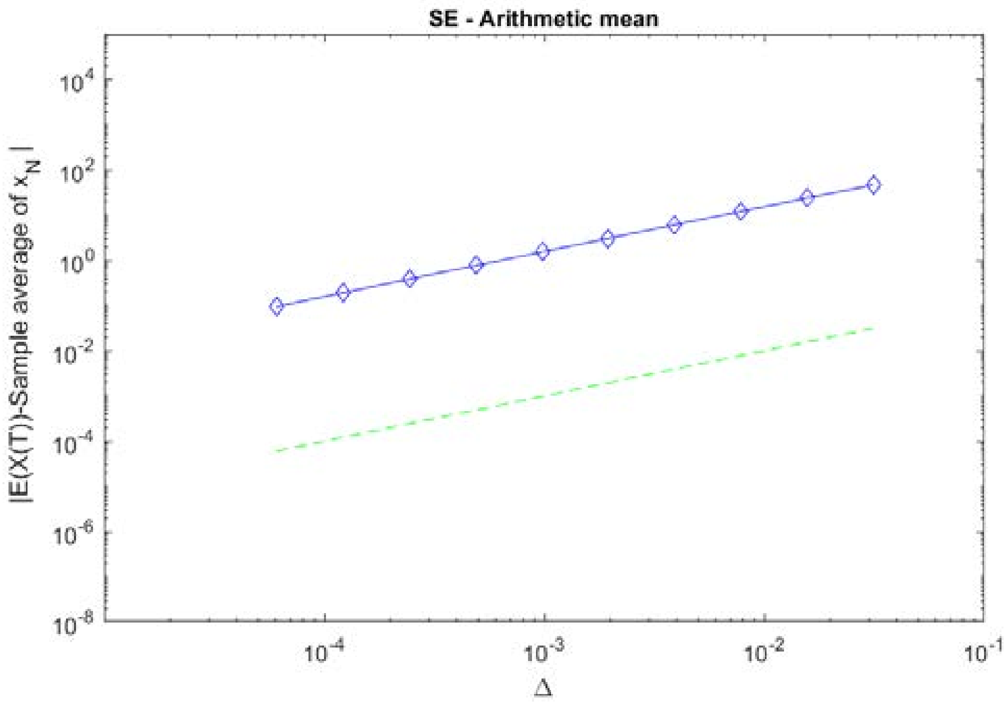

Figure 4.15: The weak error plot for SE with arithmetic approximation.

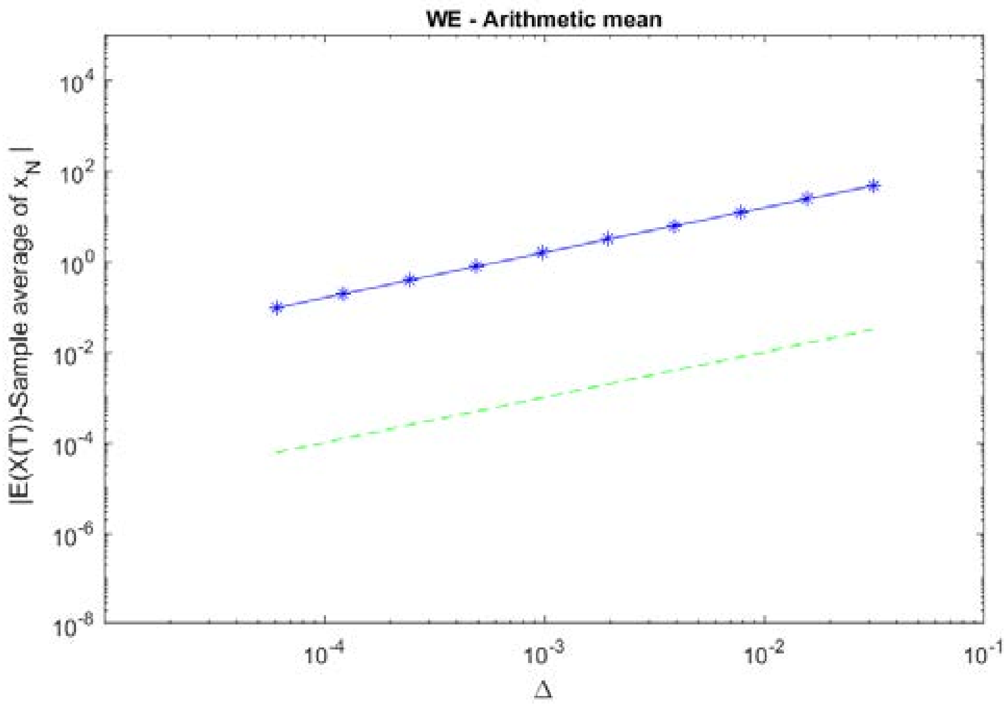

Figure 4.16: The weak error plot for WE with arithmetic approximation.

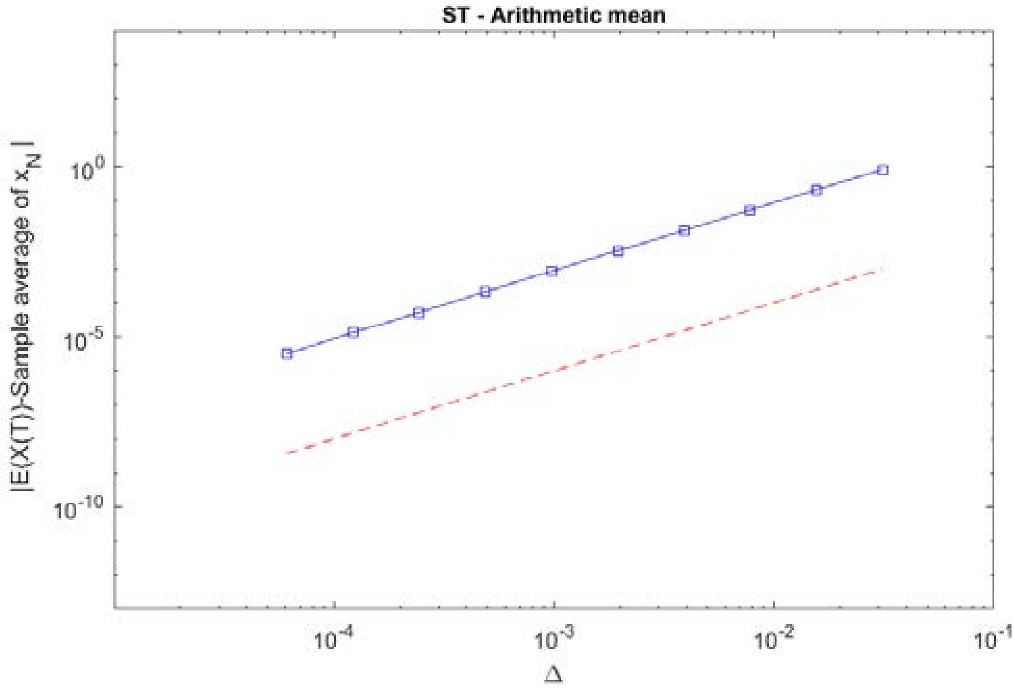

Figure 4.17: The weak error plot for ST with arithmetic approximation.

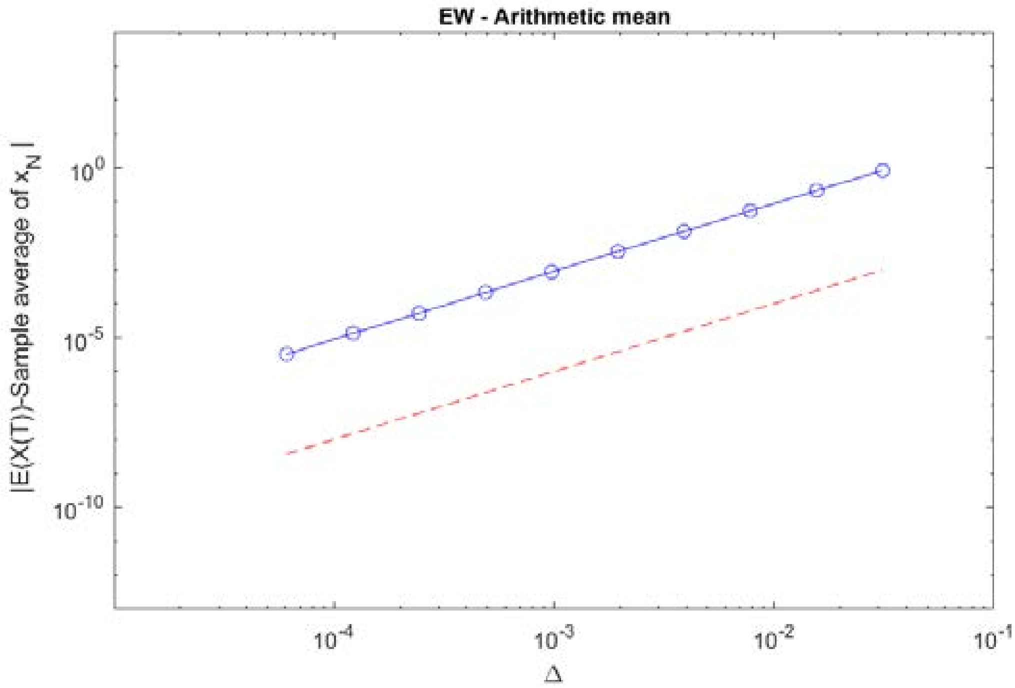

Figure 4.18: The weak error plot for EW with arithmetic approximation.

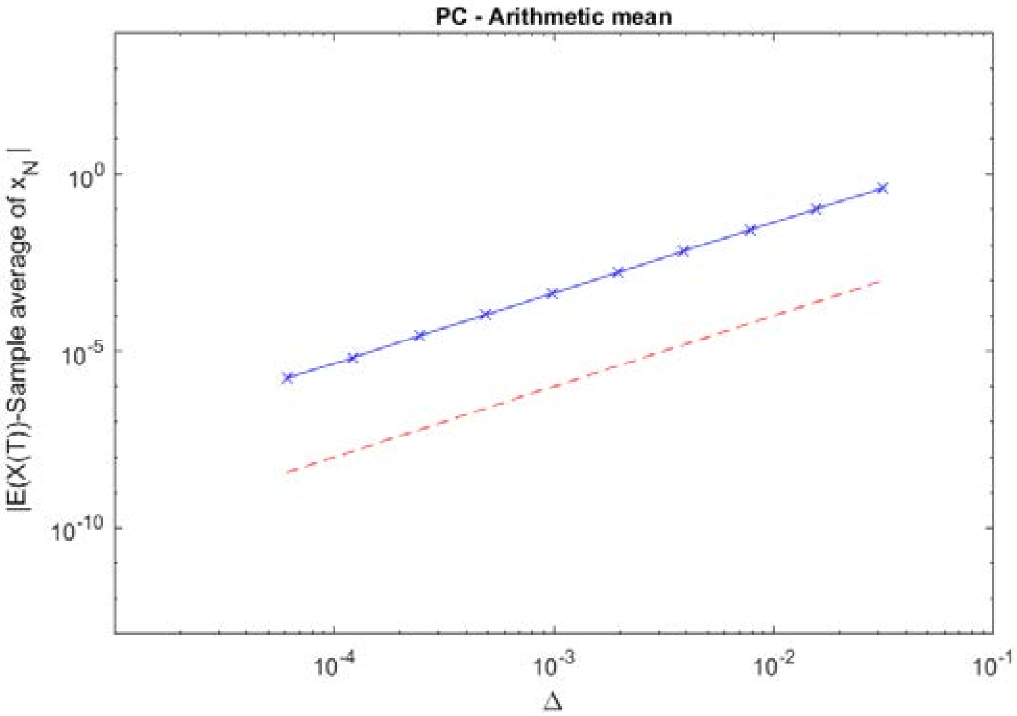

Figure 4.19: The weak error plot for PC with arithmetic approximation.

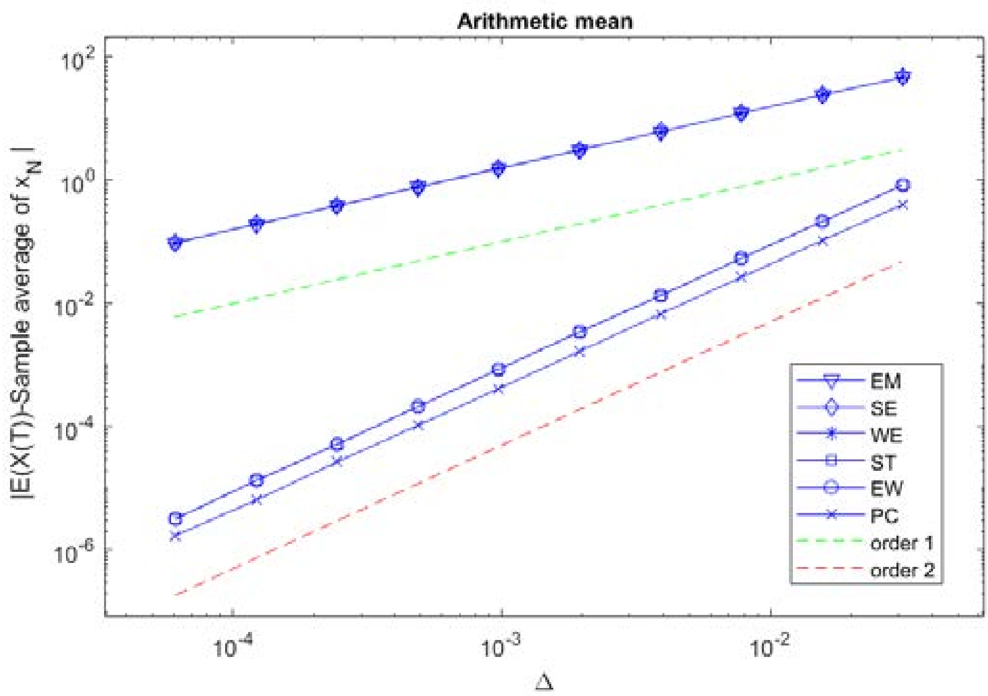

Figure 4.20: The weak error plot for six methods with arithmetic approximation.

Table 4.1 shows run time (in seconds), slopes and intercepts of the lines of best fit for the corresponding results from Figure 4.7-4.12, where we use the harmonic mean to approximate the exact solution.

Group	Scheme	Run time (s)	Slope	Intercept
A	EM	223.345087	0.9920	0.0375
	SE	267.165441	0.9933	0.0307
	WE	277.000700	0.9934	0.0411
B	ST	950.058416	1.8097	0.6273
	EW	675.070831	1.8105	0.6248
	PC	1378.320298	1.9050	2.2246

Table 4.1: Run time, Slope and Intercept (harmonic mean).

In table 4.2 shows run time (in seconds), slopes and intercepts of the lines of best fit for the corresponding results from Figure 4.14-4.19, where we use the arithmetic mean to approximate the exact solution.

Group	Scheme	Run time (s)	Slope	Intercept
A	EM	207.688084	0.9920	0.0376
	SE	238.988821	0.9933	0.0307
	WE	305.213606	0.9934	0.0411
B	ST	1184.745484	1.9966	0.0615
	EW	1104.565897	1.9965	0.0616
	PC	1514.254598	1.9883	0.0536

Table 4.2: Run time, Slope and Intercept (arithmetic mean).

According to the Table 4.1 - 4.2 for group A, the run times using the EM method are shorter than those using the SE and WE methods, respectively. In terms of the weak order of convergence, the EM, SE and WE methods have order approximately 1. For group B, the run times using the EW method are shorter than those using the ST and PC methods, respectively. In terms of the weak order of convergence, the ST, EW and PC methods have order approximately 2.

CHAPTER V

CONCLUSION

In this project, we study the GLSDE model and use six numerical methods to simulate sample paths for the GLSDE model. The selected numerical methods are Euler-Maruyama (EM), two-point simplified weak Euler (SE), three-point simplified weak Euler (WE), simplified order 2.0 weak Taylor(ST), explicit order 2.0 weak (EW) and order 2.0 weak predictor-corrector (PC) methods. We call the first three ones group A and the other three ones group B.

From the result of this work, the EM method has the best run time in group A and the EW method has the best run time in group B. In case of weak order of convergence, the WE method tends to have the best weak order of convergence in group A, and the PC method tends to have the best weak order of convergence in group B.

We can choose these numerical methods to suit our desive. For example, if we want a method that provides the best approximation with weak order 2, we should choose the PC method. However, if we need the fastest method with weak order 2, we should choose the EW method. If we want the fastest and easiest to implement method, we should choose the EM method.

As for the future work, we may try to study the sensitivity of the parameters in the GLSDE model. Moreover, stochasticity from a jump process may be added to the model.

Bibliography

1. Julia Viro. Logistic model. MAT 132, 2016.
2. Kink P. Some analysis of a stochastic logistic growth model. Stochastic Analysis and Applications, 36(2): 240-256.
3. Kloeden, Pater E. and Platen, Eckhard. Numerical solutions of stochastic differential equations. Springer-Verlag(1999): 457-510.
4. Oksendal B. Stochastic differential equations. Springer-Verlag(2000).
5. Skiadas, Christos H. Exact solutions of stochastic differential equations: Gompertz, generalized logistic and revised exponential. Methodology and Computing in Applied Probability 12.2 (2010): 261-270.
6. Tsoularis A. Analysis of logistic growth models. Res. Lett. Inf. Math. Sci, (2001)2: 3-46.

APPENDIX

APPENDIX A

The Project Proposal of Course 2301399 Project Proposal
Academic Year 2018

Project Title (Thai)	ระเบียบวิธีเชิงตัวเลขสำหรับตัวแบบโลจิสติคทั่วไปเชิงสุ่ม
Project Title (English)	Numerical methods for a stochastic generalized logistic model
Project Advisor	Raywat Tanadkithirun, Ph.D.
By	Miss Duangrudee Somphan ID 5833517823
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A.1 **Background and Rationale**

The logistic model has been used in biology and economics for a long time due to its simplicity and effectiveness. It has been developed as a stochastic differential equation (SDE) model. The so-called generalized logistic SDE (GLSDE) has the form

$$dX_t = rX_t\left(1 - \left(\frac{X_t}{K}\right)^m\right)dt + \sigma X_t dW_t \tag{1}$$

where X_t is the target process, K, m, r, σ are parameters in the model and W_t is the Wiener process. The explicit solution of the GLSDE is known [2] and has the form

$$X_t = K\Phi_t^{-1/m}\left\{\left(\frac{X_0}{K}\right)^{-m} + rm\int_0^t \Phi_s^{-1}ds\right\}^{-1/m}$$

whrer $\Phi_t = e^{\left(-rm+\frac{\sigma^2 m}{2}\right)t-\sigma m W_t}$ and $X_0 \in \mathbb{R}$.

The easiest numerical method is the Euler-Maruyama (EM) method. To apply this method to the SDE

$$dX_t = f(X_t)dt + g(X_t)dW_t,$$

we need to discretize the time domain $[0, T]$ into N equidistance subintervals. Let $t_n = n\Delta$ for all $i = 0, 1, .., N$ where $\Delta = \frac{T}{N}$. We denote x_n to be the numerical solution at time step t_n using the EM scheme

$$x_{n+1} = x_n + f(x_n)\Delta + g(x_n)\Delta W_n$$

where ΔW_n is normally distributed with mean 0 and variance Δ and $x_0 = X_0$.

For a given family of numerical methods, we need a criterion to determine the best one and that is the order of convergence. For a given numerical method, we say that the *weak order of convergence* equal to α, if there exists a constant C such that

$$|E[X_{t_n}] - E[x_n]| \leq C\Delta^\alpha$$

for all $n = 1, 2, \dots, N$ and for all sufficiently small Δ. So, the greater α is, the better the method will be.

In this work, we will use a number of numerical methods to simulate some sample paths for GLSDE. The selected methods are EM, Simplified weak Euler (SE), Simplified order 2.0 weak Taylor (ST), Explicit order 2.0 weak (EW), Implicit order 2.0 weak (IW) and Order 2.0 weak Predictor-Corrector (PC) methods. Then, we find their weak orders of convergence and compare them.

A.2 **Objectives**

1. To study how to implement some numerical methods for GLSDE.
2. To numerically find weak orders of convergence of the selected numerical methods for GLSDE.
3. To rank the performances of the selected numerical methods for GLSDE in terms of weak orders of convergence as well as the time to run each method.

A.3 **Scope**

1. The SDE model in this work is GLSDE which is given by the equation (1).
2. We will use only EM, SE, EW, PC, ST and IW methods.
    2.1 The EM scheme is given by $x_{n+1} = x_n + a\,\Delta + b\,\Delta W_n$
    where $\Delta W_n \sim N(0, \Delta)$

2.2 The SE scheme is given by $x_{n+1} = x_n + a\,\Delta + b\,\Delta\widehat{W}_n$
where $\widehat{W}_n$ is a two-point distributed random variable with

$$P(\Delta\widehat{W}_n = \pm\sqrt{\Delta}) = \frac{1}{2}$$

2.3 The WE scheme is given by $x_{n+1} = x_n + a\,\Delta + b\,\Delta\widehat{W}_n$
where $\widehat{W}_n$ is a three-point distributed random variable with

$$P(\Delta\widehat{W}_n = \pm\sqrt{3\Delta}) = \frac{1}{6}, \qquad P(\Delta\widehat{W}_n = 0) = \frac{2}{3}.$$

2.4 The ST scheme is given by

$$x_{n+1} = x_n + a\Delta + b\Delta\widehat{W}_n + \frac{1}{2}\,b\,b'\left\{(\Delta\widehat{W}_n)^2 - \Delta\right\}$$
$$+\frac{1}{2}\left(a'b + ab' + \frac{1}{2}b''b^2\right)\Delta\widehat{W}_n\Delta + \frac{1}{2}\left(aa' + \frac{1}{2}a''b^2\right)\Delta^2$$

where $\widehat{W}_n$ is a three-point distributed random variable with

$$P(\Delta\widehat{W}_n = \pm\sqrt{3\Delta}) = \frac{1}{6}, \qquad P(\Delta\widehat{W}_n = 0) = \frac{2}{3}.$$

2.5 The EW scheme is given by

$$x_{n+1} = x_n + \frac{1}{2}\left(a(\bar{\Upsilon}) + a\right)\Delta + \frac{1}{4}\left(b(\bar{\Upsilon}^+) \,+\, b(\bar{\Upsilon}^-) \,+\, 2b\right)\Delta\widehat{W}_n$$
$$+\frac{1}{4}\left(b(\bar{\Upsilon}^+) \,-\, b(\bar{\Upsilon}^-)\right)\left\{\left(\Delta\widehat{W}_n\right)^2 \,-\, \Delta\right\}\Delta^{-1/2}$$

with supporting values

$$\bar{\Upsilon} \;=\; x_n \;+\; a\,\Delta + b\,\Delta\widehat{W}_n\,,$$

and
$$\bar{\Upsilon}^{\pm} \;=\; x_n + a\,\Delta \;\pm\; b\,\sqrt{\Delta}$$

where $\widehat{W}_n$ is a three-point distributed random variable with

$$P(\Delta\widehat{W}_n = \pm\sqrt{3\Delta}) = \frac{1}{6}, \qquad P(\Delta\widehat{W}_n = 0) = \frac{2}{3}.$$

2.6 The PC scheme is given by $x_{n+1} = x_n + \dfrac{1}{2}\left\{a(\bar{x}_{n+1}) + a\right\}\Delta + \Psi_n$

with

$$\Psi_n = b\Delta\widehat{W}_n + \frac{1}{2}\,b\,b'\left\{(\Delta\widehat{W}_n)^2 \,-\, \Delta\right\} + \frac{1}{2}\left(a\,b' + \frac{1}{2}b^2b''\right)\Delta\widehat{W}_n\Delta$$

and predictor

$$\bar{x}_{n+1} = x_n + a\,\Delta + \Psi_n + \frac{1}{2}\,a'\,b\,\Delta\widehat{W}_n\Delta + \frac{1}{2}\left(a\,a' \,+ \frac{1}{2}\,a''b^2\right)\Delta^2$$

where $\widehat{W}_n$ is a three-point distributed random variable with

$$P(\Delta\widehat{W}_n = \pm\sqrt{3\Delta}) = \frac{1}{6}, \qquad P(\Delta\widehat{W}_n = 0) = \frac{2}{3}$$

3. We will search for the orders of convergence in weak sense only.

A.4 Project Activities

1. Review basic knowledge and articles on SDEs which are related to our project.

2. Find the SDE model to study.

3. Understand GLSDE and its derivation.

4. Defend the project proposal.

5. Perform simulations by using the selected six numerical methods.

6. Conclude results and write a report.

Research Plan

Project Activities	Month, 2018						Month, 2019			
	Jul.	Aug.	Sep.	Oct.	Nov.	Dec.	Jan.	Feb.	Mar.	Apr.
1. Review basic knowledge and articles on SDEs which are related to our project.										
2. Find the SDE model to study.										
3.Understand GLSDE and its derivation.										
4.Defend the project proposal.										
5.Perform simulations by using the selected six numerical methods.										
6. Conclude results and write a report.										

A.5 **Benefits**

The benefits to the student who implements this project are as follows.

1. To develop the skills to search for information and to improve thinking process.
2. To gain knowledge in SDE and know how to implement some numerical methods for a certain SDE.
3. To be able to rank the performances of a given set of numerical methods by the use of weak orders of convergence.

The benefits for users of this project are as follows.

1. Readers can choose a numerical method for GLSDE model suitably for their use depending on how fast and how accurately they need.

A.6 **Equipment**

1. Computer
2. Paper
3. Printer
4. MATLAB, Microsoft Word
5. Journals and related books

A.7 **Budget**

1. Paper A4 600 Bath
2. Printer ink 2,500 Bath
3. Stationery 400 Bath
4. Photocopy 1,500 Bath

A.8 **References**

[1] Kink P (2018) *Some analysis of a stochastic logistic growth model*. Stochastic Analysis and Applications, 36(2): 240-256.

[2]Skiadas CH (2010) *Exact solutions of stochastic differential equations: Gompertz, generalized logistic and revised exponential*. Methodology and Computing in Applied Probability 12: 261–270.

Biography

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