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ชื่อโครงการ ความสัมพันธ์ทางเดียวกันหลายตัวแปร

Multivariate Comonotonicity


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# ความสัมพันธ์ทางเดียวกันหลายตัวแปร

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# Multivariate Comonotonicity

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PEERAPONG PANYASAKULWONG: MULTIVARIATE COMONOTONICITY. ADVISOR : ASSOC. PROF.SONGKIAT SUMETKIJAKAN, Ph.D., CO-ADVISOR : ASSOC. PROF.TIPPAWAN SANTIWIPANONT, 30 pp.

We study three different definitions of comonotonicity between two d-dimensional random vectors which are strong comonotonicity, projection comonotonicity and weak comonotonicity.

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# Chapter 1
# Introduction

Nondecreasing real-valued functions on $\mathbb{R}$ serve as a good starting point for studying comonotonicity in the sense that if $x$ increases then $y = f(x)$ increases and vice versa. This is equivalent to the statement that any two points on the graph of $f$ are comparable with respect to the componentwise ordering $\leq$, that is the graph of $f$ is $\leq$-totally ordered. This approach gives rise to a definition of comonotonicity of two random variables via the comonotonicity of their support as studied in [**1**, **2**]

Comonotonicity between two $d$-dimentional random vector is much harder to define mainly because subsets of $\mathbb{R}^d$ might not be totally ordered when $d \geq 2$. In this project, three definitions of multivariate comonotonicity are studied and some basic results are given and proved.

In the second chapter and third chapter, respectively, we give some principal definitions of comonotonicity of 2 random variables and provide some basic properties. Later in chapter 3, we extend the definition of comonotonicity to $d$ random variables. Next, in chapter 4, we will study the comonotonicity of two $d$-dimensional random vectors. Equivalently, we turn our attention to $\leq$-componentwise ordering $\mathbb{R}^d \times \mathbb{R}^d$. Moreover, the comonotonicity is categorized into three types: 1.) strong comonotonicity; 2.) projection comonotonicity; and 3.) weak comonotonicity. In final chapter, we summarize the types of comonotonicity and leave some open problems to the readers.

# Chapter 2

# Preliminaries

In this chapter, we will review well-known results which are used in the following chapters.

We give notation $D := \{1, 2, \ldots, d\}$.

## 2.1 Ordering Relations

First, we introduce ordering relation which is used to define comonotonicity.

**Definition 2.1.1.** Given two nonempty, partially ordered spaces $(\mathcal{X}, \leq_{\mathcal{X}})$ and $(\mathcal{Y}, \leq_{\mathcal{Y}})$, we will denote by $\lesssim$ the product partial order on $\mathcal{X} \times \mathcal{Y}$: $(x_1, y_1) \lesssim (x_2, y_2) \Longleftrightarrow x_1 \leq_{\mathcal{X}} x_2$ and $y_1 \leq_{\mathcal{Y}} y_2$.

Note that, except when explicitly stated, we will consider the case $(\mathcal{X}, \leq_{\mathcal{X}}) = (\mathcal{Y}, \leq_{\mathcal{Y}}) = (\mathbb{R}^n, \leq)$, where $\leq$ is the natural component-wise order.

**Example 2.1.2.** In the case $n = 3$, we have $((1,2,1),(0,1,2)) \lesssim ((2,3,4),(5,2,4))$, $((1,5,1),(0,1,2)) \not\lesssim ((2,3,4),(5,2,4))$ and $((1,2,1),(6,1,2)) \not\lesssim ((2,3,4),(5,2,4))$.

**Definition 2.1.3.** A relation $\leq$ is a **total order** on a set $\mathcal{X}$ if the following properties hold for all $a, b$ and $c$ in $\mathcal{X}$:
(1) Reflexivity: $a \leq a$;
(2) Antisymmetry: $a \leq b$ and $b \leq a$ implies $a = b$;
(3) Transitivity: $a \leq b$ and $b \leq c$ implies $a \leq c$; and
(4) Comparability: $a \leq b$ or $b \leq a$.

**Definition 2.1.4.** (Extended Real Number System) Define $\bar{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ where $-\infty < x < \infty$ for any $x \in \mathbb{R}$.

## 2.2 Basics of Probability

In this section, we give definitions in Probability Theory which are used in the project.

**Definition 2.2.1.** Let $(\Omega, \mathscr{F})$ be a measurable space. A function $X : \Omega \to \mathbb{R}$ is called a **random variable** if $\{\omega \in \Omega : X(\omega) < \alpha\} \in \mathscr{F}$ for all $\alpha \in \mathbb{R}$.

**Definition 2.2.2.** Let $X_1, X_2, \ldots, X_d$ be random variables. We define a ***d*-dimensional random vector** as $\mathbf{X} = (X_1, X_2, \ldots, X_d)$.

**Definition 2.2.3.** Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space and $X$ a random variable. A function $F_X : \mathbb{R} \to [0, 1]$ defined by $F_X(x) = \mathbb{P}(\{\omega \in \Omega : X(\omega) \leq x\}) = \mathbb{P}(X \leq x)$ for all $x \in \mathbb{R}$ is called the **distribution function** of $X$.

**Definition 2.2.4.** The **joint distribution function** of random variables $X_1, X_2, \ldots, X_d$ on $(\Omega, \mathscr{F}, \mathbb{P})$ is the function $F_{X_1,X_2,\ldots,X_d} : \mathbb{R}^d \to [0, 1]$ defined by $F_{X_1,X_2,\ldots,X_d}(x_1, x_2, \ldots, x_d) = P(X_1 \leq x_1, X_2 \leq x_2, \ldots, X_d \leq x_d)$.

**Definition 2.2.5.** Given a distribution function $F : \mathbb{R} \to [0, 1]$, the **generalized inverse** is the function $F^{-1} : [0, 1] \to \bar{\mathbb{R}} = [-\infty, \infty]$ defined as $F^{-1}(y) := \sup\{x \in \mathbb{R} : F(x) \leq y\}$.

**Example 2.2.6.** Let $X$ be a random variable whose distribution function is

$$F(x) = \begin{cases} 0 & ; x < 0 \\ 1 - e^{-x} & ; x \geq 0. \end{cases}$$

Then $F^{-1}(y) = -\log(1 - y)$.

**Example 2.2.7.** Let $X$ be a random variable whose distribution function is

$$F(x) = \begin{cases} 0 & ; x < 0 \\ \frac{x^2}{2} & ; 0 < x \leq 1 \\ 0.5 & ; 1 < x \leq 2 \\ \frac{\sqrt{x}}{2} & ; 2 < x \leq 4 \\ 1 & ; x \geq 4. \end{cases}$$

Then

$$F^{-1}(y) = \begin{cases} \sqrt{2y}\,; 0 \leq y \leq 0.5 \\ 0.2 \quad ; 0.5 < y \leq \frac{1}{\sqrt{2}} \\ 4x^2 \quad ; \frac{1}{\sqrt{2}} < y < 1 \\ \infty \quad ; y = 1. \end{cases}$$

(see Fig. 2.1).

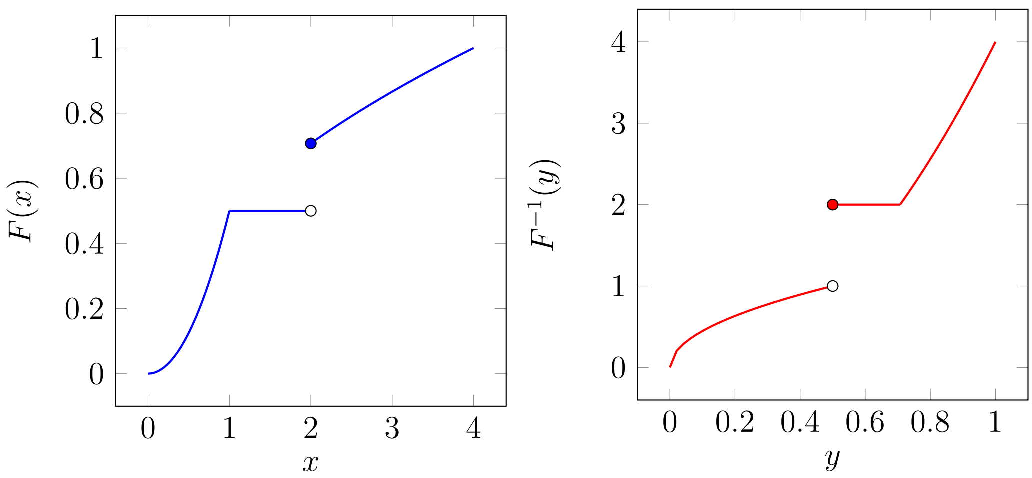


Figure 2.1: An increasing function (left) and its corresponding generalized inverse (right).

**Definition 2.2.8.** Let $\mathbf{X}$ and $\mathbf{Y}$ be $n$-dimensional random vectors (random variables). $\mathbf{X} \sim \mathbf{Y}$ means that $\mathbf{X}$ and $\mathbf{Y}$ have the same joint distribution, i.e., $F_{\mathbf{X}}(\mathbf{t}) = F_{\mathbf{Y}}(\mathbf{t})$ for all $\mathbf{t} \in \mathbb{R}^n$.

**Definition 2.2.9.** A random variable $X$ is said to have uniform distribution on $[0,1]$ ($X \sim \mathcal{U}[0,1]$) if the distribution function of $X$ is:

$$F_X(x) = \begin{cases} 0\,; x < 0 \\ x\,; 0 \leq x < 1 \\ 1\,; x \geq 1. \end{cases}$$

**Definition 2.2.10.** Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space. An event $E \in \mathscr{F}$ happens **almost surely** if $\mathbb{P}(E) = 1$.

## 2.3 Support

In the third section, we review the definition of support which is used to define comonotonicity.

**Definition 2.3.1.** Let $X_1, \ldots, X_d$ be random variables. The **support** of $(X_1, \ldots, X_d)$ is the complement of $\bigcup\{\mathcal{O} \subseteq \mathbb{R}^d : \mathcal{O} \text{ is open and } \mathbb{P}((X_1, \ldots, X_d) \in \mathcal{O}) = 0\}$ denoted by $\text{supp}(X_1, \ldots, X_d)$.

**Example 2.3.2.** Let $X$ be a random variable such that $\mathbb{P}(X = 0) = 0$. Define $Y_1 = X$ and $Y_2 = X^3$ if $X \neq 0$ but $Y_2 = 20$ if $X = 0$. Then $\text{supp}(X, Y_1) = (X, Y_1)$ (see Fig. 2.2) but $\text{supp}(X, Y_2) = \{(x, x^3) : x \in \mathbb{R}\}$ (see Fig. 2.3).

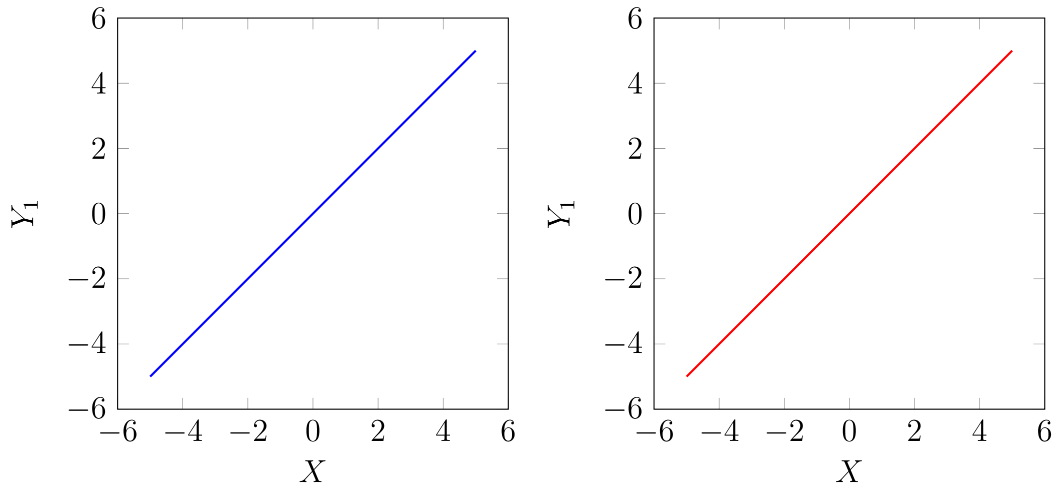


Figure 2.2: A random vector $(X, Y_1)$ (left) and support of random vector $(X, Y_1)$ (right).

**Remark 2.3.3.** Let $(X_1, \ldots, X_d), (Y_1, \ldots, Y_d)$ be random vectors. If $(X_1, \ldots, X_d) \sim (Y_1, \ldots, Y_d)$, then $\text{supp}(X_1, \ldots, X_d) = \text{supp}(Y_1, \ldots, Y_d)$.

*Proof.* Since $(\mathbf{X_1}, \mathbf{Y_1}) \sim (\mathbf{X_2}, \mathbf{Y_2})$, we have

$$\begin{aligned} supp(\mathbf{X_1}, \mathbf{Y_1}) &= [\bigcup\{\mathcal{O} \subseteq \mathbb{R}^d \times \mathbb{R}^d : \mathcal{O} \text{ is open and } \mathbb{P}((\mathbf{X_1}, \mathbf{Y_1}) \in \mathcal{O}) = 0\}]^c \\ &= [\bigcup\{\mathcal{O} \subseteq \mathbb{R}^d \times \mathbb{R}^d : \mathcal{O} \text{ is open and } \mathbb{P}((\mathbf{X_2}, \mathbf{Y_2}) \in \mathcal{O}) = 0\}]^c \\ &= supp(\mathbf{X_2}, \mathbf{Y_2}). \end{aligned}$$

□

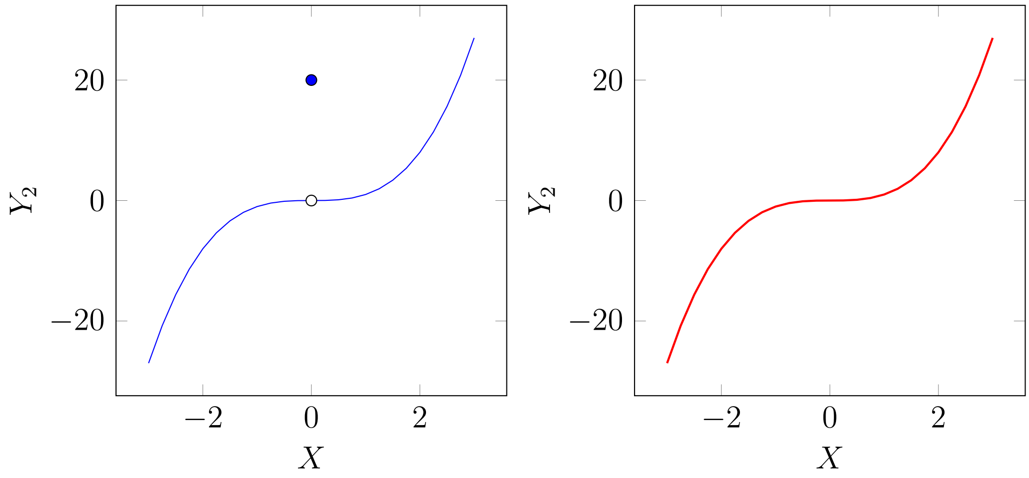


Figure 2.3: A random vector $(X, Y_2)$ (left) and support of random vector $(X, Y_2)$ (right).

**Remark 2.3.4.** Let $(X_1, \ldots, X_d)$ be a random vector. If $(x_1, \ldots, x_d) \in \text{supp}(X_1, \ldots, X_d)$, then for any $\epsilon > 0$ there exists $\omega \in \Omega$ such that $(x'_1, \ldots, x'_d) = (X_1, \ldots, X_d)(\omega)$ and $(x'_1, \ldots, x'_d) \in B((x_1, \ldots, x_d), \epsilon)$.

*Proof.* Since $(x_1, \ldots, x_d) \in \text{supp}(X_1, \ldots, X_d)$, we have $\mathbb{P}((X_1, \ldots, X_d) \in \mathcal{O}) > 0$ every open set $\mathcal{O}$. Let $\epsilon > 0$. Then $\mathbb{P}((X_1, \ldots, X_d) \in B((x_1, \ldots, x_d), \epsilon) > 0$, so the necessary condition holds. □

## 2.4 Projection

**Definition 2.4.1.** Let $E \subseteq \mathbb{R}^d$. The $(i, j)^{th}$ projection and $i^{th}$ projection of $E$ is denoted by $\pi_{ij}(E), \pi^*_i(E)$, respectively.

**Example 2.4.2.** Let $A = \Big\{(a_1, a_2, \ldots, a_{10}) : 0 \le a_i \le i \text{ where } i \in \{1, 2, \ldots, 10\}\Big\} \subseteq \mathbb{R}^{10}$. Then $\pi_{2,5}(A) = \{(x, y) : 0 \le x \le 2, 0 \le y \le 5\}$ and $\pi^*_3(A) = \{x : 0 \le x \le 3\}$.

## 2.5 Copula

Finally, we introduce copula which is a function we use in this project.

**Definition 2.5.1.** A function $C : [0, 1]^n \to [0, 1]$ is called a copula if it is the restriction onto $[0,1]^n$ of a distribution function of random variables $U_1,\ldots,U_n$ which are uniformly distributed on $(0,1)$.

We write $C(\mathbf{u}) = C(u_1,\ldots,u_n)$ for a copula have the following properties:

1. $C(\mathbf{u}) = u_i$ when $u_j = 1$ for all $j \neq i$:

2. $C(\mathbf{u}) = 0$ when $u_i = 0$ for some $i$: and

3. the $n$-increasing property, i.e., $\forall \mathbf{x},\mathbf{y} \in [0,1]^n, x_i \leq y_i,\ i = 1,\ \ldots,n$, it holds $\sum_{J\subset\{1,\ldots,n\}} (-1)^{|J|} C(a_1^J,\ldots,a_n^J) \geq 0$, where $a_i^J = \begin{cases} x_i\,; i \in J \\ y_i\,; i \notin J. \end{cases}$

An equivalent definition is the distribution function of $(F_1(X_1),\ldots,F_n(X_n))$ (see [**3**]).

**Definition 2.5.2.** The upper and lower Fréchet bounds $C_+, C_-$, respectively defined by $C_+(u_1,\ldots,u_d) = \min(u_1,\ldots,u_d)$, $C_-(u_1,\ldots,u_d) = \max\Big(\sum_{i=1}^d u_i - d + 1, 0\Big)$.

**Example 2.5.3.** For $\mathbf{u} \in [0,1]^d$, define $\Pi(\mathbf{u}) = \Pi_{i=1}^d u_i \in [0,1]$.

**Example 2.5.4.** For $\mathbf{u} \in [0,1]^d$, the Farlie-Gumbel-Morgenstern (FGM) copula for $d$-dimensional is

$$C(\mathbf{u}) = \Pi_{i=1}^d u_i\Big(1 + \sum_{k=2}^{d} \sum_{1\leq j_1\leq\ldots\leq j_k\leq d} \theta_{j_1,j_2,\ldots,j_k}(1-u_{j_1})(1-u_{j_2})\ldots(1-u_{j_k})\Big)$$

for $-1 \leq \theta_{j_1,j_2,\ldots,j_k} \leq 1$ for all $j_1,\ldots,j_k$.

# Chapter 3

# Comonotonicity

In this chapter, we will define comonotonicity and study some properties of comonotonicity.

## 3.1 Comonotonicity of two random variables

**Definition 3.1.1.** The set $\Gamma \subset \mathbb{R} \times \mathbb{R}$ is said to be *comonotonic* if it is $\lesssim$-totally ordered, i.e. if for any $(x_1, y_1), (x_2, y_2) \in \Gamma$, either $(x_1, y_1) \lesssim (x_2, y_2)$ or $(x_2, y_2) \lesssim (x_1, y_1)$.

A random vector $(X, Y)$ is called **comonotonic** if its support is comonotonic.

By Example 2.3.2, random vectors $(X, Y_1), (X, Y_2)$ are comonotonic.

Before we start with important theorem of comonotonicity, we prove the following lemmas.

**Lemma 3.1.2.** *Let $(X, Y)$ be a random vector and $\Gamma$ be the support of $(X, Y)$, then $\mathbb{P}((X, Y) \in \Gamma) = 1$.*

*Proof.* Define $\mathbb{P}_{(X,Y)}(B) = \mathbb{P}((X, Y) \in B)$. Then

$$\begin{aligned} \text{supp}(X, Y) &= [\,\bigcup\{\mathcal{O} \subseteq \mathbb{R}^2 : \mathcal{O} \text{ is open and } \mathbb{P}((X, Y) \in \mathcal{O}) = 0\}]^c \\ &= [\,\bigcup\{\mathcal{O} \subseteq \mathbb{R}^2 : \mathcal{O} \text{ is open and } \mathbb{P}_{(X,Y)}(\mathcal{O}) = 0\}]^c \\ &= \text{supp}(\mathbb{P}_{(X,Y)}) \end{aligned}$$

Thus, by proposition 3.20 in [**4**]

$$\mathbb{P}((X, Y) \in \text{supp}(X, Y)) = \mathbb{P}_{(X,Y)}(\text{supp}(X, Y)) = \mathbb{P}_{(X,Y)}(\text{supp}(\mathbb{P}_{(X,Y)})) = 1. \qquad \square$$

**Remark 3.1.3.** Let $(X, Y)$ be a random vector, $B \subseteq \mathbb{R}^2$ and $\Gamma$ be the support of $(X, Y)$, then $\mathbb{P}((X, Y) \in B) = \mathbb{P}((X, Y) \in B \cap \Gamma)$.

*Proof.* We will show that $\mathbb{P}((X,Y) \in B \setminus (B \cap \Gamma)) = 0$. By Lemma 3.1.2, $\mathbb{P}((X,Y) \in B \setminus (B \cap \Gamma)) = \mathbb{P}((X,Y) \in B \cap \Gamma^c) \leq \mathbb{P}((X,Y) \in \Gamma^c) = 0$. □

Next, we will define generalized inverse other than the stated definition.

**Lemma 3.1.4.** *Let $y$ be a real number and $F : \mathbb{R} \to [0,1]$ be a right continuous and nondecreasing function, then $F^{-1}(y) = \inf\{x \in \mathbb{R} : F(x) > y\}$.*

*Proof.* Fix $y \in [0,1]$ and let $A = \{x \in \mathbb{R} : F(x) \leq y\}$ and $B = \{x \in \mathbb{R} : F(x) > y\}$. We will prove that $\sup A = \inf B$. If $A = \emptyset$, then $B = \mathbb{R}$, so $\sup A = -\infty = \inf B$. In latter case, if $B = \emptyset$, then $A = \mathbb{R}$, so $\sup A = \infty = \inf B$.

Let $x = F^{-1}(y)$. By definition, $x = \sup A$. If $x \in A$, then $F$ is continuous at $x$ ($\because$ $F$ is already right continuous by assumption), so $x = \inf B$. Consider $x \notin A$. Then $F(x) > y$, so $x \in B$. Suppose $x \neq \inf B$, there exists $b \in B$ such that $b < x$. Then $F(b) > y$, so $b$ is an upper bound of $A$ ($\because$ $F$ is nondecreasing). But $x = \sup A$, this is a contradiction. □

**Lemma 3.1.5.** *Let $x, y$ be real numbers and $F$ be a right continuous and nondecreasing function. If $F(x) > y$, then $x \geq F^{-1}(y)$. Moreover, if $F^{-1}(y) < \infty$, then $y \leq F(F^{-1}(y))$.*

*Proof.* For the first statement, if $F(x) > y$, we are done by Lemma 3.1.4. For the second statement, assume $F^{-1}(y) < \infty$ and let $A = \{x \in \mathbb{R} : F(x) > y\}$. Since $F^{-1}(y)$ is defined by $\sup\{x \in \mathbb{R} : F(x) \leq y\}$, $A \neq \emptyset$. There exists $(x_n) \subseteq A$ such that $(x_n) \to \inf A = F^{-1}(y)$. By the right-continuity of $F$, $F(x_n)$ converge monotonically (decreasing) to $F(\inf A) = F(F^{-1}(y))$. Since $F(x_n) > y$ , $F(F^{-1}(y)) \geq y$. □

Now, we will state and prove an important theorem on comonotonicity.

**Theorem 3.1.6.** *Let $X, Y$ be random variables. The following are equivalent:*
*($a$) the random vector $(X, Y)$ is comonotonic;*
*($b$) $F_{(X,Y)}(x, y) = \min\{F_X(x), F_Y(y)\}$, for all $(x, y) \in \mathbb{R} \times \mathbb{R}$;*
*($c$) $(X, Y) \sim (F_X^{-1}(U), F_Y^{-1}(U))$, where $U \sim \mathcal{U}[0,1]$;*
*($d$) there exist a random variable $Z$ and nondecreasing functions $f_1, f_2$ such that*

*$(X,Y) \sim (f_1(Z),\ f_2(Z))$; and*

*(e) $X$ and $Y$ are almost surely nondecreasing functions of $X+Y$.*

*Proof.* We will prove that (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (a) and (a) $\Rightarrow$ (e) $\Rightarrow$ (d).

(a) $\Rightarrow$ (b). Assume that the support $\Gamma$ of $(X,Y)$ is comonotonic. Let $(x,y) \in \mathbb{R}\times\mathbb{R}$ and define $A_1 := \{(u,v) \in \Gamma : u \leq x\}$, $A_2 := \{(u,v) \in \Gamma : v \leq y\}$. We will show that $A_1 \subset A_2$ or $A_2 \subset A_1$. Suppose $A_1 \not\subset A_2$ and $A_2 \not\subset A_1$. There exist $(a,b) \in A_1 \setminus A_2$ and $(c,d) \in A_2 \setminus A_1$, so $a \leq x$ and $d \leq y$. Since $A_1, A_2 \subset \Gamma$, $(a,b) \lesssim (c,d)$ or $(c,d) \lesssim (a,b)$. If $(a,b) \lesssim (c,d)$, then $b \leq d \leq y$. So $(a,b) \in A_2$, which is a contradiction. If $(c,d) \lesssim (a,b)$, then $c \leq a \leq x$. So $(c,d) \in A_1$, which is a contradiction. So $\mathbb{P}((X,Y) \in A_1 \cap A_2) = \min\{\mathbb{P}((X,Y) \in A_1), \mathbb{P}((X,Y) \in A_2)\}$. Hence

$$\begin{aligned}
F_{(X,Y)}(x,y) &= \mathbb{P}(X \leq x, Y \leq y) \\
&= \mathbb{P}\Big((X,Y) \in (-\infty,x] \times (-\infty,y] \cap \Gamma\Big) && \text{(by Remark 3.1.3)} \\
&= \mathbb{P}((X,Y) \in A_1 \cap A_2) \\
&= \min\{\mathbb{P}((X,Y) \in A_1), \mathbb{P}((X,Y) \in A_2)\} \\
&= \min\{F_X(x),\ F_Y(y)\} && \text{(by Remark 3.1.3)}
\end{aligned}$$

(b) $\Rightarrow$ (c). Assume (b) holds. Let $U \sim \mathcal{U}[0,1]$.

By Lemma 3.1.5,

$$\begin{aligned}
\mathbb{P}(F_X^{-1}(U) \leq x,\ F_Y^{-1}(U) \leq y) &= \mathbb{P}(U \leq F_X(x),\ U \leq F_Y(y)) \\
&= \mathbb{P}(U \leq \min\{F_X(x),\ F_Y(y)\}) \\
&= \min\{F_X(x),\ F_Y(y)\} \\
&= F_{(X,\ Y)}(x,\ y)
\end{aligned}$$

Hence $(X,Y) \sim (F_X^{-1}(U), F_Y^{-1}(U))$.

(c) $\Rightarrow$ (d). Straightforward. Take $f_1 = F_X^{-1}, f_2 = F_Y^{-1}$ and $Z = U$.

(d) $\Rightarrow$ (a). Assume (d) holds. We will show that $\mathrm{supp}(X,Y)$ is comonotonic, i.e. $\mathrm{supp}(X,Y)$ is totally ordered. Clearly, the componentwise ordering is a partial order. Next, we will show that $\mathrm{supp}(X,Y)$ is comparable. Let $(x_1,y_1),(x_2,y_2) \in \mathrm{supp}(X,Y)$. By Remark 2.3.3, $(x_1,y_1),(x_2,y_2) \in \mathrm{supp}(f_1(Z),f_2(Z))$. Suppose $(x_1,y_1) \not\lesssim (x_2,y_2)$ and $(x_2,y_2) \not\lesssim (x_1,y_1)$. WLOG, we may assume $x_1 < x_2$ and $y_2 < y_1$. Let $\epsilon = \min\left\{\dfrac{x_2-x_1}{2}, \dfrac{y_1-y_2}{2}\right\}$. By Remark 2.3.4, let $(x_1',y_1'),(x_2',y_2')$ be such that $(x_1',y_1') = (f_1(z_1),f_2(z_1)) \in B_\epsilon(x_1,y_1)$ and $(x_2',y_2') = (f_1(z_2),f_2(z_2)) \in B_\epsilon(x_2,y_2)$

for some $z_1, z_2 \in Z(\Omega)$. Then $\|(f_1(z_i), f_2(z_i)) - (x_i, y_i)\| < \epsilon$ for $i = 1, 2$. Let $A := \sqrt{(f_1(z_1) - x_1)^2 + (f_2(z_1) - y_1)^2}$. We have $|f_1(z_1) - x_1| \le A < \epsilon \le \dfrac{x_2 - x_1}{2}$ and $|f_2(z_1) - y_1| \le A < \epsilon \le \dfrac{y_1 - y_2}{2}$, so $f_1(z_1) < \dfrac{x_1 + x_2}{2}$ and $f_2(z_1) > \dfrac{y_1 + y_2}{2}$. Similarly, we have $f_1(z_2) > \dfrac{x_1 + x_2}{2}$ and $f_2(z_2) < \dfrac{y_1 + y_2}{2}$. Hence $f_1(z_1) < \dfrac{x_1 + x_2}{2} < f_1(z_2)$ and $f_2(z_2) < \dfrac{y_1 + y_2}{2} < f_2(z_1)$. Since $f_1, f_2$ are nondecreasing function, $z_1 < z_2$ and $z_2 < z_1$. This is a contradiction and so $(x_1, y_1), (x_2, y_2)$ are comparable.

(a) $\Rightarrow$ (e). Assume that the support $\Gamma$ of $(X, Y)$ is comonotonic.

$$\text{There is no } (x_1, y_1), (x_2, y_2) \in \Gamma \text{ such that } x_1 < x_2 \text{ and } y_1 > y_2. \tag{1}$$

Let $Z = X + Y$ be defined on $\Omega$ and $A = \{\omega : (X(\omega), Y(\omega)) \in \Gamma\} \subseteq \Omega$. Since $\Gamma$ is the support of $(X, Y)$, we have $\mathbb{P}(A^c) = 0$. Hence it suffices to consider $Z = X + Y$ on $A$. First, we will show that $z \in Z(A)$ has almost surely unique decomposition. Assume there are $(x_1, y_1), (x_2, y_2) \in \Gamma$ such that $x_1 + y_1 = z = x_2 + y_2$. Then $x_1 - x_2 = y_2 - y_1 = -(y_1 - y_2)$, By (1), $x_1 = x_2$ and $y_1 = y_2$. Define $u(z) = x$ and $v(z) = y$. Next, we will show that $u, v$ are nondecreasing functions. Let $z_1, z_2 \in Z(A)$ such that $z_1 < z_2$, there exist $(x_1, y_1), (x_2, y_2) \in \Gamma$ such that $x_1 + y_1 = z_1 < z_2 = x_2 + y_2$. Then $x_1 - x_2 < -(y_1 - y_2)$, By (1), $x_1 - x_2 < 0$ and $y_1 - y_2 < 0$, So $u(z_1) < u(z_2)$ and $v(z_1) < v(z_2)$. Hence $X, Y$ are almost surely nondecreasing function of $X + Y$.

(e) $\Rightarrow$ (d). Straightforward by taking $f_1, f_2$ to be the nondecreasing functions and $Z$ to be $X + Y$. □

**Remark 3.1.7.** In the case of univariate marginals the following properties hold:
(1) For every pair of marginal distributions $F, G$, there exists a comonotonic random vector having these marginals.
(2) The distribution of this comonotonic random vector is unique.
(3) For any random variable $X$, the vector $(X, X)$ is comonotonic.
(4) If $(X, Y)$ is comonotonic and $F_X = F_Y$, then $X = Y$ with propability one.

Both (1) and (2) are results of Theorem 3.1.6(b) and (3) comes directly from the Definition 3.1.1. Finally, by Theorem 3.1.6(c), we find that in the special case that marginal distribution functions $F_X$ and $F_Y$ are identical, $\mathbb{P}(X = Y) = 1$.

## 3.2 Comonotonicity of $d$ random variables

We will extend the definition of comonotonicity of two random variables to an arbitrary(but finite) number of random variables.

**Definition 3.2.1.** A set $\Gamma \subset \mathbb{R}^d$ is said to be *comonotonic* if it is $\lesssim$-totally ordered, i.e. if for any $(x_1, \ldots, x_d), (y_1, \ldots, y_d) \in \Gamma$, either $(x_1, \ldots, x_d) \lesssim (y_1, \ldots, y_d)$ or $(y_1, \ldots, y_d) \lesssim (x_1, \ldots, x_d)$.

A random vector $(X_1, \ldots, X_d)$ is called **comonotonic** if its support is comonotonic.

When $d = 2$, Definiton 3.2.1 is equivalent to Definition 3.1.1.

Before we will show the theorem, we prove the following lemma.

**Lemma 3.2.2.** *Fix $(x_1, \ldots, x_d) \in \mathbb{R}^d$, $i, j \in D$ and let $A_k := \{(a_1, \ldots, a_d) \in \Gamma : a_k \leq x_k\}$ for all $k \in D$. If $\pi_{ij}(A_i) \subseteq \pi_{ij}(A_j)$, then $A_i \subseteq A_j$.*

*Proof.* Suppose $A_i \not\subseteq A_j$. There exists $(b_1, \ldots, b_d) \in A_i$ such that $(b_1, \ldots, b_d) \notin A_j$. Then $b_i \leq x_i$ but $b_j > x_j$, so $(b_i, b_j) \in \pi_{ij}(A_i)$ and $(b_i, b_j) \notin \pi_{ij}(A_j)$. Hence $\pi_{ij}(A_i) \not\subseteq \pi_{ij}(A_j)$. □

**Theorem 3.2.3.** *Let $X_1, \ldots, X_d$ be the random variables. The following are equivalent:*

*$(a)$ the random vector $(X_1, \ldots, X_d)$ is comonotonic;*

*$(b)$ $F_{(X_1,\ldots,X_d)}(x_1, \ldots, x_d) = \min\{F_{X_1}(x_1), \ldots, F_{X_d}(x_d)\}$, for all $(x_1, \ldots, x_d) \in \mathbb{R}^d$;*

*$(c)$ $(X_1, \ldots, X_d) \sim (F_{X_1}^{-1}(U), \ldots, F_{X_d}^{-1}(U))$, where $U \sim \mathcal{U}[0, 1]$;*

*$(d)$ there exist a random variable $Z$ and nondecreasing functions $f_1, \ldots, f_d$ such that $(X_1, \ldots, X_d) \sim (f_1(Z), \ldots, f_d(Z))$; and*

*$(e)$ $X_1, \ldots, X_d$ are almost surely nondecreasing functions of $X_1 + \cdots + X_d$.*

*Proof.* Proof in this theorem is similar to Theorem 3.1.6. We will prove that (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (a) and (a) $\Rightarrow$ (e) $\Rightarrow$ (d).

(a) $\Rightarrow$ (b). Assume that the support $\Gamma$ of $(X_1, \ldots, X_d)$ is comonotonic. Let $(x_1, \ldots, x_d) \in \mathbb{R}^d$ and define $A_i := \{(a_1, \ldots, a_d) \in \Gamma : \; a_i \leq x_i\}$. Since $(X_i, X_j)$ is comonotonic for all $i, j \in D$, by the proof of Theorem 3.1.6 (a$\Rightarrow$b), either $\pi_{ij}(A_i) \subset \pi_{ij}(A_j)$ or

$\pi_{ij}(A_j) \subset \pi_{ij}(A_i)$ and by Lemma 3.2.2, either $A_i \subset A_j$ or $A_j \subset A_i$ for all $i, j \in D$. Then $\subset$ is a total order on $\mathscr{A} = \{A_1, \ldots, A_d\}$. Since $\mathscr{A}$ is finite, there exists $n \in D$ such that $A_n \subseteq A_i$ for all $i \in D$. Thus $A_n = \bigcap_{i=1}^{d} A_i$. Then

$$\begin{aligned} \mathbb{P}\left((X_1, \ldots, X_d) \in \bigcap_{i=1}^{d} A_i\right) &= \mathbb{P}((X_1, \ldots, X_d) \in A_n) \\ &= \min\{\mathbb{P}((X_1, \ldots, X_d) \in A_1), \ldots, \mathbb{P}((X_1, \ldots, X_d) \in A_d)\}. \end{aligned}$$

Hence

$$\begin{aligned} F_{(X_1,\ldots,X_d)}(x_1, \ldots, x_d) &= \mathbb{P}(X_1 \leq x_1, \ldots, X_d \leq x_d) \\ &= \mathbb{P}((X_1, \ldots, X_d) \in \Pi_{i=1}^{d}(-\infty, x_i) \cap \Gamma) \quad \text{(by Remark 3.1.3)} \\ &= \mathbb{P}\left((X_1, \ldots, X_d) \in \bigcap_{i=1}^{d} A_i\right) \\ &= \min\{\mathbb{P}((X_1, \ldots, X_d) \in A_1), \ldots, \mathbb{P}((X_1, \ldots, X_d) \in A_d)\} \\ &= \min\{F_{X_1}(x_1), \ldots, F_{X_d}(x_d)\} \qquad \text{(by Remark 3.1.3)} \end{aligned}$$

(b) $\Rightarrow$ (c). Assume (b) holds. Let $U \sim \mathcal{U}[0, 1]$.

By Lemma 3.1.5, then

$$\begin{aligned} \mathbb{P}(F_{X_1}^{-1}(U) \leq x_1, \ldots, F_{X_d}^{-1}(U) \leq x_d) &= \mathbb{P}(U \leq F_{X_1}(x_1), \ldots, U \leq F_{X_d}(x_d)) \\ &= \mathbb{P}(U \leq \min\{F_{X_1}(x_1), \ldots, F_{X_d}(x_d)\}) \\ &= \min\{F_{X_1}(x_1), \ldots, F_{X_d}(x_d)\} \\ &= F_{(X_1,\ldots,X_d)}(x_1, \ldots, x_d) \end{aligned}$$

Hence $(X_1, \ldots, X_d) \sim (F_{X_1}^{-1}(U), \ldots, F_{X_d}^{-1}(U))$.

(c) $\Rightarrow$ (d). Straightforward. Take $f_i = F_{X_i}^{-1}$ for all $i \in D$ and $Z = U$.

(d) $\Rightarrow$ (a). Assume (d) holds. We will show that $\text{supp}(X_1, \ldots, X_d)$ is comonotonic, i.e. $\text{supp}(X_1, \ldots, X_d)$ is totally ordered. Clearly, the componentwise ordering is a partial order. Next, we will show that $\text{supp}(X_1, \ldots, X_d)$ is comparable. Let $(x_1, \ldots, x_d), (y_1, \ldots, y_d) \in \text{supp}(X_1, \ldots, X_d)$. By Remark 2.3.3,

$$(x_1, \ldots, x_d), (y_1, \ldots, y_d) \in \text{supp}(f_1(Z), \ldots, f_d(Z)).$$

Suppose $(x_1, \ldots, x_d) \nleq (y_1, \ldots, y_d)$ and $(y_1, \ldots, y_d) \nleq (x_1, \ldots, x_d)$. WLOG, we may assume $x_1 < y_1$ and $y_k < x_k$ for all $k \neq 1$. Let $\epsilon = \min\left\{\frac{y_1 - x_1}{2}, \frac{x_2 - y_2}{2}, \ldots, \frac{x_d - y_d}{2}\right\}$. By Remark 2.3.4, let $(a_1, \ldots, a_d) = (f_1(z_1), \ldots, f_d(z_1)) \in B_\epsilon(x_1, \ldots, x_d)$ and $(b_1, \ldots, b_d)$ $= (f_1(z_2), \ldots, f_d(z_2)) \in B_\epsilon(y_1, \ldots, y_d)$ for some $z_1, z_2 \in Z(\Omega)$. Then

$\|(f_1(z_1),\ldots,f_d(z_1))-(x_1,\ldots,x_d)\|<\epsilon$ and $\|(f_1(z_2),\ldots,f_d(z_2))-(y_1,\ldots,y_d)\|<\epsilon$. Let $A:=\sqrt{\sum_{i=1}^{d}(f_i(z_1)-x_i)^2}$. We have $|f_1(z_1)-x_1|\leq A<\epsilon\leq\frac{y_1-x_1}{2}$, so $f_1(z_1)<\frac{x_1+y_1}{2}$. For $k=2,\ldots,d$, $|f_k(z_1)-x_k|\leq A<\epsilon\leq\frac{x_k-y_k}{2}$, so $f_k(z_1)>\frac{x_k+y_k}{2}$. Similarly, we have $f_1(z_2)>\frac{x_1+y_1}{2}$ and $f_k(z_2)<\frac{x_k+y_k}{2}$ for some $k=2,\ldots,d$. Hence $f_1(z_1)<\frac{x_1+y_1}{2}<f_1(z_2)$ and $f_k(z_2)<\frac{x_k+y_k}{2}<f_k(z_1)$ for all $k\neq 1$. Since $f_1,\ldots,f_d$ are nondecreasing function, $z_1<z_2$ and $z_2<z_1$, a contradiction and so $(x_1,\ldots,x_d),(y_1,\ldots,y_d)$ in $\text{supp}(X_1,\ldots,X_d)$ are comparable.

(a) $\Rightarrow$ (e). Assume that the support $\Gamma$ of $(X_1,\ldots,X_d)$ is comonotonic.

$$\text{There is no } (x_1,\ldots,x_d),(y_1,\ldots,y_d)\in\Gamma \text{ such that } x_i<y_i \text{ and } x_j>y_j \text{ for some } i\in D \text{ for any } j\neq i. \quad (1)$$

Let $Z=X_1+\ldots+X_d$ defined on $\Omega$ and $A=\{\omega:(X_1(\omega),\ldots,X_d(\omega))\in\Gamma\}\subseteq\Omega$. Since $\Gamma$ is support of $(X_1,\ldots,X_d)$, we have $\mathbb{P}(A^c)=0$. Hence we will consider $Z=X+Y$ on $A$. First, we will show that $z\in Z(A)$ has almost surely unique decomposition. Assume there are $(x_1,\ldots,x_d),(y_1,\ldots,y_d)\in\Gamma$ such that $x_1+\ldots+x_d=z=y_1+\ldots+y_d$. Then $x_1-y_1=-[(x_2-y_2)+\ldots+(x_d-y_d)]$, By (1), $x_i=y_i$ for all $i\in D$. Let $i\in D$. Define $u_i(z)=x_i$. Next, we will show that $u_i$ are nondecreasing function. Let $z_1,z_2\in Z(A)$ such that $z_1<z_2$, there exist $(x_1,\ldots,x_d),(y_1,\ldots,y_d)\in\Gamma$ such that $x_1+\ldots+x_d=z_1<z_2=y_1+\ldots+y_d$. Then $x_1-y_1<-[(x_2-y_2)+\ldots+(x_d-y_d)]$, By (1), $x_i-y_i<0$ for all $i\in D$. So $u_i(z_1)<u_i(z_2)$ for all $i\in D$. Hence $X_1,\ldots,X_d$ are almost surely nondecreasing functions of $X_1+\ldots+X_d$.

(e) $\Rightarrow$ (d). Straightforward by taking $f_1,\ldots,f_d$ to be the nondecreasing functions and $Z$ to be $X_1+\ldots+X_d$. □

In the next chapter, we will extend the definition of comonotonicity of random variables to random vectors through componentwise ordering.

# Chapter 4

# Multivariate Comonotonicity

In this chapter, we will define the comonotonicity of random vectors and study properties similar to the previous chapter.

## 4.1 Strong comonotonicity

First, we provide the definition of strong comonotonicity which is similar to Definition 3.1.1.

**Definition 4.1.1.** A set $\Gamma \subset \mathbb{R}^{\mathrm{d}} \times \mathbb{R}^{\mathrm{d}}$ is said to be *s-comonotonic* if it is $\lesssim$-totally ordered, i.e. if for any $(\boldsymbol{x}_1, \boldsymbol{y}_1), (\boldsymbol{x}_2, \boldsymbol{y}_2) \in \Gamma$, either $(\boldsymbol{x}_1, \boldsymbol{y}_1) \lesssim (\boldsymbol{x}_2, \boldsymbol{y}_2)$ or $(\boldsymbol{x}_2, \boldsymbol{y}_2) \lesssim (\boldsymbol{x}_1, \boldsymbol{y}_1)$.

An ordered pair of $d$-random vectors $(\mathbf{X}, \mathbf{Y})$ is called **s-comonotonic** if its support is s-comonotonic.

Definition 4.1.1 is essentially Definition 3.2.1 where $\Gamma \subseteq \mathbb{R}^{2d}$ and when $d = 1$, Definition 4.1.1 reduces to Definition 3.1.1.

Now, we give an example of s-comonotonic random vector.

**Example 4.1.2.** Fix $d = 2$ consider the marginals $F_{\mathbf{X}} = C_+(F_{X_1}, F_{X_2})$ and $F_{\mathbf{Y}} = C_+(F_{Y_1}, F_{Y_2})$, for some distribution functions $F_{X_i}, F_{Y_i},\ i = 1, 2$. If $U \sim \mathcal{U}[0, 1]$, then the vector $\left(\left(F_{X_1}^{-1}(U), F_{X_2}^{-1}(U)\right), \left(F_{Y_1}^{-1}(U), F_{Y_2}^{-1}(U)\right)\right)$ is s-comonotonic since $F_{X_i}^{-1}, F_{Y_i}^{-1},\ i = 1, 2$ are nondecreasing functions.

Since the space $\mathbb{R}^d$ is not totally ordered when $d > 1$, s-comonotonicity imposes heavy constraints on the marginal distributions $F_{\mathbf{X}}$ and $F_{\mathbf{Y}}$.

**Lemma 4.1.3.** *If a set $\Gamma \subset \mathbb{R}^d \times \mathbb{R}^d$ is s-comonotonic, then the sets $\pi_i^*(\Gamma)$, $i = 1, 2$ are $\leq$-totally ordered.*

*Proof.* Let $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \pi_1^*(\Gamma)$, there exist $\boldsymbol{y}_1, \boldsymbol{y}_2 \in \mathbb{R}^d$ such that $(\boldsymbol{x}_1, \boldsymbol{y}_1), (\boldsymbol{x}_2, \boldsymbol{y}_2) \in \Gamma$. Since $\Gamma$ is $\lesssim$-totally ordered, either $(\boldsymbol{x}_1, \boldsymbol{y}_1) \lesssim (\boldsymbol{x}_2, \boldsymbol{y}_2)$ or $(\boldsymbol{x}_2, \boldsymbol{y}_2) \lesssim (\boldsymbol{x}_1, \boldsymbol{y}_1)$. In the first case we have that $\boldsymbol{x}_1 \leq \boldsymbol{x}_2$; *in the second that* $\boldsymbol{x}_2 \leq \boldsymbol{x}_1$, hence $\pi_1^*(\Gamma)$ is $\leq$-totally ordered. Similarly, we have $\pi_2^*(\Gamma)$ is $\leq$-totally ordered. □

From Lemma 4.1.3 and Theorem 3.2.3(b), it follows that an s-comonotonic random vector $(\mathbf{X}, \mathbf{Y})$ has multivariate marginals of the form

$$F_{\mathbf{X}}(x_1, \ldots, x_d) = C_+\Big(F_{X_1}(x_1), \ldots, F_{X_d}(x_d)\Big) \tag{4.1.1a}$$

$$F_{\mathbf{Y}}(y_1, \ldots, y_d) = C_+\Big(F_{Y_1}(y_1), \ldots, F_{Y_d}(y_d)\Big) \tag{4.1.1b}$$

where $F_{X_1}, \ldots, F_{X_d}$ and $F_{Y_1}, \ldots, F_{Y_d}$ are univariate distribution functions of respective random variables.

The following theorem characterizes s-comonotonicity and shows that $(\mathbf{X}, \mathbf{Y})$ is s-comonotonic if and only if the $2d$ random variables $X_1, \ldots, X_d, Y_1, \ldots, Y_d$ are all pairwise comonotonic.

**Theorem 4.1.4.** *Let* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$ *be two random vectors with respective distribution functions* $F_{\boldsymbol{X}}$ *and* $F_{\boldsymbol{Y}}$ *of the form in equation (4.1.1). The following are equivalent:*
*(a) the random vector* $(\boldsymbol{X}, \boldsymbol{Y})$ *is s-comonotonic;*
*(b)* $F_{(\boldsymbol{X},\boldsymbol{Y})}(\boldsymbol{x}, \boldsymbol{y}) = \min\{F_{\boldsymbol{X}}(\boldsymbol{x}), F_{\boldsymbol{Y}}(\boldsymbol{y})\}$, *for all* $(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^d \times \mathbb{R}^d$;
*(c)* $(\boldsymbol{X}, \boldsymbol{Y}) \sim \Big(\Big(F_{X_1}^{-1}(U), \ldots, F_{X_d}^{-1}(U)\Big), \Big(F_{Y_1}^{-1}(U), \ldots, F_{Y_d}^{-1}(U)\Big)\Big)$, *where* $U \sim \mathcal{U}[0, 1]$;
*(d) there exist a random variable* $Z$ *and nondecreasing functions* $f_1, \ldots, f_d, g_1, \ldots, g_d$ *such that* $(\boldsymbol{X}, \boldsymbol{Y}) \sim ((f_1(Z), \ldots, f_d(Z)), (g_1(Z), \ldots, g_d(Z)))$; *and*
*(e) for all* $i, j \in D$, $X_i$ *and* $Y_j$ *are almost surely nondecreasing functions of* $X_i + Y_j$.

*Proof.* We will prove that (a) ⇒ (b) ⇒ (c) ⇒ (d) ⇒ (e) ⇒ (a).
(a) ⇒ (b). Assume that the support $\Gamma$ of $(\mathbf{X},\mathbf{Y})$ is s-comonotonic. Let $(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^d \times \mathbb{R}^d$ and define $A_1 := \{(\boldsymbol{u},\boldsymbol{v}) \in \Gamma : \boldsymbol{u} \leq \boldsymbol{x}\}$ and $A_2 := \{(\boldsymbol{u},\boldsymbol{v}) \in \Gamma : \boldsymbol{v} \leq \boldsymbol{y}\}$. Write $(\boldsymbol{x}, \boldsymbol{y}) = ((x_1, \ldots, x_d), (y_1, \ldots, y_d))$. We will show that either $A_1 \subset A_2$ or $A_2 \subset A_1$. Suppose $A_1 \not\subset A_2$ and $A_2 \not\subset A_1$, there exist $(\boldsymbol{a}, \boldsymbol{b}) \in A_1 \setminus A_2$ and $(\boldsymbol{c}, \boldsymbol{d}) \in A_2 \setminus A_1$, so $\boldsymbol{a} \leq \boldsymbol{x}$ and $\boldsymbol{d} \leq \boldsymbol{y}$. Since $A_1, A_2 \subset \Gamma$, $(\boldsymbol{a}, \boldsymbol{b}) \lesssim (\boldsymbol{c}, \boldsymbol{d})$ or $(\boldsymbol{c}, \boldsymbol{d}) \lesssim (\boldsymbol{a}, \boldsymbol{b})$. If $(\boldsymbol{a}, \boldsymbol{b}) \lesssim (\boldsymbol{c}, \boldsymbol{d})$, then $\boldsymbol{b} \leq \boldsymbol{d} \leq \boldsymbol{y}$. So $(\boldsymbol{a}, \boldsymbol{b}) \in A_2$, which is a contradiction. If

$(\boldsymbol{c}, \boldsymbol{d}) \lesssim (\boldsymbol{a}, \boldsymbol{b})$, then $\boldsymbol{c} \leq \boldsymbol{a} \leq \boldsymbol{x}$. So $(\boldsymbol{c}, \boldsymbol{d}) \in A_1$, which is a contradiction. Thus $\mathbb{P}((\mathbf{X}, \mathbf{Y}) \in A_1 \cap A_2) = \min\{\mathbb{P}((\mathbf{X}, \mathbf{Y}) \in A_1), \mathbb{P}((\mathbf{X}, \mathbf{Y}) \in A_2)\}$. Hence

$$\begin{aligned}
F_{(\mathbf{X},\mathbf{Y})}(\boldsymbol{x}, \boldsymbol{y}) &= \mathbb{P}(\mathbf{X} \leq \boldsymbol{x}, \mathbf{Y} \leq \boldsymbol{y}) \\
&= \mathbb{P}\left((\mathbf{X}, \mathbf{Y}) \in \left(\Pi_{i=1}^{d}(-\infty, x_i] \times \Pi_{i=1}^{d}(-\infty, y_i]\right) \cap \Gamma\right) \quad \text{(by Remark 3.1.3)} \\
&= \mathbb{P}((\mathbf{X}, \mathbf{Y}) \in A_1 \cap A_2) \\
&= \min\{\mathbb{P}((\mathbf{X}, \mathbf{Y}) \in A_1), \mathbb{P}((\mathbf{X}, \mathbf{Y}) \in A_2)\} \\
&= \min\{F_{\mathbf{X}}(\boldsymbol{x}),\ F_{\mathbf{Y}}(\boldsymbol{y})\} \quad \text{(by Lemma 3.1.3)}
\end{aligned}$$

(b) $\Rightarrow$ (c). Assume (b) holds. Let $U \sim \mathcal{U}[0, 1]$.

By Lemma 3.1.5,

$$\begin{aligned}
&\mathbb{P}\left[\left(F_{X_1}^{-1}(U), \ldots, F_{X_d}^{-1}(U)\right), \left(F_{Y_1}^{-1}(U), \ldots, F_{Y_d}^{-1}(U)\right) \leq (\boldsymbol{x}, \boldsymbol{y})\right] \\
&= \mathbb{P}\left[F_{X_1}^{-1}(U) \leq x_1, \ldots, F_{X_d}^{-1}(U) \leq x_d, F_{Y_1}^{-1}(U) \leq y_1, \ldots, F_{Y_d}^{-1}(U) \leq y_d\right] \\
&= \mathbb{P}\left[U \leq F_{X_1}(x_1), \ldots, U \leq F_{X_d}(x_d), U \leq F_{Y_1}(y_1), \ldots, U \leq F_{Y_d}(y_d)\right] \\
&= \mathbb{P}\left[U \leq \min\left(\min_{i \in D}\{F_{X_i}(x_i)\}, \min_{i \in D}\{F_{Y_i}(y_i)\}\right)\right] \\
&= \min\left(\min_{i \in D}\{F_{X_i}(x_i)\}, \min_{i \in D}\{F_{Y_i}(y_i)\}\right) \\
&= \min\left(F_{\mathbf{X}}(\boldsymbol{x}), F_{\mathbf{Y}}(\boldsymbol{y})\right) \\
&= F_{(\boldsymbol{X},\boldsymbol{Y})}(\boldsymbol{x}, \boldsymbol{y}).
\end{aligned}$$

(c) $\Rightarrow$ (d). Straightforward. Take $f_i = F_{X_i}^{-1}, g_i = F_{Y_i}^{-1}$ for all $i \in D$ and $Z = U$.

(d) $\Rightarrow$ (e). Assume (d) holds. There is a random variable Z such that $(X_i, Y_j) \sim (f_i(Z), g_j(Z))$ for all $i, j \in D$. By Theorem 3.1.6, the random vector $(X_i, Y_j)$ is comonotonic, so (e) is true.

(e) $\Rightarrow$ (a). Suppose that $(\mathbf{X}, \mathbf{Y})$ is not s-comonotonic. There exist $(\boldsymbol{x}_1, \boldsymbol{y}_1), (\boldsymbol{x}_2, \boldsymbol{y}_2) \in \text{supp}(\mathbf{X}, \mathbf{Y})$ such that $(\boldsymbol{x}_1, \boldsymbol{y}_1), (\boldsymbol{x}_2, \boldsymbol{y}_2)$ are not comparable. Write $x_k = (x_k^1, \ldots, x_k^d)$ and $y_k = (y_k^1, \ldots, y_k^d)$ for all $k \in D$. WLOG, we may assume that $x_1^i \leq x_2^i$ and $y_2^i < y_1^i$ for some $i \in D$ , so $(X_i, Y_i)$ is not comonotonic. By Theorem 3.1.6, $X_i$ and $Y_i$ are not almost surely nondecreasing functions of $X_i + Y_i$. □

**Remark 4.1.5.** The following properties hold for $d$-dimensional random vectors $\mathbf{X}$ and $\mathbf{Y}$:

(1) Given a pair of marginal distributions $F_{\mathbf{X}}, F_{\mathbf{Y}}$ of random vectors $\mathbf{X}, \mathbf{Y}$, respectively, there exists an s-comonotonic random vector having these marginals if and

only if both $\mathbf{X}$ and $\mathbf{Y}$ have copula $C_+$.

(2) The distribution of this s-comonotonic random vector is unique.

(3) Given a random vector $\mathbf{X}$, the vector $(\mathbf{X}, \mathbf{X})$ is s-comonotonic only if $\mathbf{X}$ has copula $C_+$.

(4) If $(\mathbf{X}, \mathbf{Y})$ is s-comonotonic and $F_{\mathbf{X}} = F_{\mathbf{Y}}$, then $\mathbf{X} = \mathbf{Y}$ with propability one.

Both (1) and (2) are results of Theorem 4.1.4(b) and (3) holds since s-comonotonicity has marginal of a copula $C_+$. Finally, by Theorem 4.1.4(c), we find that in the special case that all marginal distribution functions are identical, then $\mathbb{P}(\mathbf{X} = \mathbf{Y}) = 1$.

We can extend the definition of strong comonotonicity of two random vectors to an arbitrary number of random vectors(finite) where random vactors have the same copula $C_+$. Next, we will show that $(\mathbf{X}, \mathbf{X})$ is not necessary s-comonotonic.

**Example 4.1.6.** Let $X$ and $Y$ be random variables with probability density functions $f_X(x) = \mathbb{1}_{[0,1]}(x)$ and $f_Y(y) = \frac{1}{2}\mathbb{1}_{[0,1]}(y) + \mathbb{1}_{(1,\frac{3}{2}]}(y)$, respectively. Suppose that they are independent. Then the joint distribution function is

$$F_{(X,Y)}(x, y) = \begin{cases} \frac{xy}{2} & : x \in [0, 1], y \in [0, 1] \\ \frac{x}{2} + x(y-1) & : x \in [0, 1], y \in (1, \frac{3}{2}] \\ 0 & : otherwise. \end{cases}$$

Denote $\mathbf{X} = (X, Y)$. Then $F_{(\mathbf{X},\mathbf{X})}((\frac{1}{2}, 1), (1, \frac{1}{3})) = F_{(X,Y)}(\min(\frac{1}{2}, 1), \min(1, \frac{1}{3})) = F_{(X,Y)}(\frac{1}{2}, \frac{1}{3}) = \frac{1}{12}$ and $\min\left\{F_{\mathbf{X}}(\frac{1}{2}, 1), F_{\mathbf{X}}(1, \frac{1}{3})\right\} = \min\{\frac{1}{4}, \frac{1}{6}\} = \frac{1}{6}$.

Hence $F_{(\mathbf{X},\mathbf{X})}((\frac{1}{2}, 1), (1, \frac{1}{3})) \neq \min\left\{F_{\mathbf{X}}(\frac{1}{2}, 1), F_{\mathbf{X}}(1, \frac{1}{3})\right\}$.

By Theorem 4.1.4(b), $(\mathbf{X}, \mathbf{X})$ is not s-comonotonic.

## 4.2 Projection comonotonicity

We consider a weaker concept of comonotonicity according to which the vector $(\mathbf{X}, \mathbf{X})$ is always comonotonic. For each $i \in D$, let $A_i, B_i$ be measurable subsets of the real line. Given a set $\Gamma \subset \left(\Pi_{i=1}^d A_i\right) \times \left(\Pi_{i=1}^d B_i\right)$, for each $i \in D$ we denote the projection of $\Gamma$ on the space $A_i \times B_i$ by $\pi_i^*(\Gamma)$.

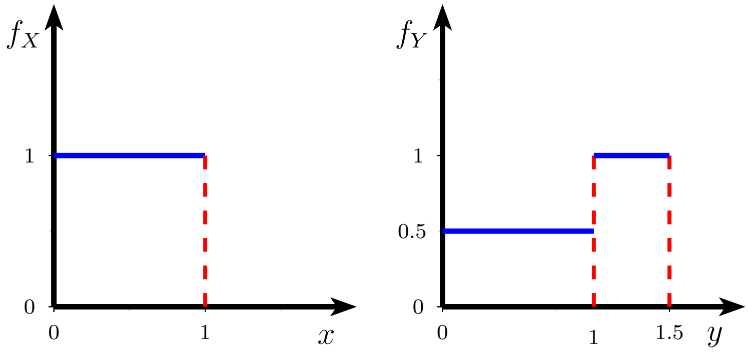


Figure 4.1: Probability density function of $f_X$ and $f_Y$, respectively.

**Definition 4.2.1.** The set $\Gamma \subset \left(\Pi_{i=1}^d A_i\right) \times \left(\Pi_{i=1}^d B_i\right)$ is said to be $\pi$-*comonotonic* if, for all $i \in D$, $\pi_i^*(\Gamma)$ is comonotonic as a subset of $A_i \times B_i$.

A random vector $(\mathbf{X}, \mathbf{Y})$ is called **$\pi$-comonotonic** if its support is $\pi$-comonotonic.

When $d = 1$, Definition 4.2.1 is equivalent to Definitions 3.1.1 and 4.1.1. When $d > 1$, an s-comonotonic random vector is $\pi$-comonotonic, but not vice versa.

**Example 4.2.2.** Let $F_{\mathbf{X}} = F_{\mathbf{Y}} = C_-$ and $U \sim \mathcal{U}[0, 1]$. Then the random vector $((U, 1-U), (U, 1-U))$ has bivariate marginals $F_{\mathbf{X}}, F_{\mathbf{Y}}$, and is $\pi$-comonotonic, but not s-comonotonic since the copula of $(U, 1-U)$ is not $C_+$ (see Fig 4.2).

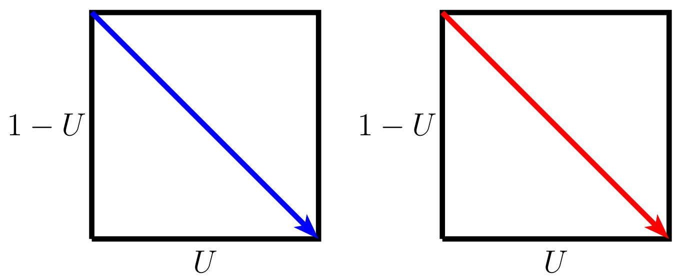


Figure 4.2: The random vector $(\mathbf{X}, \mathbf{Y})$ where $\mathbf{X} = \mathbf{Y} = (U, 1-U)$.

Next, we find the marginal of $\pi$-comonotonicity.

**Lemma 4.2.3.** *If the random vector* $(\boldsymbol{X}, \boldsymbol{Y})$ *is* $\pi$*-comonotonic, then its marginal distribution functions* $F_{\boldsymbol{X}}$ *and* $F_{\boldsymbol{Y}}$ *have a common copula.*

*Proof.* By the Sklar's theorem, for any random vector $(Z_1, \ldots, Z_d)$ there exists a copula $C$ such that $F_{\mathbf{Z}}(z_1, \ldots, z_d) = C(F_{Z_1}(z_1), \ldots, F_{Z_d}(z_d))$. Then

$$\begin{aligned} F_{\mathbf{Z}}(z_1,\ldots,z_d) &= C(F_{Z_1}(z_1),\ldots,F_{Z_d}(z_d)) \\ &= \mathbb{P}\Big(U_1 \le F_{Z_1}(z_1),\ldots,U_d \le F_{Z_d}(z_d)\Big) \\ &= \mathbb{P}\Big(F_{Z_1}^{-1}(U_1) \le z_1,\ldots,F_{Z_d}^{-1}(U_d) \le z_d\Big) \end{aligned}$$

So $(Z_1,\ldots,Z_d) \sim (F_{Z_1}^{-1}(U_1),\ldots,F_{Z_d}^{-1}(U_d))$. If $(X_1,\ldots,X_d) \sim (F_{X_1}^{-1}(U_1),\ldots,F_{X_d}^{-1}(U_d))$ there is a copula $C$ such that $F_{\mathbf{X}}(x_1,\ldots,x_d) = C(F_{X_1}(x_1),\ldots,F_{X_d}(x_d))$.
Assume that the random vector $(\mathbf{X},\mathbf{Y})$ is $\pi$-comonotonic. Then the random vector $(X_i,Y_i) := \pi_i(\mathbf{X},\mathbf{Y})$ is comonotonic. By Theorem 3.1.6, for each $i \in D$, there exists $U_i \sim \mathcal{U}[0,1]$ such that $(X_i,Y_i) \sim \Big(F_{X_i}^{-1}(U_i),F_{Y_i}^{-1}(U_i)\Big)$. As a consequence $(X_1,\ldots,X_d) \sim \Big(F_{X_1}^{-1}(U_1),\ldots,F_{X_d}^{-1}(U_d)\Big)$ and $(Y_1,\ldots,Y_d) \sim \Big(F_{Y_1}^{-1}(U_1),\ldots,F_{Y_d}^{-1}(U_d)\Big)$. Hence $F_{\mathbf{X}}$ and $F_{\mathbf{Y}}$ have a common copula. □

It follows from Lemma 4.2.3 that a $\pi$-comonotonic random vector $(\mathbf{X},\mathbf{Y})$ has multivariate marginals of the form

$$F_{\mathbf{X}}(x_1,\ldots,x_d) = C\Big(F_{X_1}(x_1),\ldots,F_{X_d}(x_d)\Big) \tag{4.2.1a}$$

$$F_{\mathbf{Y}}(y_1,\ldots,y_d) = C\Big(F_{Y_1}(y_1),\ldots,F_{Y_d}(y_d)\Big) \tag{4.2.1b}$$

for some univariate distribution functions $F_{X_1},\ldots,F_{X_d},F_{Y_1},\ldots,F_{Y_d}$ and a copula $C$.

**Remark 4.2.4.** If $\mathbf{X}$ and $\mathbf{Y}$ are s-comonotonic, then $C$ in equation (4.2.1) is the upper Fréchet bound $C_+$.

Now, we will show a property which will be useful in proving the theorem.

**Remark 4.2.5.** Let $\mathbf{X},\mathbf{Y}$ be two random vectors in $\mathbb{R}^d$ and for each $i \in D, U_i \sim \mathcal{U}[0,1]$. Then

$$\begin{aligned} &\mathbb{P}\Big[F_{X_1}^{-1}(U_1) \le x_1,\ldots,F_{X_d}^{-1}(U_d) \le x_d, F_{Y_1}^{-1}(U_1) \le y_1,\ldots,F_{Y_d}^{-1}(U_d) \le y_d\Big] \\ &= \mathbb{P}\Big[U_1 \le F_{X_1}(x_1),\ldots,U_d \le F_{X_d}(x_d), U_1 \le F_{Y_1}(y_1),\ldots,U_d \le F_{Y_d}(y_d)\Big] \\ &= \mathbb{P}\Big[U_1 \le \min\{F_{X_1}(x_1),F_{Y_1}(y_1)\},\ldots,U_d \le \min\{F_{X_d}(x_d),F_{Y_d}(y_d)\}\Big] \\ &= C\Big(\min\{F_{X_1}(x_1),F_{Y_1}(y_1)\},\ldots,\min\{F_{X_d}(x_d),F_{Y_d}(y_d)\}\Big). \end{aligned}$$

The following theorem characterizes $\pi$-comonotonicity and shows that $(\mathbf{X},\mathbf{Y})$ is $\pi$-comonotonic if and only if $\mathbf{X}$ and $\mathbf{Y}$ have the same copula and every pair $(X_i,Y_i)$ is comonotonic.

**Theorem 4.2.6.** *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two random vectors in $\mathbb{R}^d$ with respective distribution $F_{\boldsymbol{X}}$ and $F_{\boldsymbol{Y}}$ of the form equation (4.2.1). The following are equivalent:*

*(a) the random vector $(\boldsymbol{X}, \boldsymbol{Y})$ is $\pi$-comonotonic;*

*(b) there exists a copula $C$ such that*
$F_{(\boldsymbol{X},\boldsymbol{Y})}(\boldsymbol{x}, \boldsymbol{y}) = C\Big(\min\{F_{X_1}(x_1), F_{Y_1}(y_1)\}, \dots, \min\{F_{X_d}(x_d), F_{Y_d}(y_d)\}\Big)$
*for all $(\boldsymbol{x}, \boldsymbol{y}) = ((x_1, \dots, x_d), (y_1, \dots, y_d)) \in \mathbb{R}^d \times \mathbb{R}^d$;*

*(c) $(\boldsymbol{X}, \boldsymbol{Y}) \sim \Big(\Big(F^{-1}_{X_1}(U_1), \dots, F^{-1}_{X_d}(U_d)\Big), \Big(F^{-1}_{Y_1}(U_1), \dots, F^{-1}_{Y_d}(U_d)\Big)\Big)$, where $\mathbf{U} = (U_1, \dots, U_d)$ is a random vector having distribution $C$;*

*(d) there exist a random vector $\boldsymbol{Z} = (Z_1, \dots, Z_d)$ and nondecreasing functions $f_1, \dots, f_d, g_1, \dots, g_d$ such that $(\boldsymbol{X}, \boldsymbol{Y}) \sim ((f_1(Z_1), \dots, f_d(Z_d)), (g_1(Z_1), \dots, g_d(Z_d))$;*

*(e) for each $i \in D$, $X_i$ and $Y_i$ are almost surely nondecreasing functions of $X_i + Y_i$.*

*Proof.* We will prove that (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (a) and (d) $\Leftrightarrow$ (e).

(a) $\Rightarrow$ (b). Assume that the random vector $(\mathbf{X}, \mathbf{Y})$ is $\pi$-comonotonic. From the proof of Lemma 4.2.3, there exists $U_i \sim \mathcal{U}[0,1]$ such that $(X_i, Y_i) \sim \Big(F^{-1}_{X_i}(U_i), F^{-1}_{Y_i}(U_i)\Big)$ for all $i \in D$. Then

$$\begin{aligned}
F_{(\mathbf{X},\mathbf{Y})}(\boldsymbol{x}, \boldsymbol{y}) &= \mathbb{P}[\mathbf{X} \le \boldsymbol{x}, \mathbf{Y} \le \boldsymbol{y}] \\
&= \mathbb{P}\Big[X_1 \le x_1, \dots, X_d \le x_d, Y_1 \le y_1, \dots, Y_d \le y_d\Big] \\
&= \mathbb{P}\Big[F^{-1}_{X_1}(U_1) \le x_1, \dots, F^{-1}_{X_d}(U_d) \le x_d, F^{-1}_{Y_1}(U_1) \le y_1, \dots, F^{-1}_{Y_d}(U_d) \le y_d\Big] \\
&= C\Big(\min\{F_{X_1}(x_1), F_{Y_1}(y_1)\}, \dots, \min\{F_{X_d}(x_d), F_{Y_d}(y_d)\}\Big) \text{ (By Remark 4.2.5).}
\end{aligned}$$

(b) $\Rightarrow$ (c). Assume (b) holds. Then

$$\begin{aligned}
&\mathbb{P}\Big[\Big(F^{-1}_{X_1}(U_1), \dots, F^{-1}_{X_d}(U_d)\Big), \Big(F^{-1}_{Y_1}(U_1), \dots, F^{-1}_{Y_d}(U_d)\Big) \le (\boldsymbol{x}, \boldsymbol{y})\Big] \\
&= \mathbb{P}\Big[F^{-1}_{X_1}(U_1) \le x_1, \dots, F^{-1}_{X_d}(U_d) \le x_d, F^{-1}_{Y_1}(U_1) \le y_1, \dots, F^{-1}_{Y_d}(U_d) \le y_d\Big] \\
&= C\Big(\min\{F_{X_1}(x_1), F_{Y_1}(y_1)\}, \dots, \min\{F_{X_d}(x_d), F_{Y_d}(y_d)\}\Big) \qquad \text{(By Remark 4.2.5)} \\
&= F_{(\mathbf{X},\mathbf{Y})}(\boldsymbol{x}, \boldsymbol{y}).
\end{aligned}$$

(c) $\Rightarrow$ (d). Straightforward. Take $f_i = F^{-1}_{X_i}, g_i = F^{-1}_{Y_i}$ and $Z_i = U_i$ for all $i \in D$.

(d) $\Rightarrow$ (a). Assume (d) holds. We will show that $\text{supp}(\mathbf{X}, \mathbf{Y})$ is comonotonic, i.e. $\text{supp}(\mathbf{X}, \mathbf{Y})$ is totally ordered. Clearly, $\lesssim$ is a partial order on $\mathbb{R}^{2d}$. Next, we will show that $\text{supp}(\mathbf{X}, \mathbf{Y})$ is comparable. Let $(\boldsymbol{x}_1, \boldsymbol{y}_1), (\boldsymbol{x}_2, \boldsymbol{y}_2) \in \text{supp}(\mathbf{X}, \mathbf{Y})$. By Remark 2.3.3, then $(\boldsymbol{x}_1, \boldsymbol{y}_1), (\boldsymbol{x}_2, \boldsymbol{y}_2) \in \text{supp}((f_1(Z_1), \dots, f_d(Z_d)), (g_1(Z_1), \dots, g_d(Z_d))$. Write

$\boldsymbol{x}_i = (x_i^1, \ldots, x_i^d)$ and $\boldsymbol{y}_i = (y_i^1, \ldots, y_i^d)$ for $i = 1, 2$. Suppose $(\boldsymbol{x}_1, \boldsymbol{y}_1) \not\lessgtr (\boldsymbol{x}_2, \boldsymbol{y}_2)$ and $(\boldsymbol{x}_2, \boldsymbol{y}_2) \not\lessgtr (\boldsymbol{x}_1, \boldsymbol{y}_1)$. WLOG, we may assume $x_1^1 < x_2^1$, $y_2^1 < y_1^1$ but $x_1^m < x_2^m$, $y_1^m < y_2^m$ for all $m \neq 1$. Let $\epsilon = \min\left\{\frac{x_2^1 - x_1^1}{2}, \ldots, \frac{x_2^d - x_1^d}{2}, \frac{y_1^1 - y_2^1}{2}, \frac{y_2^2 - y_1^2}{2}, \ldots, \frac{y_2^d - y_1^d}{2}\right\}$.
By Remark 2.3.4 let $(\boldsymbol{x}_1', \boldsymbol{y}_1'), (\boldsymbol{x}_2', \boldsymbol{y}_2')$ such that
$(\boldsymbol{x}_1', \boldsymbol{y}_1') = ((f_1(z_1), \ldots, f_d(z_d)), (g_1(z_1), \ldots, g_d(z_d))) \in B_\epsilon(\boldsymbol{x}_1, \boldsymbol{y}_1)$ and
$(\boldsymbol{x}_2', \boldsymbol{y}_2') = ((f_1(z_1'), \ldots, f_d(z_d')), (g_1(z_1'), \ldots, g_d(z_d'))) \in B_\epsilon(\boldsymbol{x}_2, \boldsymbol{y}_2)$
for some $z_1, \ldots, z_d, z_1', \ldots, z_d' \in Z(\Omega)$. Proof is similar to Theorem 3.2.3(d $\Rightarrow$ a), we have $f_1(z_1) < \frac{x_1^1 + x_2^1}{2} < f_1(z_1')$ and $g_1(z_1') < \frac{y_1 + y_2}{2} < g_1(z_1)$. Since $f_1, g_1$ are nondecreasing function, $z_1 < z_1'$ and $z_1' < z_1$, a contradiction and so $(\boldsymbol{x}_1, \boldsymbol{y}_1), (\boldsymbol{x}_2, \boldsymbol{y}_2)$ are comparable.

(d) $\Leftrightarrow$ (e). Note that (d) holds if and only if for each $i \in D$, $(X_i, Y_i)$ is comonotonic if and only if $X_i$ and $Y_i$ are almost surely nondecreasing functions of $X_i + Y_i$ $\square$

**Corollary 4.2.7.** *Let (4.1.1) holds. Then* $(\boldsymbol{X}, \boldsymbol{Y})$ *is* $\pi$*-comonotonic if and only if it is s-comonotonic.*

**Remark 4.2.8.** The following properties hold:
(1) Given a pair of marginal distributions $F, G$ there exists a $\pi$-comonotonic random vectors having these marginals if and only if both $\mathbf{X}$ and $\mathbf{Y}$ have the same copula.
(2) The distribution of this $\pi$-comonotonic random vector is unique.
(3) Given a random vector $\mathbf{X}$, the vector $(\mathbf{X}, \mathbf{X})$ is $\pi$-comonotonic.
(4) If $(\mathbf{X}, \mathbf{Y})$ is $\pi$-comonotonic and $F_{\mathbf{X}} = F_{\mathbf{Y}}$, then $\mathbf{X} = \mathbf{Y}$ with propability one.

Both (1) and (2) are a results of Theorem 4.2.6(b) and (3) holds since $\mathbf{X}, \mathbf{X}$ have the same copula. Finally, by Theorem 4.2.6(c), we have $\mathbb{P}(\mathbf{X} = \mathbf{Y}) = 1$.

## 4.3 Weak comonotonicity

In this section, we will reduce condition of comonotonicity of random vectors based on componentwise order.

**Definition 4.3.1.** A set $\Gamma \subset \mathbb{R}^d \times \mathbb{R}^d$ is said to be *w-comonotonic* if $\boldsymbol{x}_1 \leq \boldsymbol{x}_2 \Longleftrightarrow \boldsymbol{y}_1 \leq \boldsymbol{y}_2$ for any $(\boldsymbol{x}_1, \boldsymbol{y}_1), (\boldsymbol{x}_2, \boldsymbol{y}_2) \in \Gamma$.

A random vector $(\mathbf{X}, \mathbf{Y})$ is called **w-comonotonic** if its support is w-comonotonic.

When $d = 1$, Definition 4.3.1 is equivalent to Definitions 3.1.1, 4.1.1 and 4.2.1. When $d > 1$, a $\pi$-comonotonic random vector is also w-comonotonic, but not vice versa.

**Example 4.3.2.** Assume that the distribution of $\mathbf{X} := (X_1, X_2)$ is a nonsymmetric copula $C$, and define the linear transformation $T : [0,1]^2 \to [0,1]^2$ by $T(x_1, x_2) := (x_2, x_1)$. For $(a_1, a_2), (b_1, b_2) \in [0,1]^2$, $(a_1, a_2) \le (b_1, b_2) \Leftrightarrow (a_2, a_1) \le (b_2, b_1) \Leftrightarrow T(a_1, a_2) \le T(b_1, b_2)$. Then the vector $(\mathbf{X}, T(\mathbf{X}))$ is w-comonotonic. Denoting by $C_T$ the distribution of $T(\mathbf{X}) = (X_2, X_1)$, we have $C_T(u_1, u_2) = \mathbb{P}[T(X_1, X_2) \le (u_1, u_2)] = \mathbb{P}[X_2 \le u_1, X_1 \le u_2] = C(u_2, u_1)$. Since $C$ is nonsymmetric, we have $C \neq C_T$. Then $\mathbf{X}, T(\mathbf{X})$ do not have the same copula, so $(\mathbf{X}, T(\mathbf{X}))$ is not $\pi$-comonotonic (see Fig. 4.3).

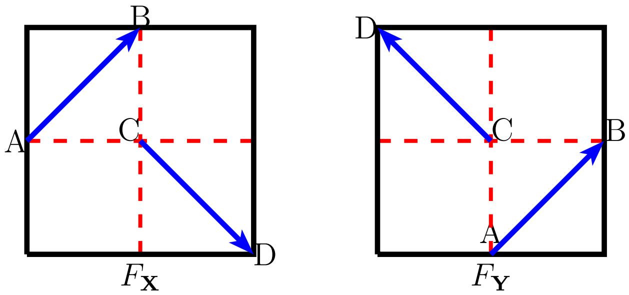


Figure 4.3: The support of a w-comonotonic which is not $\pi$-comonotonic.

Definition 4.3.1 still does not guarantee the existence of a w-comonotonic random vector for an arbitrary choice of the multivariate marginals.

**Remark 4.3.3.** Let $(\mathbf{X}, \mathbf{Y})$ be w-comonotonic. We assume that the marginal $F_{\mathbf{X}}$ has copula $C_+$. Then $F_{\mathbf{X}}(\boldsymbol{x}) = C_+(F_{X_1}(x_1), \ldots, F_{X_d}(x_d)) = \min\{F_{X_1}(x_1), \ldots, F_{X_d}(x_d)\}$, so $\mathbf{X}$ is comonotonic, hence each element in $\mathbf{X}$ is comparable. By definition of w-comonotonicity, each element in $\mathbf{Y}$ is comparable, so $\mathbf{Y}$ is comonotonic. Hence $F_{\mathbf{Y}}$ has copula $C_+$.

Moreover, Definition 4.3.1 does not assure uniqueness of a w-comonotonic random vector having fixed multivariate marginals.

**Example 4.3.4.** Let $d = 2$ and $F_{\mathbf{X}} = F_{\mathbf{Y}} = C_-$. For $\mathbf{Y} = \mathbf{X}$, the random vector $(\mathbf{X}, \mathbf{X})$ is w-comonotonic (obvious by Definition 4.3.1), but it is not the only one with marginals $F_{\mathbf{X}}, F_{\mathbf{Y}}$. Moreover, for $\mathbf{Y} = \mathbf{1} - \mathbf{X}$, $(\mathbf{X}, \mathbf{1} - \mathbf{X})$, where $\mathbf{1} := (1, 1)$, is w-comonotonic too since no pair of points in $(\mathbf{X}, \mathbf{1} - \mathbf{X})$ is comparable (see Fig. 4.4).

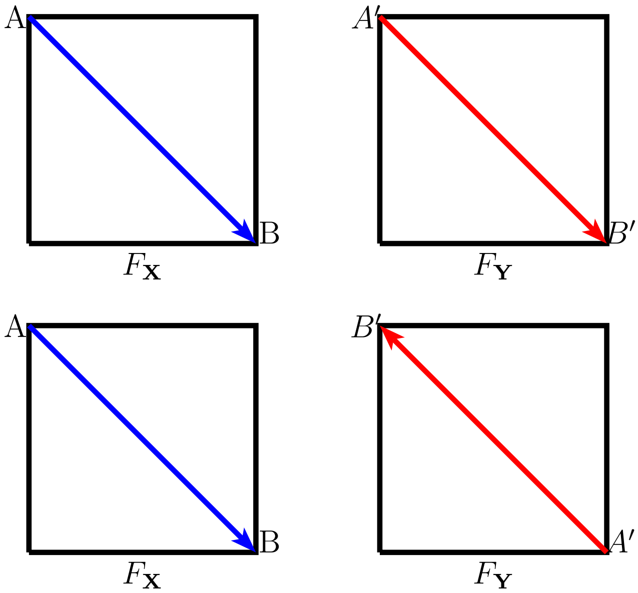


Figure 4.4: The support of two w-comonotonic vectors having the same marginals.

**Theorem 4.3.5.** *Suppose that* $(\mathcal{X}, \leq_{\mathcal{X}})$ *and* $(\mathcal{Y}, \leq_{\mathcal{Y}})$ *are two partially ordered spaces containing at least two distinct points. If for any* $\gamma_1 \subset \mathcal{X}, \gamma_2 \subset \mathcal{Y}$ *it is possible to define a w-comonotonic set* $\Gamma \subset \mathcal{X} \times \mathcal{Y}$ *having* $\gamma_1$ *and* $\gamma_2$ *as its projections, then at least one of the following statement is true:*
*(a)* $(\mathcal{X}, \leq_{\mathcal{X}})$ *and* $(\mathcal{Y}, \leq_{\mathcal{Y}})$ *are totally ordered spaces;*
*(b) any set* $\Gamma \subset \mathcal{X} \times \mathcal{Y}$ *is w-comonotonic.*

*Proof.* Suppose that it is possible to find in one of the two spaces, say $\mathcal{X}$, two points $\boldsymbol{x}_1, \boldsymbol{x}_2$ with $\boldsymbol{x}_1 \leq_{\mathcal{X}} \boldsymbol{x}_2$ and in the other space $\mathcal{Y}$ two distinct points $\boldsymbol{y}_1, \boldsymbol{y}_2$ such that neither $\boldsymbol{y}_1 \leq \boldsymbol{y}_2$ nor $\boldsymbol{y}_2 \leq \boldsymbol{y}_1$ holds.
Choose $\gamma_1 := \{\boldsymbol{x}_1, \boldsymbol{x}_2\}$ and $\gamma_2 := \{\boldsymbol{y}_1, \boldsymbol{y}_2\}$. Note that $\gamma_1$ is a totally ordered subset of $\mathcal{X}$, while no pairs of vectors are $\leq_{\mathcal{Y}}$-comparable in $\gamma_2$. By definition of w-comonotonicity, it is not possible to find a w-comonotonic set $\Gamma \subset \mathcal{X} \times \mathcal{Y}$ with

projections $\gamma_1$ and $\gamma_2$.

Then if $\mathcal{X}$ has two points $\boldsymbol{x}_1, \boldsymbol{x}_2$ with $\boldsymbol{x}_1 \leq_{\mathcal{X}} \boldsymbol{x}_2$, then $\mathcal{Y}$ is totally order space and implying $\mathcal{X}$ is totally order space. Hence (a) holds. In this latter case, if $\mathcal{X}$ has two distinct points $\boldsymbol{x}_1, \boldsymbol{x}_2$ such that neither $\boldsymbol{x}_1 \leq_{\mathcal{X}} \boldsymbol{x}_2$ nor $\boldsymbol{x}_2 \leq_{\mathcal{X}} \boldsymbol{x}_1$, then no pairs of vectors in $\mathcal{Y}$ are comparable, so is $\mathcal{X}$. Thus any set $\Gamma \subset \mathcal{X} \times \mathcal{Y}$ is w-comonotonic, i.e., (b) holds. □

**Remark 4.3.6.** The following properties hold:

(1) Given a pair of marginal distributions $F_{\mathbf{X}}, F_{\mathbf{Y}}$, the existence of a w-comonotonic random vector having these marginals is not always assured. (see Remark 4.3.3)

(2) In general, the distribution of a w-comonotonic random vector with fixed marginals is not unique. (see Example 4.3.4)

(3) Given a random vector $\mathbf{X}$, the vector $(\mathbf{X}, \mathbf{X})$ is w-comonotonic.

(4) If $(\mathbf{X}, \mathbf{Y})$ is w-comonotonic and $F_{\mathbf{X}} = F_{\mathbf{Y}}$, then $\mathbf{X}$ is not necessarily equal to $\mathbf{Y}$ with propability one. (see Remark 4.3.3)

# Chapter 5
# Conclusion

In our project, we have the following results:

1. If a random vector $(\mathbf{X}, \mathbf{Y})$ is strong comonotonic, then both random vectors $\mathbf{X}, \mathbf{Y}$ have copula $C_+$ and the $2d$ random variables $X_1, \ldots, X_d, Y_1, \ldots, Y_d$ are pairwise comonotonic.

2. If a random vector $(\mathbf{X}, \mathbf{Y})$ is projection comonotonic, then both random vector $\mathbf{X}, \mathbf{Y}$ have the same copula and every pair $(X_i, Y_i)$ is comonotonic for any $i \in D$.

3. Every concept of comonotonicity including the definition of weak comonotonicity on a partially ordered space, has to drop either that partially ordered space is totally ordered or that every subset of product of partially ordered space is weak comonotonic.

4. Strong comonotonicity implies projection comonotonicity and projection comonotonicity implies weak comonotonicity.

**Open Question**

1. Is it possible to investigate the Comonotonicity without using componentwise ordering?

# References

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# Appendix A

แบบเสนอหัวข้อโครงงาน รายวิชา 2301399 Project Proposal ปีการศึกษา 2560

| | |
|---|---|
| ชื่อโครงงาน (ภาษาไทย) | ความสัมพันธ์ทางเดียวกันหลายตัวแปร |
| ชื่อโครงงาน (ภาษาอังกฤษ) | Multivariate Comonotonicity |
| อาจารย์ที่ปรึกษา | รศ.ดร. ทรงเกียรติ สุเมธกิจการ |
| | รศ. ทิพวัลย์ สันติวิภานนท์ |
| ผู้ดำเนินการ | นายพีรพงศ์ ปัญญาสกุลวงศ์ เลขประจำตัวนิสิต 5833537323 |
| | สาขาวิชา คณิตศาสตร์ ภาควิชา คณิตศาสตร์และวิทยาการคอมพิวเตอร์ |
| | คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย |

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**หลักการและเหตุผล**

การนิยามความสัมพันธ์ทางเดียวกันระหว่างสองตัวแปรสุ่มนั้นนิยามได้อย่างตรงไปตรงมาและค่อนข้างง่ายต่อการศึกษาและทำความเข้าใจ อย่างไรก็ตามเมื่อพิจารณาเวกเตอร์สุ่มซึ่งประกอบขึ้นด้วยอย่างน้อยสองตัวแปรสุ่ม ความสัมพันธ์ระหว่างสองเวกเตอร์สุ่มมีความซับซ้อนและไม่เป็นธรรมชาติในการจำแนกว่าเป็นความสัมพันธ์ทางเดียวกันหรือไม่ เราจึงสนใจศึกษาวิธีต่างๆ ในการเขียนนิยามและตีความความสัมพันธ์ทางเดียวกันระหว่างสองเวกเตอร์สุ่ม รวมทั้งสมบัติและความเกี่ยวข้องระหว่างความสัมพันธ์ทางเดียวกันเหล่านี้ โดยบทนิยามทั้งหมดจะอ้างอิงจากบทความ Multivariate comonotonicity โดย Giovanni Puccetti และ Marco Scarsini

**วัตถุประสงค์**

ศึกษาสมบัติของความสัมพันธ์ทางเดียวกันของสองเวกเตอร์สุ่มและเปรียบเทียบความแตกต่างของความสัมพันธ์ทางเดียวกันทั้งสามประเภท

**ขอบเขตของโครงงาน**

เปรียบเทียบความแตกต่างของความสัมพันธ์ทางเดียวกันทั้งสามประเภทคือ ความสัมพันธ์ทางเดียวอย่างเข้ม (strong comonotonicity), ความสัมพันธ์ทางเดียวแบบภาพฉาย (projection comonotonicity) และความสัมพันธ์ทางเดียวอย่างอ่อน (weak comonotonicity)

**วิธีการดำเนินงาน**

1. ทบทวนความรู้พื้นฐานทางด้านความน่าจะเป็นและศึกษาความรู้พื้นฐานเกี่ยวกับ copula
2. ศึกษานิยามและสมบัติพื้นฐานของความสัมพันธ์ทางเดียวกันของหลายตัวแปร
3. ศึกษาเปรียบเทียบความสัมพันธ์ทางเดียวกันประเภทต่างๆ
4. สรุปและเขียนรูปเล่มโครงงาน

**เวลาดำเนินงาน**

| | เดือน | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | ปี 2560 | | | | | ปี 2561 | | | |
| | ส.ค. | ก.ย. | ต.ค. | พ.ย. | ธ.ค. | ม.ค. | ก.พ. | มี.ค. | เม.ย. |
| ทบทวนความรู้พื้นฐานที่จำเป็น | ✓ | ✓ | | | | | | | |
| ศึกษานิยามและสมบัติของ Comontonicity | | ✓ | ✓ | ✓ | | | | | |
| เปรียบเทียบประเภทต่างๆของ Comonotonicity | | | | ✓ | ✓ | ✓ | ✓ | | |
| สรุปและเขียนรูปเล่มโครงงาน | | | | | | | ✓ | ✓ | ✓ |

**ประโยชน์ที่คาดว่าจะได้รับ**

1. ได้รับความรู้เกี่ยวกับความสัมพันธ์ทางเดียว
2. ได้ฝึกการอ่านงานวิจัยและการสืบค้นข้อมูลจากแหล่งอ้างอิงเพิ่มเติม
3. สามารถนำไปใช้ศึกษาทางด้านการเงินได้

# Appendix B

## Biography

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Mathematics Program

Department of Mathematics and Computer Science

Faculty of Science, Chulalongkorn University