โครงการ

การเรียนการสอนเพื่อเสริมประสบการณ์

ชื่อโครงการ	ค่าเฉพาะระยะทางและพลังงานระยะทางของกราฟเคย์เลย์ยูนิแทรี Distance spectra and distance energy of unitary Cayley graphs
ชื่อนิสิต	นายชานนท์ ทองประยูร 5833510323
ภาควิชา	คณิตศาสตร์และวิทยาการคอมพิวเตอร์ สาขาวิชา คณิตศาสตร์
ปีการศึกษา	2561

คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย

ค่าเฉพาะระยะทางและพลังงานระยะทางของกราฟเคย์เลย์ยูนิแทรี

นายชานนท์ ทองประยูร

โครงงานนี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรวิทยาศาสตรบัณฑิต

สาขาวิชาคณิตศาสตร์ ภาควิชาคณิตศาสตร์และวิทยาการคอมพิวเตอร์

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ปีการศึกษา 2561

Distance spectra and distance energy of unitary Cayley graphs

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A Project Submitted in Partial Fulfillment of the Requirements
for the Degree of Bachelor of Science Program in Mathematics
Department of Mathematics and Computer Science
Faculty of Science
Chulalongkorn University
Academic Year 2018

หัวข้อโครงงาน	ค่าเฉพาะระยะทางและพลังงานระยะทางของกราฟเคย์เลย์ยูนิแทรี
โดย	นาย ชานนท์ ทองประยูร เลขประจำตัว 5833510323
สาขาวิชา	คณิตศาสตร์
อาจารย์ที่ปรึกษาโครงงาน	ศาสตราจารย์ ดร.ยศนันต์ มีมาก

ภาควิชาคณิตศาสตร์และวิทยาการคอมพิวเตอร์ คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย อนุมัติให้นับโครงงานฉบับนี้เป็นส่วนหนึ่ง ของการศึกษาตามหลักสูตรปริญญาบัณฑิต ในรายวิชา 2301499 โครงงานวิทยาศาสตร์ (Senior Project)

. . . . หัวหน้าภาควิชาคณิตศาสตร์และวิทยาการคอมพิวเตอร์

(ศาสตราจารย์ ดร.กฤษณะ เนียมมณี)

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. . . . อาจารย์ที่ปรึกษาโครงงาน

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. . . . กรรมการ

(ผู้ช่วยศาสตราจารย์ ดร.พงษ์เดช มนทกานติรัตน์)

ชานนท์ ทองประยูร: ค่าเฉพาะระยะทางและพลังงานระยะทางของกราฟเคย์เลย์ยูนิแทรี (DISTANCE SPECTRA AND DISTANCE ENERGY OF UNITARY CAYLEY GRAPHS) อ.ที่ปรึกษาโครงงาน: ศ.ดร.ยศนันต์ มีมาก, 32 หน้า

เราคำนวนค่าเฉพาะระยะทางและพลังงานระยะทางของกราฟเคย์เลย์ยูนิแทรีและกราฟกำกัดของยูนิแทรีเคย์เลย์จากการเหนี่ยวนำโดยการส่งกำลังสองซึ่งนิยามเหนือริงเฉพาะที่จำกัด R นอกจากนี้เรายังคำนวนค่าเฉพาะระยะทางและพลังงานระยะทางของกราฟเทนเซอร์ระหว่างทางเดินที่มีความยาวสองกับกราฟเคย์เลย์ยูนิแทรีเหนือ R

ภาควิชา คณิตศาสตร์และวิทยาการคอมพิวเตอร์ ลายมือชื่อนิสิต ....

สาขาวิชา . คณิตศาสตร์ . ลายมือชื่อ อ.ที่ปรึกษาโครงงาน ....

ปีการศึกษา . . . . 2561 . . . .

5833510323 : MAJOR MATHEMATICS.
KEYWORDS: DISTANCE SPECTRA, DISTANCE ENERGY

CHANON THONGPRAYOON: DISTANCE SPECTRA AND DISTANCE ENERGY OF UNITARY CAYLEY GRAPHS. ADVISOR: PROF. YOTSANAN MEEMARK, PH.D., 32 PP.

In this project, we compute the distance spectra and distance energy of the unitary Cayley graph and the restricted unitary Cayley graph induced from the square mapping defined over a finite local ring R. Furthermore, we investigate the distance spectra and distance energy of the tensor graph of the path of length two and the unitary Cayley graph over R.

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Field of Study . . Mathematics . . Advisor's Signature . Yotsanan Meemark
Academic Year . . . . . . 2018 . . . . .

Acknowledgments

I would like to express my gratitude to Professor Yotsanan Meemark, Ph.D. who is my project advisor for his help and advices. I receive valuable knowledge and experiences throughout working in this project. It is my pleasure to have an opportunity to work with him. Also, I would like to express my thanks to project committee: Professor Patanee Udomkavanich, Ph.D. and Assistant Professor Pongdate Montagantirud, Ph.D. for their great suggestions and comments. Lastly, I would like to thank my family and my friends for their support and encouragement.

Contents

Chapter 1

Preliminaries

In this project, rings always contain the identity $1 \neq 0$.

1.1 Some Background in Ring Theory

We begin by providing some facts about commutative rings which are used in this project.

Definition 1.1. Let u be an element in a ring. Then u is a **unit** of R if there exists a $v \in R$ such that $uv = vu = 1$. Also, the set of all units in R is denoted by $R^{\times}$. It forms a group under the multiplication of R.

Definition 1.2. Let R be a commutative ring and $I \subseteq R$. Then I is said to be an **ideal** of R if it satisfies the following properties: (i) $(I, +)$ is a subgroup of $(R, +)$; (ii) for any $r \in R$ and $a \in I$, then $ar \in I$.

Definition 1.3. An ideal M of a ring R is said to be **maximal** if $M \neq R$ and for any ideal J of R,

$$M \subseteq J \subseteq R \Rightarrow J = M \text{ or } J = R.$$

Definition 1.4. A commutative ring is said to be **local** if it has a unique maximal ideal.

Theorem 1.5. *If R is a local ring with unique maximal ideal M, then $R^{\times} = R \smallsetminus M$.*

Theorem 1.6. *Let R be a finite commutative ring. Then $R \cong R_1 \times R_2 \times \dots R_s$ where R_i is a local ring for $i \in \{1, 2, \dots, s\}$.*

1.2 Some Background in Graph Theory

In this section, we provide some definitions of graphs, spectra and energy.

Definition 1.7. A **graph** G is an ordered pair $(V(G), E(G))$ where $V(G)$ is the set of all vertices in G and $E(G) = \{\{v, w\} : v, w \in V(G)\}$ is the set of all edges in G.

Definition 1.8. Let G and H be graphs. The **cartesian product** of G and H, denoted by $G \times H$, is a graph whose vertex set is $V(G) \times V(H)$ and edge set is $\{\{(v, w), (v', w')\} : v = v'$ and $\{w, w'\} \in E(H)$ or $\{v, v'\} \in V(G)$ and $w = w'\}$.

Definition 1.9. Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be graphs. The **tensor product** of G and H, denoted by $G \otimes H$, is a graph whose vertex set is $V(G) \times V(H)$ and the edge set is $\{\{(v, w), (v', w')\} : \{v, v'\} \in E(G)$ and $\{w, w'\} \in E(H)\}$.

Definition 1.10. If v is a vertex of a graph G, then the **degree** of v is the number of edges connecting v and denoted by $\deg v$.

Definition 1.11. A graph G is **regular** if every vertex of G has the same degree.

Definition 1.12. A graph G is **strongly regular** with parameters (n, k, λ, μ) if G has n vertices with regularity k and every pair of adjacent vertices and every pair of non-adjacent vertices have λ and μ common neighbors, respectively.

Definition 1.13. Let G be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. The **adjacency matrix** $A(G) = [a_{ij}]$ is an $n \times n$ matrix where $a_{ij} = 1$ if the v_i is adjacent to the v_j; otherwise, $a_{ij} = 0$ for all $i, j \in \{1, 2, \ldots, n\}$.

Definition 1.14. The **distance** of two vertices v and w in a graph G is the least number of edges connecting them. It is denoted by $d(v, w)$. Also, $d(v, v) = 0$ for all $v \in V(G)$. The **diameter** of a graph G is $\text{diam}(G) = \max\{d(v, w) : v, w \in V(G)\}$.

Definition 1.15. Let G be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. The **distance matrix** $D(G) = [d_{ij}]$ of a graph G is also an $n \times n$ matrix where d_{ij} is the distance from v_i to v_j for all $i, j \in \{1, 2, \ldots, n\}$.

Definition 1.16. A matrix is said to be **circulant** if its first row generates the other rows by shifting the first row to the right or shifting the first row to the left.

Example 1.17. Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix}$$

Then, A is a circulant matrix whose the first row is $(1, 2, 3)$.

Definition 1.18. An eigenvalue of the adjacency matrix of a graph G is called a **spectrum** of G. The set of all spectra of G is denoted by Spec(G). Also, if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are spectra of the adjacemcy matrix of a graph G with multiplicities $m_1, m_2, \ldots, m_n$, respectively, then we write Spec$(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ m_1 & m_2 & \cdots & m_n \end{pmatrix}$ for the spectrum of G.

Definition 1.19. A graph with n vertices is said to be **complete** if every pair of distinct vertices has an edge connecting them. We let K_n denote the complete graph with n vertices.

Theorem 1.20. Spec$(K_n) = \begin{pmatrix} n-1 & -1 \\ 1 & n-1 \end{pmatrix}$.

Definition 1.21. The **energy** of a graph G, denoted by $E(G)$, is defined as the sum of the absolute value of each spectrum of G counting multiplicities.

Definition 1.22. An eigenvalue of the distance matrix of a graph G is called a **distance spectrum** of G. The set of all distance spectra of G is denoted by Spec$_D(G)$. Similarly, if $\mu_1, \mu_2, \ldots, \mu_n$ are distance spectra of the distance matrix of a graph G with multiplicities $k_1, k_2, \ldots, k_n$, respectively, we write Spec$_D(G) = \begin{pmatrix} \mu_1 & \mu_2 & \cdots & \mu_n \\ k_1 & k_2 & \cdots & k_n \end{pmatrix}$ for the distance spectrum of G.

Definition 1.23. The **distance energy** of a graph G, denoted by $DE(G)$, is defined as the sum of the absolute value of each distance spectrum of G counting multiplicities.

1.3 Spectra of Unitary Cayley Graphs

In this section, we collect the previous results on the spectra and energies of unitary Cayley graphs from [1, 2, 3, 4, 6, 7, 8].

Definition 1.24. Let R be a finite commutative ring. The **unitary Cayley graph** of R is a graph whose vertex set is R and the edge set is $\{\{a,b\} : a,b \in R$ and $a-b \in R^{\times}\}$. The unitary Cayley graph of R is denoted by G_R.

Definition 1.25. Let R be a finite commutative ring. A subgraph H_R of G_R is called the **restricted unitary Cayley graph induced from the square mapping** if the vertex set of H_R is R and the edge set is $\{\{a,b\} : a,b \in R$ and $a-b \in K_R(R^{\times 2})\}$, where $K_R = \{a \in R^{\times} : a^2 = 1\}$ and $(R^{\times})^2 = \{a^2 : a \in R^{\times}\}$.

Definition 1.26. Let q be a prime power such that $q \equiv 1 \ mod \ 4$. The **Paley graph** is the graph whose vertex set is the finite field of order q, denoted by $\mathbb{F}_q$, and the edge set is $\{\{a,b\} : a,b \in R$ and $a-b \in (\mathbb{F}_q^{\times})^2\}$. The Payley graph over $\mathbb{F}_q$ is denoted by $H_{\mathbb{F}q}$.

Proposition 1.27. *[2]. We have the following properties of unitary Cayley graphs.*

1. *Let R be a finite commutative ring such that $R \cong R_1 \times R_2 \times \cdots \times R_s$ where R_i is a local ring for $i \in \{1,2,\ldots,s\}$. Then $G_R \cong G_{R_1} \otimes G_{R_2} \otimes \cdots \otimes G_{R_s}$.*
2. *If R is a finite local ring with maximal ideal M, then G_R is a complete multipartite graph whose partite sets are the cosets of M. In particular, if F is a finite field, then G_F is the complete graph on $|F|$ elements.*

Proposition 1.28. *If R is a finite local ring, then* $\text{diam}(G_R) = 2$*.*

Proof. From Proposition 1.27 (2), G_R is a complete multipartite graph, so its diameter is 2. □

Theorem 1.29. *[7]. Let R be a finite local ring with maximal ideal M of size m. Then*

$$\text{Spec}(G_R) = \begin{pmatrix} |R|-m & -m & 0 \\ 1 & \frac{|R|}{m}-1 & \frac{|R|}{m}(m-1) \end{pmatrix}$$

and $E(G_R) = 2(|R| - m)$. In particular, if F is a field, then

$$\text{Spec}(G_F) = \begin{pmatrix} |F|-1 & -1 \\ 1 & |F|-1 \end{pmatrix}$$

and $E(G_F) = 2(|F| - 1)$.

Theorem 1.30. *[6]. Let A be an $n \times n$ circulant matrix with the first row $(a_0, \dots, a_{n-1})$. Then the spectrum λ_j of A are*

$$\lambda_j = a_0 + a_1\omega_j + a_2{\omega_j}^2 + \cdots + a_{n-1}{\omega_j}^{n-1}$$

where $\omega_j = e^{\frac{2\pi i j}{n}}$ for all $j \in \{1, 2, \dots, n\}$.

Lemma 1.31. *[3]. Let $A = \begin{pmatrix} A_0 & A_1 \\ A_1 & A_0 \end{pmatrix}$ be a 2×2 symmetric matrix. Then the spectra of A can be obtained from the spectra of $A_0 + A_1$ and $A_0 - A_1$.*

Lemma 1.32. *[4]. Let q be a prime power such that $q \equiv 1 \ (mod\ 4)$. The Paley graph $H_{\mathbb{F}_q}$ is a strongly regular graph with parameters $(q, \frac{q-1}{2}, \frac{q-5}{4}, \frac{q-1}{4})$ and $\text{Spec}(H_{\mathbb{F}_q}) = \begin{pmatrix} \frac{q-1}{2} & \frac{\sqrt{q}-1}{2} & \frac{-\sqrt{q}-1}{2} \\ 1 & \frac{q-1}{2} & \frac{q-1}{2} \end{pmatrix}$.*

Proposition 1.33. *If q is a prime power such that $q \equiv 1 \ (mod\ 4)$, then the Paley graph $H_{\mathbb{F}_q}$ is of diameter two.*

Proof. It follows from the fact that $H_{\mathbb{F}_q}$ is strongly regular so any two vertices has a common neighbor. □

Proposition 1.34. *[1]. Let G be a regular graph of degree ρ with n vertices. If* diam(G) *is at most 2, then*

$$\text{Spec}_D(G) = \{2n - 2 - \rho\} \cup \{-\lambda - 2 : \lambda \in \text{Spec}(G) \text{ and } \lambda \neq \rho\}.$$

Theorem 1.35. *[8] Let R be a finite local ring with unique maximal ideal M of size m and of characteristic an odd prime power.*

1. *-1 is a square in R if and only if $\frac{|R|}{m} \equiv 1 \ mod\ 4$.*

2. *If -1 is not a square in R, then $H_R = G_R$.*

3. *If -1 is a square in R, then $H_R \cong H_{R/M} \otimes \dot{\mathcal{K}}_m$ where $\dot{\mathcal{K}}_m$ is the m-complete graph with a loop on each vertex. Moreover, we obtain the spectra of H_R as follows:*

$$\text{Spec}(H_R) = \begin{pmatrix} \frac{|R|-m}{2} & \frac{m(\sqrt{\frac{|R|}{m}}-1)}{2} & \frac{m(-\sqrt{\frac{|R|}{m}}-1)}{2} & 0 \\ 1 & \frac{\frac{|R|}{m}-1}{2} & \frac{\frac{|R|}{m}-1}{2} & m \end{pmatrix}.$$

1.4 Our Objectives

Illic [6] established the distance matrix and computed the distance energy of the unitary Cayley graph $G_{\mathbb{Z}_n}$ in terms of the Euler ϕ-function. Gopal [5] obtained the distance spectra of the cartesian graph $G \times K_2$ where G is a regular graph of diameter 1 or 2.

Hence, we turn our interests to distance spectra and distance energy of G_R and H_R where R is a finite local ring. Furthermore, we also investigate the distance spectra of $K_2 \otimes G_R$. The results are presented in the next chapter. Section 2.1 covers the work on G_R and $K_2 \otimes G_R$ and Section 2.2 discussed the work on H_R defined in Section 1.3.

Chapter 2

Results

2.1 Distance Spectra and Distance Energy of G_R

In the first section, we begin by computing distance spectra of G_R, followed by $K_2 \otimes G_R$.

Theorem 2.1. *Let R be a finite local ring with maximal ideal M of size m.*

1. *If $m = 1$, then* $\text{Spec}_D(G_R) = \begin{pmatrix} |R|-1 & -1 \\ 1 & |R|-1 \end{pmatrix}$.

2. *If $m > 1$, then* $\text{Spec}_D(G_R) = \begin{pmatrix} |R|+m-2 & m-2 & -2 \\ 1 & \frac{|R|}{m}-1 & \frac{|R|}{m}(m-1) \end{pmatrix}$.

Proof. If $m = 1$, then $M = \{0\}$, so R is a field. It follows from [2] that G_R is a complete graph. Then

$$\text{Spec}_D(G_R) = \begin{pmatrix} |R|-1 & -1 \\ 1 & |R|-1 \end{pmatrix}.$$

Now, assume that $m > 1$. Write $M = \{0, x_2, x_3, \ldots, x_m\}$ and $q = \frac{|R|}{m}$. Then

$$R/M = \{M, a_2 + M, a_3 + M, \ldots, a_q + M\}.$$

Thus,

$$R = \dot{\bigcup_{1 \le i \le q}} a_i + M = \dot{\bigcup_{1 \le j \le m}} \{x_j, a_2 + x_j, a_3 + x_j, \ldots, a_q + x_j\}.$$

By Proposition 1.27 (2), G_R is a complete multipartite graph whose partite sets are the cosets of M. So, for any vertices v and w of G_R, we have

$$d(v,w) = \begin{cases} 0 & \text{if } v = w, \\ 1 & \text{if } v \text{ and } w \text{ are from different cosets,} \\ 2 & \text{if } v \neq w \text{ and } v, w \text{ are in the same cosets.} \end{cases}$$

For $j \in \{1, 2, \ldots, m\}$, we consider the distance matrix,

$$A_j = \begin{array}{c} \begin{matrix} x_j & a_2 + x_j & a_3 + x_j & \cdots & a_{q-1} + x_j & a_q + x_j \end{matrix} \\ \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 0 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 0 \end{pmatrix} \end{array} \begin{matrix} x_j \\ a_2 + x_j \\ a_3 + x_j \\ \\ a_{q-1} + x_j \\ a_q + x_j \end{matrix} := A$$

and for $j, k \in \{1, \ldots, m\}$ and $j \neq k$, we have the distance matrix

$$B_{jk} = \begin{array}{c} \begin{matrix} x_k & a_2 + x_k & a_3 + x_k & \cdots & a_{q-1} + x_k & a_q + x_k \end{matrix} \\ \begin{pmatrix} 2 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 2 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 2 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 2 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 2 \end{pmatrix} \end{array} \begin{matrix} x_j \\ a_2 + x_j \\ a_3 + x_j \\ \\ a_{q-1} + x_j \\ a_q + x_j \end{matrix} := B.$$

The distance matrix of G_R can be presented using the above two matrices A and B as follows:

$$D(G_R) = \begin{pmatrix} A_1 & B_{12} & B_{13} & \cdots & B_{1m} \\ B_{21} & A_2 & B_{23} & \cdots & B_{2m} \\ \vdots & \vdots & \vdots & & \vdots \\ B_{m1} & B_{m2} & B_{m3} & \cdots & A_m \end{pmatrix} = \begin{pmatrix} A & B & B & \cdots & B \\ B & A & B & \cdots & B \\ \vdots & \vdots & \vdots & & \vdots \\ B & B & B & \cdots & A \end{pmatrix}.$$

Since A and B are circulant matrices, it follows that $D(G_R)$ is a circulant matrix. Let $\omega = e^{\frac{2\pi i}{|R|}}$ and $\omega_j = \omega^j$ for all $j \in \{0, 1, \ldots, |R|-1\}$. From Theorem 1.30, we obtain the distance spectra

$$\begin{aligned}\mu_j = \; & 0 + \omega_j + \cdots + \omega_j^{q-1} + 2\omega_j^q + \omega_j^{q+1} + \cdots + \omega_j^{2q-1} + \cdots + 2\omega_j^{(m-1)q} \\ & + \omega_j^{(m-1)q+1} + \cdots + \omega_j^{mq-1} \\ = & \sum_{k=0}^{|R|-1} \omega_j^k + \sum_{l=0}^{m-1} \omega_j^{ql} - 2\end{aligned}$$

for all $j \in \{0, 1, \ldots, |R|-1\}$. We may distinguish three cases.

Case 1. $j = 0$. Then $\omega_0 = 1$ and $\mu_0 = |R| + m - 2$.

Case 2. $j = km$ for some $k \in \{1, 2, \ldots, q-1\}$. Then $\omega_j^q = \omega^{jq} = \omega^{kmq} = \omega^{k|R|} = 1$, so $\mu_j = 0 + m - 2 = m - 2$.

Case 3. j is not a multiple of m. Then $\omega_j^q \neq 1$, so

$$\sum_{l=0}^{m-1} \omega_j^{ql} = \frac{1 - \omega_j^{qm}}{1 - \omega_j^q} = \frac{1 - 1}{1 - \omega_j^q} = 0.$$

Thus, $\mu_j = 0 + 0 - 2 = -2$.

Therefore, the above computation gives

$$\text{Spec}_D(G_R) = \begin{pmatrix} |R| + m - 2 & m - 2 & -2 \\ 1 & \frac{|R|}{m} - 1 & \frac{|R|}{m}(m-1) \end{pmatrix}.$$

This completes the proof. □

Corollary 2.2. *Let R be a finite local ring with maximal ideal M of size m. Then the distance energy of G_R is*

$$DE(G_R) = \begin{cases} 2(|R| - 1) & \text{if } m = 1, \\ 4(|R| - \frac{|R|}{m}) & \text{if } m > 1. \end{cases}$$

Proof. If $m = 1$, then $\text{Spec}_D(G_R) = \begin{pmatrix} |R| - 1 & -1 \\ 1 & |R| - 1 \end{pmatrix}$, so

$$DE(G_R) = ||R| - 1| + (|R| - 1)|-1| = 2(|R| - 1).$$

If $m > 1$, then Spec$_D(G_R)$=$\begin{pmatrix} |R|+m-2 & m-2 & -2 \\ 1 & \frac{|R|}{m}-1 & \frac{|R|}{m}(m-1) \end{pmatrix}$, so

$$\begin{aligned} DE(G_R) &= |R|+m-2+|R|-\frac{2|R|}{m}-m+2+\frac{2|R|}{m}(m-1) \\ &= 4(|R|-\frac{|R|}{m}) \end{aligned}$$

as desired. □

Theorem 2.3. *Let R be a finite local ring with maximal ideal M of size m and K_2 the path of length 2.*

1. *If $m = 1$, then*

$$\text{Spec}_D(K_2 \otimes G_R) = \begin{pmatrix} 3(|R|-1) & |R|-5 & -4 & 0 \\ 1 & 1 & |R|-1 & |R|-1 \end{pmatrix}.$$

2. *If $m > 1$, then*

$$\text{Spec}_D(K_2 \otimes G_R) = \begin{pmatrix} 3|R|+2m-2 & |R|-2m-2 & 2m-2 & -2m-2 & -2 \\ 1 & 1 & \frac{|R|}{m}-1 & \frac{|R|}{m}-1 & 2(|R|-\frac{|R|}{m}) \end{pmatrix}.$$

Proof. Similar to the proof of Theorem 2.1, we may write $M = \{0, x_2, x_3, \ldots, x_m\}$, $q = \frac{|R|}{m}$. Then

$$R/M = \{M, a_2 + M, a_3 + M, \ldots, a_q + M\}.$$

Thus,

$$R = \dot{\bigcup_{1\le i\le q}} a_i + M = \dot{\bigcup_{1\le j\le m}} \{x_j, a_2 + x_j, a_3 + x_j, \ldots, a_q + x_j\}.$$

By Proposition 1.27 (2), G_R is a complete multipartite graph whose partite sets are the costes of M, for any vertices (x, v) and (y, w) of $K_2 \otimes G_R$, we have

$$d((x,v),(y,w)) = \begin{cases} 0 & \text{if } (x,v) = (y,w), \\ 1 & \text{if } x \neq y \text{ and } v, w \text{ are from different cosets,} \\ 2 & \text{if } x = y \text{ and } v \neq w, \\ 3 & \text{if } x \neq y \text{ and } v, w \text{ are from the same cosets.} \end{cases}$$

For $x \in V(K_2)$ and $j \in \{1, 2, \ldots, m\}$, we consider the distance matrix

$$A_j = \begin{array}{c} \begin{array}{ccccc} (x, x_j) & (x, a_2 + x_j) & \cdots & (x, a_{q-1} + x_j) & (x, a_q + x_j) \end{array} \\ \left(\begin{array}{ccccc} 0 & 2 & \cdots & 2 & 2 \\ 2 & 0 & \cdots & 2 & 2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 2 & 2 & \cdots & 0 & 2 \\ 2 & 2 & \cdots & 2 & 0 \end{array} \right) \end{array} \begin{array}{c} (x, x_j) \\ (x, a_2 + x_j) \\ \vdots \\ (x, a_{q-1} + x_j) \\ (x, a_q + x_j) \end{array} := A,$$

for $x \in V(K_2)$ and $j, k \in \{1, 2, \ldots, m\}$ with $a_i + x_j \neq a_i + x_k$ for $i \in \{1, 2, \ldots, q\}$, we have the distance matirx,

$$B_{jk} = \begin{array}{c} \begin{array}{ccccc} (x, x_k) & (x, a_2 + x_k) & \cdots & (x, a_{q-1} + x_k) & (x, a_q + x_k) \end{array} \\ \left(\begin{array}{ccccc} 2 & 2 & \cdots & 2 & 2 \\ 2 & 2 & \cdots & 2 & 2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 2 & 2 & \cdots & 2 & 2 \\ 2 & 2 & \cdots & 2 & 2 \end{array} \right) \end{array} \begin{array}{c} (x, x_j) \\ (x, a_2 + x_j) \\ \vdots \\ (x, a_{q-1} + x_j) \\ (x, a_q + x_j) \end{array} := B,$$

and for $x, y \in V(K_2)$ with $x \neq y$ and $j, k \in \{1, 2, \ldots, m\}$, we have the distance matrix

$$C_{jk} = \begin{array}{c} \begin{array}{ccccc} (x, x_k) & (x, a_2 + x_k) & \cdots & (x, a_{q-1} + x_k) & (x, a_q + x_k) \end{array} \\ \left(\begin{array}{ccccc} 3 & 1 & \cdots & 1 & 1 \\ 1 & 3 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 3 & 1 \\ 1 & 1 & \cdots & 1 & 3 \end{array} \right) \end{array} \begin{array}{c} (y, x_j) \\ (y, a_2 + x_j) \\ \vdots \\ (y, a_{q-1} + x_j) \\ (y, a_q + x_j) \end{array} := C.$$

If $m = 1$, then the distance matrix of $K_2 \otimes G_R$ is $D(K_2 \otimes G_R) = \begin{pmatrix} A_1 & C_{11} \\ C_{11} & A_1 \end{pmatrix}$. Let $\theta = e^{\frac{2\pi i}{|R|}}$ and $\theta_j = \theta^j$ for $j \in \{0, 1, \ldots, |R| - 1\}$. We know from Lemma 1.31 that the distance spectra are obtained by computing the spectra of $A_1 + C_{11}$ and $A_1 - C_{11}$

which are both circulant. From, Theorem 1.30, we obtain the distance spectra of $A_1 + C_{11}$ and $A_1 - C_{11}$, denoted by λ_j and ρ_j, respectively:

$$\lambda_j = 3\sum_{k=0}^{|R|-1} \theta_j^k \quad \text{and} \quad \rho_j = \sum_{l=0}^{|R|-1} \theta_j^l - 4$$

for $j \in \{0, 1, \ldots, |R|-1\}$. If $j = 0$, then $\lambda_0 = 3|R|$ and $\rho_0 = |R| - 4$. If $j \neq 0$, then

$$\sum_{k=0}^{|R|-1} \theta_j^k = \sum_{l=0}^{|R|-1} \theta_j^l = 0,$$

so $\lambda_j = 0$ and $\rho_j = -4$. Consequently,

$$\text{Spec}_D(K_2 \otimes G_R) = \begin{pmatrix} 3|R| & |R|-4 & -4 & 0 \\ 1 & 1 & |R|-1 & |R|-1 \end{pmatrix}.$$

If $m > 1$, then we obtain the distance matirx

$$D(K_2 \otimes G_R) = \begin{pmatrix} A_1 & B_{12} & B_{13} & \cdots & B_{1m} & C_{11} & C_{12} & C_{13} & \cdots & C_{1m} \\ B_{21} & A_2 & B_{23} & \cdots & B_{2m} & C_{21} & C_{22} & C_{23} & \cdots & C_{2m} \\ B_{31} & B_{32} & A_3 & \cdots & B_{3m} & C_{31} & C_{32} & C_{33} & \cdots & C_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ B_{m1} & B_{m2} & B_{m3} & \cdots & A_m & C_{m1} & C_{m2} & C_{m3} & \cdots & C_{mm} \\ C_{11} & C_{12} & C_{13} & \cdots & C_{1m} & A_1 & B_{12} & B_{13} & \cdots & B_{1m} \\ C_{21} & C_{22} & C_{23} & \cdots & C_{2m} & B_{21} & A_2 & B_{23} & \cdots & B_{2m} \\ C_{31} & C_{32} & C_{33} & \cdots & C_{3m} & B_{31} & B_{32} & A_3 & \cdots & B_{3m} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ C_{m1} & C_{m2} & C_{m3} & \cdots & C_{mm} & B_{m1} & B_{m2} & B_{m3} & \cdots & A_m \end{pmatrix}.$$

Hence,

$$D(K_2 \otimes G_R) = \begin{pmatrix} A & B & B & \cdots & B & C & C & C & \cdots & C \\ B & A & B & \cdots & B & C & C & C & \cdots & C \\ B & B & A & \cdots & B & C & C & C & \cdots & C \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ B & B & B & \cdots & A & C & C & C & \cdots & C \\ C & C & C & \cdots & C & A & B & B & \cdots & B \\ C & C & C & \cdots & C & B & A & B & \cdots & B \\ C & C & C & \cdots & C & B & B & A & \cdots & B \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ C & C & C & \cdots & C & B & B & B & \cdots & A \end{pmatrix}.$$

Then, our distance matrix is of the form $\begin{pmatrix} A_0 & A_1 \\ A_1 & A_0 \end{pmatrix}$. By Lemma 1.31, the distance spectra can be obtained by computing the eigenvalues of the matrices $A_0 + A_1$ and $A_0 - A_1$ whcih are $|R| \times |R|$ circulant matrices. Thus, we compute the spectra of $A_0 - A_1$, then $A_0 + A_1$ from their first row. First, let ω=$e^{\frac{2\pi i}{|R|}}$ and ω_j=ω^j for all $j \in \{0, 1, \ldots, |R| - 1\}$. For each $j \in \{0, 1, \ldots, |R| - 1\}$, let μ_j denote a spectrum of $A_0 - A_1$, and κ_j denote a spectrum of $A_0 + A_1$. From Theorem 1.30, we obtain the distance spectra

$$\begin{aligned} \mu_j = & -3 + \omega_j + \cdots + \omega_j^{q-1} + (-1)\omega_j^q + \omega_j^{q+1} + \cdots + \omega_j^{2q-1} + \cdots + (-1)\omega_j^{(m-1)q} \\ & + \omega_j^{(m-1)q+1} + \cdots + \omega_j^{mq-1} \\ = & \sum_{k=0}^{|R|-1} \omega_j^k - 2\sum_{l=0}^{m-1} \omega_j^{ql} - 2 \end{aligned}$$

and

$$\begin{aligned} \kappa_j &= 3 + 3\omega_j + \cdots + 3\omega_j^{q-1} + 5\omega_j^q + 3\omega_j^{q+1} + \cdots + 3\omega_j^{2q-1} + \cdots + 5\omega_j^{(m-1)q} \\ &\quad + 3\omega_j^{(m-1)q+1} + \cdots + 3\omega_j^{mq-1} \\ &= 3\sum_{k=0}^{|R|-1} \omega_j^k + 2\sum_{l=0}^{m-1} \omega_j^{ql} - 2 \end{aligned}$$

for all $j \in \{1, 2, \ldots, |R| - 1\}$. We now distinguish j into three cases.

Case 1. $j = 0$. Then $\omega_0 = 1$ so $\mu_0 = |R| - 2m - 2$, $\kappa_0 = 3|R| + 2m - 2$.

Case 2. $j = km$ for some $k \in \{1, 2, \ldots, q - 1\}$. Then $\omega_j^q = \omega^{jq} = \omega^{kmq} = \omega^{|R|k} = 1$. Thus, $\mu_j = -2m - 2$ and $\kappa_j = 2m - 2$.

Case 3. j is not a multiple of m. Then $\omega_j^q \neq 1$.

$$\sum_{l=0}^{m-1} \omega_j^{ql} = \frac{1 - \omega_j^{qm}}{1 - \omega_j^q} = \frac{1 - 1}{1 - \omega_j^q}, \sum_{k=0}^{m-1} \omega_j^{qk} = \frac{1 - \omega_j^{qm}}{1 - \omega_j^q} = \frac{1 - 1}{1 - \omega_j^q} = 0.$$

Thus, $\omega_j = -2$ and $\kappa_j = -2$.

Therefore, we have

$$\text{Spec}_D(K_2 \otimes G_R) = \begin{pmatrix} 3|R| + 2m - 2 & |R| - 2m - 2 & 2m - 2 & -2m - 2 & -2 \\ 1 & 1 & \frac{|R|}{m} - 1 & \frac{|R|}{m} - 1 & 2(|R| - \frac{|R|}{m}) \end{pmatrix}.$$

This completes the proof. □

Corollary 2.4. *Let R be a finite local ring with maximal ideal M of size m and K_2 the path of length 2.*

1. *If $m = 1$, then* $DE(K_2 \otimes G_R) = \begin{cases} 8|R| - 8 & \text{if } |R| \geq 4, \\ 6|R| & \text{if } |R| = 1, 2 \text{ or } 3. \end{cases}$

2. *If $m > 1$, then* $DE(K_2 \otimes G_R) = \begin{cases} 12|R| - 4m - 4\frac{|R|}{m} - 4 & \text{if } |R| \geq 2m + 2, \\ 10|R| - 4\frac{|R|}{m} & \text{if } |R| < 2m + 2. \end{cases}$

Proof. If $m = 1$, then

$$\text{Spec}_D(K_2 \otimes G_R) = \begin{pmatrix} 3|R| & |R| - 4 & -4 & 0 \\ 1 & 1 & |R| - 1 & |R| - 1 \end{pmatrix}.$$

Case 1. $|R| \geq 4$. Then $||R| - 4| = |R| - 4$, so

$$DE(K_2 \otimes G_R) = 3|R| + (|R| - 4) + (|R| - 1)|-4| = 8|R| - 8.$$

Case 2. $|R| < 4$. Then $||R| - 4| = 4 - |R|$, so

$$DE(K_2 \otimes G_R) = 3|R| + (4 - |R|) + (|R| - 1)|-4| = 6|R|.$$

If $m > 1$, then

$$\text{Spec}_D(K_2 \otimes G_R) = \begin{pmatrix} 3|R| + 2m - 2 & |R| - 2m - 2 & 2m - 2 & -2m - 2 & -2 \\ 1 & 1 & \frac{|R|}{m} - 1 & \frac{|R|}{m} - 1 & 2(|R| - \frac{|R|}{m}) \end{pmatrix}.$$

$$\begin{aligned} DE(K_2 \otimes G_R)) &= |3|R| + 2m - 2| + ||R| - 2m - 2| + (\frac{|R|}{m} - 1)|2m - 2| + \\ &\quad (\frac{|R|}{m} - 1)|-2m - 2| + (2(|R| - \frac{|R|}{m}))|-2| \\ &= (3|R| + 2m - 2) + (||R| - 2m - 2|) + (2|R| - 2\frac{|R|}{m} - 2m + 2) + \\ &\quad (2|R| + 2\frac{|R|}{m} - 2m - 2) + (4|R| - 4\frac{|R|}{m}) \\ &= 11|R| - 2m - 4\frac{|R|}{m} - 2 + ||R| - 2m - 2|. \end{aligned}$$

Case 1. $|R| \geq 2m + 2$. We have $||R| - 2m - 2| = |R| - 2m - 2$. Then

$$\begin{aligned} DE(G_{K_2} \otimes G_R)) &= 11|R| - 2m - 4\frac{|R|}{m} - 2 + ||R| - 2m - 2| \\ &= 11|R| - 2m - 4\frac{|R|}{m} - 2 + |R| - 2m - 2 \\ &= 12|R| - 4m - 4\frac{|R|}{m} - 4. \end{aligned}$$

Case 2. $|R| < 2m + 2$. We have $||R| - 2m - 2| = 2m + 2 - |R|$. Then

$$\begin{aligned} DE(G_{K_2} \otimes G_R)) &= 11|R| - 2m - 4\frac{|R|}{m} - 2 + ||R| - 2m - 2| \\ &= 11|R| - 2m - 4\frac{|R|}{m} - 2 + 2m + 2 - |R| \\ &= 10|R| - 4\frac{|R|}{m}. \end{aligned}$$

Hence, we have the corollary. □

2.2 Distance Spectra and Distance Energy of H_R

Throughout the second section, we obtain the distance spectra of Paley graphs, H_R and $H_{\mathbb{Z}_n}$.

Theorem 2.5. *Let q be a prime power such that $q \equiv 1 \ mod\ 4$. Then*

$$\text{Spec}_D(H_{\mathbb{F}_q}) = \begin{pmatrix} \frac{3}{2}(q-1) & \frac{-3-\sqrt{q}}{2} & \frac{-3+\sqrt{q}}{2} \\ 1 & \frac{q-1}{2} & \frac{q-1}{2} \end{pmatrix}.$$

Proof. By Lemma 1.32, $H_{\mathbb{F}_q}$ is regular of degree $\frac{q-1}{2}$ and $\text{Spec}(H_{\mathbb{F}_q}) = \begin{pmatrix} \frac{q-1}{2} & \frac{\sqrt{q}-1}{2} & \frac{-\sqrt{q}-1}{2} \\ 1 & \frac{q-1}{2} & \frac{q-1}{2} \end{pmatrix}$. We know from Proposition 1.33 that $\text{diam}(H_{\mathbb{F}_q}) = 2$. Therefore, by Proposition 1.34, $\text{Spec}_D(H_{\mathbb{F}_q}) = \{2n-2-\frac{q-1}{2}\} \cup \{-\lambda-2 : \lambda \in \text{Spec}(H_{\mathbb{F}_q}) \text{ and } \lambda \neq \frac{q-1}{2}\}$. Hence,

$$\text{Spec}_D(H_{\mathbb{F}_q}) = \begin{pmatrix} 2q-2-\frac{q-1}{2} & \frac{1-\sqrt{q}}{2}-2 & \frac{\sqrt{q}+1}{2}-2 \\ 1 & \frac{q-1}{2} & \frac{q-1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3}{2}(q-1) & \frac{-3-\sqrt{q}}{2} & \frac{-3+\sqrt{q}}{2} \\ 1 & \frac{q-1}{2} & \frac{q-1}{2} \end{pmatrix}$$

as desired. □

Corollary 2.6. *Let q be a prime power such that $q \equiv 1 \ mod\ 4$. Then*

$$DE(H_{\mathbb{F}_q}) = \begin{cases} \frac{1}{4}(2q^{\frac{3}{2}} + 6q - 2\sqrt{q} - 6) & \text{if } q \geq 9, \\ 6 + 2\sqrt{5} & \text{if } q = 5. \end{cases}$$

Proof. By Theorem 2.5,

$$DE(H_{\mathbb{F}_q}) = \left|\frac{3}{2}(q-1)\right| + (\frac{q-1}{2})\left|\frac{-3-\sqrt{q}}{2}\right| + (\frac{q-1}{2})\left|\frac{-3+\sqrt{q}}{2}\right|$$

If $q \geq 9$, then

$$\begin{aligned} DE(H_{\mathbb{F}_q}) &= \frac{3q-3}{2} + (\frac{q-1}{2})(\frac{\sqrt{q}+3}{2}) + (\frac{q-1}{2})(\frac{\sqrt{q}-3}{2}) \\ &= \frac{3q-3}{2} + \frac{q^{\frac{3}{2}}+3q-\sqrt{q}-3}{4} + \frac{q^{\frac{3}{2}}-3q-\sqrt{q}+3}{4} \\ &= \frac{1}{4}(2q^{\frac{3}{2}} + 6q - 2\sqrt{q} - 6). \end{aligned}$$

If $q = 5$, then $DE(H_{\mathbb{F}_5}) = 6 + 2\sqrt{5}$. □

Lemma 2.7. *Let G be a graph with n vertices and $\dot{\mathcal{K}}_m$ denote the m-complete graph with a loop on each vertex. If G is regular of degree k, then $A(G \otimes \dot{\mathcal{K}}_m)$ is an $mn \times mn$ matrix of the form*

$$A(G \otimes \dot{\mathcal{K}}_m) = \begin{pmatrix} A(G) & A(G) & \cdots & A(G) \\ A(G) & A(G) & \cdots & A(G) \\ \vdots & \vdots & & \vdots \\ A(G) & A(G) & \cdots & A(G) \end{pmatrix}.$$

Moreover, $G \otimes \dot{\mathcal{K}}_m$ is regular of degree mk.

Proof. Let (u_1, v_1) and (u_2, v_2) be a pair of vertices of $G \otimes \dot{\mathcal{K}}_m$. Since v_1 and v_2 is adjacent, whether (u_1, v_1) and (u_2, v_2) are adjacent to each other or not depending on u_1, u_2. Consequently, we obtain the above $A(G \otimes \dot{\mathcal{K}}_m)$. □

Theorem 2.8. *Let R be a finite local ring with maximal ideal M of size m and characteristic of an odd prime power. If -1 is not a square in R, then $H_R = G_R$ (by Theorem 1.35), so $DE(H_R) = DE(G_R)$. If -1 is a square in R, then*

$$\text{Spec}_D(H_R) = \begin{pmatrix} \frac{3|R|+m-4}{2} & -\frac{m(\sqrt{\frac{|R|}{m}}-1)+4}{2} & \frac{m(\sqrt{\frac{|R|}{m}}+1)-4}{2} & -2 \\ 1 & \frac{\frac{|R|}{m}-1}{2} & \frac{\frac{|R|}{m}-1}{2} & m \end{pmatrix} \text{ and}$$

$$DE(H_R) = \begin{cases} \frac{1}{2}(3|R| + 5m - 4 + (|R| - m)\sqrt{\frac{|R|}{m}}) & \text{if } m(\sqrt{\frac{|R|}{m}} + 1) \geq 4, \\ \frac{1}{2}(3|R| + 5m - 4 + (\frac{|R|}{m} - 1)(4 - m)) & \text{if } m(\sqrt{\frac{|R|}{m}} + 1) < 4. \end{cases}$$

Proof. Assume that -1 is a square in R. Then by Theorem 1.35, $H_R \cong H_{R/M} \otimes \dot{\mathcal{K}}_m$. Note that $H_{R/M}$ is a Paley graph. Moreover, we obtain the spectra of H_R as follows:

$$\text{Spec}(H_R) = \begin{pmatrix} \frac{|R|-m}{2} & \frac{m(\sqrt{\frac{|R|}{m}}-1)}{2} & \frac{m(-\sqrt{\frac{|R|}{m}}-1)}{2} & 0 \\ 1 & \frac{\frac{|R|}{m}-1}{2} & \frac{\frac{|R|}{m}-1}{2} & m \end{pmatrix}.$$

Since $H_{R/M}$ is a Paley graph, it is regular of degree $\frac{\frac{|R|}{m}-1}{2}$. By Lemma 2.7, H_R is a regular graph of degree $\frac{|R|-m}{2}$. From Proposition 1.34, $\text{Spec}_D(H_R) = \{2|R|-2-\frac{|R|-m}{2}\}$ $\cup \{-\lambda - 2 : \lambda \in \text{Spec}(G) \text{ and } \lambda \neq \frac{|R|-m}{2}\}$. Then the distance spectra of H_R is given by

$$\text{Spec}_D(H_R) = \begin{pmatrix} \frac{3|R|+m-4}{2} & -\frac{m(\sqrt{\frac{|R|}{m}}-1)+4}{2} & \frac{m(\sqrt{\frac{|R|}{m}}+1)-4}{2} & -2 \\ 1 & \frac{\frac{|R|}{m}-1}{2} & \frac{\frac{|R|}{m}-1}{2} & m \end{pmatrix}.$$

Next, we compute the distance energy of H_R.

Case 1. $m(\sqrt{\frac{|R|}{m}}+1) \geq 4$. We have $\left|\frac{m(\sqrt{\frac{|R|}{m}}+1)-4}{2}\right| = \frac{m(\sqrt{\frac{|R|}{m}}+1)-4}{2}$.

$$
\begin{aligned}
DE(H_R) = &\left|\frac{3|R|+m-4}{2}\right| + (\frac{\frac{|R|}{m}-1}{2})\left|-\frac{m(\sqrt{\frac{|R|}{m}}-1)+4}{2}\right| + (\frac{\frac{|R|}{m}-1}{2})\left|\frac{m(\sqrt{\frac{|R|}{m}}+1)-4}{2}\right| \\
&+ m|-2| \\
&= \frac{3|R|+5m-4}{2} + (\frac{\frac{|R|}{m}-1}{4})(m(\sqrt{\frac{|R|}{m}}-1)+4) + (\frac{\frac{|R|}{m}-1}{4})(m(\sqrt{\frac{|R|}{m}}+1)-4) \\
&= \frac{3|R|+5m-4}{2} + (\frac{\frac{|R|}{m}-1}{4})(2m\sqrt{\frac{|R|}{m}}) \\
&= \frac{3|R|+5m-4}{2} + (\frac{|R|-m}{2})\sqrt{\frac{|R|}{m}} \\
&= \frac{1}{2}(3|R|+5m-4+(|R|-m)\sqrt{\frac{|R|}{m}}).
\end{aligned}
$$

Case 2. $m(\sqrt{\frac{|R|}{m}}+1) < 4$. We have $\left|\frac{m(\sqrt{\frac{|R|}{m}}+1)-4}{2}\right| = \frac{4-m(\sqrt{\frac{|R|}{m}}+1)}{2}$.

$$
\begin{aligned}
DE(H_R) = &\left|\frac{3|R|+m-4}{2}\right| + (\frac{\frac{|R|}{m}-1}{2})\left|-\frac{m(\sqrt{\frac{|R|}{m}}-1)+4}{2}\right| + (\frac{\frac{|R|}{m}-1}{2})\left|\frac{m(\sqrt{\frac{|R|}{m}}+1)-4}{2}\right| \\
&+ m|-2| \\
&= \frac{3|R|+5m-4}{2} + (\frac{\frac{|R|}{m}-1}{4})(m(\sqrt{\frac{|R|}{m}}-1)+4) + (\frac{\frac{|R|}{m}-1}{4})(4-m(\sqrt{\frac{|R|}{m}}+1)) \\
&= \frac{3|R|+5m-4}{2} + (\frac{\frac{|R|}{m}-1}{4})(8-2m) \\
&= \frac{1}{2}(3|R|+5m-4+(\frac{|R|}{m}-1)(4-m)).
\end{aligned}
$$

This completes the proof. □

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The Project Proposal of Course 2301399 Project Proposal Academic Year 2018

Project Title (Thai)	ค่าเฉพาะระยะทางและพลังงานระยะทางของกราฟยูนิแทรีเคย์เลย์
Project Title (English)	Distance spectra and distance energy of unitary Cayley graphs
Project Advisor	Professor Yotsanan Meemark, Ph.D.
By	Mr. Chanon Thongprayoon ID 5833510323
	Mathematics, Department of Mathematics and Computer Science
	Faculty of Science, Chulalongkorn University

Background and Rationale

A commutative ring with identity 1 is said to be a **local ring** if it has a unique maximal ideal. Let R be a finite commutative ring with identity 1 and $S \subseteq R$. The **Cayley graph** of R, denoted by $Cay(R, S)$, is a graph whose vertex set is R and the edge set is $\{\{a, b\} : a, b \in R \text{ and } a - b \in S\}$. If $S = R^{\times}$, then $Cay(R, R^{\times})$ is called the **unitary Cayley graph** of R. Also $Cay(R, R^{\times})$ is denoted by G_R. Here, $R^{\times}$ denotes the group of units of R. It follows from [1] that if R is a finite local ring, then G_R is a complete multi-partite graph whose partite sets are the cosets of M where M is the maximal ideal of R.

The **adjacency matrix** $A = [a_{ij}]$ of a graph G with n vertices is a $n \times n$ matrix where $a_{ij} = 1$ if the ith vertex is adjacent to the jth vertex; otherwise, $a_{ij} = 0$ for all $i, j \in \{1, 2, ..., n\}$. An eigenvalue of A is called a **spectrum** of G. The **distance** of the ith vertex and the jth vertex is the least number of edges connecting them. The **distance matrix** $D = [d_{ij}]$ of a graph G is also an $n \times n$ matrix where d_{ij} is the distance from ith vertex to jth vertex and $d_{ij} = 0$ if $i = j$ for all $i, j \in \{1, 2, ..., n\}$ and the eigenvalue of D is called the **distance spectrum** of the graph G.

The **energy** of a graph G, denoted by $E(G)$, is the sum of its absolute values of spectra. Similarly, the **distance energy** of the graph G, denoted by $DE(G)$, is the sum of its absolute values of distance spectra.

Akhtar et al. [2] studied and obtained all spectra of the unitary Cayley graph of a finite commutative ring R. Consequently, the energy is computed. Later, A.Illic. [1] established the distance matrix and computed the distance energy of $G_{\mathbb{Z}_n}$ in terms of Euler function as follows: 1. $2(p-1)$ where n is a prime p; 2. $4(n-2)$ if n is a power of 2; 3. $2(2n+\phi(n)(2^{k-1}-1)-m-2+\prod_{i=1}^{k}(2-p_i))$ where $m = p_1p_2...p_k$ is the maximal square-free divisor of n which is an odd composite number; 4. $2n-2m+\phi(n)2^{k-1}-(2+2\phi(n))-(2-2\phi(n))+(\frac{5n}{2}-2(\phi(n)+1))+\left|2(\phi(n)-1)-\frac{n}{2}\right|)$ where n is even with odd prime divisor.

In this project, we focus on computing the distance energy of the unitary Cayley graphs of finite local rings. We also plan to extend our domain to some finite commutative rings.

Objectives

To compute the distance spectra and the distance energy of the unitary Cayley graphs of a finite local ring and a finite commutative ring.

Project Activities

1. Study the articles [1] and [3].
2. Review the elementary knowledge in Abstract Algebra and Graph Theory.
3. Construct the distance matrix and compute the distance spectrum of the unitary Cayley graph over finite local rings.
4. Continue to work on the distance matrix and the distance energy of G_R where R is a finite commutative ring.
5. Write a report.

Activities Table

Project Activities	August 2018 - April 2019								
	Aug	Sep	Oct	Nov	Dec	Jan	Feb	Mar	Apr
1.Study the articles [1] and [3].	■	■							
2.Review the elementary knowledge in Abstract Algebra and Graph Theory.		■	■						
3.Construct the distance matrix and compute the distance spectrum of the unitary Cayley graph over finite local rings.				■	■				
4.Continue to work on the distance matrix and the distance energy of G_R where R is a finite commutative ring.						■	■		
5.Write a report.								■	■

Benefits

To obtain the general results of the distance energy of the unitary Cayley graphs whose vertices are elements in finite local rings and some finite commutative rings.

Equipment

1. Computer
2. Printer
3. Stationery
4. Paper

References

[1] D. Kiani, M. Aghaei, Y. Meemark, B. Suntornpoch., Energy of unitary Cayley graphs and gcd-graphs, *Linear Algebra and its Application* 435 (2011) 1336–1343.

[2] R. Akhtar, M. Boggers, T. Jackson-Henderson, I. Jimenez, R. Karpman, A. Kinzel, D. Pritikin, On the unitary Cayley graph of a finite ring, *The Electronic Journal of Combinatorics* (2009).

[3] A. Illic., Distance spectra and distance energy of integral circulant graphs, *Linear Algebra and its Application* (2010) 1005–1014.

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