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Perfect codes in unitary Cayley graphs

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กรวิชญ์ อนุตตรา: **รหัสสมบูรณ์ในกราฟเคย์เลย์ยูนิแทรี** (PERFECT CODES IN UNITARY CAYLEY GRAPHS) อ.ที่ปรึกษาโครงงาน: **ศ.ดร.ยศนันต์ มีมาก**, 28 หน้า

ในโครงงานนี้ เราได้เงื่อนไขที่จำเป็นและเพียงพอที่จะมีกรุปย่อย H ของ $\mathbb{Z}_n$ เป็นรหัสสมบูรณ์ในกราฟ $\mathrm{Cay}(\mathbb{Z}_n, \mathbb{Z}_n^{\times})$ และเราแสดงด้วยว่ากรุปย่อย $SL_2(\mathbb{F}_q)$, U, A และ K_δ เป็นรหัสสมบูรณ์ของ $GL_2(\mathbb{F}_q)$ และ พิจารณาต่อว่ากรุปย่อยเหล่านี้เป็นรหัสสมบูรณ์ทั้งหมดของ $GL_2(\mathbb{F}_q)$ หรือไม่

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In this project, we find a criterion in which there exists a subgroup H of $\mathbb{Z}_n$ being a perfect code in the graph $\mathrm{Cay}(\mathbb{Z}_n, \mathbb{Z}_n^\times)$. We also show that the subgroups $\mathrm{SL}_2(\mathbb{F}_q)$, U, A and K_δ are perfect codes of $\mathrm{GL}_2(\mathbb{F}_q)$ and determine if they are total perfect codes.

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Contents

Chapter 1

Perfect codes of graphs

Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a simple undirected graph on n vertices. For $u, v \in V(\Gamma)$ and $u \neq v$. The **distance** of u and v, denoted by $d(u, v)$, is the number of edges of a shortest path connecting them. if $u = v$, $d(u, v) = 0$. Let t be a positive integer and C a subset of $V(\Gamma)$. We say that C is a **perfect t-code** in Γ if for every vertex $v \in V(\Gamma)$ there exists a unique $c \in C$ such that $d(c, v) \leq t$. A perfect 1-code is called a **perfect code**. In addition, C is a **total perfect code** in Γ if for every vertex $v \in V(\Gamma)$ there exists a unique $c \in C$ such that $d(c, v) = 1$. In other words, C is a total perfect code in Γ if every vertex of $V(\Gamma)$ has exactly one neighbor in C.

Lemma 1.1. [5] *If C is a total perfect code of Γ, then $|C|$ is even.*

Proof. By the above definition of total perfect code, C is a total perfect code in Γ if every vertex of $V(\Gamma)$ has exactly one neighbor in C. So for each $c \in C$ there is unique $c' \in C$ such that $c \neq c'$ and $d(c, c') = 1$. We can pair elements in C and so $|C|$ is even. □

The concept of perfect codes in graphs were developed from the work of Biggs [1]. In Coding Theory, codes that attain the Hamming bound are said to be perfect. The q-ary perfect codes of length n are precisely the perfect 1-codes in the Hamming graph $H(n, q)$. The vertex set of $H(n, q)$ is $\mathbb{F}_q^{\times}$ and two words are adjacent if they have Hamming distance one.

We shall be interested in perfect codes in Cayley graphs.

Let G be a group and S a nonempty subset of G such that $e \notin S$ and $S = S^{-1}$. The **Cayley graph** Cay(G, S) **of G with respect the connection set** S is the graph with vertex set G such that for any $x, y \in G$, x and y are adjacent if and only if $yx^{-1} \in S$. Since $e \notin S$ and $S^{-1} = S$, the graph is undirected and has no loops.

Let C be a nonempty subset of a group G. Then C is a **perfect code** (respectively, **total perfect code**) of G if C is a perfect code (respectively, total perfect code) in some Caley graph of G. That is, there is a nonempty subset S of G with $e \notin S$ and $S = S^{-1}$ such that C is a perfect code (respectively, total perfect code) in Cay(G, S).

Let H be a subgroup of a group G. If we choose a subset $\{x_\alpha\}$ of G such that G is a disjoint union of that right cosets $\{x_\alpha H\}$, then $\{x_\alpha\}$ is called a **left transversal** of right coset representatives of H in G. The right transversal can be defined in an analogous way. Huang et al. [5] showed the following criteria.

Theorem 1.2. [5] *Let H be a subgroup of a group G. Then*
(a) H is a perfect code in Cay(G, S) *if and only if $S \cup \{e\}$ is a left transversal of H in G.*
(b) H is a total perfect code in Cay(G, S) *if and only if S is a left transversal of H in G.*

Theorem 1.3. [5] *Let G be a group and H a normal subgroup of G. Then*
(a) H is a perfect of G if and only if for any $g \in G$, $g^2 \in H$ implies $(gh)^2 = e$ for some $h \in H$.
(b) H is a total perfect of G if and only if $|H|$ is even and for any $g \in G$, $g^2 \in H$ implies $(gh)^2 = e$ for some $h \in H$.

Theorem 1.4. [5] *Let G be a cyclic group and H a subgroup of G. Then H is a perfect code of G if and only if either $|H|$ or $|G/H|$ is odd.*

In what follows, we shall use Huang's results to study perfect codes in the graph Cay$(\mathbb{Z}_n, \mathbb{Z}_n^\times)$ and perfect codes of $\mathrm{GL}_2(\mathbb{F}_q)$ in Chapters 2 and 4, respectively. Chapter 3 presents all subgroups that we shall study in Chapter 4.

Chapter 2

Perfect codes in Cay$(\mathbb{Z}_n, \mathbb{Z}_n^\times)$

Let $n \geq 2$ be a positive integer and consider the group $G = (\mathbb{Z}_n, +)$. Feng et al. [3] gave a necessary and sufficient condition for the graph Cay$(\mathbb{Z}_n, S)$ to admit a perfect code as follows.

Theorem 2.1. [3] *Let n be a positive integer and p be an odd prime. Then*
(a) A Cay$(\mathbb{Z}_n, S)$ *with $|S| = p-1$ admits a perfect code if and only if $p \mid n$ and $s \not\equiv s' \mod p$ for all distinct $s, s' \in S \cup \{0\}$.*
(b) A Cay$(\mathbb{Z}_n, S)$ *with $|S| = p$ admits a total perfect code if and only if $p \mid n$ and $s \not\equiv s' \mod p$ for all distinct $s, s' \in S$.*

Theorem 2.2. [3] *Let n, l be positive integers and p be an odd prime such that $p^l \mid n$ but $p^{l+1} \nmid n$. Then*
(a) A Cay$(\mathbb{Z}_n, S)$ *with $|S| = p^l - 1$ admits a perfect code if and only if $s \not\equiv s' \mod p^l$ for all distinct $s, s' \in S \cup \{0\}$.*
(b) A Cay$(\mathbb{Z}_n, S)$ *with $|S| = p^l$ admits a total perfect code if and only if $s \not\equiv s' \mod p^l$ for all distinct $s, s' \in S$.*

Now, we concern about $S = \mathbb{Z}_n^\times$. The graph Cay$(\mathbb{Z}_n, \mathbb{Z}_n^\times)$ is called the **unitary Cayley graph of $\mathbb{Z}_n$**. Its vertex set is $\mathbb{Z}_n$ and $a, b \in \mathbb{Z}_n$ are adjacent if gcd$(a-b, n) = 1$. Write $\phi(n) = |\mathbb{Z}_n^\times|$, the cardinality of the group of units of $\mathbb{Z}_n$. It is well known that $\phi(n)$ is even for all $n \geq 3$.

Theorem 2.3. *Let n be a positive integer and $n \geq 3$. Then there exists a subgroup H of $\mathbb{Z}_n$ such that H is a perfect code in* Cay$(\mathbb{Z}_n, \mathbb{Z}_n^\times)$ *if and only if $\phi(n) + 1 \mid n$.*

Proof. Assume that H is a subgroup of $\mathbb{Z}_n$ such that H is a perfect code in Cay$(\mathbb{Z}_n, \mathbb{Z}_n^\times)$. By Theorem 1.2 (a), $\mathbb{Z}_n^\times \cup \{0\}$ is a left transversal of H in $\mathbb{Z}_n$, so $|\mathbb{Z}_n^\times \cup \{0\}| = \dfrac{|\mathbb{Z}_n|}{|H|}$. Thus, $(\phi(n)+1)|H| = n$, so $\phi(n)+1 \mid n$.

Conversely, assume that $\phi(n)+1 \mid n$. Then $n = (\phi(n)+1)d$ for some $d \in \mathbb{N}$. Let $H = d\mathbb{Z}_n$. Then H is subgroup of $\mathbb{Z}_n$ of order $\dfrac{n}{d} = \phi(n)+1$. Since $\phi(n)+1$ is odd, H is a perfect code of $\mathbb{Z}_n$ by Theorem 1.4. □

Theorem 2.4. *If $n = p$ or $2p$ for an odd prime p, then $\phi(n)+1 \mid n$.*

Proof. Since p is an odd prime, $\phi(p) = p-1$ and $\phi(2p) = p-1$, so $\phi(p)+1 \mid p$ and $\phi(2p)+1 \mid 2p$. □

Gay [4] asked for other $n > 2$ such that $\phi(n)+1$ divides n, called Schinzel's problem. This problem is related to a problem of Lehmer [6]. Some progression on this problem can be found in [2]. We suspect that there are no other $n > 2$ such that $\phi(n)+1$ divides n.

Chapter 3

Subgroups of $\mathrm{GL}_2(\mathbb{F}_q)$

Let q be a prime power and let $\mathbb{F}_q$ denote the finite field of q elements. Consider the **general linear group** of 2×2 matrices over $\mathbb{F}_q$ given by

$$\mathrm{GL}_2(\mathbb{F}_q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \ a, b, c, d \in \mathbb{F}_q \text{ and } ad \neq bc \right\}.$$

It is well known that $|\mathrm{GL}_2(\mathbb{F}_q)| = (q^2 - 1)(q^2 - q)$. Define $\varphi : \mathrm{GL}_2(\mathbb{F}_q) \to \mathbb{F}_q \smallsetminus \{0\}$ by $\varphi(A) = \det(A)$ for all $A \in \mathrm{GL}_2(\mathbb{F}_q)$. Then φ is an onto homomorphism with kernel

$$\mathrm{SL}_2(\mathbb{F}_q) = \Big\{ A \mid A \in \mathrm{GL}_2(\mathbb{F}_q) \text{ and } \det(A) = 1 \Big\}.$$

Thus, $\mathrm{SL}_2(\mathbb{F}_q)$ is a normal subgroup of $\mathrm{GL}_2(\mathbb{F}_q)$ of cardinality

$$|\mathrm{SL}_2(\mathbb{F}_q)| = \frac{|\mathrm{GL}_2(\mathbb{F}_q)|}{|\mathbb{F}_q \smallsetminus \{0\}|} = \frac{(q^2-1)(q^2-q)}{q-1} = q(q^2-1).$$

It is called the **special linear group** of 2×2 matrices over $\mathbb{F}_q$. Let

$$U = \left\{ \begin{pmatrix} s & 0 \\ 0 & 1 \end{pmatrix} : s \in \mathbb{F}_q \smallsetminus \{0\} \right\} \text{ and } A = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbb{F}_q \right\}.$$

Next, consider q is odd. Then there is an element δ in $\mathbb{F}_q$ which is not a square in $\mathbb{F}_q$. Let

$$K_\delta = \left\{ \begin{pmatrix} x & y\delta \\ y & x \end{pmatrix} : x, y \in \mathbb{F}_q \text{ not both zero} \right\}.$$

Then $|U| = q - 1$, $|A| = q$ and $|K_\delta| = q^2 - q$. We proceed to show that they are subgroups of $\mathrm{GL}_2(\mathbb{F}_q)$.

Proposition 3.1. *The above sets U, A and K_δ are subgroups of* $\mathrm{GL}_2(\mathbb{F}_q)$.

Proof. It is obvious that U, A and K_δ contain $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Let $a, x \in \mathbb{F}_q \smallsetminus \{0\}$. Then

$$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} x^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} ax^{-1} & 0 \\ 0 & 1 \end{pmatrix} \in U.$$

Let $b, y \in \mathbb{F}_q$. Then

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & b-y \\ 0 & 1 \end{pmatrix} \in A.$$

Assume that q is odd. Let $a, b \in \mathbb{F}_q$ not both zero and $x, y \in \mathbb{F}_q$ not both zero. Since δ is not a square, $a^2 - b^2\delta$ and $x^2 - y^2\delta$ are nonzero, so $K_\delta \subseteq \mathrm{GL}_2(\mathbb{F}_q)$. Then

$$\begin{aligned} \begin{pmatrix} a & b\delta \\ b & a \end{pmatrix} \begin{pmatrix} x & y\delta \\ y & x \end{pmatrix}^{-1} &= \frac{1}{x^2 - y^2\delta} \begin{pmatrix} a & b\delta \\ b & a \end{pmatrix} \begin{pmatrix} x & -y\delta \\ -y & x \end{pmatrix} \\ &= \frac{1}{x^2 - y^2\delta} \begin{pmatrix} ax - by\delta & (bx - ay)\delta \\ bx - ay & ax - by\delta \end{pmatrix} \in K_\delta. \end{aligned}$$

Therefore, U, A and K_δ are subgroups of $\mathrm{GL}_2(\mathbb{F}_q)$. □

Chapter 4

Perfect codes of $\mathrm{GL}_2(\mathbb{F}_q)$

In this chapter, we show that the subgroups of $\mathrm{GL}_2(\mathbb{F}_q)$ in Chapter 3 are perfect codes by using Huang's criteria [5]. We also determine if they are total perfect codes.

Let q be a prime power. Since $\mathrm{SL}_2(\mathbb{F}_q)$ is a normal subgroup of $\mathrm{GL}_2(\mathbb{F}_q)$ and we can show that its cardinality is even, we may use Theorem 1.3 to prove that it is a total perfect code of $\mathrm{GL}_2(\mathbb{F}_q)$ as follows.

Theorem 4.1. *$\mathrm{SL}_2(\mathbb{F}_q)$ is a total perfect code of $\mathrm{GL}_2(\mathbb{F}_q)$.*

Proof. Let $A \in \mathrm{GL}_2(\mathbb{F}_q)$ such that $\det(A^2) = 1$. Then $\det(A) = \pm 1$.
Case 1. $\det(A) = 1$. Then $A \in \mathrm{SL}_2(\mathbb{F}_q)$. Since $\mathrm{SL}_2(\mathbb{F}_q)$ is a subgroup of $\mathrm{GL}_2(\mathbb{F}_q)$, there is a $B \in \mathrm{SL}_2(\mathbb{F}_q)$ such that $AB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then $(AB)^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.
Case 2. $\det(A) = -1$. Choose $C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{F}_q)$. Thus, $\det(CA) = \det(C)\det(A) = (-1)(-1) = 1$, so $CA \in \mathrm{SL}_2(\mathbb{F}_q)$. Since $\mathrm{SL}_2(\mathbb{F}_q)$ is a subgroup of $\mathrm{GL}_2(\mathbb{F}_q)$, there is a $B \in \mathrm{SL}_2(\mathbb{F}_q)$ such that $(CA)B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then

$$AB = C^{-1}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } (AB)^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore, there is a $B \in \mathrm{SL}_2(\mathbb{F}_q)$ such that $(AB)^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Thus, $\mathrm{SL}_2(\mathbb{F}_q)$ is a perfect code of $\mathrm{GL}_2(\mathbb{F}_q)$ by Theorem 1.3 (a). We also have

$$|\,\mathrm{SL}_2(\mathbb{F}_q)| = \frac{|\,\mathrm{GL}_2(\mathbb{F}_q)|}{|\mathbb{F}_q \smallsetminus \{0\}|} = \frac{(q^2-1)(q^2-q)}{q-1} = q(q^2-1),$$

so $|\,\mathrm{SL}_2(\mathbb{F}_{q^k})|$ is even. Hence, $\mathrm{SL}_2(\mathbb{F}_q)$ is a total perfect code of $\mathrm{GL}_2(\mathbb{F}_q)$ by Theorem 1.3 (b). □

Next, we study the subgroups U, A and K_δ of $\mathrm{GL}_2(\mathbb{F}_q)$ in Proposition 3.1. Note that these subgroups may not be normal in $\mathrm{GL}_2(\mathbb{F}_q)$. So we cannot use Theorem 1.3. In order to apply Theorem 1.2, we study their left transversals.

Lemma 4.2. *For* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \mathrm{GL}_2(\mathbb{F}_q)$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} U = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} U \text{ if and only if } b' = b,\ d' = d,\ a' = as, \text{ and}$$
$$c' = cs \text{ for some } s \in \mathbb{F}_q \smallsetminus \{0\}.$$

Proof. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \mathrm{GL}_2(\mathbb{F}_q)$. Assume that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} U = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} U$. Then

$$\begin{aligned}\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} &= \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \\ &= \frac{1}{ad-bc}\begin{pmatrix} a'd-bc' & b'd-bd' \\ ac'-a'c & ad'-b'c \end{pmatrix} \in U.\end{aligned}$$

It follows that

$$b'd = bd',\ ac' = a'c,\ ad'-b'c = ad-bc \text{ and } a'd-bc' = (ad-bc)s$$

for some $s \in \mathbb{F}_q \smallsetminus \{0\}$. Then $(ad'-b'c)b = (ad-bc)b$. Since $b'd = bd'$, we have

$$(ad-bc)b' = (ad-bc)b,$$

so $b' = b$ because $ad-bc \neq 0$. Also, $(a'd-bc')a = (ad-bc)as$ impiles $(ad-bc)a' = (ad-bc)as$. Since $ad-bc \neq 0$, we get $a' = as$. Similarly, we can show that $d' = d$ and $c' = cs$.

Conversely, assume that $b' = b,\ d' = d,\ a' = as$ and $c' = cs$ for some $s \in \mathbb{F}_q \smallsetminus \{0\}$.

Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} as & b \\ cs & d \end{pmatrix}$$
$$= \frac{1}{ad-bc} \begin{pmatrix} (ad-bc)s & 0 \\ 0 & ad-bc \end{pmatrix} = \begin{pmatrix} s & 0 \\ 0 & 1 \end{pmatrix} \in U$$

Therefore, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} U = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} U.$ □

Lemma 4.2 says that the coset representatives of U in $\mathrm{GL}_2(\mathbb{F}_q)$ are parametrized by the second column of the representatives. Now, let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{F}_q)$.

Case 1. $b = 0$. Then $a \neq 0$ and

$$\begin{pmatrix} a & 0 \\ c & d \end{pmatrix} U = \begin{pmatrix} as & 0 \\ cs & d \end{pmatrix} U \text{ for all } s \in \mathbb{F}_q \smallsetminus \{0\}.$$

Thus, let $s = -1/a$, we may choose its representative to be $\begin{pmatrix} -1 & 0 \\ \bar{c} & d \end{pmatrix}$ where $\bar{c} = -c/a$.

Case 2. $b \neq 0$ and $d = 0$. Then $c \neq 0$ and

$$\begin{pmatrix} a & b \\ c & 0 \end{pmatrix} U = \begin{pmatrix} as & b \\ cs & 0 \end{pmatrix} U \text{ for all } s \in \mathbb{F}_q \smallsetminus \{0\}.$$

Thus, let $s = -1/cb$, we have the representative $\begin{pmatrix} \bar{a} & b \\ -b^{-1} & 0 \end{pmatrix}$ where $\bar{a} = -a/cb$.

Case 3. $b \neq 0$ and $d \neq 0$. We consider the following subcases.

Subcase 3.1. $c = 0$. Then $a \neq 0$ and

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} U = \begin{pmatrix} as & b \\ 0 & d \end{pmatrix} U \text{ for all } s \in \mathbb{F}_q \smallsetminus \{0\}.$$

Since $a \neq 0$, let $s = 1/a$, we have the representative $\begin{pmatrix} 1 & b \\ 0 & d \end{pmatrix}$.

Subcase 3.2. $c \neq 0$ and $a = 0$. Then

$$\begin{pmatrix} 0 & b \\ c & d \end{pmatrix} U = \begin{pmatrix} 0 & b \\ cs & d \end{pmatrix} U \text{ for all } s \in \mathbb{F}_q \smallsetminus \{0\}.$$

Since $c \neq 0$, let $s = -1/bc$, we have the representative $\begin{pmatrix} 0 & b \\ -b^{-1} & d \end{pmatrix}$.

Subcase 3.3. $c \neq 0$ and $a \neq 0$. Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} U = \begin{pmatrix} as & b \\ cs & d \end{pmatrix} U \text{ for all } s \in \mathbb{F}_q \setminus \{0\}.$$

Since $a \neq 0$ and $c \neq 0$, let $s = \dfrac{-1}{ad-bc}$, we have the representative $\begin{pmatrix} \bar{a} & b \\ \bar{c} & d \end{pmatrix}$ where $\bar{a} = \dfrac{-a}{ad-bc}$ and $\bar{c} = \dfrac{-c}{ad-bc}$. Hence, we have shown:

Theorem 4.3. *If*

$$W_1 = \left\{ \begin{pmatrix} -1 & 0 \\ c & d \end{pmatrix} : c, d \in \mathbb{F}_q \text{ and } d \neq 0 \right\}, W_2 = \left\{ \begin{pmatrix} a & b \\ -b^{-1} & 0 \end{pmatrix} : a, b \in \mathbb{F}_q \text{ and } b \neq 0 \right\},$$

$$W_3 = \left\{ \begin{pmatrix} 1 & b \\ 0 & d \end{pmatrix} : b, d \in \mathbb{F}_q \setminus \{0\} \right\}, W_4 = \left\{ \begin{pmatrix} 0 & b \\ -b^{-1} & d \end{pmatrix} : b, d \in \mathbb{F}_q \setminus \{0\} \right\} \text{ and}$$

$$W_5 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{F}_q \setminus \{0\} \text{ and } ad - bc = -1 \right\},$$

then $W_1 \dot\cup W_2 \dot\cup W_3 \dot\cup W_4 \dot\cup W_5$ is a left transversal of U in $\mathrm{GL}_2(\mathbb{F}_q)$*. Moreover, if q is odd, this transversal does not contain* $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Theorem 4.4. *Let q be a prime power. Then*
(a) If q is odd, then U is a total perfect code of $\mathrm{GL}_2(\mathbb{F}_q)$.
(b) If q is even, then U is a perfect code of $\mathrm{GL}_2(\mathbb{F}_q)$ *but it is not a total perfect code.*

Proof. Let $S = W_1 \dot\cup W_2 \dot\cup W_3 \dot\cup W_4 \dot\cup W_5$ be the left transversal of U in $\mathrm{GL}_2(\mathbb{F}_q)$ as in Theorem 4.3. Next, we show that $S = S^{-1}$. Let $A \in W_1$. Then $A = \begin{pmatrix} -1 & 0 \\ c & d \end{pmatrix}$ for some $c, d \in \mathbb{F}_q$ and $d \neq 0$. Thus,

$$A^{-1} = \frac{-1}{d} \begin{pmatrix} d & 0 \\ -c & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ cd^{-1} & d^{-1} \end{pmatrix}$$

is in W_1, so $W_1 = W_1^{-1}$. Let $B \in W_3$, then $B = \begin{pmatrix} 1 & b \\ 0 & d \end{pmatrix}$ for some $b, d \in \mathbb{F}_q \setminus \{0\}$. Thus,

$$B^{-1} = \frac{1}{d}\begin{pmatrix} d & -b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -bd^{-1} \\ 0 & d^{-1} \end{pmatrix}$$

is in W_3, so $W_3 = W_3^{-1}$. Let $C \in W_2$. Then $C = \begin{pmatrix} a & b \\ -b^{-1} & 0 \end{pmatrix}$ for some $a, b \in \mathbb{F}_q$ and $b \neq 0$. Thus,

$$C^{-1} = \begin{pmatrix} 0 & -b \\ b^{-1} & a \end{pmatrix}$$

is in W_4, so $W_4 = W_2^{-1}$. Conversely, Let $D \in W_4$. Then $D = \begin{pmatrix} 0 & b \\ -b^{-1} & d \end{pmatrix}$ for some $b, d \in \mathbb{F}_q$ and $b \neq 0$. Thus,

$$D^{-1} = \begin{pmatrix} d & -b \\ b^{-1} & 0 \end{pmatrix}$$

is in W_2, so $D \in W_2^{-1}$ and we have $W_2 = W_4^{-1}$. let $E \in W_5$, then $E = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $a, b, c, d \in \mathbb{F}_q \setminus \{0\}$ and $ad - bc = -1$. Then

$$E^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} -d & b \\ c & -a \end{pmatrix}$$

and $ad - bc = -1$, so $E^{-1} \in W_5$. Hence, $W_5 = W_5^{-1}$. Therefore, $S = S^{-1}$.

Assume that q is odd. Then the left transversal S of U does not contain $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $S = S^{-1}$. By Theorem 1.2 (b), U is a total perfect code of Cay(GL$_2(\mathbb{F}_q), S)$.

Finally, we assume that q is even. Then $1 = -1$ in $\mathbb{F}_q$, so $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in S$. Choose $S' = S \setminus \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then $S' = S'^{-1}$ and $S' \cup \{e\} = S$ is a left transversal od U in GL$_2(\mathbb{F}_q)$, so U is a perfect code in Cay(GL, S') by Theorem 1.2 (a). Since $|U| = q-1$ is odd, U is not a total perfect code by Lemma 1.1. □

Lemma 4.5. *For* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \mathrm{GL}_2(\mathbb{F}_q)$ *,*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} A = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} A \text{ if and only if } a' = a,\ c' = c,\ b' = b + at,$$

$$\text{and } d' = d + ct \text{ for some } t \in \mathbb{F}_q.$$

Proof. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \mathrm{GL}_2(\mathbb{F}_q)$. Assume that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} A = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} A$. Then

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} &= \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \\ &= \frac{1}{ad - bc} \begin{pmatrix} a'd - bc' & b'd - bd' \\ ac' - a'c & ad' - b'c \end{pmatrix} \in A. \end{aligned}$$

It follows that

$$a'd - bc' = ad - bc,\ ac' = a'c,\ ad' - b'c = ad - bc \text{ and } b'd - bd' = (ad - bc)t$$

for some $t \in \mathbb{F}_q$. Then $(a'd - bc')c = (ad - bc)c$. Since $a'c = ac'$, we have

$$(ad - bc)c' = (ad - bc)c,$$

so $c' = c$ because $ad - bc \neq 0$. Also, $(ad' - b'c)b = (ad - bc)b$ implies

$$a(b'd - (ad - bc)t) - b'cb = (ad - bc)b.$$

Since $ad - bc \neq 0$, we get $b' = b + at$. Similarly, we can show that $a = a'$ and $d' = d + ct$.

Conversely, we assume that $a' = a$, $c' = c$, $b' = b + at$ and $d' = d + ct$ for some $t \in \mathbb{F}_q$. Then

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} &= \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} a & b + at \\ c & d + ct \end{pmatrix} \\ &= \frac{1}{ad - bc} \begin{pmatrix} ad - bc & (ad - bc)t \\ 0 & ad - bc \end{pmatrix} \\ &= \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \in A. \end{aligned}$$

Therefore, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} A = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} A.$ □

Lemma 4.5 says that the right coset representatives of A in $\mathrm{GL}_2(\mathbb{F}_q)$ are parametrized by the first column of the representatives. Now, let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{F}_q)$. Then we distinguish two cases, namely, $a = 0$ and $a \neq 0$.
Case 1. $a = 0$. Then $c \neq 0$ and

$$\begin{pmatrix} 0 & b \\ c & d \end{pmatrix} A = \begin{pmatrix} 0 & b \\ c & d+ct \end{pmatrix} A \text{ for all } t \in \mathbb{F}_q.$$

Thus, let $t = -d/c$, we may choose its representative to be $\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$.
Case 2. $a \neq 0$. Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} A = \begin{pmatrix} a & b+at \\ c & d+ct \end{pmatrix} A \text{ for all } t \in \mathbb{F}_q.$$

Since $a \neq 0$, let $t = -b/a$, we have the representative $\begin{pmatrix} a & 0 \\ c & \bar{d} \end{pmatrix}$ where $\bar{d} = d - \frac{bc}{a} = \frac{ad-bc}{a} \neq 0$. Hence, we have shown:

Theorem 4.6. *If*

$$V_1 = \left\{ \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} : b, c \in \mathbb{F}_q \setminus \{0\} \right\} \textit{ and } V_2 = \left\{ \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} : a, d \in \mathbb{F}_q \setminus \{0\} \textit{ and } c \in \mathbb{F}_q \right\},$$

then $V_1 \dot{\cup} V_2$ is a left transversal of A in $\mathrm{GL}_2(\mathbb{F}_q)$*.*

Theorem 4.7. *Let q be a prime power. Then*
(a) If q is odd, then A is a perfect code of $\mathrm{GL}_2(\mathbb{F}_q)$ *but is not a total perfect code.*
(b) If q is even, then A is a total perfect code of $\mathrm{GL}_2(\mathbb{F}_q)$*.*

Proof. Let $S = \left(V_1 \dot{\cup} V_2\right)$ be the left transversal of A in $\mathrm{GL}_2(\mathbb{F}_q)$ in Theorem 4.6. Next, we show that $S = S^{-1}$. Let $A \in V_1$. Then $A = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$ for some $b, c \in \mathbb{F}_q \setminus \{0\}$. Thus,

$$A^{-1} = \frac{-1}{bc} \begin{pmatrix} 0 & -b \\ -c & 0 \end{pmatrix} = \begin{pmatrix} 0 & c^{-1} \\ b^{-1} & 0 \end{pmatrix}$$

is in V_1, so $V_1 = V_1^{-1}$. Let $B \in V_2$. Then $B = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}$ for some $a, c, d \in \mathbb{F}_q$ and $a, d \neq 0$. Thus,

$$B^{-1} = \frac{1}{ad}\begin{pmatrix} d & 0 \\ -c & a \end{pmatrix} = \begin{pmatrix} a^{-1} & 0 \\ -c(ad)^{-1} & d^{-1} \end{pmatrix}$$

is in V_2, so $V_2 = V_2^{-1}$. Therefore, $S = S^{-1}$.

Assume q is odd. Let $S' = S \setminus \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$. Then $S' = S'^{-1}$ and $S' \cup \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ is a left transversal of A is $\mathrm{GL}_2(\mathbb{F}_q)$ by Theorem 4.6. By Theorem 1.2 (a), A is a perfect code of Cay$(GL_2(\mathbb{F}_q), S')$. Since $|A| = q$ is odd, A is not a total perfect code of $GL_2(\mathbb{F}_q)$ by Lemma 1.1.

Finally, we assume that q is even. Since $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in A$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, so $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1}$. Let $S'' = \left(S \setminus \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \right) \cup \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\}$. Then $S'' = S''^{-1}$ and S'' is a left transversal of A in $\mathrm{GL}_2(\mathbb{F}_q)$. By Theorem 1.2 (b), A is a total perfect code of Cay$(GL_2(\mathbb{F}_q), S'')$. □

Finally, we assume that q is an odd prime power and let δ be a nonsquare element in $\mathbb{F}_q$. We now study the subgroup K_δ. First, we determine its left transversal.

Theorem 4.8. *The set*

$$\left\{ \begin{pmatrix} a & b \\ 0 & -1 \end{pmatrix} K_\delta : a, b \in \mathbb{F}_q \text{ and } a \neq 0 \right\}$$

consists of all right cosets of K_δ in $\mathrm{GL}_2(\mathbb{F}_q)$*. Hence,*

$$T = \left\{ \begin{pmatrix} a & b \\ 0 & -1 \end{pmatrix} : a, b \in \mathbb{F}_q \text{ and } a \neq 0 \right\}$$

is a left transversal of K_δ in $\mathrm{GL}_2(\mathbb{F}_q)$*. Moreover, T does not contain* $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Proof. Let $a, b, c, d \in \mathbb{F}_q$ and $a, c \neq 0$ be such that $\begin{pmatrix} a & b \\ 0 & -1 \end{pmatrix} K_\delta = \begin{pmatrix} c & d \\ 0 & -1 \end{pmatrix} K_\delta$. Then

$$\begin{aligned} \begin{pmatrix} a & b \\ 0 & -1 \end{pmatrix}^{-1} \begin{pmatrix} c & d \\ 0 & -1 \end{pmatrix} &= \frac{-1}{a} \begin{pmatrix} -1 & -b \\ 0 & a \end{pmatrix}^{-1} \begin{pmatrix} c & d \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} a^{-1} & ba^{-1} \\ 0 & -1 \end{pmatrix} \begin{pmatrix} c & d \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} ca^{-1} & (d-b)a^{-1} \\ 0 & 1 \end{pmatrix} \in K_\delta. \end{aligned}$$

It follows that $ca^{-1} = 1$ and $(d-b)a^{-1} = 0$. Since $a \neq 0$, $a = c$ and $b = d$. Thus,

$$\left| \left\{ \begin{pmatrix} a & b \\ 0 & -1 \end{pmatrix} K_\delta : a, b \in \mathbb{F}_q \text{ and } a \neq 0 \right\} \right| = q(q-1) = q^2 - q.$$

Also,

$$[\mathsf{GL}_2(\mathbb{F}_q) : K_\delta] = \frac{|\,\mathsf{GL}_2(\mathbb{F}_q)|}{|K_\delta|} = \frac{(q^2-1)(q^2-q)}{q^2-1} = q^2 - q.$$

Hence, the set $\left\{ \begin{pmatrix} a & b \\ 0 & -1 \end{pmatrix} K_\delta : a, b \in \mathbb{F}_q \text{ and } a \neq 0 \right\}$ consists of all right cosets of K_δ in $\mathsf{GL}_2(\mathbb{F}_q)$. Therefore, T is a left transversal of K_δ in $\mathsf{GL}_2(\mathbb{F}_q)$. Since q is odd, $1 \neq -1$ in $\mathbb{F}_q$ so T does not contain $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. □

Theorem 4.9. *K_δ is a total perfect code of* $\mathsf{GL}_2(\mathbb{F}_q)$*.*

Proof. Let $S = T$. By Proposition 4.8, S is a left transversal of K_δ in $\mathsf{GL}_2(\mathbb{F}_q)$. Next, we show that $S = S^{-1}$. Let $\begin{pmatrix} a & b \\ 0 & -1 \end{pmatrix} \in S$. Then $a \neq 0$ and

$$\begin{pmatrix} a & b \\ 0 & -1 \end{pmatrix}^{-1} = \frac{-1}{a} \begin{pmatrix} -1 & -b \\ 0 & a \end{pmatrix} = \begin{pmatrix} a^{-1} & ba^{-1} \\ 0 & -1 \end{pmatrix}$$

is also in S. By Theorem 1.2, K_δ is a total perfect code of Cay$(GL_2(\mathbb{F}_q), S)$. □

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[2] G.L. Cohen and S.L. Segal, A note concerning those n for which $\phi(n)+1$ divides n. *Fib. Quart.*, 27 (1989) 285–286.

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กลุ่ม MATH02 สอบวันอังคารที่ 20 พฤศจิกายน 2561 เวลา 14.00 น. ห้อง 608/6
เอกสารนี้ได้รับการอนุมัติจากอาจารย์ที่ปรึกษาโครงงานแล้ว
ลงชื่อ ....
(วันที่ ....)

The Project Proposal of Course 2301399 Project Proposal Academic Year 2018

Project Title (Thai)	รหัสสมบูรณ์ในกราฟเคย์เลย์ยูนิแทรี
Project Title (English)	Perfect codes in unitary Cayley graphs
Project Advisor	Professor Dr.Yotsanan Meemark
By	Mr. Korawich Anuttra ID 5833502323
	Mathematics, Department of Mathematics and Computer Science, Chulalongkorn University

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Background and Rationale and Scope

Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a simple undirected graph on n vertices. For $u, v \in V(\Gamma)$ and $u \neq v$. The **distance** of u and v, denoted by $d(u, v)$, is the number of edges of a shortest path connecting them. If $u = v$, $d(u, v) = 0$. Let t be a positive integer and C a subset of $V(\Gamma)$. We say that C is a **perfect t-code** in Γ if for every vertex $v \in V(\Gamma)$ there exists a unique $c \in C$ such that $d(c, v) \leq t$. A perfect 1-code is called a **perfect code**.
In addition, C is a **total perfect code** in Γ if for every vertex $v \in V(\Gamma)$ there exists a unique $c \in C$ such that $d(c, v) = 1$. In other words, C is a total perfect code in Γ if every vertex of $V(\Gamma)$ has exactly one neighbor in C.

Let G be a finite group and S a subset of G with $e \notin S$ and $S = S^{-1}$. The **Cayley graph** Cay(G, S) **with respect to the connection set S** is the graph with vertex set G such that $x, y \in G$ are adjacent if and only if $xy^{-1} \in S$.

Huang et al. [1] showed that for a subgroup H of G, we have
(a) H is a perfect code in Γ if and only if $S \cup \{e\}$ is a left transversal of H in G.
(b) H is a total perfect code in Γ if and only if S is a left transversal of H in G.
Later, Feng et al. [2] gave a necessary and sufficient condition for Cay$(\mathbb{Z}_n, S)$

graph of degree $p-1$ admit a perfect code and degree p admit a total perfect code, where p is an odd prime. Here, degree is $|S|$.

Let R be a finite commutative ring with identity 1. The **unitary Cayley graph of** R is the Cayley graph Cay$(R, R^{\times})$ where $R^{\times}$ is the group of units of R.

In this project we shall study a perfect code in the unitary Cayley graph of R. We plan to determine R such that a perfect code or a total perfect code in Cay$(R, R^{\times})$ exists by using the work of Huang [1] and Feng [2].

Objectives

To find some characteristics of a finite commutative ring R such that a perfect code or a total perfect code in Cay$(R, R^{\times})$ exists.

Project Activities

1. Study the work of Huang [1] and Feng [2].
2. Review basic knowledge on Number Theory, Abstract Algebra and Algebraic Graph Theory which relates to our project.
3. Use properties of a perfect code and a total perfect code of G to find some properties in S such that Cay$(\mathbb{Z}_n, S)$ exists a perfect code and a total perfect code.
4. Work on condition of a perfect code and a total perfect code in Cay$(R, R^{\times})$.
5. Write a report.

Activities Table

Project Activities	August 2018 - April 2019								
	Aug	Sep	Oct	Nov	Dec	Jan	Feb	Mar	Apr
1.Study the work of Huang [1] and Feng [2].	■								
2.Review basic knowledge on Number Theory, Abstract Algebra and Algebraic Graph Theory which relates to our project.		■							
3.Use properties of a perfect code and a total perfect code of G to find some properties in S such that Cay$(\mathbb{Z}_n, S)$ exists a perfect code and a total perfect code.			■	■	■				
4.Work on condition of a perfect code and a total perfect code in Cay$(R, R^{\times})$.						■	■		
5.Write a report.								■	■

Benefits

To obtain some characteristics of a finite commutative ring R such that a perfect code and total perfect code in Cay$(R, R^{\times})$ exists by using results of Huing and Feng.

Equipment

1. Computer
2. Paper
3. Printer
4. Stationery

Reference

[1] H. Huang, B. Xia and S. Zhou. Perfect codes in Cayley graphs, *SIAM J. Discrete Math.*, 32(2017), 548–559.

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