CHAPTER IV

GROUP DIVISIBLE DESIGNS

WITH m and $n \equiv 2 \pmod 3$, $m \neq 2$ and $n \neq 2$

In this chapter, we focus on the existence of GDDs with m and $n \equiv 2 \pmod 3$, $m \neq 2$, $n \neq 2$, $\lambda_1 \geq \lambda_2$ and $\lambda_1' \geq \lambda_2$. We prove the sufficiency of the existence problem by constructing GDDs satisfying the necessary conditions in Theorem 1.2. The investigation is separated into three sections for each (m, n) in $\{(\bar{2}, \bar{2}), (\bar{2}, \bar{5}), (\bar{5}, \bar{5})\}$.

4.1 m and $n \equiv 2 \pmod 6$, $m \neq 2$ and $n \neq 2$

First, we construct GDDs with m and $n \equiv 2 \pmod 6$, $m \neq 2$ and $n \neq 2$. This case is concluded in Theorem 4.3, requiring some small GDDs in Lemma 4.2. The following small GDDs will be used in Lemma 4.2.

Lemma 4.1. *Let* $(\lambda_1, \lambda_2) \in \{(2, 1), (4, 2)\}$. *The following* GDD*s exist:*

(i) *a* $\mathsf{GDD}(2, 2; \lambda_1, 0, \lambda_2)$,

(ii) *a* $\mathsf{GDD}(2, 8; \lambda_1, 6, \lambda_2)$ *and*

(iii) *a* $\mathsf{GDD}(8, 2; \lambda_1, 6, \lambda_2)$.

Proof. (i) Let $\mathcal{B}_1 = \{\{x_1, x_2, y_1\}, \{x_1, x_2, y_2\}\}$. Then, $(M_2, N_2, \mathcal{B}_1)$ is a $\mathsf{GDD}(2, 2; 2, 0, 1)$. Moreover, two copies of $\mathcal{B}_1$ forms a $\mathsf{GDD}(2, 2; 4, 0, 2)$.

(ii) Let $B = \{y_1, y_2\}$. From (i), there exists a $\mathsf{GDD}(2, 2; \lambda_1, 0, \lambda_2)$ on the vertex set $M_2 \cup B$, namely $(M_2, B, \mathcal{B}_1)$. Let $\mathcal{B}_2 = \{\{y_1, y_2, y_k\} : k \in \{3, 4, 5, \dots, 8\}\}$ be a collection of triangles. By Theorem 2.12, we can decompose the graph $6K_6(N_n \setminus B)$ into a collection of triangles $\mathcal{T}_1$ and $2\lambda_2 + 10$ 1-factors, say $F_{i,j}$ and $F'_{p,q}$ where

$i, p \in \{1,2\}$, $j \in \{1,2,3,\dots,\lambda_2\}$ and $q \in \{1,2,3,4,5\}$. Let $\mathcal{F}_1$ be a collection of triangles defined by

$$\mathcal{F}_1 = \{x_i + F_{i,j}, y_p + F'_{p,q} : i, p \in \{1,2\}, j \in \{1,2,3,\dots,\lambda_2\}, q \in \{1,2,3,4,5\}\}.$$

Thus, $(M_2, N_8, \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{T}_1 \cup \mathcal{F}_1)$ is our desired GDD.

(iii) Let $A = \{x_1, x_2\}$. From (i), there exists a GDD$(2,2;\lambda_1,\ 0.\lambda_2)$ on the set $A \cup N_2$, namely $(A, N_2, \mathcal{B}_1)$. Let $\mathcal{B}_2 = \{\{y_1, y_2, x_k\} : k \in \{3,4,5,\dots,8\}\}$ be a collection of triangles. By Theorem 2.12, we can decompose the graph $\lambda_1 K_6(M_m \setminus A)$ into a collection of triangles $\mathcal{T}_1$ and $2\lambda_1 + 2(\lambda_2 - 1)$ 1-factors, say $F_{i,j}$ and $F'_{p,q}$ where $i, p \in \{1,2\}$, $j \in \{1,2,3,\dots,\lambda_1\}$ and $q \in \{1,2,3,\dots,\lambda_2 - 1\}$. Let $\mathcal{F}_1$ be a collection of triangles defined by

$$\mathcal{F}_1 = \{x_i + F_{i,j}\}, y_p + F'_{p,q} : i, p \in \{1,2\}, j \in \{1,2,3,\dots,\lambda_1\}, q \in \{1,2,3.\dots.\lambda_2 - 1\}\}.$$

Hence, $(M_8, N_2, \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{T}_1 \cup \mathcal{F}_1)$ is a desired GDD. □

Lemma 4.2. *Let m and n be positive integers such that m and $n \equiv 2 \pmod 6$, $m \neq 2$ and $n \neq 2$. There exist a* GDD$(m,n;2,6,1)$ *and a* GDD$(m,n;4,6,2)$.

Proof. We write $m = 6h + 8$ and $n = 6k + 8$ for nonnegative integers $h.k$. Let $A = \{x_1, x_2\}$, $B = \{y_1, y_2\}$ and $(\lambda_1, \lambda_2) \in \{(2,1).(4,2)\}$. We consider the construction in the following six cases.

Case (i) $0 = h \leq k$. From Lemma 4.1, there exists a GDD$(8,2;\lambda_1,6,\lambda_2)$ on $M_8 \cup B$, namely $(M_8, B, \mathcal{B}_1)$. Since $k \geq 0$ and $\lambda_2 \in \{1,2\}$, by Theorem 2.12, we can decompose the graph $6K_{n-2}(N_n \setminus B)$ into a collection of triangles $\mathcal{T}_1$ and $8\lambda_2 + 12$ 1-factors, say $F_{i,j}$ and $F'_{p,q}$ where $i \in \{1,2,3,\dots,8\}$, $j \in \{1,2,3,\dots,\lambda_2\}$, $p \in \{1,2\}$ and $q \in \{1,2,3,\dots,6\}$. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be collections of triangles defined by

$$\mathcal{F}_1 = \{x_i + F_{i,j} : i \in \{1,2,3,\dots,8\}, j \in \{1.2,3,\dots,\lambda_2\}\}$$

and

$$\mathcal{F}_2 = \{y_p + F'_{p,q} : p \in \{1,2\}, q \in \{1,2,3,\ldots,6\}\}.$$

Hence, $(M_8, N_n, \mathcal{B}_1 \cup \mathcal{T}_1 \cup \mathcal{F}_1 \cup \mathcal{F}_2)$ is a desired GDD.

Case (ii) $1 = h \leq k$. In this case we let $A = \{x_1, x_2, x_3, \ldots, x_8\}$ and $B = \{y_1, y_2\}$. Let $\mathcal{B}_1 = \{\{x_k, y_1, y_2\} : k \in \{9, 10, 11, \ldots, 14\}\}$ be a collection of triangles. For $(\lambda_1, \lambda_2) = (2, 1)$, we consider the graph $2K_6(M_{14} \setminus A)$ as a $\mathsf{TS}(6;2)$, namely $(M_{14} \setminus A, \mathcal{F}_1)$. For $(\lambda_1, \lambda_2) = (4, 2)$, by Theorem 2.12, the graph $4K_6(M_{14} \setminus A)$ can be decomposed into a collection of triangles $\mathcal{T}_1$ and two 1-factors, say F_1 and F_2. Let $\mathcal{F}_1$ be a collection of triangles defined by

$$\mathcal{F}_1 = \{y_i + F_i : i \in \{1,2\}\}.$$

Since $\lambda_1 = 2\lambda_2$, by Lemma 2.5, the graph $\lambda_1 K_8(A)$ can be decomposed into $6\lambda_1 + 2\lambda_2$ 1-factors, say $T_{i,j}$ and $T'_{p,q}$ where $i \in \{9, 10, 11, \ldots, 14\}$, $j \in \{1, 2, 3, \ldots, \lambda_1\}$, $p \in \{1, 2\}$ and $q \in \{1, 2, 3, \ldots, \lambda_2\}$. Let $\mathcal{F}_2$ and $\mathcal{F}_3$ be collections of triangles defined by

$$\mathcal{F}_2 = \{x_i + T_{i,j} : i \in \{9, 10, 11, \ldots, 14\}, j \in \{1, 2, 3, \ldots, \lambda_1\}\}$$

and

$$\mathcal{F}_3 = \{y_p + T'_{p,q} : p \in \{1,2\}, q \in \{1,2,3,\ldots,\lambda_2\}\}.$$

Since $k + 1 \geq 2$, by Theorem 2.12, we can decompose the graph $6K_{n-2}(N_n \setminus B)$ into a collection of triangles $\mathcal{T}_2$ and $14\lambda_2 + 12$ 1-factors, say $H_{i,j}$ and $H'_{p,q}$ where $i \in \{1, 2, 3, \ldots, 14\}$, $j \in \{1, 2, 3, \ldots, \lambda_2\}$, $p \in \{1, 2\}$ and $q \in \{1, 2, 3, \ldots, \lambda_2\}$. Let $\mathcal{F}_4$ and $\mathcal{F}_5$ be collections of triangles defined by

$$\mathcal{F}_4 = \{x_i + H_{i,j} : i \in \{1,2,3,\ldots,14\}, j \in \{1,2,3,\ldots,\lambda_2\}\}$$

and

$$\mathcal{F}_5 = \{y_p + H'_{p,q} : p \in \{1,2\}, q \in \{1,2,3,\ldots,\lambda_2\}\}.$$

Hence, $(M_{14}, N_n, \mathcal{B}_1 \cup \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \cup \mathcal{F}_4 \cup \mathcal{F}_5 \cup \mathcal{T}_1 \cup \mathcal{T}_2)$ is a desired GDD.

Case (iii) $2 \leq h \leq k$. In this case we let $A = \{x_1, x_2, x_3, \ldots, x_8\}$ and $B = \{y_1, y_2\}$. By Lemma 4.1, there exists a GDD$(8, 2; \lambda_1, 6, \lambda_2)$ on the vertex set $A \cup B$, namely $(A, B, \mathcal{B}_1)$. Since $h \geq 2$ and $\lambda_1 = 2\lambda_2$, by Theorem 2.12, we can decompose the graph $\lambda_1 K_{m-8}(M_m \setminus A)$ into a collection of triangles $\mathcal{T}_1$ and $8\lambda_1 + 2\lambda_2$ 1-factors, say $F_{i,j}$ and $F'_{p,q}$ where $i \in \{1, 2, 3, \ldots, 8\}, j \in \{1, 2, 3, \ldots, \lambda_1\}, p \in \{1, 2\}$ and $q \in \{1, 2, 3, \ldots, \lambda_2\}$. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be collections of triangles defined by

$$\mathcal{F}_1 = \{x_i + F_{j,i} : i \in \{1, 2, 3, \ldots, 8\}, j \in \{1, 2, 3, \ldots, \lambda_1\}\}$$

and

$$\mathcal{F}_2 = \{y_p + F'_{p,q} : p \in \{1, 2\}, q \in \{1, 2, 3. \ldots, \lambda_2\}\}.$$

Again, by Theorem 2.12, we can decompose the graph $6K_{n-2}(N_n \setminus B)$ into a collection of triangles $\mathcal{T}_2$ and $\lambda_2 m + 12$ 1-factors, say $H_{i,j}$ and $H'_{p,q}$ where $i \in \{1, 2, 3, \ldots, m\}$, $j \in \{1, 2, 3, \ldots, \lambda_2\}$, $p \in \{1, 2\}$ and $q \in \{1, 2, 3, \ldots, 6\}$. Let $\mathcal{F}_3$ and $\mathcal{F}_4$ be collections of triangles defined by

$$\mathcal{F}_3 = \{x_i + H_{i,j} : i \in \{1, 2, 3, \ldots, m\}, j \in \{1, 2, 3, \ldots, \lambda_2\}\}$$

and

$$\mathcal{F}_4 = \{y_p + H'_{p,q} : p \in \{1, 2\}, q \in \{1, 2, 3, \ldots, 6\}\}.$$

Thus, $(M_m, N_n, \mathcal{B}_1 \cup \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \cup \mathcal{F}_4)$ is a desired GDD.

Case (iv) $h > k = 0$. By Lemma 4.1, there exists a GDD$(2, 8; \lambda_1, 6, \lambda_2)$ on the vertex set $A \cup N_8$, namely $(A, N_8, \mathcal{B}_1)$. Since $h \geq 1$ and $\lambda_1 = 2\lambda_2$, by Theorem 2.12, we can decompose the graph $\lambda_1 K_{m-2}(M_m \setminus A)$ into a collection of triangles $\mathcal{T}_1$ and $2\lambda_1 + 8\lambda_2$ 1-factors, say $F_{i,j}$ and $F'_{p,q}$ where $i \in \{1, 2\}$, $j \in \{1, 2, 3, \ldots, \lambda_1\}$, $p \in \{1, 2, 3, \ldots, 8\}$ and $q \in \{1, 2, 3, \ldots, \lambda_2\}$. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be collections of triangles defined by

$$\mathcal{F}_1 = \{x_i + F_{i,j} : i \in \{1, 2\}, j \in \{1, 2, 3, \ldots, \lambda_1\}\}$$

and

$$\mathcal{F}_2 = \{y_p + F'_{p,q} : p \in \{1,2,3,\ldots,8\}, q \in \{1,2,3,\ldots,\lambda_2\}\}.$$

Then, $(M_m, N_8, \mathcal{B}_1 \cup \mathcal{T}_1 \cup \mathcal{F}_1 \cup \mathcal{F}_2)$ is a desired GDD.

Case (v) $h > k = 1$. In this case we let $A = \{x_1, x_2, x_3, \ldots, x_8\}$ and $B = \{y_1, y_2\}$. From Lemma 4.1, there exists a GDD$(8, 2; \lambda_1, 6, \lambda_2)$ on the vertex set $A \cup B$, namely $(A, B, \mathcal{B}_1)$. When $h = 2$, by Theorem 2.12, we can decompose the graph $6K_{12}(N_n \setminus B)$ into a collection of triangles $\mathcal{T}_1$ and $20\lambda_2 + 12$ 1-factors, say $F_{i,j}$ and $F'_{p,q}$ where $i \in \{1,2,3,\ldots,20\}$, $j \in \{1,2,3,\ldots,\lambda_2\}, p \in \{1,2\}$ and $q \in \{1,2,3,\ldots,6\}$. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be collections of triangles defined by

$$\mathcal{F}_1 = \{x_i + F_{i,j} : i \in \{1,2,3,\ldots,20\}, j \in \{1,2,3,\ldots,\lambda_2\}\}$$

and

$$\mathcal{F}_2 = \{y_p + F'_{p,q} : p \in \{1,2\}, q \in \{1,2,3,\ldots,6\}\}.$$

Again, by Theorem 2.12, we can decompose the graph $\lambda_1 K_{12}(M_m \setminus A)$ into a collection of triangles $\mathcal{T}_2$ and $8\lambda_1 + 2\lambda_2$ 1-factors, say $H_{i,j}$ and $H'_{p,q}$ where $i \in \{1,2,3,\ldots,8\}$, $j \in \{1,2,3,\ldots,\lambda_1\}$, $p \in \{1,2\}$ and $q \in \{1,2,3,\ldots,\lambda_2\}$. Let $\mathcal{F}_3$ and $\mathcal{F}_4$ be collections of triangles defined by

$$\mathcal{F}_3 = \{x_i + H_{i,j} : i \in \{1,2,3,\ldots,8\}, j \in \{1,2,3,\ldots,\lambda_1\}\}$$

and

$$\mathcal{F}_4 = \{y_p + H'_{p,q} : p \in \{1,2\}, q \in \{1,2,3,\ldots,\lambda_2\}\}.$$

Then, $(M_{20}, N_{14}, \mathcal{B}_1 \cup \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \cup \mathcal{F}_4 \cup \mathcal{T}_1 \cup \mathcal{T}_2)$ is a GDD$(20, 14, \lambda_1, 6, \lambda_2)$.

Now assume $h \geq 3$. By Theorem 2.12, the graph $6K_{12}(N_n \setminus B)$ can be decomposed into a collection of triangles $\mathcal{T}_3$ and $8\lambda_2 + 12$ 1-factors, say $T_{i,j}$ and $T'_{p,q}$ where $i \in \{1,2,3,\ldots,8\}$, $j \in \{1,2,3,\ldots,\lambda_2\}$, $p \in \{1,2\}$ and $q \in \{1,2,3,\ldots,6\}$. Let $\mathcal{F}_5$ and $\mathcal{F}_6$ be collections of triangles defined by

$$\mathcal{F}_5 = \{x_i + T_{i,j} : i \in \{1,2,3,\ldots,8\}, j \in \{1,2,3,\ldots,\lambda_2\}\}$$

and

$$\mathcal{F}_6 = \{y_p + T'_{p,q} : p \in \{1, 2\}, q \in \{1, 2, 3, \ldots, 6\}\}.$$

Since $h \geq 3$, again by Theorem 2.12, we can decompose the graph $\lambda_1 K_{m-8}(M_m \setminus A)$ into a collection of triangles $\mathcal{T}_4$ and $8\lambda_1 + 14\lambda_2$ 1-factors, say $G_{i,j}$ and $G'_{p,q}$ where $i \in \{1, 2, 3, \ldots, 8\}$, $j \in \{1, 2, 3, \ldots, \lambda_1\}$, $p \in \{1, 2, 3, \ldots, 14\}$ and $q \in \{1, 2, 3, \ldots, \lambda_2\}$. Let $\mathcal{F}_7$ and $\mathcal{F}_8$ be collections of triangles defined by

$$\mathcal{F}_7 = \{x_i + G_{i,j} : i \in \{1, 2, 3, \ldots, 8\}, j \in \{1, 2, 3, \ldots, \lambda_1\}\}$$

and

$$\mathcal{F}_8 = \{y_p + G'_{p,q} : p \in \{1, 2, 3, \ldots, 14\}, q \in \{1, 2, 3, \ldots. \lambda_2\}\}.$$

Then, $(M_m, N_{14}, \mathcal{B}_1 \cup \mathcal{T}_3 \cup \mathcal{T}_4 \cup \mathcal{F}_5 \cup \mathcal{F}_6 \cup \mathcal{F}_7 \cup \mathcal{F}_8)$ is a desired GDD.

Case (vi) $h > k \geq 2$. In this case we let $A = \{x_1, x_2\}$ and $B = \{y_1, y_2, y_3, \ldots, y_8\}$. From Lemma 4.1, there is a GDD$(2, 8; \lambda_1, 6, \lambda_2)$ on the vertex set $A \cup B$, namely $(A, B, \mathcal{B}_1)$. Since $k \geq 2$, by Theorem 2.12, we can decompose the graph $6K_{n-8}(N_n \setminus B)$ into a collection of triangles $\mathcal{T}_1$ and $2\lambda_2 + 48$ 1-factors, say $F_{i,j}$ and $F'_{p,q}$ where $i \in \{1, 2\}$, $j \in \{1, 2, 3, \ldots, \lambda_2\}$, $p \in \{1, 2, 3, \ldots, 8\}$ and $q \in \{1, 2, , 3 \ldots, 6\}$. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be collections of triangles defined by

$$\mathcal{F}_1 = \{x_i + F_{i,j} : i \in \{1, 2\}, j \in \{1, 2, 3. \ldots, \lambda_2\}\}$$

and

$$\mathcal{F}_2 = \{y_p + F'_{p,q} : p \in \{1, 2, 3, \ldots, 8\}, q \in \{1, 2, 3, \ldots, 6\}\}.$$

Since $h > k$, again by Theorem 2.12, we can decompose the graph $\lambda_1 K_{m-2}(M_m \setminus A)$ into a collection of triangles $\mathcal{T}_2$ and $2\lambda_1 + \lambda_2 n$ 1-factors, say $H_{i,j}$ and $H'_{p,q}$ where $i \in \{1, 2\}$, $j \in \{1, 2, 3, \ldots, \lambda_1\}$, $p \in \{1, 2, 3, \ldots, n\}$ and $q \in \{1, 2, 3, \ldots, \lambda_2\}$. Let $\mathcal{F}_3$ and $\mathcal{F}_4$ be collections of triangles defined by

$$\mathcal{F}_3 = \{x_i + H_{i,j} : i \in \{1, 2\}, j \in \{1, 2, 3, \ldots, \lambda_1\}\}$$

and

$$\mathcal{F}_4 = \{y_p + H'_{p,q} : p \in \{1,2,3,\ldots,n\}, q \in \{1,2,3,\ldots,\lambda_2\}\}.$$

Hence, $(M_m, N_n, \mathcal{B}_1 \cup \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \cup \mathcal{F}_4)$ is a desired GDD. □

Now, we are in the position to establish all of our GDDs in this case.

Theorem 4.3. *Let m and n be positive integers such that m and $n \equiv 2 \pmod 6$, $m \neq 2$ and $n \neq 2$. Let λ_1, λ_1' and λ_2 be nonnegative integers such that $\lambda_1 \geq \lambda_2$ and $\lambda_1' \geq \lambda_2$. If λ_1 and $\lambda_1' \equiv 0 \pmod 2$ and $3|(\lambda_1 + \lambda_1' + \lambda_2)$, then there exists a* GDD$(m, n; \lambda_1, \lambda_1', \lambda_2)$.

Proof. The construction is depended on the values of λ_2 in the following two cases.

Case (i) λ_2 is even. Then, $6|(\lambda_1 + \lambda_1' + \lambda_2)$. We regard λ_1, λ_1' and λ_2 as integers modulo 6 to determine all possible values of $(\lambda_1, \lambda_1', \lambda_2)$. Note that a GDD$(m, n; \lambda_1, \lambda_1', \lambda_2)$ is equivalent to a GDD$(n, m; \lambda_1', \lambda_1, \lambda_2)$. Thus, we have that

$$(\lambda_1, \lambda_1', \lambda_2) \in \{(\bar{0},\bar{0},\bar{0}), (\bar{2},\bar{2},\bar{2}), (\bar{4},\bar{4},\bar{4}), (\bar{4},\bar{0},\bar{2}), (\bar{2},\bar{4},\bar{0}), (\bar{0},\bar{2},\bar{4})\}.$$

If $(\lambda_1, \lambda_1', \lambda_2) \in \{(\bar{0},\bar{0},\bar{0}), (\bar{2},\bar{2},\bar{2}), (\bar{4},\bar{4},\bar{4})\}$, then we can apply Theorem 2.3 to obtain a TS$(m; \lambda_1 - \lambda_2)$, a TS$(n; \lambda_1' - \lambda_2)$ and a TS$(m + n; \lambda_2)$. Then, by Lemma 3.1, these cases are done. Now, to construct GDDs when $(\lambda_1, \lambda_1', \lambda_2) \in \{(\bar{4},\bar{0},\bar{2}), (\bar{2},\bar{4},\bar{0}), (\bar{0},\bar{2},\bar{4})\}$, we define the notation

$$(\lambda_1, \lambda_1', \lambda_2) \to (\lambda_1 + 2, \lambda_1' + 2, \lambda_2 + 2)$$

to denote that if a GDD$(m, n; \lambda_1, \lambda_1', \lambda_2)$ exists, then a GDD$(m, n; \lambda_1 + 2, \lambda_1' + 2, \lambda_2 + 2)$ exists by applying Lemma 3.1 with a TS$(m + n; 2)$, or, equivalently, a GDD$(m, n; 2, 2, 2)$. Note from condition (iv) in Theorem 1.2 that $\lambda_2 \neq 0$ except the case when $(\lambda_1, \lambda_1', \lambda_2) = (\bar{0},\bar{0},\bar{0})$.

Now, to construct all GDDs in each case $(\lambda_1, \lambda_1', \lambda_2) = (\bar{a}, \bar{b}, \bar{c})$, it suffices to construct only the smallest one. The larger GDDs in each case can be simply obtained by applying Lemma 3.1 (i) to combine the smallest GDD with a $\mathsf{TS}(m+n;6a)$ where $a \in \mathbb{N}$, which is equivalent to $\mathsf{GDD}(m,n;6a,6a,6a)$. The diagram

$$(4,6,2) \to (6,8,4) \to (8,10,6)$$

shows that a $\mathsf{GDD}(m,n;4,6,2)$ existing from Lemma 4.2 provides the smallest GDDs of the remaining two cases. These values are the smallest ones because we have that $\lambda_2 \neq 0, \lambda_1 \geq \lambda_2$ and $\lambda_1' \geq \lambda_2$. This completes the proof.

Case (ii) λ_2 is odd. From $3|(\lambda_1+\lambda_1'+\lambda_2)$, we regard λ_1, λ_1' and λ_2 as integers modulo 6. The possible values of $(\lambda_1, \lambda_1', \lambda_2)$ are in $\{(2,0,1),(4,4,1),(0,0,3),(4,2,3),(0,4,5),(2,2,5)\}$. Similar to Case (i), the following diagrams show that a $\mathsf{GDD}(m,n;2,6,1)$ existing from Lemma 4.2 and a $\mathsf{GDD}(m,n;4,4,1)$ existing from Lemma 2.13 (v) yield the rest of our desired GDDs and this completes the proof.

$$(2,6,1) \to (4,8,3) \to (6,10,5)$$

$$(4,4,1) \to (6,6,3) \to (8,8,5)$$

□

4.2 $m \equiv 2 \pmod 6$, $n \equiv 5 \pmod 6$ and $m \neq 2$

In this section, we consider GDDs when $m \neq 2, m \equiv 2 \pmod 6$ and $n \equiv 5 \pmod 6$. The main construction is provided in Theorem 4.6. In details, we give a method to construct our desired GDDs from certain small GDDs, which are a $\mathsf{GDD}(m,n;3,2,1)$ and a $\mathsf{GDD}(m,n;5,3,1)$ obtained in Lemmas 4.4 and 4.5, respectively.

Lemma 4.4. *Let m and n be positive integers such that $m \neq 2$, $m \equiv 2 \pmod 6$ and $n \equiv 5 \pmod 6$. There exists a* $\mathsf{GDD}(m,n;3,2,1)$.

Proof. We write $m = 6h+8$ and $n = 6k+5$ for nonnegative integers h and k. Let $A = \{x_1, x_2\}$ and $B = \{y_1, y_2, y_3, y_4, y_5\}$. First, we let $\mathcal{B}_1 = \{\{x_1, x_2, y_1\}, \{x_1, x_2, y_2\}, \{x_1, x_2, y_3\}, \{x_1, y_4, y_5\}, \{x_2, y_4, y_5\}, \{y_1, y_2, y_4\}, \{y_2, y_3, y_4\}, \{y_1, y_3, y_4\}, \{y_1, y_2, y_5\}, \{y_2, y_3, y_5\}, \{y_1, y_3, y_5\}\}$. Then, $(A, B, \mathcal{B}_1)$ is a GDD$(2,5;3,2,1)$. We consider the construction in the following four cases.

Case (i) $h+1 < k$. By Theorem 2.12, we can decompose the graph $3K_{m-2}(M_m \setminus A)$ into a collection of triangles $\mathcal{T}_1$ and eleven 1-factors, say $F_{i,j}$ and F'_p where $i \in \{1,2\}, j \in \{1,2,3\}$ and $p \in \{1,2,3,4,5\}$. Let $\mathcal{F}_1$ be a collection of triangles defined by

$$\mathcal{F}_1 = \{x_i + F_{i,j}, y_p + F'_p : i \in \{1,2\}, j \in \{1,2,3\}, p \in \{1,2,3,4,5\}\}$$

Since $1 \leq h+1 < k$, again by Theorem 2.12, we can decompose the graph $2K_{n-5}(N_n \setminus B)$ into a collection of triangles $\mathcal{T}_2$ and $6h+18$ 1-factors, say H_i and $H'_{p,q}$ where $i \in \{1,2,3,\ldots,6h+8\}, p \in \{1,2,3,4,5\}$ and $q \in \{1,2\}$. Let $\mathcal{F}_2$ be a collection of triangles defined by

$$\mathcal{F}_2 = \{x_i + H_i, y_p + H'_{p,q} : i \in \{1,2,3,\ldots,6h+8\}, p \in \{1,2,3,4,5\}, q \in \{1,2\}\}.$$

Then, $(M_m, N_n, \mathcal{B}_1 \cup \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{F}_1 \cup \mathcal{T}_2)$ is a desired GDD.

Case (ii) $h+1 \geq k, k \neq 1$. By Theorem 2.12, we can decompose the graph $3K_{m-2}(M_m \setminus A)$ into a collection of triangles $\mathcal{T}_1$ and $6k+11$ 1-factors, say $F_{i,j}$ and F'_p where $i \in \{1,2\}, j \in \{1,2,3\}$ and $p \in \{1,2,3,\ldots,6k+5\}$. Let $\mathcal{F}_1$ be a collection of triangles defined by

$$\mathcal{F}_1 = \{x_i + F_{i,j}, y_p + F'_p : i \in \{1,2\}, j \in \{1,2,3\}, p \in \{1,2,3,\ldots,6k+5\}\}.$$

If $k = 0$, the construction is done here and $(M_m, N_5, \mathcal{B}_1 \cup \mathcal{T}_1 \cup \mathcal{F}_1)$ is a desired GDD. Now assume $k \neq 0$, then $k \geq 2$. Again by Theorem 2.12, we can decompose the graph $2K_{n-5}(N_n \setminus B)$ into a collection of triangles $\mathcal{T}_2$ and 12 1-factors, say

H_i and $H'_{p,q}$ where $i, q \in \{1,2\}$ and $p \in \{1,2,3,4,5\}$. Let $\mathcal{F}_2$ be a collection of triangles defined by

$$\mathcal{F}_2 = \{x_i + H_i, y_p + H'_{p,q} : i, q \in \{1,2\}, p \in \{1,2,3,4,5\}\}.$$

Hence, $(M_m, N_n, \mathcal{B}_1 \cup \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{F}_1 \cup \mathcal{F}_2)$ is a desired GDD.

Case (iii) $h + 1 = k = 1$. In this case, we let $B = \{y_1, y_2, y_3\}$. By Theorem 2.9, each copy of K_8 in $3K_8(M_8)$ can be considered as a maximum packing of order 8 having a 1-factor as the leave, say $(M_8, \mathcal{T}_j, \mathcal{L}_j)$ for $j \in \{1,2,3\}$. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be collections of triangles defined by

$$\mathcal{F}_1 = \{y_j + \mathcal{L}_j : j \in \{1,2,3\}\}$$

and

$$\mathcal{F}_2 = \{\{y_1, y_2, y_3\}, \{y_1, y_2, y_3\}\}.$$

By Theorem 2.5, the graph $2K_8(N_{11} \setminus B)$ can be decomposed into 14 1-factors, say H_i and $H'_{p,q}$ where $i \in \{1,2,3,\ldots,8\}, p \in \{1,2,3\}$ and $q \in \{1,2\}$. Let $\mathcal{F}_3$ be a collection of triangles defined by

$$\mathcal{F}_3 = \{x_i + H_i, y_p + H'_{p,q} : i \in \{1,2,3,\ldots,8\}, p \in \{1,2,3\}, q \in \{1,2\}\}.$$

Hence, $(M_8, N_{11}, \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3 \cup \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3)$ is a desired GDD.

Case (iv) $h + 1 > k = 1$. In this case, we let $A = \{x_1, x_2\}$ and $B = \{y_1, y_2, y_3\}$. Let $\mathcal{B}_1 = \{\{x_1, x_2, y_1\}, \{x_1, x_2, y_2\}, \{x_1, x_2, y_3\}, \{y_1, y_2, y_3\}, \{y_1, y_2, y_3\}\}$ be a collection of triangles. By Theorems 2.9 and 2.5, a copy of K_8 in $2K_8(N_{11} - B)$ can be considered as a maximum packing of order 8, $(M_8, \mathcal{T}_1, \mathcal{L})$, having a 1-factor as the leave; and the other copy can be decomposed into seven 1-factors. Thus, there are a total of eight 1-factors, denoted by F_i and $F'_{p,q}$ where $i, q \in \{1,2\}$ and $p \in \{1,2,3\}$. Let $\mathcal{F}_1$ be a collection of triangles defined by

$$\mathcal{F}_1 = \{x_i + F_i, y_p + F'_{p,q} : i, q \in \{1,2\}, p \in \{1,2,3\}\}.$$

Since $h+1 \geq 2$, by Theorem 2.12, we can decompose the graph $3K_{m-2}(M_m \setminus A)$ into a collection of triangles $\mathcal{T}_2$ and 17 1-factors, say $H_{i,j}$ and H'_p where $i \in \{1,2\}, j \in \{1,2,3\}$ and $p \in \{1,2,3,\ldots,11\}$. Let $\mathcal{F}_2$ be a collection of triangles defined by

$$\mathcal{F}_2 = \{x_i + H_{i,j}, y_p + H'_p : i \in \{1,2\}, j \in \{1,2,3\}, p \in \{1,2,3,\ldots,11\}\}.$$

Thus, $(M_m, N_n, \mathcal{B}_1 \cup \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{F}_1 \cup \mathcal{F}_2)$ is a desired GDD. $\square$

Lemma 4.5. *Let m and n be positive integers such that $m \neq 2$, $m \equiv 2 \pmod 6$ and $n \equiv 5 \pmod 6$. There exists a* $\mathsf{GDD}(m,n;5,3,1)$.

Proof. We write $m = 6h+8$ and $n = 6k+5$ for nonnegative integers h and k. Let $A = \{x_1, x_2\}$ and $B = \{y_1, y_2, y_3, \ldots, y_5\}$. First, we note that the set $\mathcal{B}_1 = \{\{x_1, x_2, y_i\} : i \in \{1,2,3,4,5\}\}$ forms a $\mathsf{GDD}(2,5;5,0,1)$, namely $(A, B, \mathcal{B}_1)$. By Theorem 2.3, there exists a $\mathsf{TS}(5;3)$ on B. Thus, by Lemma 3.1, there exists a $\mathsf{GDD}(2,5;5,3,1)$, namely $(A, B, \mathcal{B}_2)$. We separate the construction in the following three cases.

Case (i) $h+1 \geq k, k \neq 1$. By Theorem 2.12, we can decompose the graph $5K_{m-2}(M_m \setminus A)$ into a collection of triangles $\mathcal{T}_1$ and $6k+15$ 1-factors, say $F_{i,j}$ and F'_p where $i \in \{1,2\}$, $j \in \{1,2,3,4,5\}$ and $p \in \{1,2,3,\ldots,6k+5\}$. Let $\mathcal{F}_1$ be a collection of triangles defined by

$$\mathcal{F}_1 = \{x_i + F_{i,j}, y_p + F'_p : i \in \{1,2\}, j \in \{1,2,3,4,5\}, p \in \{1,2,3,\ldots,6k+5\}\}.$$

If $k=0$, then the construction is done here and $(M_m, N_5, \mathcal{B}_2 \cup \mathcal{T}_1 \cup \mathcal{F}_1)$ is a desired GDD. Assume that $k \neq 0$, then $k \geq 2$. Again by Theorem 2.12, we can decompose the graph $3K_{6k}(N_n \setminus B)$ into a collection of triangles $\mathcal{T}_2$ and 17 1-factors, say H_i and $H'_{p,q}$ where $i \in \{1,2\}$, $p \in \{1,2,3,4,5\}$ and $q = \{2,3,4\}$. Let $\mathcal{F}_2$ be a collection of triangles defined by

$$\mathcal{F}_2 = \{x_i + H_i, y_p + H'_{p,q} : i \in \{1,2\}, p \in \{1,2,3,4,5\}, q = \{2,3,4\}\}.$$

Hence, $(M_m, N_n, \mathcal{B}_2 \cup \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{F}_1 \cup \mathcal{F}_2)$ is a desired GDD.

Case (ii) $h+1 \geq k = 1$. Applying Theorem 2.12, we can decompose the graph $5K_{m-2}(M_m \setminus A)$ in the same way as the previous case and obtain the collections of triangles $\mathcal{T}_1$ and $\mathcal{F}_1$. With the same theorem, the graph $3K_6(N_{11} \setminus B)$ can be decomposed into a collection of triangles $\mathcal{T}_2$ and seven 1-factors, say $F_{i,1}$ and $F_{j,2}$ where $i \in \{1, 2\}$ and $j \in \{1, 2, 3, 4, 5\}$. Let $\mathcal{F}_2$ be a collection of triangles defined by

$$\mathcal{F}_2 = \{x_i + F_{i,1}, y_j + F_{j,2} : i \in \{1, 2\}, j \in \{1, 2, 3, 4, 5\}\}.$$

By Theorem 2.7, the graph $3K_5(B)$ can be decomposed into six 2-factors, say $C_6, C_7, C_8, \ldots, C_{11}$. Let $\mathcal{C}$ be a collection of triangles defined by

$$\mathcal{C} = \{y_i + C_i : i \in \{6, 7, 8, \ldots, 11\}\}.$$

Hence, $(M_m, N_{11}, \mathcal{B}_1 \cup \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{F}_1 \cup \mathcal{F}_1 \cup \mathcal{C})$ is a desired GDD.

Case (iii) $h+1 < k$. Since $h+1 \geq 1$, by Theorem 2.12, we can decompose the graph $5K_{m-2}(M_m \setminus A)$ into a collection of triangles $\mathcal{T}_1$ and 15 1-factors, say $F_{i,j}$ and F'_p where $i \in \{1, 2\}$ and $j, p \in \{1, 2, 3, 4, 5\}$. Let $\mathcal{F}_1$ be a collection of triangles defined by

$$\mathcal{F}_1 = \{x_i + F_{i,j}, y_p + F'_p : i \in \{1, 2\}, j, p \in \{1, 2, 3, 4, 5\}\}.$$

Since $1 \leq h+1 < k$, again by Theorem 2.12, we can decompose the graph $3K_{6k}(N_n \setminus B)$ into a collection of triangles $\mathcal{T}_2$ and $6h+23$ 1-factors, say H_i and $H'_{p,q}$ where $i \in \{1, 2, 3, \ldots, 6h+8\}$, $p \in \{1, 2, 3, 4, 5\}$ and $q \in \{2, 3, 4\}$. Let $\mathcal{F}_2$ be a collection of triangles defined by

$$\mathcal{F}_2 = \{x_i + H_i, y_p + H'_{p,q} : i \in \{1, 2, 3, \ldots, 6h+8\}, p \in \{1, 2, 3, 4, 5\}, q \in \{2, 3, 4\}\}.$$

Hence, $(M_m, N_n, \mathcal{B}_2 \cup \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{F}_1 \cup \mathcal{F}_2)$ is a desired GDD. $\square$

Theorem 4.6. *Let m and n be positive integers such that $m \neq 2$, $m \equiv 2 \pmod 6$ and $n \equiv 5 \pmod 6$. Let λ_1, λ_1' and λ_2 be nonnegative integers such that $\lambda_1 \geq \lambda_2$ and $\lambda_1' \geq \lambda_2$. If $\lambda_1 \equiv \lambda_2 \pmod 2$ and $3|(\lambda_1 + \lambda_1' + \lambda_2)$, then there exists a* $\mathsf{GDD}(m, n; \lambda_1, \lambda_1', \lambda_2)$.

Proof. From the assumption, we regard λ_1, λ_1' and λ_2 as integers modulo 6 and determine all possible values of $(\lambda_1, \lambda_1', \lambda_2)$. We display all 36 cases in Table 4.1. Note from condition (iv) in Theorem 1.2 that $\lambda_2 \neq 0$ except the cases when $(m, n) \in \{(\bar{0}, \bar{0}), (\bar{0}, \bar{3})\}$.

(λ_1, λ_1')						λ_2
$(\bar{0}, \bar{0})$	$(\bar{0}, \bar{3})$	$(\bar{2}, \bar{1})$	$(\bar{2}, \bar{4})$	$(\bar{4}, \bar{2})$	$(\bar{4}, \bar{5})$	$\bar{0}$
$(\bar{1}, \bar{1})$	$(\bar{1}, \bar{4})$	$(\bar{3}, \bar{2})$	$(\bar{3}, \bar{5})$	$(\bar{5}, \bar{0})$	$(\bar{5}, \bar{3})$	$\bar{1}$
$(\bar{0}, \bar{1})$	$(\bar{0}, \bar{4})$	$(\bar{2}, \bar{2})$	$(\bar{2}, \bar{5})$	$(\bar{4}, \bar{0})$	$(\bar{4}, \bar{3})$	$\bar{2}$
$(\bar{1}, \bar{2})$	$(\bar{1}, \bar{5})$	$(\bar{3}, \bar{0})$	$(\bar{3}, \bar{3})$	$(\bar{5}, \bar{1})$	$(\bar{5}, \bar{4})$	$\bar{3}$
$(\bar{0}, \bar{2})$	$(\bar{0}, \bar{5})$	$(\bar{2}, \bar{0})$	$(\bar{2}, \bar{3})$	$(\bar{4}, \bar{1})$	$(\bar{4}, \bar{4})$	$\bar{4}$
$(\bar{1}, \bar{0})$	$(\bar{1}, \bar{3})$	$(\bar{3}, \bar{1})$	$(\bar{3}, \bar{4})$	$(\bar{5}, \bar{2})$	$(\bar{5}, \bar{5})$	$\bar{5}$

Table 4.1: All possible values of $(\lambda_1, \lambda_1', \lambda_2)$

It is easy to see that GDDs with $(\lambda_1, \lambda_1', \lambda_2) \in \{(\bar{k}, \bar{k}, \bar{k}), (\bar{k}, \overline{k+3}, \bar{k}) : k \in \{0, 1, 2, 3, 4, 5\}\}$ exist by applying Theorem 2.3 and Lemma 3.1. For other 24 cases, we use the right arrow

$$(\lambda_1, \lambda_1', \lambda_2) \Rightarrow (\lambda_1 + 1, \lambda_1' + 1, \lambda_2 + 1)$$

to denote that if a $\mathsf{GDD}(m, n; \lambda_1, \lambda_1', \lambda_2)$ exists, then a $\mathsf{GDD}(m, n; \lambda_1 + 1, \lambda_1' + 1, \lambda_2 + 1)$ exists by applying Lemma 3.1 with a $\mathsf{TS}(m + n; 1)$, which is equivalent to a $\mathsf{GDD}(m, n; 1, 1, 1)$; and use the down arrow

$$\begin{array}{c}(\lambda_1, \lambda_1', \lambda_2)\\ \Downarrow \\ (\lambda_1, \lambda_1' + 3, \lambda_2)\end{array}$$

to denote that if a $\mathsf{GDD}(m, n; \lambda_1, \lambda_1', \lambda_2)$ exists, then a $\mathsf{GDD}(m, n; \lambda_1, \lambda_1' + 3, \lambda_2)$ exists by applying Lemma 3.1 with a $\mathsf{TS}(n; 3)$, or, equivalently, a $\mathsf{GDD}(m, n; 0, 3, 0)$.

Now, to construct all GDDs in each case $(\lambda_1, \lambda_1', \lambda_2) = (\bar{a}, \bar{b}, \bar{c})$, it suffices to construct only the smallest one. The larger GDDs in each case can be simply obtained by applying Lemma 3.1 (i) to combine the smallest GDD with a $\mathsf{TS}(m + n; 6a)$ where $a \in \mathbb{N}$, which is equivalent to $\mathsf{GDD}(m, n; 6a, 6a, 6a)$. The following diagram shows that a $\mathsf{GDD}(m, n; 3, 2, 1)$ existing from Lemma 4.4 provides some of those smallest GDDs. These values are the smallest ones because we have that $\lambda_2 \neq 0$, $\lambda_1 \geq \lambda_2$ and $\lambda_1' \geq \lambda_2$.

$$\begin{array}{ccccccccccc}(3,2,1) & \Rightarrow & (4,3,2) & \Rightarrow & (5,4,3) & \Rightarrow & (6,5,4) & \Rightarrow & (7,6,5) & \Rightarrow & (8,7,6)\\ \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\ (3,5,1) & \Rightarrow & (4,6,2) & \Rightarrow & (5,7,3) & \Rightarrow & (6,8,4) & \Rightarrow & (7,9,5) & \Rightarrow & (8,10,6)\end{array}$$

Moreover, two copies of a $\mathsf{GDD}(m, n; 3, 2, 1)$ form a $\mathsf{GDD}(m, n; 6, 4, 2)$. The following diagram shows that a $\mathsf{GDD}(m, n; 6, 4, 2)$ provides some of those smallest GDDs.

$$\begin{array}{ccccccccc}(6,4,2) & \Rightarrow & (7,5,3) & \Rightarrow & (8,6,4) & \Rightarrow & (9,7,5) & \Rightarrow & (10,8,6)\\ \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\ (6,7,2) & \Rightarrow & (7,8,3) & \Rightarrow & (8,9,4) & \Rightarrow & (9,10,5) & \Rightarrow & (10,11,6)\end{array}$$

Lastly, the following diagram shows that a GDD$(m, n; 5, 3, 1)$ existing from Lemma 4.5 yields a GDD$(m, n; 5, 6, 1)$.

$$(5,3,1)$$

$$\Downarrow$$

$$(5,6,1)$$

Therefore, all of our smallest GDDs for the remaining 24 cases are obtained. Thus, our construction is completed. $\square$

4.3 m and $n \equiv 5 \pmod 6$

In this section, we consider the existence of GDDs when m and $n \equiv 5 \pmod 6$. The main proof is shown in Theorem 4.9. Similar to the previous section, most of our desired GDDs in this case can be obtained from one case of them, which is a GDD$(m, n; 3, 4, 2)$. Lemma 4.7 gives a construction of such GDD when $m = n$ while Lemma 4.8 shows the cases when $m \neq n$.

Lemma 4.7. *Let n be a positive integer such that $n \equiv 5 \pmod 6$. There exists a* GDD$(n, n; 3, 4, 2)$.

Proof. We write $n = 6k + 5$ for a nonnegative integer k. The construction is separated in the following four cases.

Case (i) $k = 0$. By Theorem 2.7, the graph $4K_5(N_5)$ can be decomposed into eight 2-factors, say C_1, C_2, C_3, C_4 and C_5, and three cycles, namely $(y_1y_2y_3y_4y_5)$, $(y_1y_3y_5y_2y_4)$ and $(y_1y_3y_5y_2y_4)$. The graph obtained from the union of these 3 cycles can be decomposed into a collection of triangles $\mathcal{T}$ as follows:

$$\mathcal{T} = \{\{y_1, y_3, y_4\}, \{y_2, y_4, y_5\}, \{y_1, y_3, y_5\}, \{y_1, y_2, y_4\}, \{y_2, y_3, y_5\}\}.$$

Moreover, let $\mathcal{C}$ be a collection of triangles defined by

$$\mathcal{C} = \{x_i + C_i : i \in \{1, 2, 3, 4, 5\}\}.$$

We note that $(M_5, N_5, \mathcal{T} \cup \mathcal{C})$ is a $\mathsf{GDD}(5, 5; 0, 4, 2)$. Moreover, the graph $3K_5(M_5)$ can be considered as a $\mathsf{TS}(5; 3)$, namely $(M_5, \mathcal{B})$. Thus, $(M_5, N_5, \mathcal{T} \cup \mathcal{C} \cup \mathcal{B})$ is a $\mathsf{GDD}(5, 5; 3, 4, 2)$.

Case (ii) $k = 1$. Let $A = \{x_1, x_2, x_3\}$. By Theorem 2.5, the graph $3K_8(M_{11} \setminus A)$ can be decomposed into 21 1-factors, say $F_1, F_2, F_{i,j}$ and F'_p where $i, j \in \{1, 2, 3\}$ and $p \in \{2, 3, 4, \ldots, 11\}$. Let $\mathcal{F}_1$ be a collection of triangles defined by

$$\mathcal{F}_1 = \{x_i + F_{i,j}, y_p + F'_p, y_1 + F_1, y_1 + F_2 : i, j \in \{1, 2, 3\}, p \in \{2, 3, 4, \ldots, 11\}\}.$$

Besides, two copies of K_{10} in the graph $4K_{10}(N_{11} \setminus \{y_1\})$ can be considered as a $\mathsf{TS}(10; 2)$, namely $(N_{11} \setminus \{y_1\}, \mathcal{B}_1)$. For the other two copies, by Theorem 2.5, we can decompose $2K_{10}(N_{11} \setminus \{y_1\})$ into 18 1-factors, say H_i, H'_j and $H_{p,q}$ where $i \in \{4, 5, 6, \ldots, 11\}$, $j \in \{1, 2, 3, 4\}$, $p \in \{1, 2, 3\}$ and $q \in \{1, 2\}$. Let $\mathcal{F}_2$ and $\mathcal{F}_3$ be collections of triangles defined by

$$\mathcal{F}_2 = \{x_i + H_i, x_p + H_{p,q} : i \in \{4, 5, 6, \ldots, 11\}, p \in \{1, 2, 3\}, q \in \{1, 2\}\}$$

and

$$\mathcal{F}_3 = \{y_1 + H'_j : j \in \{1, 2, 3, 4\}\}.$$

Lastly, we let $\mathcal{B}_2 = \{\{x_1, x_2, y_1\}, \{x_2, x_3, y_1\}, \{x_1, x_3, y_1\}, \{x_1, x_2, x_3\}, \{x_1, x_2, x_3\}\}$ be a collection of triangles on $A \cup \{y_1\}$. Hence, $(M_{11}, N_{11}, \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \cup \mathcal{B}_1 \cup \mathcal{B}_2)$ is a desired GDD.

Case (iii) $k = 2$. Note that the graph $3K_{17}(M_{17})$ can be considered as a $\mathsf{TS}(17; 3)$, namely $(M_{17}, \mathcal{B})$. By Theorem 2.7, the graph $4K_{17}(N_{17})$ can be decomposed into 32 2-factors, say $C_1, C_2, C_3, \ldots, C_{17}$ and 15 cycles, namely 3 copies of cycle in $\{(y_1y_2y_3\cdots y_{17}),(y_1y_8y_{15}\cdots y_{12}),(y_1y_9y_{17}\cdots y_{11})\}$ and 2 copies of each cycle in $\{(y_1y_3y_5\cdots y_{17}),(y_1y_4y_7\cdots y_{16}),(y_1y_6y_{11}\cdots y_{14})\}$.

Let $\mathcal{C}$ be a collection of triangles defined by $\mathcal{C}=\{x_i+C_i : i\in\{1,2,3,\ldots,17\}\}$. Besides. the graph obtained by the union of those specified 15 cycles can be decomposed into a collection of triangles $\mathcal{T}$ as follows:

$$\begin{aligned}\mathcal{T}=\Big\{&\{y_1,y_2,y_9\},\{y_2,y_3,y_{10}\},\{y_3,y_4,y_{11}\},\ldots,\{y_{17},y_1,y_8\},\\&\{y_1,y_2,y_9\},\{y_2,y_3,y_{10}\},\{y_3,y_4,y_{11}\},\ldots,\{y_{17},y_1,y_8\},\\&\{y_1,y_2,y_9\},\{y_2,y_3,y_{10}\},\{y_3,y_4,y_{11}\},\ldots,\{y_{17},y_1,y_8\},\\&\{y_1,y_3,y_6\},\{y_2,y_4,y_7\},\{y_3,y_5,y_8\},\ldots,\{y_{17},y_2,y_5\},\\&\{y_1,y_3,y_6\},\{y_2,y_4,y_7\},\{y_3,y_5,y_8\},\ldots,\{y_{17},y_2,y_5\}\Big\}.\end{aligned}$$

Hence, $(M_{17},N_{17},\mathcal{B}\cup\mathcal{C}\cup\mathcal{T})$ is a desired GDD.

Case (iv) $k\geq 3$. In this case, we let $A=\{x_1,x_2,x_3,x_4,x_5\}$ and $B=\{y_1,y_2,y_3,y_4,y_5\}$. From Case (i), there exists a GDD$(5,5;3,4,2)$ on $A\cup B$, namely $(A,B,\mathcal{B}_1)$. By Theorem 2.12, the graph $3K_{6k}(M_n\setminus A)$ can be decomposed into a collection of triangles $\mathcal{T}_1$ and $6k+25$ 1-factors, say F'_i and $F_{p,q}$ where $i\in\{1,2,3,\ldots,6k\}$ and $p,q\in\{1,2,3,4,5\}$. Let $\mathcal{F}_1$ be a collection of triangles defined by

$$\mathcal{F}_1=\{y_i+F'_i,y_p+F_{p,q} : i\in\{1,2,\ldots,6k\},p,q\in\{1,2,3,4,5\}\}.$$

Again, by Theorem 2.12, we can decompose the graph $4K_{6k}(N_n\setminus B)$ into a collection of triangles $\mathcal{T}_2$ and $6k+30$ 1-factors, say H_i, $H_{s,t}$ and $H'_{p,q}$ where $i\in\{1,2,3,\ldots,6h\}$, $s,p\in\{1,2,3,4,5\}$, $t\in\{2,3\}$ and $q\in\{4,5,6,7\}$. Let $\mathcal{F}_2$ and $\mathcal{F}_3$ be collectiona of triangles defined by

$$\mathcal{F}_2=\{x_i+H_i,x_s+H_{s,t} : i\in\{1,2,3,\ldots,6h\},s\in\{1,2,3,4,5\},t\in\{2,3\}\}$$

and

$$\mathcal{F}_3=\{y_p+H'_{p,q} : p\in\{1,2,3,4,5\},q\in\{4,5,6,7\}\}.$$

Hence, $(M_m, N_n, \mathcal{B}_1 \cup \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3)$ is a desired GDD. □

Lemma 4.8. *Let m and n be positive integers such that m and $n \equiv 5 \pmod 6$. Then, there exists a* GDD$(m, n; 3, 4, 2)$.

Proof. We write $m = 6h + 5$ and $n = 6k + 5$ for nonnegative integers h and k. Let $A = \{x_1, x_2, x_3, x_4, x_5\}$ and $B = \{y_1, y_2, y_3, y_4, y_5\}$. First, we note that from the proof of Case (i) in Lemma 4.7, there exist a GDD$(5, 5; 0, 4, 2)$ and a GDD$(5, 5; 3, 4, 2)$ on $A \cup B$, namely $(A, B, \mathcal{B}_1)$ and $(A, B, \mathcal{B}_2)$, respectively. The construction of our desired GDD is separated in the following 7 cases.

Case (i) $2 \leq h < k$ or $h = 0$ and $k \geq 2$. By Theorem 2.12, we can decompose the graph $4K_{6k}(N_n \setminus B)$ into a collection of triangles $\mathcal{T}_1$ and $12h + 30$ 1-factors, say $H_{i,j}$ and $H'_{p,q}$ where $i \in \{1, 2, 3, \ldots, 6h + 5\}$, $j \in \{1, 2\}$, $p \in \{1, 2, 3, 4, 5\}$ and $q \in \{3, 4, 5, 6\}$. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be collections of triangles defined by

$$\mathcal{F}_1 = \{x_i + H_{i,j} : i \in \{1, 2, 3, \ldots, 6h + 5\}, j \in \{1, 2\}\}$$

and

$$\mathcal{F}_2 = \{y_p + H'_{p,q} : p \in \{1, 2, 3, 4, 5\}, q \in \{3, 4, 5, 6\}\}.$$

If $h = 0$, then the proof is done here and $(M_m, N_n, \mathcal{B}_2 \cup \mathcal{T}_1 \cup \mathcal{F}_1 \cup \mathcal{F}_2)$ is a desired GDD. Now, assume that $h \geq 2$. Again by Theorem 2.12, the graph $3K_{6h}(M_m \setminus A)$ can be decomposed into a collection of triangles $\mathcal{T}_2$ and 25 1-factors, say $F_{i,j}$ and $F'_{p,q}$ where $i, p \in \{1, 2, 3, 4, 5\}$, $j \in \{1, 2, 3\}$ and $q \in \{4, 5\}$. Let $\mathcal{F}_3$ be a collection of triangles defined by

$$\mathcal{F}_3 = \{x_i + F_{i,j}, y_l + F'_{p,q} : i, p \in \{1, 2, 3, 4, 5\}, j \in \{1, 2, 3\}, q \in \{4, 5\}\}.$$

Thus, $(M_m, N_n, \mathcal{B}_2 \cup \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3)$ is a desired GDD.

Case (ii) $1 = h < k$. By Theorem 2.7, the graph $3K_5(A)$ can be decomposed into six 2-factors, say $C_6, C_7, C_8, \ldots, C_{11}$. Let $\mathcal{C}$ be a collection of triangles defined by

$$\mathcal{C} = \{x_i + C_i : i \in \{6, 7, 8, \ldots, 11\}\}.$$

By Theorem 2.12, we can decompose the graph $3K_6(M_{11} \setminus A)$ into a collection of triangles $\mathcal{T}_1$ and 15 1-factors, say F_i and $F_{p,q}$ where $i, p \in \{1, 2, 3, 4, 5\}$ and $q \in \{2, 3\}$. Let $\mathcal{F}_1$ be a collection of triangles defined by

$$\mathcal{F}_1 = \{x_i + F_i, y_p + F_{p,q} : i, p \in \{1, 2, 3, 4, 5\}, q \in \{2, 3\}\}.$$

Again, by Theorem 2.12, we can decompose the graph $4K_{6k}(N_n \setminus B)$ into a collection of triangles $\mathcal{T}_2$ and $12h + 30$ 1-factors, say $H_{i,j}$ and $H'_{p,q}$ where $i \in \{1, 2, 3, \ldots, 6h + 5\}$, $j \in \{1, 2\}$, $p \in \{1, 2, 3, 4, 5\}$ and $q \in \{3, 4, 5, 6\}$. Let $\mathcal{F}_2$ and $\mathcal{F}_3$ be collections of triangles defined by

$$\mathcal{F}_2 = \{x_i + H_{i,j} : i \in \{1, 2, 3, \ldots, 6h + 5\}, j \in \{1, 2\}\}$$

and

$$\mathcal{F}_3 = \{y_p + H'_{p,q} : p \in \{1, 2, 3, 4, 5\}, q \in \{3, 4, 5, 6\}\}.$$

Thus, $(M_{11}, N_n, \mathcal{B}_1 \cup \mathcal{C} \cup \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3)$ is a desired GDD.

Case (iii) $h = 0$ and $k = 1$. By Theorem 2.7, the graph $3K_5(M_5)$ can be decomposed into six 2-factors, say $C_6, C_7, C_8, \ldots, C_{11}$ Let $\mathcal{C}$ be a collection of triangles defined by

$$\mathcal{C} = \{y_i + C_i : i \in \{6, 7, 8, \ldots, 11\}\}.$$

By Theorem 2.5, we can decompose the graph $4K_6(N_{11} \setminus B)$ into 20 1-factors, say $F_{i,j}$ where $i \in \{1, 2, 3, 4, 5\}$ and $j \in \{1, 2, 3, 4\}$. Let $\mathcal{F}_1$ be a collection of triangles defined by

$$\mathcal{F}_1 = \{y_i + F_{i,j} : i \in \{1, 2, 3, 4, 5\}, j \in \{1, 2, 3, 4\}\}.$$

Hence, $(M_5, N_{11}, \mathcal{B}_1 \cup \mathcal{C} \cup \mathcal{F}_1)$ is a desired GDD.

Case (iv) $h > k \geq 2$ or $h \geq 2$ and $k = 0$. By Theorem 2.12, we can decompose the graph $3K_{6h}(M_m \setminus A)$ into a collection of triangles $\mathcal{T}_1$ and $12k + 25$ 1-factors, say $H_{i,j}$ and $H'_{p,q}$ where $i \in \{1, 2, 3, 4, 5\}$, $j \in \{3, 4, 5\}$, $p \in \{1, 2, 3, \ldots, 6k+5\}$ and $q \in \{1, 2\}$. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be collections of triangles defined by

$$\mathcal{F}_1 = \{x_i + H_{i,j} : i \in \{1, 2, 3, 4, 5\}, j \in \{3, 4, 5\}\}$$

and

$$\mathcal{F}_2 = \{y_p + H'_{p,q} : p \in \{1, 2, 3, \ldots, 6k + 5\}, q \in \{1, 2\}\}.$$

If $k = 0$, then the proof is done here and $(M_m, N_n, \mathcal{B}_2 \cup \mathcal{T}_1 \cup \mathcal{F}_1 \cup \mathcal{F}_2)$ is a desired GDD. Now, assume that $k \geq 2$. Again by Theorem 2.12, the graph $4K_{6k}(N_n \setminus B)$ can be decomposed into a collection of triangles $\mathcal{T}_2$ and 30 1-factors, say $F_{i,j}$ and $F'_{p,q}$ where $i, p \in \{1, 2, 3, 4, 5\}$, $j \in \{1, 2\}$ and $q \in \{3, 4, 5, 6\}$. Let $\mathcal{F}_3$ be a collection of triangles defined by

$$\mathcal{F}_3 = \{x_i + F_{i,j}, y_p + F'_{p,q} : i, p \in \{1, 2, 3, 4, 5\}, j \in \{1, 2\}, q \in \{3, 4, 5, 6\}\}.$$

Hence, $(M_m, N_n, \mathcal{B}_2 \cup \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3)$ is a desired GDD.

Case (v) $h \geq 3$ and $k = 1$. By Theorem 2.7, the graph $3K_5(A)$ can be decomposed into six 2-factors, say $C_6, C_7, C_8, \ldots, C_{11}$ Let $\mathcal{C}$ be a collection of triangles defined by

$$\mathcal{C} = \{y_i + C_i : i \in \{6, 7, 8, \ldots, 11\}\}.$$

By Theorem 2.5, we can decompose the graph $4K_6(N_{11} \setminus B)$ into 20 1-factors, say $F_{i,j}$ where $i \in \{1, 2, 3, 4, 5\}$ and $j \in \{1, 2, 3, 4\}$. Let $\mathcal{F}_1$ be a collection of triangles defined by

$$\mathcal{F}_1 = \{y_i + F_{i,j} : i \in \{1, 2, 3, 4, 5\}, j \in \{1, 2, 3, 4\}\}.$$

By Theorem 2.12, we can decompose the graph $3K_{6h}(M_m \setminus A)$ into a collection of triangles $\mathcal{T}_1$ and 37 1-factors, say $H_{i,j}$ and $H'_{p,q}$ where $i \in \{1, 2, 3, 4, 5\}$, $j \in \{3, 4, 5\}$,

$p \in \{1, 2, 3, \ldots, 11\}$ and $q \in \{1, 2\}$. Let $\mathcal{F}_2$ and $\mathcal{F}_3$ be collections of triangles defined by

$$\mathcal{F}_2 = \{x_i + H_{i,j} : i \in \{1, 2, 3, 4, 5\}, j \in \{3, 4, 5\}\}$$

and

$$\mathcal{F}_3 = \{y_p + H'_{p,q} : p \in \{1, 2, 3, \ldots, 11\}, q \in \{1, 2\}\}.$$

Therefore, $(M_m, N_{11}, \mathcal{B}_1 \cup \mathcal{C} \cup \mathcal{T}_1 \cup \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3)$ is a desired GDD.

Case (vi) $h = 2$ and $k = 1$. Note that the graph $3K_{17}(M_{17})$ can be considered as a $\mathsf{TS}(17; 3)$, namely $(M_{17}, \mathcal{B}_3)$. By Theorem 2.7, the graph $4K_{11}(N_{11})$ can be decomposed into 20 2-factors, say $C_1, C_2, C_3, \ldots, C_{17}$ and the three cycles, namely $(y_1y_2y_3y_4y_5y_6y_7y_8y_9y_{10}y_{11})$, $(y_1y_4y_7y_{10}y_2y_5y_8y_{11}y_3y_6y_9)$ and $(y_1y_3y_5y_7y_9y_{11}y_2y_4\ y_6y_8y_{10})$. Let $\mathcal{C}$ be a collection of triangles defined by

$$\mathcal{C} = \{x_i + C_i : i \in \{1, 2, 3, \ldots, 17\}\}.$$

Besides, the graph obtained from the union of above specified three cycles can be decomposed into a collection of triangles $\mathcal{T}$ as follows:

$$\begin{aligned}\mathcal{T} = \{&\{y_6, y_7, y_9\}, \{y_7, y_8, y_{10}\}, \{y_8, y_9, y_{11}\}, \{y_9, y_{10}, y_1\}, \{y_{10}, y_{11}, y_2\},\\ &\{y_{11}, y_1, y_3\}, \{y_1, y_2, y_4\}, \{y_2, y_3, y_5\}, \{y_3, y_4, y_6\}, \{y_4, y_5, y_7\}, \{y_5, y_6, y_8\}\}.\end{aligned}$$

Hence, $(M_{17}, N_{11}, \mathcal{B}_3 \cup \mathcal{C} \cup \mathcal{T})$ is a desired GDD.

Case (vii) $h = 1$ and $k = 0$. By Theorem 2.7, the graph $3K_5(A)$ can be decomposed into six 2-factors, say $C_6, C_7, C_8, \ldots, C_{11}$. Let $\mathcal{C}$ be a collection of triangles defined by

$$\mathcal{C} = \{x_i + C_i : i \in \{6, 7, 8, \ldots, 11\}\}.$$

By Theorem 2.5, we can decompose the graph $3K_6(M_{11} \setminus A)$ into 15 1-factors, say F_i and $F_{p,q}$ where $i, p \in \{1, 2, 3, 4, 5\}$ and $q \in \{2, 3\}$. Let $\mathcal{F}_1$ be a collection of triangles defined by

$$\mathcal{F}_1 = \{x_i + F_i, y_p + F_{p,q} : i, p \in \{1, 2, 3, 4, 5\}, q \in \{2, 3\}\}.$$

Hence, $(M_{11}, N_5, \mathcal{B}_1 \cup \mathcal{C} \cup \mathcal{F}_1)$ is a desired GDD. $\square$

Theorem 4.9. *Let m and n be positive integers such that m and $n \equiv 5 \pmod 6$. Let λ_1, λ_1' and λ_2 be nonnegative integers such that $\lambda_1 \geq \lambda_2$ and $\lambda_1' \geq \lambda_2$. If $\lambda_2 \equiv 0 \pmod 2$ and $\lambda_1 + \lambda_1' + \lambda_2 \equiv 0 \pmod 3$, then there exists a* $\mathsf{GDD}(m, n; \lambda_1, \lambda_1', \lambda_2)$.

Proof. We regard λ_1, λ_1' and λ_2 as integers modulo 6 and examine the necessary conditions to see all possible values of $(\lambda_1, \lambda_1', \lambda_2)$. We display them in Table 4.2.

λ_2	(λ_1, λ_1')					
$\bar{0}$	$(\bar{0},\bar{0})$	$(\bar{0},\bar{3})$	$(\bar{1},\bar{2})$	$(\bar{1},\bar{5})$	$(\bar{2},\bar{4})$	$(\bar{4},\bar{5})$
	$(\bar{3},\bar{3})$	$(\bar{3},\bar{0})$	$(\bar{2},\bar{1})$	$(\bar{5},\bar{1})$	$(\bar{4},\bar{2})$	$(\bar{5},\bar{4})$
$\bar{2}$	$(\bar{0},\bar{1})$	$(\bar{0},\bar{4})$	$(\bar{1},\bar{3})$	$(\bar{2},\bar{2})$	$(\bar{2},\bar{5})$	$(\bar{3},\bar{4})$
	$(\bar{1},\bar{0})$	$(\bar{4},\bar{0})$	$(\bar{3},\bar{1})$	$(\bar{5},\bar{5})$	$(\bar{5},\bar{2})$	$(\bar{4},\bar{3})$
$\bar{4}$	$(\bar{0},\bar{2})$	$(\bar{0},\bar{5})$	$(\bar{1},\bar{1})$	$(\bar{1},\bar{4})$	$(\bar{2},\bar{3})$	$(\bar{3},\bar{5})$
	$(\bar{2},\bar{0})$	$(\bar{5},\bar{0})$	$(\bar{4},\bar{4})$	$(\bar{4},\bar{1})$	$(\bar{3},\bar{2})$	$(\bar{5},\bar{3})$

Table 4.2: All possible values of $(\lambda_1, \lambda_1', \lambda_2)$

Note from condition (iv) in Theorem 1.2 that $\lambda_2 \neq 0$ except the cases when $(\lambda_1, \lambda_1') \in \{(\bar{0},\bar{0}), (\bar{0},\bar{3}), (\bar{3},\bar{0}), (\bar{3},\bar{3})\}$. Since λ_2 is even, we observe that if $\lambda_1 - \lambda_2 \equiv 0$ or $3 \pmod 6$ and $\lambda_1' - \lambda_2 \equiv 0$ or $3 \pmod 6$, then we can obtain our desired GDDs by applying Lemma 3.1 to combine a $\mathsf{TS}(m+n; \lambda_2)$, a $\mathsf{TS}(m; \lambda_1 - \lambda_2)$ and a $\mathsf{TS}(n; \lambda_1' - \lambda_2)$, which exist from Theorem 2.3. Thus, the following cases are done:

(i) $\lambda_1, \lambda_1' \equiv 0$ or $3 \pmod 6$, $\lambda_2 \equiv 0 \pmod 6$,

(ii) $\lambda_1, \lambda_1' \equiv 2$ or $5 \pmod 6$, $\lambda_2 \equiv 2 \pmod 6$ and

(iii) $\lambda_1, \lambda_1' \equiv 1$ or $4 \pmod 6$, $\lambda_2 \equiv 4 \pmod 6$.

For the remaining of our desired GDDs, it suffices to construct only the smallest values of possible $(\lambda_1, \lambda_1', \lambda_2)$ in each case because the larger ones in each case can be simply obtained by applying Lemma 3.1 (i) to combine the smallest GDD with a $\mathsf{TS}(m+n;6a)$ where $a \in \mathbb{N}$, which is equivalent to $\mathsf{GDD}(m,n;6a,6a,6a)$. We use the curve arrow

$$\begin{tikzcd}
(\lambda_1, \lambda_1', \lambda_2) \arrow[r, bend left] & (\lambda_1 + 2, \lambda_1' + 2, \lambda_2 + 2)
\end{tikzcd}$$

to denote that if a $\mathsf{GDD}(m,n;\lambda_1,\lambda_1',\lambda_2)$ exists, then a $\mathsf{GDD}(m,n;\lambda_1+2,\lambda_1'+2,\lambda_2+2)$ exists by applying Lemma 3.1 with a $\mathsf{TS}(m+n;2)$, or, equivalently, a $\mathsf{GDD}(m,n;2,2,2)$. Also, we use the down arrow

$$\begin{tikzcd}
(\lambda_1, \lambda_1', \lambda_2) \arrow[d, Rightarrow] \\
(\lambda_1, \lambda_1' + 3, \lambda_2)
\end{tikzcd}$$

to denote that if a $\mathsf{GDD}(m,n;\lambda_1,\lambda_1',\lambda_2)$ exists, then a $\mathsf{GDD}(m,n;\lambda_1,\lambda_1'+3,\lambda_2)$ exists. Lastly, we use the long right arrow

$$(\lambda_1, \lambda_1', \lambda_2) \Longrightarrow (\lambda_1 + 3, \lambda_1', \lambda_2)$$

to denote that if a $\mathsf{GDD}(m,n;\lambda_1,\lambda_1',\lambda_2)$ exists, then a $\mathsf{GDD}(m,n;\lambda_1+3,\lambda_1',\lambda_2)$ exists by applying Lemma 3.1 with a $\mathsf{TS}(m;3)$, or, equivalently, a $\mathsf{GDD}(m,n;3,0,0)$.

Therefore, the following diagram shows that a $\mathsf{GDD}(m,n;3,4,2)$ obtained from Lemma 4.8 yields all of the smallest GDDs for the remaining cases. These values are the smallest ones because we have that $\lambda_2 \neq 0, \lambda_1 \geq \lambda_2$ and $\lambda_1' \geq \lambda_2$. This completes the proof.

$$\begin{tikzcd}
(3,4,2) \arrow[r, Rightarrow] \arrow[d, Rightarrow] \arrow[rr, bend left] & (6,4,2) \arrow[d, Rightarrow] & (5,6,4) \arrow[r, Rightarrow] \arrow[d, Rightarrow] \arrow[rr, bend left] & (8,6,4) \arrow[d, Rightarrow] & (7,8,6) \arrow[r, Rightarrow] \arrow[d, Rightarrow] & (10,8,6) \arrow[d, Rightarrow] \\
(3,7,2) \arrow[r, Rightarrow] & (6,7,2) & (5,9,4) \arrow[r, Rightarrow] & (8,9,4) & (7,11,6) \arrow[r, Rightarrow] & (10,11,6)
\end{tikzcd}$$

□