CHAPTER IV
EXAMPLES

Our first example shows the case where the growth in equality condition in Corollary 3.5 is equalities with linearly independent continued fractions of simple shape.

Example 1. 1) Consider the continued fractions in $\mathbf{F}$, $\alpha_1 = [x^1, x^3, x^5, x^7, \ldots]$, $\alpha_2 = [x^2, x^4, x^6, x^8 \ldots]$. Corollary 3.5 shows that α_1, α_2 and 1 are linearly independent over $\mathbb{F}_q(x)$.
2) Consider the N continued fractions in $\mathbf{F}$, $\alpha_1 = [x^{N+1}, x^{N^2+1}, x^{N^3+1}, \ldots]$, $\alpha_2 = [x^{N+2}, x^{N^2+2}, x^{N^3+2}, \ldots], \ldots$, $\alpha_N = [x^{2N}, x^{N^2+N}, x^{N^3+N}, \ldots]$. We see that $|a_{n,j+1}| = q^{N^{n+1}+j+1} = q|a_{n,j}|$. Let $\varepsilon > 1$. Choose $N_0 \in \mathbb{N}$ such that for all $n \geq N_0$, we get $n \log n > \varepsilon$. Then we have (3.16) of Corollary 3.4. To verify (3.17) of Corollary 3.4, choose $N_1 > N_0$ such that for all $n \geq N_1$, $q^{N^n+N+1} \geq 2q^{N^2}$. Then

$$\begin{aligned} |a_{n,N}|^{N-1}\left(1+\frac{1}{n}\right) &= \left(q^{N^{n+1}+N}\right)^{N-1}\left(1+\frac{1}{n}\right) \\ &= |a_{n+1,1}|q^{-N^{n+1}+N^2-N-1}\left(1+\frac{1}{n}\right) \\ &< |a_{n+1,1}|. \end{aligned}$$

By Corollary 3.4, $\alpha_1, \alpha_2, \ldots, \alpha_N$ and 1 are linearly independent over $\mathbb{F}_q(x)$.

Our next example is quite interesting. In the case of real numbers, it involves several earlier works, see e.g. [3] and [7], where certain real series are shown to have explicit continued fraction expansions.

Let $\beta > 1$ be a real irrational number and $g \in \mathbb{F}_q[x] \backslash \mathbb{F}_q$. Define

$$\alpha_g(\beta) := \sum_{j=1}^{\infty} (g-1) g^{-[j\beta]} = \sum_{\nu=1}^{\infty} c_\nu g^{-\nu} \in \mathbf{F}, \tag{4.1}$$

where $c_\nu = 0$ if there is no $j \in \mathbb{N}$ with $[j\beta] = \nu$, and $c_\nu = g-1$ if there is $j \in \mathbb{N}$ with $[j\beta] = \nu$. The following auxiliary result is Lemma 1 of [7].

Lemma 4.1. *Let* $[b_0, b_1, b_2, \ldots]$ *be the simple continued fraction of* $1/\beta$ *and* p_n/q_n *its nth convergent. Then the* $c'_\nu s$ *introduced above are determined as follows:*

1. $c_{kb_1} = g - 1$ *for* $k = 1, 2, \ldots, b_2$*, and* $c_\nu = 0$ *for all other* $\nu \leq b_1 b_2 + 1 = q_2$.

2. *If* $n \geq 2$ *and* $q_n < \nu \leq q_{n+1}$*, then* $c_\nu = c_{r_n(\nu)}$*, where* $r_n(\nu)$ *denotes the smallest positive residue of* ν *(mod* q_n*).*

Just as in the case of real continued fractions, [7], the continued fraction expansion of the lacunary series $\alpha_g(\beta)$ in **F** can also be found explicitly as in the following theorem.

Theorem 4.2. *Let* $\beta > 1$ *be a real irrational number. Let* $[b_0, b_1, b_2, \ldots]$ *be the simple continued fraction of* $1/\beta$*,* p_n/q_n *its nth convergent and let* $g \in \mathbb{F}_q[x] \backslash \mathbb{F}_q$*. If*

$$A_0 := gb_0, \qquad A_n := g^{q_{n-2}} \sum_{i=0}^{b_n - 1} g^{iq_{n-1}} \quad (n \geq 1), \tag{4.2}$$

then $\alpha_g(\beta) = [A_0, A_1, A_2, \ldots]$.

Proof. With A_n defined above, for $n \geq 0$, let

$$P_{-2} = 0,\ P_{-1} = 1,\ P_n = A_n P_{n-1} + P_{n-2};\ \ Q_{-2} = 1,\ Q_{-1} = 0,\ Q_n = A_n Q_{n-1} + Q_{n-2}. \tag{4.3}$$

We claim that, for $n \geq 1$,

$$P_n = \sum_{\nu=1}^{q_n} \frac{c_\nu}{g-1} g^{q_n - \nu}, \quad Q_n = \sum_{\nu=1}^{q_n} g^{q_n - \nu}. \tag{4.4}$$

Since the verifications for both P_n and Q_n are the same, we only provide that of the former. When $n = 1$, since $b_0 = 0$, $A_0 = 0$, then $P_0 = A_0 P_{-1} + P_{-2} = 0$, and so $P_1 = A_1 P_0 + P_{-1} = 1$. Since $q_1 = b_1 q_0 + q_{-1} = b_1$, by Lemma 4.1

$$\sum_{\nu=1}^{q_1} \frac{c_\nu}{g-1} g^{q_1 - \nu} = \sum_{\nu=1}^{b_1} \frac{c_\nu}{g-1} g^{b_1 - \nu} = \sum_{\nu=1}^{b_1 - 1} \frac{c_\nu}{g-1} g^{b_1 - \nu} + \frac{c_{b_1}}{g-1} = 0 + \frac{g-1}{g-1} = 1,$$

i.e., the claim holds for $n = 1$. When $n = 2$, we have $q_0 = 1, q_1 = b_1, P_0 = 0$, and $P_1 = 1$. By (4.2), (4.3) and Lemma 4.1,

$$\begin{aligned} P_2 &= \sum_{i=0}^{b_2 - 1} g^{iq_1 + 1} = g^{b_2 b_1 + 1 - b_1} + g^{b_2 b_1 + 1 - 2b_1} + \ldots + g^{b_1 + 1} + g^1 \\ &= g^{q_2 - b_1} + g^{q_2 - 2b_1} + \ldots + g^{q_2 - (b_2 - 1)b_1} + g^{q_2 - b_2 b_1} = \sum_{k=1}^{b_2} g^{q_2 - kb_1} = \sum_{\nu=1}^{q_2} \frac{c_\nu}{g-1} g^{q_2 - \nu}, \end{aligned}$$

which yields (4.4) for $n = 2$. Now let $n \geq 2$ and assume (4.4) is true up to n. Then with (4.3), (4.4) and $q_{n+1} = b_{n+1}q_n + q_{n-1}$, we get

$$P_{n+1} = A_{n+1}P_n + P_{n-1} = \sum_{\lambda=0}^{b_{n+1}-1} \sum_{\nu=1}^{q_n} \frac{c_\nu}{g-1} g^{(1+\lambda)q_n + q_{n-1} - \nu} + \sum_{\nu=1}^{q_{n-1}} \frac{c_\nu}{g-1} g^{b_{n+1}q_n + q_{n-1} - \nu - b_{n+1}q_n}$$

$$= \sum_{\lambda=0}^{b_{n+1}-1} \sum_{\nu=1}^{q_n} \frac{c_\nu}{g-1} g^{q_{n+1} - (\nu + \lambda q_n)} + \sum_{\nu=1}^{q_{n-1}} \frac{c_\nu}{g-1} g^{q_{n+1} - (\nu + b_{n+1}q_n)}$$

$$= \sum_{\mu=1}^{b_{n+1}q_n} \frac{c_\mu}{g-1} g^{q_{n+1}-\mu} + \sum_{\mu=1+b_{n+1}q_n}^{q_{n+1}} \frac{c_\mu}{g-1} g^{q_{n+1}-\mu} = \sum_{\nu=1}^{q_{n+1}} \frac{c_\nu}{g-1} g^{q_{n+1}-\nu},$$

and the claim is verified. Observe that

$$\sum_{\lambda=0}^{b_{n+1}-1} \sum_{\nu=1}^{q_n} \frac{c_\nu}{g-1} g^{q_{n+1} - (\nu + \lambda q_n)} = \sum_{\mu=1}^{b_{n+1}q_n} \frac{c_\mu}{g-1} g^{q_{n+1}-\mu}.$$

On the other hand, since

$$Q_n = 1 + g + \ldots + g^{q_n - 2} + g^{q_n - 1} = \frac{g^{q_n} - 1}{g - 1}$$

and $[0, A_1, A_2, \ldots, A_n] = P_n/Q_n$, we have

$$\left| \alpha_g(\beta) - \frac{P_n}{Q_n} \right| = \left| \alpha_g(\beta) - \frac{g^{q_n}}{g^{q_n} - 1} \sum_{\nu=1}^{q_n} c_\nu g^{-\nu} \right|$$

$$= \left| \left(1 - \frac{g^{q_n}}{g^{q_n} - 1} \right) \sum_{\nu=1}^{q_n} c_\nu g^{-\nu} + \sum_{\nu=q_n+1}^{\infty} c_\nu g^{-\nu} \right| = \left| g^{-q_n} \right| \left| (g-1)\, g^{-b_1} \right| \to 0 \quad (n \to \infty).$$

□

Example 2. Let $\beta > 1$ be a real irrational number, $[b_0, b_1, b_2, \ldots]$ and p_n/q_n be the real simple continued fraction of $1/\beta$ and its nth convergent. Let $g_1, g_2 \in \mathbb{F}_q[x] \backslash \mathbb{F}_q$ be such that $|g_1| > |g_2|$. Assume that the real simple continued fraction of $1/\beta$ has unbounded partial quotients. Let $[A_0, A_1, A_2, \ldots, A_n, \ldots]$ and $C_n(g_1)/D_n(g_1)$ be the continued fraction of $\alpha_{g_1}(\beta)$ and its nth convergent. Let $[B_0, B_1, B_2, \ldots, B_n, \ldots]$ and $C_n(g_2)/D_n(g_2)$ be the continued fraction of $\alpha_{g_2}(\beta)$ and its nth convergent. By Theorem 4.2,

$$A_0 = g_1 b_0 = 0, \quad A_n = \frac{g_1^{q_n} - g_1^{q_{n-2}}}{g_1^{q_{n-1}} - 1}, \quad D_n(g_1) = \frac{g_1^{q_n} - 1}{g_1 - 1} \quad (n \geq 1)$$

and

$$B_0 = g_2 b_0 = 0, \quad B_n = \frac{g_2^{q_n} - g_2^{q_{n-2}}}{g_2^{q_{n-1}} - 1}, \quad D_n(g_2) = \frac{g_2^{q_n} - 1}{g_2 - 1} \quad (n \geq 1).$$

Since the continued fraction of $1/\beta$ has unbounded partial quotients, there is a sequence of positive integers (n_k) such that $b_{n_k+1} \to \infty \quad (k \to \infty)$. Let $n_0 = 0$. Choose $\varepsilon_{n_0} = 1, \varepsilon_{n_k} = q \quad (k \geq 1)$ and for all $k \geq 0$, choose $c_n = q \ (n_k < n < n_{k+1})$. Then we have (3.3) and (3.4) of Theorem 3.3. To verify (3.5) and (3.6), choose

$$\delta_{n_0} = 1, \delta_{n_k} = \frac{|B_{n_k+1}|}{|A_{n_k}|} \quad (k \geq 1)$$

and for all $k \geq 0, \ n_k < n < n_{k+1}$, choose

$$d_n = \frac{|B_{n+1}|}{|A_n|}.$$

Then

$$\begin{aligned}\prod_{i=0}^{k}(d_{n_i+1}d_{n_i+2}\cdots d_{n_{i+1}-1}\delta_{n_{i+1}}) &= \left|\frac{B_2}{A_1}\cdots\frac{B_{n_1}}{A_{n_1-1}}\frac{B_{n_1+1}}{A_{n_1}}\right|\cdot\left|\frac{B_{n_1+2}}{A_{n_1+1}}\cdots\frac{B_{n_2}}{A_{n_2-1}}\frac{B_{n_2+1}}{A_{n_2}}\right|\cdots \\ &\cdots\left|\frac{B_{n_k+2}}{A_{n_k+1}}\cdots\frac{B_{n_{k+1}}}{A_{n_{k+1}-1}}\frac{B_{n_{k+1}+1}}{A_{n_{k+1}}}\right| \\ &= \frac{|D_{n_{k+1}}(g_2)||B_{n_{k+1}+1}|}{|D_{n_{k+1}}(g_1)||B_1|} \\ &= \frac{1}{|B_1|}\left|\frac{g_2}{g_1}\right|^{q_{n_{k+1}}-1}|g_2|^{(B_{n_{k+1}+1}-1)q_{n_{k+1}}+q_{n_{k+1}}-1} \\ &\geq \frac{1}{|B_1|}\left|\frac{g_2}{g_1}\right|^{q_{n_{k+1}}-1}|g_2|^{(B_{n_{k+1}+1}-1)(q_{n_{k+1}}-1)} \\ &= \frac{1}{|B_1|}\left|\frac{g_2^{B_{n_{k+1}+1}}}{g_1}\right|^{q_{n_{k+1}}-1} \\ &\to \infty \quad (k \to \infty).\end{aligned}$$

By Theorem 3.3, $\alpha_{g_1}(\beta), \alpha_{g_2}(\beta)$ and 1 are linearly independent over $\mathbb{F}_q(x)$.

Next we show that $\alpha_{g_1}(\beta), \alpha_{g_2}(\beta)$ are algebraically independent over $\mathbb{F}_q(x)$. Since $|D_n(g_1)| = |g_1|^{q_n-1} \geq q^{q_n-1}|g_2|^{q_n-1}$, choose $f_2(n) = q^{q_n-1} \to \infty \quad (n \to \infty)$. Then $|D_n(g_1)| \geq f_2(n)|D_n(g_2)|$. Since the continued fraction of $1/\beta$ has unbounded partial quotients, there is a subsequence (n_k) such that $b_{n_k} \to \infty \quad (k \to \infty)$. Take $f_1(k) = \left[(b_{n_k+1} - 1)\log_{|g_1|}|g_2|\right]$. Then $f_1(k) \to \infty \quad (k \to \infty)$. Moreover we obtain

$$|D_{n_k}(g_1)|^{f_1(k)} = |g_1|^{(q_{n_k}-1)f_1(k)} \leq |g_1|^{q_{n_k}(b_{n_k+1}-1)} \leq |g_1|^{q_{n_k+1}-q_{n_k}} = |A_{n_k+1}|.$$

Theorem 3.16 implies that $\alpha_{g_1}(\beta)$ and $\alpha_{g_2}(\beta)$ are algebraically independent over $\mathbb{F}_q(x)$.

The last example comes from one of our recent results, Theorem 8 of [27], which displays another type of lacunary series having explicit continued fractions.

Theorem 4.3. *Let I be a fixed positive integer, $\{k_i\}_{i\geq 1}$ a sequence of positive integers, $\{a_i\}_{i\geq I}$ a sequence of nonzero polynomials over $\mathbb{F}_q$, subject to the condition that if $I=1$, then a_1 and those a_i $(i\geq 2)$ for which $k_i=2$ are non-constant polynomials over $\mathbb{F}_q$. Let the sequence $\{Q_i\}_{i\geq 1}$ be defined by*

$$Q_1=1,\ Q_2,\ Q_3,...,Q_I\in\mathbb{F}_q[x]\backslash\mathbb{F}_q;\quad Q_u=a_{u-1}Q_{u-1}^{k_{u-1}}Q_{u-2}^{k_{u-2}}...Q_{u-I}^{k_{u-I}}\quad (u\geq I+1),$$

and let

$$C(u)=\sum_{i=1}^{u}\frac{1}{Q_i}\qquad (u\in\mathbb{N}).$$

Assume that

1. *if $I\geq 2$, then $Q_2|\ Q_3|\ ...|\ Q_I$;*
2. *$k_i\geq 2$ $(i\geq I)$.*

If $C(u)=[b_0;b_1,...,b_n]$ $(u\geq I+1)$, then there exists $\beta\in\mathbb{F}_q\backslash\{0\}$ such that

$$C(u+1)=[b_0;b_1,...,b_n,\beta s_u,-b_n,-b_{n-1},...,-b_1],$$

where $s_u=\frac{a_uQ_u^{k_u-1}}{a_{u-1}Q_{u-I}^{k_{u-I}}}$. Moreover, if $Q_2,\ Q_3,\dots,Q_I$ and all polynomials a_i $(i\geq I)$ are monic, then $\beta=(-1)^n\ \{l.c.(b_nb_{n-1}\cdots b_1)\}^{-2}$, where $l.c.(p)$ denotes the leading coefficient of $p(x)\in\mathbb{F}_q[x]$.

Example 3. Let $j=1,2$. Let $E_j\in\mathbb{F}_q[x]$ be monic with $\deg E_j\geq 1$ and $|E_1|>|E_2|$. Let $I=1$, $k_\nu=\nu+1$ $(\nu\geq 1)$. Let $a_{1,j}=E_j^2$ and $a_{\nu,j}=1$ $(\nu\geq 2)$. Let $Q_{1,j}=1$. Define $Q_{u,j}=a_{u-1,j}Q_{u-1,j}^{k_{u-1}}$ and $s_{u,j}=\frac{a_uQ_u^{k_u-1}}{a_{u-1}Q_{u-1}^{k_{u-1}}}$ $(u\geq 2)$. Then

$$\begin{aligned}
Q_{2,j}&=E_j^{2!}, & s_{2,j}&=E_j^{2!}\\
Q_{3,j}&=E_j^{3!}, & s_{3,j}&=E_j^{3!2}\\
Q_{4,j}&=E_j^{4!}, & s_{4,j}&=E_j^{4!3}\\
&\vdots & &\vdots\\
Q_{u,j}&=E_j^{u!}, & s_{u,j}&=E_j^{u!(u-1)}.
\end{aligned}$$

Define $C_j(1)=1$, $C_j(u)=\sum_{i=1}^{u}\frac{1}{Q_{1,j}}$ and $C_j(\infty)=\sum_{i=1}^{\infty}\frac{1}{Q_{i,j}}$ $(j=1,2)$.
Here $C_j(1)=1=[1]$, $C_j(2)=1+1/E_j=[1,E_j]$. Thus, by Theorem 4.3, we have

$$\begin{aligned}
&C_j(3)=[1,E_j^{2!},-E_j^{2!},-E_j^{2!}],\\
&C_j(4)=[1,E_j^{2!},-E_j^{2!},-E_j^{2!},-E_j^{3!2},E_j^{2!},E_j^{2!},-E_j^{2!}],\\
&C_j(5)=[1,E_j^{2!},-E_j^{2!},-E_j^{2!},-E_j^{3!2},E_j^{2!},E_j^{2!},-E_j^{2!},-E_j^{4!3},\\
&\qquad E_j^{2!},-E_j^{2!},-E_j^{2!},E_j^{3!2},E_j^{2!},E_j^{2!},-E_j^{2!}],\\
&\qquad\vdots
\end{aligned}$$

Let a subsequence $(n_k)_{k=0}^{\infty}$ be defined by $n_0=0, n_1=1$, and $n_k=2n_{k-1}+1$ $(k\ge 2.)$
Write

$$\frac{P_{u,j}}{Q_{u,j}}:=C_j(u)=\frac{E_j^{u!}+E_j^{u!-2!}+\ldots+1}{E_j^{u!}}.$$

We see that $\gcd(P_{u,j},Q_{u,j})=1$ so that $P_{u,j}/Q_{u,j}$ is associated with the n_{u-1}th continued fraction of $C_j(\infty)$. In general, write $C_1(\infty)=[\alpha_0,\alpha_1,\alpha_2,\ldots,\alpha_n,\ldots]$, and $C_2(\infty)=[\beta_0,\beta_1,\beta_2,\ldots,\beta_n,\ldots]$. Choose $\varepsilon_{n_k}=q=c_n$ for all $k\ge 0$ and $n_k<n<n_{k+1}$. Then we have (3.3) and (3.4) of Theorem 3.3. To verify (3.5) and (3.6), for all $k\ge 0$ and $n_k<n<n_{k+1}$, choose

$$\delta_{n_k}=\frac{|\beta_{n_k+1}|}{|\alpha_{n_k}|}\quad\text{and}\quad d_n=\frac{|\beta_{n+1}|}{|\alpha_n|}.$$

Then

$$\begin{aligned}
\prod_{i=0}^{N}(d_{n_i+1}d_{n_i+2}\cdots d_{n_{i+1}-1}\delta_{n_{i+1}}) &= \left|\frac{\beta_2}{\alpha_1}\right|\cdot\left|\frac{\beta_3}{\alpha_2}\frac{\beta_4}{\alpha_3}\right|\cdot\left|\frac{\beta_5}{\alpha_4}\frac{\beta_6}{\alpha_5}\frac{\beta_7}{\alpha_6}\frac{\beta_8}{\alpha_7}\right|\cdots\\
&\quad\cdots\left|\frac{\beta_{n_N+2}}{\alpha_{n_N+1}}\cdots\frac{\beta_{n_{N+1}-1}}{\alpha_{n_{N+1}-1}}\frac{\beta_{n_{N+1}}}{\alpha_{n_{N+1}}}\right|_{i=N}\\
&=\frac{|Q_{N+2,2}||s_{N+2,2}|}{|Q_{N+2,1}||E_2|^{2!}}=\frac{1}{|E_2|^{2!}}\left(\frac{|E_2|^{(N+3)!-1}}{|E_1|}\right)^{(N+2)!}\\
&\to\infty\quad(N\to\infty).
\end{aligned}$$

By Theorem 3.3, $C_1(\infty), C_2(\infty)$ and 1 are linearly independent.

We have

$$|Q_{u,j}|=|E_j|^{u!}\to\infty\quad(u\to\infty),$$

and

$$\left|C_j(\infty)-\frac{P_{u,j}}{Q_{u,j}}\right|=\left|\sum_{i=u+1}^{\infty}\frac{1}{Q_{i,j}}\right|=\frac{1}{|Q_{u+1,j}|}=\frac{1}{|E_j|}^{(u+1)!}.$$

Then

$$\frac{|C_1(\infty) - P_{u,1}/Q_{u,1}|}{|C_2(\infty) - P_{u,2}/Q_{u,2}|} = \left|\frac{E_2}{E_1}\right|^{(u+1)!} \to 0 \quad (u \to 0).$$

To verify (3.30) of Theorem 3.15, let $M > 0$. Since $(u+1) \to \infty \quad (u \to \infty)$, there exists N_0 such that for all $u \geq N_0$

$$|E_1|^{u+1} > |E_1|^M \text{ and } |E_2|^{u+1} > |E_1 E_2|^M.$$

Then, for all $u \geq N_0$,

$$\frac{1}{|E_1|^{\frac{(u+1)!}{u!}}} < \frac{1}{|E_1|^M} \text{ and } \frac{1}{|E_2|^{\frac{(u+1)!}{u!}}} < \frac{1}{|E_1 E_2|^M}$$

and so

$$\left|C_1(\infty) - \frac{P_{u,1}}{Q_{u,1}}\right| < \frac{1}{|Q_{u,1}|^M} \text{ and } \left|C_2(\infty) - \frac{P_{u,2}}{Q_{u,2}}\right| < \frac{1}{|Q_{u,1} Q_{u,2}|^M}.$$

By Theorem 3.15, $C_1(\infty), C_2(\infty)$ are algebraically independent over $\mathbb{F}_q(x)$.